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p-adic Estimates of the Number of Permutation Representations

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Dedicated to Professor Tomoyuki Yoshida on the occasion of his 70-th birthday

Abstract

Let p be a prime, and let G be a finite abelian p-group of type $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum \lambda_i = s$. Set $u = \max\{\lambda_1, [(s+1)/2]\}$ and $v = s - u$. For each nonnegative integer n, let $h_n(G)$ be the number of homomorphisms from G to the symmetric group S_n on n letters. Except for the case where $p = 2$ and $u + \delta_{v0} \leq v + 1$, δ the Kronecker delta, or $p = 3$ and $u = v \geq 1$, there exist p-adic analytic functions $f_r(X)$ for $r = 0, 1, \dots, p^{u+1} - 1$ and a polynomial $\eta(X)$ with integer coefficients such that for any nonnegative integer y, $h_{p^{u+1}y+r}(G) = p^{\{\sum_{j=1}^{u} p^j - (u-v)\}y} f_r(y) \prod_{j=1}^{y} \eta(j)$ and $\mathrm{ord}_p(h_{p^{u+1}y+r}(G)) = \{\sum_{j=1}^{u} p^j - (u - v)\}y + \mathrm{ord}_p(f_r(y))$. If $p = 2$, $\lambda_3 = 0$, and $u = v \geq 1$ or if $p = 3$ and $u = v \geq 1$, then $h_n(G)$ has analogous properties. Under the assumption that $\lambda_3 = 0$, some results for the number of permutation representations of G in the wreath product of a cyclic group of order p with S_n are also presented.

1 Introduction

Let A be a group which contains only a finite number of subgroups of index d for each positive integer d, and let $h_n(A)$ be the number of homomorphisms from A to the symmetric group S_n on n letters. We set $h_0(A) = 1$ and define a formal power series $E_A(X)$ by

$$E_A(X) = \sum_{n=0}^{\infty} \frac{h_n(A)}{n!} X^n.$$

According to Wohlfahrt [21], such an exponential generating function is expressed by

$$E_A(X) = \exp\left(\sum_{B \leq_f A} \frac{1}{|A : B|} X^{|A:B|} \right), \tag{1.1}$$

where the summation $\sum_{B \leq_f A}$ runs over all subgroups B of A of finite index $|A : B|$.

Throughout the paper, p stands for a prime. Given a nonzero integer a, let $\mathrm{ord}_p(a)$ be the exponent of p in the decomposition of a into prime factors. The Wohlfahrt formula (1.1) is useful in exploring $\mathrm{ord}_p(h_n(A))$. For a nonnegative integer s, we denote by C_{p^s} a cyclic

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group of order p^s. By Eq.(1.1), $E_{C_{p^s}}(X) = \exp(\sum_{k=0}^{s} X^{p^k}/p^k)$. For a real number x, let $[x]$ be the largest integer not exceeding x. As a property of $h_n(C_{p^s})$, it is known that

$$\mathrm{ord}_p(h_n(C_{p^s})) \geq \sum_{j=1}^{s} \left[\frac{n}{p^j}\right] - s\left[\frac{n}{p^{s+1}}\right]$$

for any nonnegative integer n, or equivalently,

$$\mathrm{ord}_p(h_{p^{s+1}y+r}(C_{p^s})) \geq \left(\sum_{j=1}^{s} p^j - s\right) y + \mathrm{ord}_p(r!)$$

for any nonnegative integers y and r with $r < p^{s+1}$ (see [1, 4, 5, 7, 9, 10]).

We denote by $\mathbb{Z}_p$ the ring of p-adic integers. Given $x = \sum_{i=0}^{\infty} x_i p^i \in \mathbb{Z}_p - \{0\}$ with $0 \leq x_i \leq p-1$, let $\mathrm{ord}_p(x)$ be the first index i such that $x_i \neq 0$. We define $\mathrm{ord}_p(0)$ to be the symbol ∞ for the sake of convenience. Let $\mathbb{Z}_p[[X]]$ be the ring of formal power series in an indeterminate X with p-adic integer coefficients and $\mathbb{Z}_p\langle X\rangle$ the subring consisting of all $\sum_{n=0}^{\infty} a_n X^n \in \mathbb{Z}_p[[X]]$ such that $\mathrm{ord}_p(a_n) \to \infty$ $(n \to \infty)$.

There exist p-adic analytic functions $\eta(X)$ and $f_r(X)$ for $r = 0, 1, \dots, p^2 - 1$ contained in $\mathbb{Z}_p\langle X\rangle$ such that for any nonnegative integer y,

$$h_{p^2y+r}(C_p) = p^{(p-1)y} f_r(y) \prod_{j=1}^{y} \eta(j) \tag{1.2}$$

and $\mathrm{ord}_p(h_{p^2y+r}(C_p)) = (p-1)y + \mathrm{ord}_p(f_r(y))$ (cf. [8, 15]). The purpose of this paper is to give an extension of this fact. Let G be a finite abelian p-group. Apart from certain exceptional cases, we are successful in finding a p-adic estimate of $h_n(G)$ such as Eq.(1.2). The formal power series $E_p(X) := \exp(\sum_{k=0}^{\infty} X^{p^k}/p^k)$, whose coefficients are p-adic integers, is called the Artin-Hasse exponential (see, *e.g.*, [11, Chapter IV §2] and [16, VII.2]). Using $E_p(X)$ and $E_{C_{p^s}}(X)$, Conrad [2] presented some p-adic properties of $h_n(C_{p^s})$. We attempt to adapt his methods for the study of $h_n(G)$ on the basis of some facts given in [12, 20].

A sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers containing only finitely many nonzero terms with $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\sum \lambda_i = s$ is called the type of a finite abelian p-group G of order p^s if G is isomorphic to the direct product

$$C_{p^{\lambda_1}} \times C_{p^{\lambda_2}} \times \cdots$$

of cyclic p-groups $C_{p^{\lambda_i}}$ of order p^{λ_i} for $i = 1, 2, \dots$.

From now on, we suppose that G is a finite abelian p-group of order p^s and of type $\lambda = (\lambda_1, \lambda_2, \dots)$. Incidentally, $\sum \lambda_i = s$ and λ is a partition of s. We define

$$u := \max\left\{\lambda_1, \left[\frac{s+1}{2}\right]\right\} \quad \text{and} \quad v := \min\left\{s - \lambda_1, \left[\frac{s}{2}\right]\right\}.$$

Obviously, $u \geq v \geq 0$. Since $s = [s/2] + [(s+1)/2]$, it follows that $s = u + v$. We set

$$\tau_p^{(u,v)}(n) = \begin{cases} \displaystyle\sum_{j=1}^{u} \left[\frac{n}{2^j}\right] + \left[\frac{n}{2^{u+2}}\right] - \left[\frac{n}{2^{u+3}}\right] & \text{if } p = 2 \text{ and } u = v \geq 1, \\ \displaystyle\sum_{j=1}^{u} \left[\frac{n}{p^j}\right] - (u-v)\left[\frac{n}{p^{u+1}}\right] & \text{otherwise} \end{cases}$$

for each nonnegative integer n, and set

$$\kappa_p(u,v) = \begin{cases} u+3 & \text{if } p=2 \text{ and } u=v\geq 1, \\ u+2 & \text{if } p=2 \text{ and } u=v+1\geq 2, \text{ or if } p=3 \text{ and } u=v\geq 1, \\ u+1 & \text{otherwise.} \end{cases}$$

The following main theorem of this paper is concerned with a p-adic estimate of $h_n(G)$.

Theorem 1.1 *Except for the case where $p=2$ and either $u+\delta_{v0}=v+1$ or $\lambda_3\geq 1$ and $u=v\geq 2$, there exist p-adic analytic functions $f_r(X)$ for $r=0, 1, \ldots, p^{\kappa_p(u,v)}-1$ contained in $\mathbb{Z}_p\langle X\rangle$ and a polynomial $\eta(X)$ of degree $p^{\kappa_p(u,v)}-1$ with integer coefficients such that for any nonnegative integer y,*

$$h_{p^{\kappa_p(u,v)}y+r}(G) = p^{\tau_p^{(u,v)}(p^{\kappa_p(u,v)}y)} f_r(y) \prod_{j=1}^{y} \eta(j)$$

and

$$\mathrm{ord}_p(h_{p^{\kappa_p(u,v)}y+r}(G)) = \tau_p^{(u,v)}(p^{\kappa_p(u,v)}y) + \mathrm{ord}_p(f_r(y)).$$

Moreover, $\mathrm{ord}_p(h_n(G)) \geq \tau_p^{(u,v)}(n)$ *for any nonnegative integer n.*

The second assertion of Theorem 1.1 is presented in [13, Theorem 25, Theorem 26] and [20, Theorem 1.1]. By exploring $\mathrm{ord}_p(h_n(G))$, we confirm [13, Theorem 25] and [20, Corollary 8.2] (cf. [18, Theorem 1.4]) with relation to the case where equality holds, namely,

Theorem 1.2 *The following statements hold.*

(1) *Except for the case where $p=2$ and $u=v\geq 1$, $\mathrm{ord}_p(h_n(G)) = \tau_p^{(u,v)}(n)$ for each nonnegative integer n such that $n\equiv 0 \pmod{p^{u+1}}$.*

(2) *Assume that $p=2$ and that $u=v\geq 1$.*

  (i) *If $\lambda_3=0$, then $\mathrm{ord}_2(h_n(G)) = \tau_2^{(u,v)}(n)$ for each nonnegative integer n such that $n\equiv 0,\ 2^{u+1}$, or $2^{u+2} \pmod{2^{u+3}}$.*

  (ii) *If $\lambda_3\geq 1$, then $\mathrm{ord}_2(h_n(G)) = \tau_2^{(u,v)}(n)$ for each nonnegative integer n such that $n\equiv 0,\ 2^{u+1}$, or $2^{u+1}+2^{u+2} \pmod{2^{u+3}}$.*

We set $P = C_{p^u}\times C_{p^v}$. When $\lambda_3=0$, G is identified with P. In this paper we also investigate the number of permutation representations of P in wreath products. Let $C_p^{(n)}$ denote the direct product of n copies of C_p, and let $C_p \wr S_n$ denote the semidirect product

$$C_p^{(n)} \rtimes S_n = \{(x_1, x_2, \ldots, x_n)\sigma \mid (x_1, x_2, \ldots, x_n)\in C_p^{(n)} \text{ and } \sigma\in S_n\}$$

with multiplication given by

$$(g_1, \ldots, g_n)\sigma(h_1, \ldots, h_n)\tau = (g_1h_{\sigma^{-1}(1)}, \ldots, g_nh_{\sigma^{-1}(n)})\sigma\tau$$

which is called the wreath product of C_p with S_n. Let $h_n(A; C_p)$ be the number of homomorphisms from A to $C_p\wr S_n$, and set $h_0(A;C_p)=1$. By [14, Corollary 1],

$$E_A(X;C_p) := \sum_{n=0}^{\infty} \frac{h_n(A;C_p)}{p^n n!} X^n = \exp\left(\sum_{B\leq_f A} \frac{|\mathrm{Hom}(B,C_p)|}{p|A:B|} X^{|A:B|}\right), \tag{1.3}$$

where $|\mathrm{Hom}(B, C_p)|$ is the number of homomorphisms from B to C_p (cf. [17, Corollary 3.1]).

When $v = 0$, the p-adic properties of $h_n(P; C_p)$ are explored in [12]. For example,

$$\mathrm{ord}_p(h_n(C_p; C_p)) = n - \left[\frac{n}{p}\right] \quad \text{and} \quad \mathrm{ord}_p(h_n(C_{p^s}; C_p)) \geq \sum_{j=0}^{s-1} \left[\frac{n}{p^j}\right] - s\left[\frac{n}{p^s}\right] \quad (s > 1)$$

for any nonnegative integer n. Although it is not easy to handle the exponential formula of $E_G(X; C_p)$, we successfully make use of Eq.(1.3) with $A = P$ to analyze $h_n(P; C_p)$; see Section 7 for $E_P(X; C_p)$. The behaviour of $h_n(P; C_p)$ is similar to that of $h_n(P)$. In particular, we obtain sharp bounds for $\mathrm{ord}_p(h_n(P; C_p))$, namely,

Theorem 1.3 *The following statements hold.*

(1) *Assume that either $p \geq 2$ and $u \geq v + 1 \geq 1$ or $p > 2$ and $u = v \geq 1$. Then*

$$\mathrm{ord}_p(h_n(P; C_p)) \geq \sum_{j=0}^{u-1} \left[\frac{n}{p^j}\right] - (u - v)\left[\frac{n}{p^u}\right]$$

for any nonnegative integer n, and equality holds if $n \equiv 0 \pmod{p^u}$.

(2) *Assume that $p = 2$ and $u = v \geq 1$. Then*

$$\mathrm{ord}_2(h_n(P; C_2)) \geq \sum_{j=0}^{u-1} \left[\frac{n}{2^j}\right] + \left[\frac{n}{2^{u+1}}\right] - \left[\frac{n}{2^{u+2}}\right]$$

for any nonnegative integer n, and equality holds if $n \equiv 0$, 2^u, or $2^{u+1} \pmod{2^{u+2}}$.

The paper is organized as follows. In Sections 2–4 we explore $h_n(P)$ with $P = C_{p^u} \times C_{p^v}$ under the assumption that $\lambda_3 = 0$. Section 5 is devoted to the research of $h_n(G)$ under the assumption that λ_3 is not necessarily 0. In Section 6 we derive Theorems 1.1 and 1.2 from a series of theorems given in Sections 3–5. Section 7 is devoted to the study of $h_n(P; C_p)$. A series of theorems on a p-adic estimate of $h_n(P; C_p)$ brings Theorem 1.3. In Section 8 we present a p-adic estimate of $h_n(P; C_p)$ which is inspired by Theorem 1.1. The behaviour of the p-adic power series $E_G(X)$ and $E_P(X; C_p)$ is also analyzed. We conclude the paper with a theorem on their discs of convergence which includes [20, Corollary 1.1].

2 Preliminaries

In this section we explore the Wohlfahrt formula (1.1) with $A = P = C_{p^u} \times C_{p^v}$, and prepare fundamental facts for the p-adic properties of $h_n(P)$. Given an integer k, we set

$$[k]_p = \begin{cases} \displaystyle\sum_{j=0}^{k-1} p^j & \text{if } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each nonnegative integer k, let $N_P(k)$ be the number of subgroups of order p^k in P. Then by [19, Proposition 5.3] ([20, Proposition 3.2]),

$$N_P(k) = \begin{cases} [k+1]_p & \text{if } 0 \leq k < v, \\ [v+1]_p & \text{if } v \leq k \leq u, \\ [u+v-k+1]_p & \text{if } u < k \leq u+v. \end{cases}$$

Since $N_P(k) = N_P(u+v-k)$ for any k with $0 \le k \le u+v$, it follows from Eq.(1.1) that

$$\sum_{n=0}^{\infty} \frac{h_n(P)}{n!} X^n = \exp\left(\sum_{k=0}^{v-1} \frac{[k+1]_p}{p^k} X^{p^k} + \sum_{k=v}^{u} \frac{[v+1]_p}{p^k} X^{p^k} + \sum_{k=u+1}^{u+v} \frac{[u+v-k+1]_p}{p^k} X^{p^k} \right).$$

Remark 2.1 Concerning the above formula, the denominator $p^n n!$ on the left side of the first formula in [19, Proposition 5.5] should be replaced by $n!$.

Definition 2.2 We define a formal power series $E_p^{(v)}(X) = \sum_{n=0}^{\infty} c_n^{(v)} X^n$ by

$$E_p^{(v)}(X) = \sum_{n=0}^{\infty} c_n^{(v)} X^n = \exp\left(\sum_{k=0}^{v-1} \frac{[k+1]_p}{p^k} X^{p^k} + \sum_{k=v}^{\infty} \frac{[v+1]_p}{p^k} X^{p^k} \right).$$

Proposition 2.3 *For any nonnegative integer n, $\mathrm{ord}_p(c_n^{(v)}) \ge 0$. Equivalently,*

$$E_p^{(v)}(X) - 1 \in X\mathbb{Z}_p[[X]].$$

Proof. The proposition follows from [3, Proposition 1] (see also [11, p. 97, Exercise 18] and [16, XII.2.3, Theorem (Dieudonnè-Dwork)]). □

We use Proposition 2.3 without mention. Observe that

$$\sum_{n=0}^{\infty} \frac{h_n(P)}{n!} X^n = \left(\sum_{n=0}^{\infty} c_n^{(v)} X^n \right) \exp\left(-\sum_{i=0}^{\infty} \frac{[v+1]_p - [v-i]_p}{p^{u+i+1}} X^{p^{u+i+1}} \right).$$

Let r be a nonnegative integer less than p^{u+1}. The above formula implies that

$$\sum_{y=0}^{\infty} \frac{h_{p^{u+1}y+r}(P)}{(p^{u+1}y+r)!} X^{p^{u+1}y} = \left(\sum_{j=0}^{\infty} c_{p^{u+1}j+r}^{(v)} X^{p^{u+1}j} \right) \times \exp\left(-\sum_{i=0}^{\infty} \frac{[v+1]_p - [v-i]_p}{p^{u+i+1}} X^{p^{u+i+1}} \right), \tag{2.1}$$

because

$$\exp\left(-\sum_{i=0}^{\infty} \frac{[v+1]_p - [v-i]_p}{p^{u+i+1}} X^{p^{u+i+1}} \right) - 1 \in X^{p^{u+1}} \mathbb{Q}_p[[X]],$$

where $\mathbb{Q}_p$ is the field of p-adic numbers. Substituting $-p^{u-v+1}X$ for $X^{p^{u+1}}$, we have

$$\sum_{y=0}^{\infty} \frac{h_{p^{u+1}y+r}(P)}{(p^{u+1}y+r)!} (-p^{u-v+1}X)^y = \exp(X) \left(\sum_{j=0}^{\infty} c_{p^{u+1}j+r}^{(v)} (-p^{u-v+1}X)^j \right) \times \exp\left(-\sum_{i=1}^{\infty} \frac{[v+1]_p - [v-i]_p}{p^{u+i+1}} (-p^{u-v+1}X)^{p^i} \right). \tag{2.2}$$

The p-adic properties of $h_n(P)$ are explored on the basis of Eqs.(2.1) and (2.2), including Theorems 1.1 and 1.2. In order to analyze $h_n(P)$, we require three fundamental lemmas. The following lemma is well-known (see [11, p. 7, Exercise 14] and [16, V.3.1., Lemma]).

Lemma 2.4 *Let n be a positive integer, and suppose that $n = n_0 + n_1 p + n_2 p^2 + \cdots$, where $n_0, n_1, n_2, \ldots$ are nonnegative integers less than p. Set $s_p(n) = n_0 + n_1 + n_2 + \cdots$. Then*

$$\operatorname{ord}_p(n!) = \sum_{j=1}^{\infty} \left[\frac{n}{p^j} \right] = \frac{n - s_p(n)}{p-1} \leq \frac{n-1}{p-1}.$$

Given $g(X) = \sum_{n=0}^{\infty} g_n X^n \in \mathbb{Z}_p[[X]]$, we denote by $g(X) + p^{k_1} X^{k_2} \mathbb{Z}_p\langle X \rangle$, where k_1 and k_2 are nonnegative integers, the set of all formal power series $f(X) = \sum_{n=0}^{\infty} f_n X^n$ such that $f(X) - g(X) \in p^{k_1} X^{k_2} \mathbb{Z}_p\langle X \rangle$. We quote [12, Lemma 3.2, Lemma 3.3], namely,

Lemma 2.5 *Let k be a positive integer, and let a be a p-adic integer such that $\operatorname{ord}_p(a) = k$. Except for the case where $p = 2$ and $k = 1$,*

$$\exp(aX) \in 1 + aX + \frac{a^2}{2} X^2 + \frac{a^3}{6} X^3 + p^{2k+1} X^4 \mathbb{Z}_p\langle X \rangle.$$

Lemma 2.6 *Let $\sum_{n=0}^{\ell} m_n X^n$ be a polynomial of degree ℓ with coefficients in $\mathbb{Z}_p$, and let $\sum_{n=1}^{\infty} w_n X^n \in p^k X \mathbb{Z}_p\langle X \rangle$, k a nonnegative integer. Define a sequence $\{d_n\}_{n=0}^{\infty}$ by $d_0 = m_0$ and $d_n = m_n + w_n$ for all positive integers n, where $m_{\ell+1} = m_{\ell+2} = \cdots = 0$. Then there exists a p-adic analytic function $g(X)$ contained in $\mathbb{Z}_p\langle X \rangle$ such that*

$$\sum_{n=0}^{\infty} \frac{g(n)}{n!} X^n = \exp(X) \sum_{n=0}^{\infty} d_n X^n \quad \textit{and} \quad g(X) \in \sum_{i=0}^{\ell} m_i X^{\underline{i}} + p^k X \mathbb{Z}_p\langle X \rangle,$$

where

$$X^{\underline{i}} = \begin{cases} X(X-1)\cdots(X-i+1) & \textit{if } i \geq 1, \\ 1 & \textit{if } i = 0. \end{cases}$$

3 p-adic properties

The proof of Theorem 1.1 requires a series of theorems on a p-adic estimate of $h_n(G)$. In this section we study some p-adic properties of $h_n(P)$ obtained by Eq.(2.2). Define

$$F_p^{(u,v)}(X) := \exp\left(-\sum_{i=1}^{\infty} \frac{[v+1]_p - [v-i]_p}{p^{u+i+1}} (-p^{u-v+1} X)^{p^i} \right),$$

so that by Eqs.(2.1) and (2.2),

$$\sum_{y=0}^{\infty} \frac{h_{p^{u+1}y+r}(P)}{(p^{u+1}y+r)!} X^{p^{u+1}y} = \left(\sum_{j=0}^{\infty} c^{(v)}_{p^{u+1}j+r} X^{p^{u+1}j} \right) \times \exp\left(-\frac{1}{p^{u-v+1}} X^{p^{u+1}} \right) F_p^{(u,v)}\left(-\frac{1}{p^{u-v+1}} X^{p^{u+1}} \right) \tag{3.1}$$

and

$$\sum_{y=0}^{\infty} \frac{h_{p^{u+1}y+r}(P)}{(p^{u+1}y+r)!} (-p^{u-v+1} X)^y = \exp(X) \left(\sum_{j=0}^{\infty} c^{(v)}_{p^{u+1}j+r} (-p^{u-v+1} X)^j \right) F_p^{(u,v)}(X). \tag{3.2}$$

Remark 3.1 When $v = 0$, Eq.(3.2) was given by Conrad [2].

The following proposition is a model for the p-adic properties of $h_n(P)$.

Proposition 3.2 *Let $a_1, a_2, \ldots, a_\ell$ be p-adic integers contained in $p\mathbb{Z}_p$, and assume that*

$$F_p^{(u,v)}(X) \in 1 + \sum_{i=1}^{\ell} a_i X^i + p^{u-v+2} X \mathbb{Z}_p\langle X \rangle.$$

Let r be a nonnegative integer less than p^{u+1}. Then there exists a p-adic analytic function $g_r(X)$ contained in $\mathbb{Z}_p\langle X\rangle$ such that

$$g_r(y) = \frac{h_{p^{u+1}y+r}(P)}{(p^{u+1}y+r)!}(-p^{u-v+1})^y y! \tag{3.3}$$

for any nonnegative integer y and

$$g_r(X) \in c_r^{(v)}\left(1 + \sum_{i=1}^{\ell} a_i X^{\underline{i}}\right) - c_{p^{u+1}+r}^{(v)} p^{u-v+1} X + p^{u-v+2} X \mathbb{Z}_p\langle X \rangle.$$

Proof. The proposition follows from Lemma 2.6 and Eq.(3.2). □

Under the hypotheses of Proposition 3.2, it follows from Lemma 2.4 and Eq.(3.3) that

$$\mathrm{ord}_p(h_{p^{u+1}y+r}(P)) \geq \sum_{j=1}^{u}\left[\frac{p^{u+1}y+r}{p^j}\right] - (u-v)y = \left\{\sum_{j=1}^{u} p^j - (u-v)\right\} y + \mathrm{ord}_p(r!).$$

From Proposition 3.2, we realize that an investigation of $F_p^{(u,v)}(X)$ might bring a nice p-adic estimate of $h_n(P)$. Indeed, some results are obtained in this manner. Define

$$\overline{F}_p^{(u,v)}(X) := \exp\left(-\sum_{i=2}^{\infty} \frac{[v+1]_p - [v-i]_p}{p^{u+i+1}}(-p^{u-v+1}X)^{p^i}\right),$$

so that

$$F_p^{(u,v)}(X) = \exp\left(-\frac{[v+1]_p - [v-1]_p}{p^{u+2}}(-p^{u-v+1}X)^p\right)\overline{F}_p^{(u,v)}(X).$$

By exploring these functions, we present a series of theorems on a p-adic estimate of $h_n(P)$.

Apart from the case where either $p = 2$ and $u + \delta_{v0} \leq v + 2$ or $p = 3$ and $u = v \geq 1$, the formal power series $F_p^{(u,v)}(X)$ satisfies the condition required in Proposition 3.2.

We begin with the study of p-adic properties of $h_n(P)$ under the assumption that $p \geq 3$.

Lemma 3.3 *Suppose that $p \geq 3$. If i is an integer greater than 1, then*

$$\mathrm{ord}_p\left(\frac{[v+1]_p - [v-i]_p}{p^{u+i+1}} p^{(u-v+1)p^i}\right) \geq u - v + 2.$$

In particular,

$$\overline{F}_p^{(u,v)}(X) \in 1 + p^{u-v+2} X^{p^2} \mathbb{Z}_p\langle X \rangle.$$

Proof. We have $p^i = (1+p-1)^i \geq i(p-1)+p$ and $\mathrm{ord}_p([v+1]_p - [v-i]_p) \geq v-i$. Hence

$$\begin{aligned}\mathrm{ord}_p\left(\frac{[v+1]_p - [v-i]_p}{p^{u+i+1}}\, p^{(u-v+1)p^i}\right) &\geq v-i+(u-v+1)p^i-(u+i+1)\\ &\geq i(u-v+1)(p-1)-2i+(u-v+1)(p-1)\\ &\geq (u-v+1)(p-1)\\ &\geq u-v+2,\end{aligned}$$

which is the first assertion. The second assertion follows from the first one and Lemma 2.5. □

Lemma 3.4 *The following statements hold.*

(1) *If $p=3$ and $u+\delta_{v0}=v+1$, then*

$$F_3^{(u,v)}(X) \in 1 + 3^{u-v+1}X^3 + 3^{u-v+2}X\mathbb{Z}_3\langle X\rangle.$$

(2) *If $p=3$ and $u+\delta_{v0}>v+1$ or if $p>3$, then*

$$F_p^{(u,v)}(X) \in 1 + p^{u-v+2}X\mathbb{Z}_p\langle X\rangle.$$

Proof. Since $\mathrm{ord}_p([v+1]_p - [v-1]_p) = v-1+\delta_{v0}$, it follows that

$$\mathrm{ord}_p\left(\frac{[v+1]_p - [v-1]_p}{p^{u+2}}\, p^{(u-v+1)p}\right) = (u-v+1)(p-2)+u+\delta_{v0}-v-1.$$

In particular, if $p=3$ and $u+\delta_{v0}=v+1$, then

$$\exp\left(-\frac{[v+1]_3 - [v-1]_3}{3^{u+2}}(-3^{u-v+1}X)^3\right) = \exp(3^{u-v+1}X^3 + (1-\delta_{v0})3^{u-v+2}X^3).$$

Hence the assertion is a consequence of Lemmas 2.5 and 3.3. □

Remark 3.5 Since $u \geq v$, it follows that $u+\delta_{v0} \leq v$ if and only if $u=v\geq 1$.

The case where $p=3$ and $u=v\geq 1$ is avoided in the following theorem (see Section 4).

Theorem 3.6 *Suppose that either $p\geq 3$ and $u+\delta_{v0}\geq v+1$ or $p>3$ and $u=v\geq 1$. Let r be a nonnegative integer less than p^{u+1}. Then there exists a p-adic analytic function $g_r(X)$ contained in $\mathbb{Z}_p\langle X\rangle$ such that*

$$g_r(y) = \frac{h_{p^{u+1}y+r}(P)}{(p^{u+1}y+r)!}(-p^{u-v+1})^y y!$$

for any nonnegative integer y and

$$g_r(X) \in \widehat{g}_r(X) + p^{u-v+2}X\mathbb{Z}_p\langle X\rangle,$$

where

$$\widehat{g}_r(X) = \begin{cases} c_r^{(0)}(1+3X^{\underline{3}}) - c_{3+r}^{(0)}3X & \text{if } p=3 \text{ and } u=v=0,\\ c_r^{(v)}(1+3^2X^{\underline{3}}) - c_{3^{u+1}+r}^{(v)}3^2X & \text{if } p=3 \text{ and } u=v+1\geq 2,\\ c_r^{(v)} - c_{p^{u+1}+r}^{(v)}p^{u-v+1}X & \text{otherwise.}\end{cases}$$

Proof. The theorem follows from Proposition 3.2 and Lemma 3.4. □

We turn to the study of p-adic properties of $h_n(P)$ in the case where $p = 2$.

Lemma 3.7 *Suppose that $p = 2$. If i is an integer greater than 1, then*

$$\mathrm{ord}_2\left(\frac{[v+1]_2 - [v-i]_2}{2^{u+i+1}}\, 2^{(u-v+1)2^i}\right) \geq (4i-5)(u-v) + 2\delta_{v0} + 2i - 5.$$

Moreover, if $u + \delta_{v0} \geq v + 2$, then

$$\overline{F}_2^{(u,v)}(X) \in 1 + 2^{2(u+\delta_{v0}-v-1)+(u-v+1)} X\mathbb{Z}_2\langle X\rangle.$$

Proof. We have $2^i \geq 4(i-1)$ and $\mathrm{ord}_2([v+1]_2 - [v-i]_2) \geq v - i + 2\delta_{v0}$. Hence

$$\begin{aligned}\mathrm{ord}_2\left(\frac{[v+1]_2 - [v-i]_2}{2^{u+i+1}}\, 2^{(u-v+1)2^i}\right) &\geq v - i + 2\delta_{v0} + (u-v+1)2^i - (u+i+1) \\ &= (2^i - 1)(u-v+1) + 2\delta_{v0} - 2i \\ &\geq (4i-5)(u-v+1) + 2\delta_{v0} - 2i.\end{aligned}$$

The first assertion is an immediate consequence of this inequality. The second assertion follows from the first one and Lemma 2.5. □

Lemma 3.8 *Suppose that $p = 2$ and $u + \delta_{v0} \geq v + 3$. Then*

$$F_2^{(u,v)}(X) \in \begin{cases} 1 - 2^u X^2 + 2^{2u-1}X^4 + 2^{2u+1}X\mathbb{Z}_2\langle X\rangle & \text{if } v = 0, \\ 1 - 3\cdot 2^{u-v-1}X^2 + 9\cdot 2^{2u-2v-3}X^4 + 2^{2u-2v-1}X\mathbb{Z}_2\langle X\rangle & \text{if } v \geq 1. \end{cases}$$

Proof. Since $u + \delta_{v0} - v - 1 \geq 2$, it follows from Lemma 2.5 that

$$\begin{aligned}&\exp\left(-\frac{[v+1]_2 - [v-1]_2}{2^{u+2}}(-2^{u-v+1}X)^2\right) \\ &\qquad \in \begin{cases} 1 - 2^u X^2 + 2^{2u-1}X^4 + 2^{2u+1}X\mathbb{Z}_2\langle X\rangle & \text{if } v = 0, \\ 1 - 3\cdot 2^{u-v-1}X^2 + 9\cdot 2^{2u-2v-3}X^4 + 2^{2u-2v-1}X\mathbb{Z}_2\langle X\rangle & \text{if } v \geq 1. \end{cases}\end{aligned}$$

Hence the assertion is a consequence of Lemma 3.7. □

The next theorem, together with Theorem 3.6, is a conclusion deduced from of Eq.(3.2).

Theorem 3.9 *Suppose that $p = 2$ and $u + \delta_{v0} \geq v + 3$. Let r be a nonnegative integer less than 2^{u+1}. Then there exists a 2-adic analytic function $g_r(X)$ contained in $\mathbb{Z}_2\langle X\rangle$ such that*

$$g_r(y) = \frac{h_{2^{u+1}y+r}(P)}{(2^{u+1}y+r)!}(-2^{u-v+1})^y y!$$

for any nonnegative integer y and

$$g_r(X) \in \widehat{g}_r(X) + 2^{u-v+2}X\mathbb{Z}_2\langle X\rangle,$$

where

$$\widehat{g}_r(X) = \begin{cases} c_r^{(0)}(1 - 2^u X^{\underline{2}} + 2^{2u-1}X^{\underline{4}}) - c_{2^{u+1}+r}^{(0)} 2^{u+1}X & \text{if } v = 0, \\ c_r^{(v)}(1 - 3\cdot 2^{u-v-1}X^{\underline{2}} + 9\cdot 2^{2u-2v-3}X^{\underline{4}}) - c_{2^{u+1}+r}^{(v)} 2^{u-v+1}X & \text{if } v \geq 1. \end{cases}$$

Proof. The theorem follows from Proposition 3.2 and Lemma 3.8. □

If either $p = 2$ and $u + \delta_{v0} \leq v + 2$ or $p = 3$ and $u = v \geq 1$, then Eq.(3.2) does not work. The rest of this section is mainly devoted to the case where $p = 2$ and $u + \delta_{v0} = v + 2$.

Lemma 3.10 *Suppose that $p = 2$ and $u + \delta_{v0} = v + 2$. Then*

$$\exp(-2X)F_2^{(u,v)}(X) \in \begin{cases} 1 - 2X - 4X^4 + 8X\mathbb{Z}_2\langle X\rangle & \text{if } v = 0, \\ 1 - 2X - 4X^2 - 4X^4 + 8X^5 + 8X^8 + 16X\mathbb{Z}_2\langle X\rangle & \text{if } v \geq 1. \end{cases}$$

Proof. If $v = 0$, then the assertion is given in the proof of [12, Theorem 4.4]. Observe that

$$\exp\left(-\frac{[v+1]_2 - [v-1]_2}{2^{u+2}}(-2^{u-v+1}X)^2\right) = \begin{cases} \exp(-2X^2) & \text{if } v = 0, \\ \exp(-6X^2) & \text{if } v \geq 1 \end{cases}$$

and

$$\exp(-2X)F_2^{(u,v)}(X) = \begin{cases} \exp(-2X + 2X^2 + 4X^4)\exp(-4X^2 - 4X^4)\overline{F}_2^{(u,0)}(X) & \text{if } v = 0, \\ \exp(-2X + 2X^2 + 4X^4)\exp(-8X^2 - 4X^4)\overline{F}_2^{(u,v)}(X) & \text{if } v \geq 1. \end{cases}$$

Then by Eq.(1.1) and Lemmas 2.4 and 3.7, we have

$$\exp(-2X)F_2^{(u,v)}(X)$$
$$\in \begin{cases} \left(\sum_{n=0}^{\infty} \frac{h_n(C_4)}{n!}(-2X)^n\right)\exp(-4X^2 - 4X^4) + 16X\mathbb{Z}_2\langle X\rangle & \text{if } v = 0, \\ \left(\sum_{n=0}^{\infty} \frac{h_n(C_4)}{n!}(-2X)^n\right)\exp(-8X^2 - 4X^4) + 32X\mathbb{Z}_2\langle X\rangle & \text{if } v \geq 1. \end{cases}$$

Moreover, it follows from Lemma 2.4 and Theorem 3.9 that

$$\sum_{n=0}^{\infty} \frac{h_n(C_4)}{n!}(-2X)^n \in 1 - 2X + 4X^2 + 16X\mathbb{Z}_2\langle X\rangle$$

(see the proof of [12, Theorem 4.4]). Combining these facts with Lemma 2.5, we have

$$\exp(-2X)F_2^{(u,v)}(X)$$
$$\in \begin{cases} (1 - 2X + 4X^2)(1 - 4X^2 - 4X^4) + 8X\mathbb{Z}_2\langle X\rangle & \text{if } v = 0, \\ (1 - 2X + 4X^2)(1 - 8X^2 - 4X^4 + 8X^8) + 16X\mathbb{Z}_2\langle X\rangle & \text{if } v \geq 1. \end{cases}$$

This completes the proof. □

There is a slight difference between the proof of Theorem 3.9 and that of the following theorem, because the latter requires Lemma 3.10.

Theorem 3.11 *Suppose that $p=2$ and $u+\delta_{v0}=v+2$. Let r be a nonnegative integer less than 2^{u+1}. Then there exists a 2-adic analytic function $g_r(X)$ contained in $\mathbb{Z}_2\langle X\rangle$ such that*

$$g_r(y)=\frac{h_{2^{u+1}y+r}(P)}{(2^{u+1}y+r)!}\,2^{(u-v+1)y}y!$$

for any nonnegative integer y and

$$g_r(X)\in \widehat{g}_r(X)+2^{u-v+2}X\mathbb{Z}_2\langle X\rangle,$$

where

$$\widehat{g}_r(X)=\begin{cases} c_r^{(0)}(1-2X-4X^{\underline{4}})+c_{4+r}^{(0)}4X & \text{if } v=0,\\ c_r^{(v)}(1-2X-4X^{\underline{2}}-4X^{\underline{4}}+8X^{\underline{5}}+8X^{\underline{8}})+c_{2^{u+1}+r}^{(v)}8X & \text{if } v\geq 1.\end{cases}$$

Proof. Substituting $-X$ for X in Eq.(3.2), we have

$$\sum_{y=0}^{\infty}\frac{h_{2^{u+1}y+r}(P)}{(2^{u+1}y+r)!}(2^{u-v+1}X)^y=\exp(X)\exp(-2X)\left(\sum_{j=0}^{\infty}c_{2^{u+1}j+r}^{(v)}(2^{u-v+1}X)^j\right)F_2^{(u,v)}(X).$$

Moreover, it follows from Lemma 3.10 that

$$\exp(-2X)\left(\sum_{j=0}^{\infty}c_{2^{u+1}j+r}^{(v)}(2^{u-v+1}X)^j\right)F_2^{(u,v)}(X)$$
$$\in\begin{cases} c_r^{(0)}(1-2X-4X^4)+c_{4+r}^{(0)}4X+8X\mathbb{Z}_2\langle X\rangle & \text{if } v=0,\\ c_r^{(v)}(1-2X-4X^2-4X^4+8X^5+8X^8)+c_{2^{u+1}+r}^{(v)}8X+16X\mathbb{Z}_2\langle X\rangle & \text{if } v\geq 1.\end{cases}$$

Hence the assertion is a consequence of Lemma 2.6. □

Remark 3.12 The assertions of Theorems 3.6, 3.9, and 3.11 with $v=0$ are given in [12].

The following corollary contains a result in the case where $p=2$ and $u=v=0$.

Corollary 3.13 ([9]) *Let r be a nonnegative integer less than p^{u+1}. If $v=0$, then*

$$\mathrm{ord}_p(h_n(P))\geq \tau_p^{(u,0)}(n)=\sum_{j=1}^{u}\left[\frac{n}{p^j}\right]-u\left[\frac{n}{p^{u+1}}\right]$$

for any nonnegative integer n, and

$$h_{p^{u+1}y+r}(P)\equiv(-1)^y\frac{(p^{u+1}y+r)!}{p^{(u+1)y}y!}\,c_r^{(0)}\pmod{p^{\tau_p^{(u,0)}(p^{u+1}y+r)+u+1}}$$

for any nonnegative integer y.

Proof. The assertion follows from Lemma 2.4 and Theorems 3.6, 3.9, and 3.11 except for the case where $p=2$ and $u=0$. Suppose that $p=2$ and $u=0$. By Definition 2.2,

$$\sum_{n=0}^{\infty}\frac{1}{n!}X^n=\left(\sum_{j=0}^{\infty}c_j^{(0)}X^j\right)\exp\left(-\sum_{k=1}^{\infty}\frac{1}{2^k}X^{2^k}\right),$$

and hence

$$\sum_{y=0}^{\infty} \frac{1}{(2y+r)!} X^{2y} = \left(\sum_{j=0}^{\infty} c_{2j+r}^{(0)} X^{2j} \right) \exp \left(- \sum_{k=1}^{\infty} \frac{1}{2^k} X^{2^k} \right).$$

Substituting $-2X$ for X^2, we have

$$\sum_{y=0}^{\infty} \frac{1}{(2y+r)!} (-2X)^y = \left(\sum_{j=0}^{\infty} c_{2j+r}^{(0)} (-2X)^j \right) \exp(X - X^2) \exp \left(- \sum_{k=2}^{\infty} \frac{1}{2^{k+1}} (-2X)^{2^k} \right).$$

(This formula is just Eq.(3.2).) Consequently, it follows from Lemmas 2.4, 2.5, and 3.7 that

$$\begin{aligned} 1 &\equiv \frac{(2y+r)!}{(-2)^y y!} \sum_{j=0}^{y} c_{2j+r}^{(0)} \frac{(-2)^j y!}{(y-j)!} \sum_{i=0}^{[(y-j)/2]} (-1)^i 2^i \frac{(y-j)!}{(y-j-2i)! \cdot 2^i \cdot i!} \\ &\equiv (-1)^y \frac{(2y+r)!}{2^y y!} c_r^{(0)} \pmod 2 \end{aligned}$$

for any nonnegative integer y. The proof is now complete. □

4 Sporadic cases

The preceding theorems on a p-adic estimate of $h_n(P)$ does not deal with some exceptional cases. In the case where $p = 2$ and $u = v = 0$, we know nothing but Corollary 3.13. The remaining cases are as follows: (i) $p = 3$ and $u = v \geq 1$; (ii) $p = 2$ and $u = v + 1 \geq 2$; (iii) $p = 2$ and $u = v \geq 1$. In this section we explore the p-adic properties of $h_n(P)$ in each of these sporadic cases; no 2-adic analytic function completely controls $h_n(P)$ under the assumption that $p = 2$ and $u = v + 1 \geq 2$, however.

Except for the case where $p \leq 3$ and $u = v \geq 1$ or $p = 2$ and $u = v + 1 \geq 2$, the p-adic properties of $h_n(P)$ are deduced from Eq.(3.2). In the above exceptional cases (i)–(iii), we can make good use of the formulae (4.2), (4.5), and (4.8). In order to obtain such formulae, the following three subsections begin with reformations of Eq.(2.1).

4A Case (i)

Suppose that $p = 3$ and $u = v \geq 1$. Then by Eq.(2.1),

$$\begin{aligned} \sum_{y=0}^{\infty} \frac{h_{3^{u+1}y+r}(P)}{(3^{u+1}y+r)!} X^{3^{u+1}y} &= \left(\sum_{j=0}^{\infty} c_{3^{u+1}j+r}^{(v)} X^{3^{u+1}j} \right) \exp \left(-\frac{1}{3} X^{3^{u+1}} - \frac{1}{3^4} X^{3^{u+2}} \right) \\ &\times \exp \left(-\frac{2}{3^4} X^{3^{u+2}} \right) \exp \left(-\frac{1}{3^2} X^{3^{u+2}} - \sum_{i=2}^{\infty} \frac{[v+1]_3 - [v-i]_3}{3^{u+i+1}} X^{3^{u+i+1}} \right). \end{aligned}$$

To show Eq.(4.2) above Theorem 4.1, we define

$$H_3(X) := \exp \left(-3^2 X - \sum_{i=2}^{\infty} \frac{[v+1]_3 - [v-i]_3}{3^{u+i+1}} (3^4 X)^{3^{i-1}} \right)$$

and

$$\sum_{j=0}^{\infty} w_j X^{3^{u+1}j} := \left(\sum_{j=0}^{\infty} c_{3^{u+1}j+r}^{(v)} X^{3^{u+1}j} \right) \exp\left(-\frac{1}{3}\, X^{3^{u+1}} - \frac{1}{3^4}\, X^{3^{u+2}} \right), \tag{4.1}$$

so that

$$\sum_{y=0}^{\infty} \frac{h_{3^{u+1}y+r}(P)}{(3^{u+1}y+r)!}\, X^{3^{u+1}y} = \left(\sum_{j=0}^{\infty} w_j X^{3^{u+1}j} \right) \exp\left(-\frac{2}{3^4}\, X^{3^{u+2}} \right) H_3 \left(\frac{1}{3^4}\, X^{3^{u+2}} \right).$$

Let q be a nonnegative integer less than 3. The above formula implies that

$$\begin{aligned} &\sum_{y=0}^{\infty} \frac{h_{3^{u+1}(3y+q)+r}(P)}{(3^{u+1}(3y+q)+r)!}\, X^{3^{u+1}(3y+q)} \\ &\qquad = \left(\sum_{j=0}^{\infty} w_{3j+q} X^{3^{u+1}(3j+q)} \right) \exp\left(-\frac{2}{3^4}\, X^{3^{u+2}} \right) H_3 \left(\frac{1}{3^4}\, X^{3^{u+2}} \right). \end{aligned}$$

Substituting $3^{4/3}X$ for $X^{3^{u+1}}$, we have

$$\begin{aligned} &\sum_{y=0}^{\infty} \frac{h_{3^{u+1}(3y+q)+r}(P)}{(3^{u+1}(3y+q)+r)!}\, 3^{4y+q} X^{3y} (3^{1/3}X)^q \\ &\qquad = \left(\sum_{j=0}^{\infty} w_{3j+q} 3^{4j+q} X^{3j} (3^{1/3}X)^q \right) \exp(-2X^3) H_3 \left(X^3 \right). \end{aligned}$$

Here $3^{4/3}$ should be considered as an element of a universal 3-adic field (see [16, III]). Now omit $(3^{1/3}X)^q$ and substitute X for X^3. Then we have

$$\sum_{y=0}^{\infty} \frac{h_{3^{u+1}(3y+q)+r}(P)}{(3^{u+1}(3y+q)+r)!}\, 3^{4y+q} X^y = \exp(X) \left(\sum_{j=0}^{\infty} w_{3j+q} 3^{4j+q} X^j \right) \exp(-3X) H_3(X). \tag{4.2}$$

Theorem 4.1 *Suppose that $p = 3$ and $u = v \geq 1$. Let r be a nonnegative integer less than 3^{u+1} and q a nonnegative integer less than 3. Then there exists a 3-adic analytic function $g_{q,r}(X)$ contained in $\mathbb{Z}_3\langle X \rangle$ such that*

$$g_{q,r}(y) = \frac{h_{3^{u+1}(3y+q)+r}(P)}{(3^{u+1}(3y+q)+r)!}\, 3^{4y+q} y!$$

for any nonnegative integer y and

$$g_{q,r}(X) \in \widehat{g}_{q,r}(X) + 3^2 X \mathbb{Z}_3\langle X \rangle,$$

where

$$\widehat{g}_{q,r}(X) = \begin{cases} c_r^{(v)} \left(1 - 3^3 \dfrac{h_9(C_3)}{9!} X^{\underline{3}} \right) & \text{if } q = 0, \\ c_r^{(v)} \left(-1 + 3X + 3^3 \dfrac{h_{10}(C_3)}{10!} X^{\underline{3}} \right) + 3c_{3^{u+1}+r}^{(v)} & \text{if } q = 1, \\ c_r^{(v)} \left(\dfrac{1}{2} - 3^3 \dfrac{h_{11}(C_3)}{11!} X^{\underline{3}} \right) - 3c_{3^{u+1}+r}^{(v)} + 3^2 c_{2\cdot 3^{u+1}+r}^{(v)} & \text{if } q = 2. \end{cases}$$

Proof. The theorem is proved by exploring the factor $\sum_{j=0}^{\infty} w_{3j+q}3^{4j+q}X^j$ on the right hand side of Eq.(4.2). Substituting $3^{4/3}X$ for $X^{3^{u+1}}$ in Eq.(4.1), we have

$$\sum_{j=0}^{\infty} w_j 3^{4j/3}X^j = \left(\sum_{j=0}^{\infty} c^{(v)}_{3^{u+1}j+r}3^{4j/3}X^j\right)\exp(-3^{1/3}X - X^3).$$

This, combined with Eq.(1.1), shows that

$$\sum_{j=0}^{\infty} w_j 3^{4j/3}X^j = \left(\sum_{j=0}^{\infty} c^{(v)}_{3^{u+1}j+r}3^{4j/3}X^j\right)\left(\sum_{n=0}^{\infty} \frac{h_n(C_3)}{n!}(-3^{1/3}X)^n\right).$$

Let j be a nonnegative integer. By the above equation, we have

$$w_{3j+q}3^{4j+q}3^{q/3} = \sum_{i=0}^{3j+q} c^{(v)}_{3^{u+1}(3j+q-i)+r}3^{4j+q-i}3^{(q-i)/3}\frac{h_i(C_3)}{i!}(-3^{1/3})^i,$$

whence

$$w_{3j+q}3^{4j+q} = \sum_{i=0}^{3j+q}(-1)^i c^{(v)}_{3^{u+1}(3j+q-i)+r}3^{4j+q-i}\frac{h_i(C_3)}{i!}. \tag{4.3}$$

Let i be a nonnegative integer, and denote by $i_0, i_1, i_2, \dots$ nonnegative integers less than 3 with $i = i_0 + 3i_1 + 3^2 i_2 + \cdots$. Then it follows from Lemma 2.4 and Corollary 3.13 that

$$\begin{aligned}\mathrm{ord}_3\left(\frac{h_i(C_3)}{i!}\right) &\geq \left[\frac{i}{3}\right] - \left[\frac{i}{3^2}\right] - \frac{i - i_0 - i_1 - i_2 - \cdots}{2}\\ &= i_1 + \frac{2i - 2i_0 - 6i_1}{9} - \frac{i - i_0 - i_1 - i_2 - \cdots}{2}\\ &= -\frac{5}{18}i + \frac{5}{18}i_0 + \frac{5}{6}i_1 + \frac{i_2 + i_3 + \cdots}{2},\end{aligned}$$

and thus

$$\mathrm{ord}_3\left(3^{4j+q-i}\frac{h_i(C_3)}{i!}\right) \geq \begin{cases} 4j + q - i & \text{if } i \leq 8 \leq 3j + q,\\ 3j + q - i + \frac{1}{6}j - \frac{5}{18}(q - i_0) + \frac{1}{2} & \text{if } 9 \leq i \leq 3j + q.\end{cases}$$

After all, if $j \geq 1$, then except for the case where $i = 3j + q$ with $j = 1$ or $j = 3$, we have

$$\mathrm{ord}_3\left(c^{(v)}_{3^{u+1}(3j+q^i)+r}3^{4j+q-i}\frac{h_i(C_3)}{i!}\right) \geq \max\left\{\frac{j+3}{6}, 2\right\}.$$

This, combined with Eq.(4.3), shows that

$$\sum_{j=0}^{\infty} w_{3j+q}3^{4j+q}X^j = \sum_{j=0}^{\infty}\left(\sum_{i=0}^{3j+q}(-1)^i c^{(v)}_{3^{u+1}(3j+q-i)+r}3^{4j+q-i}\frac{h_i(C_3)}{i!}\right)X^j,$$

$$\sum_{j=1}^{\infty} w_{3j+q}3^{4j+q}X^j \in (-1)^{3+q}c^{(v)}_r 3\frac{h_{3+q}(C_3)}{(3+q)!}X + (-1)^{9+q}c^{(v)}_r 3^3\frac{h_{9+q}(C_3)}{(9+q)!}X^3 + 3^2 X\mathbb{Z}_3\langle X\rangle,$$

and

$$\begin{aligned}\sum_{j=0}^{\infty} w_{3j+q} 3^{4j+q} X^j &= \sum_{i=0}^{q} (-1)^i c^{(v)}_{3^{u+1}(q-i)+r} 3^{q-i} \frac{h_i(C_3)}{i!} + \sum_{j=1}^{\infty} w_{3j+q} 3^{4j+q} X^j \\ &\in \widetilde{g}_{q,r}(X) + 3^2 X \mathbb{Z}_3\langle X \rangle,\end{aligned}$$

where

$$\widetilde{g}_{q,r}(X) = \begin{cases} c_r^{(v)} \left(1 - \frac{3}{2}\, X - 3^3 \frac{h_9(C_3)}{9!}\, X^3\right) & \text{if } q = 0, \\ c_r^{(v)} \left(-1 + 3^3 \frac{h_{10}(C_3)}{10!}\, X^3\right) + 3c^{(v)}_{3^{u+1}+r} & \text{if } q = 1, \\ c_r^{(v)} \left(\frac{1}{2} - \frac{21}{40}\, X - 3^3 \frac{h_{11}(C_3)}{11!}\, X^3\right) - 3c^{(v)}_{3^{u+1}+r} + 3^2 c^{(v)}_{2\cdot 3^{u+1}+r} & \text{if } q = 2. \end{cases}$$

Consequently, since $\exp(-3X)H_3(X) \in 1 - 3X + 3^2 X\mathbb{Z}_3\langle X \rangle$ by Lemmas 2.5 and 3.3, the assertion follows from Lemma 2.6, Corollary 3.13, and Eq.(4.2). This completes the proof. □

4B Case (ii)

Suppose that $p = 2$ and $u = v + 1 \geq 2$. By Eq.(2.1),

$$\begin{aligned}\sum_{y=0}^{\infty} \frac{h_{2^{u+1}y+r}(P)}{(2^{u+1}y+r)!}\, X^{2^{u+1}y} &= \left(\sum_{j=0}^{\infty} c^{(v)}_{2^{u+1}j+r} X^{2^{u+1}j}\right) \exp\left(-\frac{1}{2^2}\, X^{2^{u+1}} + \frac{1}{2^5}\, X^{2^{u+2}}\right) \\ &\times \exp\left(\frac{1}{2^5}\, X^{2^{u+2}}\right) \exp\left(-\frac{1}{2^2}\, X^{2^{u+2}} - \sum_{i=2}^{\infty} \frac{[v+1]_2 - [v-i]_2}{2^{u+i+1}}\, X^{2^{u+i+1}}\right).\end{aligned}$$

To show Eq.(4.5) above Theorem 4.2, we define

$$H_2(X) := \exp\left(-2^3 X - \sum_{i=2}^{\infty} \frac{[v+1]_2 - [v-i]_2}{2^{u+i+1}} (2^5 X)^{2^{i-1}}\right)$$

and

$$\sum_{j=0}^{\infty} w_j X^{2^{u+1}j} := \left(\sum_{j=0}^{\infty} c^{(v)}_{2^{u+1}j+r} X^{2^{u+1}j}\right) \exp\left(-\frac{1}{2^2}\, X^{2^{u+1}} + \frac{1}{2^5}\, X^{2^{u+2}}\right), \tag{4.4}$$

so that

$$\sum_{y=0}^{\infty} \frac{h_{2^{u+1}y+r}(P)}{(2^{u+1}y+r)!}\, X^{2^{u+1}y} = \left(\sum_{j=0}^{\infty} w_j X^{2^{u+1}j}\right) \exp\left(\frac{1}{2^5}\, X^{2^{u+2}}\right) H_2\left(\frac{1}{2^5}\, X^{2^{u+2}}\right).$$

Let q be a nonnegative integer less than 2. The above formula implies that

$$\begin{aligned}&\sum_{y=0}^{\infty} \frac{h_{2^{u+1}(2y+q)+r}(P)}{(2^{u+1}(2y+q)+r)!}\, X^{2^{u+1}(2y+q)} \\ &\qquad = \left(\sum_{j=0}^{\infty} w_{2j+q} X^{2^{u+1}(2j+q)}\right) \exp\left(\frac{1}{2^5}\, X^{2^{u+2}}\right) H_2\left(\frac{1}{2^5}\, X^{2^{u+2}}\right).\end{aligned}$$

Substituting $2^{5/2}X$ for $X^{2^{u+1}}$, we have

$$\sum_{y=0}^{\infty} \frac{h_{2^{u+1}(2y+q)+r}(P)}{2^{u+1}(2y+q)+r)!}\, 2^{5y+2q} X^{2y} (2^{1/2}X)^q$$

$$= \left(\sum_{j=0}^{\infty} w_{2j+q} 2^{5j+2q} X^{2j} (2^{1/2}X)^q \right) \exp(X^2) H_2(X^2).$$

Now omit $(2^{1/2}X)^q$ and substitute X for X^2. Then

$$\sum_{y=0}^{\infty} \frac{h_{2^{u+1}(2y+q)+r}(P)}{(2^{u+1}(2y+q)+r)!}\, 2^{5y+2q} X^{y} = \exp(X) \left(\sum_{j=0}^{\infty} w_{2j+q} 2^{5j+2q} X^{j} \right) H_2(X). \tag{4.5}$$

Theorem 4.2 *Suppose that $p = 2$ and $u = v + 1 \geq 2$. Let r be a nonnegative integer less than 2^{u+1} and q a nonnegative integer less than 2. Define a polynomial $g_{q,r}(X)$ by*

$$g_{q,r}(X) = \begin{cases} c_r^{(v)}(1 - X - X^2) & \text{if } q = 0, \\ -c_r^{(v)}(1 + X - X^2) & \text{if } q = 1. \end{cases}$$

Then for any nonnegative integer y,

$$\frac{h_{2^{u+1}(2y+q)+r}(P)}{(2^{u+1}(2y+q)+r)!}\, 2^{5y+2q} y! \equiv g_{q,r}(y) \pmod{2^2}.$$

Proof. The theorem is proved by exploring the factor $\sum_{j=0}^{\infty} w_{2j+q} 2^{5j+2q} X^j$ on the right hand side of Eq.(4.5). Substituting $2^{5/2}X$ for $X^{2^{u+1}}$ in Eq.(4.4), we have

$$\sum_{j=0}^{\infty} w_j 2^{5j/2} X^j = \left(\sum_{j=0}^{\infty} c^{(v)}_{2^{u+1}j+r} 2^{5j/2} X^j \right) \exp\left(-2^{1/2}X + X^2 \right).$$

This, combined with Eq.(1.1), shows that

$$\sum_{j=0}^{\infty} w_j 2^{5j/2} X^j = \left(\sum_{j=0}^{\infty} c^{(v)}_{2^{u+1}j+r} 2^{5j/2} X^j \right) \left(\sum_{n=0}^{\infty} \frac{h_n(C_2)}{n!} (-2^{1/2}X)^n \right).$$

Let j be a nonnegative integer. By the above equation, we have

$$w_{2j+q} 2^{5j+2q} 2^{q/2} = \sum_{i=0}^{2j+q} c^{(v)}_{2^{u+1}(2j+q-i)+r} 2^{5j+2(q-i)} 2^{(q-i)/2} \frac{h_i(C_2)}{i!} (-2^{1/2})^i,$$

whence

$$w_{2j+q} 2^{5j+2q} = \sum_{i=0}^{2j+q} (-1)^i c^{(v)}_{2^{u+1}(2j+q-i)+r} 2^{5j+2(q-i)} \frac{h_i(C_2)}{i!}. \tag{4.6}$$

For an integer i with $0 \leq i \leq 2j + q$, it follows from Lemma 2.4 and Corollary 3.13 that

$$\mathrm{ord}_2 \left(\frac{h_i(C_2)}{i!} \right) \geq \left[\frac{i}{2} \right] - \left[\frac{i}{2^2} \right] - (i - s_2(i)) = -\left[\frac{i+1}{2} \right] - \left[\frac{i}{2^2} \right] + s_2(i),$$

and thus

$$\operatorname{ord}_2\left(2^{5j+2(q-i)}\frac{h_i(C_2)}{i!}\right) \geq 2(2j+q-i) - \left[\frac{j}{2}\right] - \left[\frac{q+1}{2}\right] + s_2(i).$$

Consequently, $w_{2j+q}2^{5j+2q} \geq -[j/2]+1$ if $j \geq 1$. We now define

$$\sum_{j=0}^{\infty} \widetilde{w}_j X^j := \left(\sum_{j=0}^{\infty} w_{2j+q}2^{5j+2q}X^j\right) H_2(X).$$

By Lemmas 2.5 and 3.7, $H_2(X) \in 1 + 2^2X\mathbb{Z}_2\langle X\rangle$. This, combined with the above fact, implies that $\operatorname{ord}_2(\widetilde{w}_j) \geq -[j/2]+1$ for any positive integer j. By Eq.(4.5),

$$\frac{h_{2^{u+1}(2y+q)+r}(P)}{(2^{u+1}(2y+q)+r)!}\,2^{5y+2q}y! = \sum_{j=0}^{y}\frac{y!}{(y-j)!}\,\widetilde{w}_j$$

for any nonnegative integer y. If i and j are integers and if $i \geq j \geq 4$, then

$$\operatorname{ord}_2\left(\frac{i!}{(i-j)!}\right) = \operatorname{ord}_2(i(i-1)\cdots(i-j+1)) \geq 1 + \left[\frac{j}{2}\right].$$

Moreover, by Lemma 2.4, Corollary 3.13, and Eq.(4.6), we have

$$\begin{aligned}\sum_{j=0}^{3} w_{2j+q}2^{5j+2q}X^j &= \sum_{j=0}^{3}\left(\sum_{i=0}^{2j+q}(-1)^i c^{(v)}_{2^{u+1}(2j+q-i)+r}2^{5j+2(q-i)}\frac{h_i(C_2)}{i!}\right)X^j \\ &\in \widetilde{g}_{q,r}(X) + 2^2X\mathbb{Z}_2\langle X\rangle\end{aligned}$$

and

$$\sum_{j=0}^{3}\widetilde{w}_jX^j \in \widetilde{g}_{q,r}(X) + 2^2X\mathbb{Z}_2\langle X\rangle,$$

where

$$\widetilde{g}_{q,r}(X) = \begin{cases} c^{(v)}_r\left(1 + 2X + \dfrac{5}{3}X^2 + 2^3\dfrac{h_6(C_2)}{6!}X^3\right) & \text{if } q = 0, \\ -c^{(v)}_r\left(1 + \dfrac{13}{15}X^2 + 2^3\dfrac{h_7(C_2)}{7!}X^3\right) + 2^2c^{(v)}_{2^{u+1}+r} & \text{if } q = 1. \end{cases}$$

Hence it follows from Lemma 2.4 and Corollary 3.13 that for any nonnegative integer y,

$$\begin{aligned}\frac{h_{2^{u+1}(2y+q)+r}(P)}{(2^{u+1}(2y+q)+r)!}\,2^{5y+2q}y! &= \sum_{j=0}^{y}\frac{y!}{(y-j)!}\,\widetilde{w}_j \\ &\equiv \widetilde{w}_0 + y\widetilde{w}_1 + y(y-1)\widetilde{w}_2 + y(y-1)(y-2)\widetilde{w}_3 \\ &\equiv \begin{cases} c^{(v)}_r(1-y-y^2) \pmod{2^2} & \text{if } q = 0, \\ -c^{(v)}_r(1+y-y^2) \pmod{2^2} & \text{if } q = 1. \end{cases}\end{aligned}$$

This completes the proof. □

4C Case (iii)

Suppose that $p=2$ and $u=v\geq 1$. Then by Eq.(2.1),

$$\sum_{y=0}^{\infty}\frac{h_{2^{u+1}y+r}(P)}{(2^{u+1}y+r)!}\,X^{2^{u+1}y}=\left(\sum_{j=0}^{\infty}c^{(v)}_{2^{u+1}j+r}X^{2^{u+1}j}\right)$$
$$\times\exp\left(-\frac{1}{2}\,X^{2^{u+1}}-\frac{3}{2^3}\,X^{2^{u+2}}+\frac{11}{2^6}\,X^{2^{u+3}}+\frac{1}{2^{10}}\,X^{2^{u+4}}\right)\exp\left((-1)^{1-\delta_{u1}}\frac{1}{2^6}\,X^{2^{u+3}}\right)$$
$$\times\exp\left(-\frac{3}{2^3}\,X^{2^{u+3}}-\frac{1}{2^{10}}\,X^{2^{u+4}}-\sum_{i=3}^{\infty}\frac{[v+1]_2-[v-i]_2}{2^{u+i+1}}\,X^{2^{u+i+1}}\right).$$

To show Eq.(4.8) above Theorem 4.3, we define

$$T_2(X):=\exp\left(3\cdot 2^3X-2^2X^2-\sum_{i=3}^{\infty}\frac{[v+1]_2-[v-i]_2}{2^{u+i+1}}(-2^6X)^{2^{i-2}}\right)$$

and

$$\sum_{j=0}^{\infty}w_jX^{2^{u+1}j}:=\left(\sum_{j=0}^{\infty}c^{(v)}_{2^{u+1}j+r}X^{2^{u+1}j}\right)\times\exp\left(-\frac{1}{2}\,X^{2^{u+1}}-\frac{3}{2^3}\,X^{2^{u+2}}+\frac{11}{2^6}\,X^{2^{u+3}}+\frac{1}{2^{10}}\,X^{2^{u+4}}\right),\tag{4.7}$$

so that

$$\sum_{y=0}^{\infty}\frac{h_{2^{u+1}y+r}(P)}{(2^{u+1}y+r)!}\,X^{2^{u+1}y}=\left(\sum_{j=0}^{\infty}w_jX^{2^{u+1}j}\right)\exp\left((-1)^{1-\delta_{u1}}\frac{1}{2^6}\,X^{2^{u+3}}\right)T_2\left(-\frac{1}{2^6}\,X^{2^{u+3}}\right).$$

Let q be a nonnegative integer less than 4. The above formula implies that

$$\sum_{y=0}^{\infty}\frac{h_{2^{u+1}(4y+q)+r}(P)}{(2^{u+1}(4y+q)+r)!}\,X^{2^{u+1}(4y+q)}$$
$$=\left(\sum_{j=0}^{\infty}w_{4j+q}X^{2^{u+1}(4j+q)}\right)\exp\left((-1)^{1-\delta_{u1}}\frac{1}{2^6}\,X^{2^{u+3}}\right)T_2\left(-\frac{1}{2^6}\,X^{2^{u+3}}\right).$$

Substituting $2^{3/2}X$ for $X^{2^{u+1}}$, we have

$$\sum_{y=0}^{\infty}\frac{h_{2^{u+1}(4y+q)+r}(P)}{(2^{u+1}(4y+q)+r)!}\,2^{6y+q}X^{4y}(2^{1/2}X)^q$$
$$=\left(\sum_{j=0}^{\infty}w_{4j+q}2^{6j+q}X^{4j}(2^{1/2}X)^q\right)\exp((-1)^{1-\delta_{u1}}X^4)T_2(-X^4).$$

Now omit $(2^{1/2}X)^q$ and substitute $(-1)^{1-\delta_{u1}}X$ for X^4. Then

$$\begin{aligned}\sum_{y=0}^{\infty}&\frac{h_{2^{u+1}(4y+q)+r}(P)}{(2^{u+1}(4y+q)+r)!}\,2^{6y+q}((-1)^{1-\delta_{u1}}X)^y\\&=\exp(X)\left(\sum_{j=0}^{\infty}w_{4j+q}2^{6j+q}((-1)^{1-\delta_{u1}}X)^j\right)T_2((-1)^{\delta_{u1}}X).\end{aligned}\tag{4.8}$$

Theorem 4.3 *Suppose that $p=2$ and $u=v\geq 1$. Let r be a nonnegative integer less than 2^{u+1} and q a nonnegative integer less than 4. Then there exists a 2-adic analytic function $g_{q,r}(X)$ contained in $\mathbb{Z}_2\langle X\rangle$ such that*

$$g_{q,r}(y)=\frac{h_{2^{u+1}(4y+q)+r}(P)}{(2^{u+1}(4y+q)+r)!}(-1)^{(1-\delta_{u1})y}2^{6y+q}y!$$

for any nonnegative integer y and

$$g_{q,r}(X)\in\widehat{g}_{q,r}(X)+2^2X\mathbb{Z}_2\langle X\rangle,$$

where

$$\widehat{g}_{q,r}(X)=\begin{cases}c_r^{(v)}(1+(-1)^{1-\delta_{u1}}2X) & \text{if } q=0,\\ -c_r^{(v)}(1+(-1)^{1-\delta_{u1}}2X)+2c^{(v)}_{2^{u+1}+r} & \text{if } q=1,\\ -c_r^{(v)}(1-(-1)^{1-\delta_{u1}}2X)-2c^{(v)}_{2^{u+1}+r}+2^2c^{(v)}_{2^{u+2}+r} & \text{if } q=2,\\ \dfrac{2^2}{3}c_r^{(v)}(1-(-1)^{1-\delta_{u1}}X)-2c^{(v)}_{2^{u+1}+r}-2^2c^{(v)}_{2^{u+2}+r}+2^3c^{(v)}_{2^{u+2}+2^{u+1}+r} & \text{if } q=3.\end{cases}$$

Proof. The theorem is proved by exploring the factor $\sum_{j=0}^{\infty}w_{4j+q}2^{6j+q}((-1)^{1-\delta_{u1}}X)^j$ on the right hand side of Eq.(4.8). Substituting $2^{3/2}X$ for $X^{2^{u+1}}$ in Eq.(4.7), we have

$$\sum_{j=0}^{\infty}w_j2^{3j/2}X^j=\left(\sum_{j=0}^{\infty}c^{(v)}_{2^{u+1}j+r}2^{3j/2}X^j\right)\exp\left(-2^{1/2}\,X-3X^2+11X^4+4X^8\right).$$

This, combined with Eq.(1.1), shows that

$$\begin{aligned}\sum_{j=0}^{\infty}w_j2^{3j/2}X^j&=\left(\sum_{j=0}^{\infty}c^{(v)}_{2^{u+1}j+r}2^{3j/2}X^j\right)\left(\sum_{n=0}^{\infty}\frac{h_n(C_8)}{n!}(-2^{1/2}X)^n\right)\\&\qquad\times\left(\sum_{n=0}^{\infty}\frac{h_n(C_2)}{n!}(-2^2X^2)^n\right)\left(\sum_{n=0}^{\infty}\frac{h_n(C_2)}{n!}(2X^4)^n\right),\end{aligned}$$

because

$$\begin{aligned}&\exp\left(-2^{1/2}X-3X^2+11X^4+4X^8\right)\\&\quad=\exp\left(-2^{1/2}X+X^2+X^4+2X^8\right)\exp\left(-4X^2+8X^4\right)\exp\left(2X^4+2X^8\right)\\&\quad=\left(\sum_{n=0}^{\infty}\frac{h_n(C_8)}{n!}(-2^{1/2}X)^n\right)\left(\sum_{n=0}^{\infty}\frac{h_n(C_2)}{n!}(-2^2X^2)^n\right)\left(\sum_{n=0}^{\infty}\frac{h_n(C_2)}{n!}(2X^4)^n\right).\end{aligned}$$

Let j be a nonnegative integer. By the above equation, we have

$$w_{4j+q}2^{6j+q}2^{q/2} = \sum_{i=0}^{4j+q} c^{(v)}_{2^{u+1}(4j+q-i)+r}2^{6j+q-i}2^{(q-i)/2}\sum_{k=0}^{[i/2]}\frac{h_{i-2k}(C_8)}{(i-2k)!}(-2^{1/2})^{i-2k} \times \sum_{z=0}^{[k/2]}\frac{h_{k-2z}(C_2)}{(k-2z)!}(-2^2)^{k-2z}\frac{h_z(C_2)}{z!}\,2^z,$$

and thus

$$w_{4j+q}2^{6j+q} = \sum_{i=0}^{4j+q}\sum_{k=0}^{[i/2]}\sum_{z=0}^{[k/2]} w_j(i,k,z), \tag{4.9}$$

where

$$w_j(i,k,z) = (-1)^{i-k}c^{(v)}_{2^{u+1}(4j+q-i)+r}2^{6j+q-i+k-3z}\frac{h_{i-2k}(C_8)}{(i-2k)!}\cdot\frac{h_{k-2z}(C_2)}{(k-2z)!}\cdot\frac{h_z(C_2)}{z!}.$$

Let i, k, and z be nonnegative integers, and suppose that $i \le 4j+q$, $k \le [i/2]$, and $z \le [k/2]$. Then it follows from Lemma 2.4 and Corollary 3.13 that

$$\mathrm{ord}_2(w_j(i,k,z)) \ge 6j+q-2i+\sum_{\ell=1}^{3}\left[\frac{i-2k}{2^\ell}\right]-3\left[\frac{i-2k}{2^4}\right]+\left[\frac{k-2z}{2}\right]-\left[\frac{k-2z}{2^2}\right] +\left[\frac{z}{2}\right]-\left[\frac{z}{2^2}\right]+2k-2z+s_2(i-2k)+s_2(k-2z)+s_2(z).$$

Since $[m/2^\ell] \ge 2[m/2^{\ell+1}]$ for any nonnegative integers m and ℓ, we have

$$\mathrm{ord}_2(w_j(i,k,z)) \ge 6j+q-i-\left[\frac{i+1}{2}\right]+\left[\frac{i-2k}{2^2}\right]-\left[\frac{i-2k}{2^4}\right]+\left[\frac{k-2z}{2^2}\right]+\left[\frac{z}{2}\right]-\left[\frac{z}{2^2}\right] +k-2z+s_2(i-2k)+s_2(k-2z)+s_2(z).$$

Moreover, if $i \ge 3j \ge 3\cdot 2^4$, then $[i/2^4] \ge [3j/2^4] \ge [j/2^4]+2$, and hence

$$\left[\frac{i-2k}{2^2}\right]-\left[\frac{i-2k}{2^4}\right]+\left[\frac{k-2z}{2^2}\right]+\left[\frac{z}{2}\right]-\left[\frac{z}{2^2}\right] \ge \left[\frac{i}{2^4}\right]-2 \ge \left[\frac{j}{2^4}\right].$$

We explore $\mathrm{ord}_2(w_j(i,k,z))$ in each of the following cases **(a)**–**(h)**. Assume that $j \ge 1$.

(a) If $i \le 3j-1$, then

$$\mathrm{ord}_2(w_j(i,k,z)) \ge 6j+q-i-\left[\frac{i+1}{2}\right] \ge \left[\frac{3j+1}{2}\right]+1.$$

(b) If $3j \le i \le 4j \le 4j+q-1$, then

$$\mathrm{ord}_2(w_j(i,k,z)) \ge 6j+q-i-\left[\frac{i+1}{2}\right]+\left[\frac{j}{2^4}\right]+1 \ge \left[\frac{j}{2^4}\right]+2.$$

(c) If $4j+1 \le i \le 4j+q-1$, then either $i=4j+1$ or $i=4j+2$, and hence

$$\mathrm{ord}_2(w_j(i,k,z)) \ge 6j+q-i-\left[\frac{i+1}{2}\right]+\left[\frac{j}{2^4}\right]+2 \ge \left[\frac{j}{2^4}\right]+2.$$

(d) If $i = 4j + q$, $i - 2k \neq 0$, $k - 2z \neq 0$, and $z \neq 0$, then

$$\mathrm{ord}_2(w_j(i,k,z)) \geq 6j + q - i - \left[\frac{i+1}{2}\right] + \left[\frac{j}{2^4}\right] + 4 \geq \left[\frac{j}{2^4}\right] + 2.$$

(e) If $i = 4j + q$ and $k = z = 0$, then

$$\mathrm{ord}_2(w_j(i,k,z)) \geq 6j + q - i - \left[\frac{i+1}{2}\right] + \left[\frac{i}{2^2}\right] - \left[\frac{i}{2^4}\right] + 1 + \delta_{q1} + \delta_{q2} + 2\delta_{q3} \geq \left[\frac{j}{2}\right] + 2.$$

(f) If $i = 4j + q$, $i - 2k \neq 0$, and either $k = k - 2z \neq 0$ or $1 \leq k/2 = z \leq j - 1$, then

$$\mathrm{ord}_2(w_j(i,k,z)) \geq 6j + q - i - \left[\frac{i+1}{2}\right] + \left[\frac{j}{2^4}\right] + 2 + \delta_{q1} + \delta_{q2} + 2\delta_{q3} \geq \left[\frac{j}{2^4}\right] + 2.$$

(g) If $i = 4j + q$ and $k/2 = z = j$, then

$$\mathrm{ord}_2(w_j(i,k,z)) \geq 6j + q - i - \left[\frac{i+1}{2}\right] + \left[\frac{j}{2}\right] - \left[\frac{j}{2^2}\right] + 1 + \delta_{q1} + \delta_{q2} + 2\delta_{q3} \geq \left[\frac{j+2}{2^2}\right] + 1.$$

(h) If $i = 4j + q$, $i - 2k = 0$, and $k - 2z \neq 0$, then $i/2 = k = 2j + \delta_{q2}$, and hence

$$\mathrm{ord}_2(w_j(i,k,z)) \geq 6j + q - i - \left[\frac{i+1}{2}\right] + \left[\frac{j}{2^4}\right] + 2 + \delta_{q2} \geq \left[\frac{j}{2^4}\right] + 2.$$

After all, if $j \geq 1$, then $\mathrm{ord}_2(w_j(i,k,z)) \geq [j/2^4] + 2$ except for the case where $j = 1$, $i = 4 + q$, $k = 2$, and $z = 1$ (see **(g)**). This, combined with Eq.(4.9), shows that

$$\sum_{j=0}^{\infty} w_{4j+q} 2^{6j+q} X^j \in \sum_{i=0}^{q} \sum_{k=0}^{[i/2]} w_0(i,k,0) + w_1(4+q,2,1)X + 2^2 X\mathbb{Z}_2\langle X\rangle$$

and

$$\begin{aligned}
&\sum_{i=0}^{q} \sum_{k=0}^{[i/2]} w_0(i,k,0) + w_1(4+q,2,1)X \\
&\quad = \sum_{i=0}^{q} \sum_{k=0}^{[i/2]} (-1)^{i-k} c^{(v)}_{2^{u+1}(q-i)+r} 2^{q-i+k} \frac{h_{i-2k}(C_8)}{(i-2k)!} \cdot \frac{h_k(C_2)}{k!} + (-1)^q c^{(v)}_r 2 \frac{h_q(C_8)}{q!} X \\
&\quad = \begin{cases}
c^{(v)}_r (1+2X) & \text{if } q = 0, \\
-c^{(v)}_r (1+2X) + 2c^{(v)}_{2^{u+1}+r} & \text{if } q = 1, \\
-c^{(v)}_r (1-2X) - 2c^{(v)}_{2^{u+1}+r} + 2^2 c^{(v)}_{2^{u+2}+r} & \text{if } q = 2, \\
\dfrac{2^2}{3} c^{(v)}_r (1-X) - 2c^{(v)}_{2^{u+1}+r} - 2^2 c^{(v)}_{2^{u+2}+r} + 2^3 c^{(v)}_{2^{u+2}+2^{u+1}+r} & \text{if } q = 3.
\end{cases}
\end{aligned}$$

Consequently, since $T_2((-1)^{\delta_{u1}} X) \in 1 + 2^2 X\mathbb{Z}_2\langle X\rangle$ by Lemmas 2.5 and 3.7, the assertion follows from Lemma 2.6 and Eq.(4.8). This completes the proof. □

Example 4.4 Suppose that $p = 2$ and $P = C_2 \times C_2$. We calculate $\mathrm{ord}_2(h_n(P))$. For each nonnegative integer n, we write $c_n = c_n^{(1)}$ for the sake of shortness. Observe that

$$\begin{aligned}\sum_{n=0}^{\infty} c_n X^n &= \exp\left(X + \frac{3}{2}X^2 + \frac{3}{4}X^4 + \sum_{k=3}^{\infty} \frac{3}{2^k} X^{2^k}\right) \\ &= 1 + X + 2X^2 + \frac{5}{3}X^3 + \frac{8}{3}X^4 + \frac{32}{15}X^5 + \frac{121}{45}X^6 + \frac{634}{315}X^7 + \cdots\end{aligned}$$

and

$$c_0 = c_1 = 1,\ c_2 = 2, \quad c_3 = \frac{5}{3}, \quad c_4 = \frac{8}{3}, \quad c_5 = \frac{32}{15}, \quad c_6 = \frac{121}{45}, \quad c_7 = \frac{634}{315}.$$

Let y be a nonnegative integer, and let $g_{q,r}(X)$ be the 2-adic analytic function which is given in Theorem 4.3 with $u = v = 1$, where r and q is nonnegative integers less than 4. Then

$$\frac{h_{16y+4q+r}(P)}{(16y+4q+r)!} 2^{6y+q} y! = g_{q,r}(y) \equiv \begin{cases} c_r(1+2y) \pmod 4 & \text{if } q = 0, \\ -c_r(1+2y) + 2c_{4+r} \pmod 4 & \text{if } q = 1, \\ -c_r(1-2y) - 2c_{4+r} \pmod 4 & \text{if } q = 2, \\ -2c_{4+r} \pmod 4 & \text{if } q = 3. \end{cases} \tag{4.10}$$

Note that by Lemma 2.4,

$$\mathrm{ord}_2\left(\frac{(16y+4q+r)!}{2^{6y+q}y!}\right) = \tau_2^{(1,1)}(16y+4q+r) = 9y + 2q + \left[\frac{q}{2}\right] + \left[\frac{r}{2}\right].$$

From Eq.(4.10), we can obtain [20, Theorem 7.1(2)] with $u = 1$, namely,

$$h_{16y+4q+r}(P) \equiv \frac{(-1)^y(16y+4q+r)!}{2^{6y}y!(4q+r)!} h_{4q+r}(P) \pmod{2^{\tau_2^{(1,1)}(16y+4q+r)+2}}.$$

Set $e_n(P) = \mathrm{ord}_2(h_n(P)) - \tau_2^{(1,1)}(n)$ for $n = 1, 2, \cdots$. Then $e_{16y+4q+r}(P) = \mathrm{ord}_2(g_{q,r}(y))$. We are interested in $e_{16y+4q+r}(P)$. Observe that $\mathrm{ord}_2(c_r) \geq 1$ if and only if $r = 2$ and that $\mathrm{ord}_2(c_{4+r}) = 0$ if and only if $r = 2$. In the case where $r \neq 2$, Eq.(4.10) yields

$$\begin{aligned} e_{16y+4q+r}(P) &= 0 \quad && \text{if } q = 0,\ q = 1, \text{ or } q = 2, \\ e_{16y+4q+r}(P) &\geq 2 && \text{if } q = 3. \end{aligned}$$

With the help of Mathematica, we have the following.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$e_n(P)$	0	0	1	0	0	0	3	0	0	0	3	0	2	6	1	2

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$e_n(P)$	0	0	1	0	0	0	5	0	0	0	4	0	7	2	1	3

n	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
$e_n(P)$	0	0	1	0	0	0	3	0	0	0	3	0	2	3	1	2

What about $e_{16y+4q+r}(P)$ in the case where $r = 2$? If $r = 2$, then

$$\frac{h_{16y+4q+2}(P)}{(16y+4q+2)!}2^{6y+q}y! = g_{q,2}(y) \equiv \begin{cases} 2 \pmod 4 & \text{if } q = 0, \\ -2 + 2 \cdot \dfrac{121}{45} = 8 \cdot \dfrac{19}{45} \pmod 4 & \text{if } q = 1, \\ -2 - 2 \cdot \dfrac{121}{45} = -4 \cdot \dfrac{83}{45} \pmod 4 & \text{if } q = 2, \\ -2 \cdot \dfrac{121}{45} \pmod 4 & \text{if } q = 3, \end{cases}$$

and hence

$$\begin{aligned} e_{16y+4q+2}(P) &= 1 \quad \text{if } q = 0 \text{ or } q = 3, \\ e_{16y+4q+2}(P) &\geq 2 \quad \text{if } q = 1 \text{ or } q = 2. \end{aligned}$$

To explain the reason why $e_{16y+6}(P)$ and $e_{16y+10}(P)$ with $y = 0, 1, 2$ are greater than 2, we calculate $g_{1,2}(y)$ and $g_{2,2}(y)$ with $y = 0, 1, 2$. By Eq.(4.8) with $u = q = 1$ and $r = 2$,

$$\sum_{y=0}^{\infty} \frac{h_{16y+6}(P)}{(16y+6)!} 2^{6y+1} X^y = \exp(X) \left(\sum_{j=0}^{\infty} w_{4j+1} 2^{6j+1} X^j \right) T_2(-X). \tag{4.11}$$

The 2-adic analytic function $g_{1,2}(X)$ is obtained by applying Lemma 2.6 to this formula. Observe that by Lemma 2.5, $T_2(X) \in 1 + 3 \cdot 2^3 X + 2^2 X^2 \mathbb{Z}_2\langle X \rangle$. We also have

$$\sum_{j=0}^{\infty} w_{4j+1} 2^{6j+1} X^j \in w_1 2 + w_5 2^7 X + 2^2 X^2 \mathbb{Z}_2\langle X \rangle$$

(see the proof of Theorem 4.3). By Eq.(4.9) with $u = v = q = 1$ and $r = 2$,

$$w_1 2 = -c_2 + c_6 2 \quad \text{and} \quad w_5 2^7 = \sum_{i=0}^{5} \sum_{k=0}^{[i/2]} \sum_{z=0}^{[k/2]} w_1(i,k,z),$$

where

$$w_1(i,k,z) = (-1)^{i-k} c_{22-4i} 2^{7-i+k-3z} \frac{h_{i-2k}(C_8)}{(i-2k)!} \cdot \frac{h_{k-2z}(C_2)}{(k-2z)!} \cdot \frac{h_z(C_2)}{z!}.$$

Since $c_2 = 2$ and $c_6 = 121/45$, it turns out that

$$\begin{aligned} w_1(0,0,0) &= c_{22} 2^7, & w_1(1,0,0) &= -c_{18} 2^6, \\ w_1(2,0,0) &= c_{14} 2^5, & w_1(2,1,0) &= -c_{14} 2^6, \\ w_1(3,0,0) &= -c_{10} 2^4 \cdot \frac{2}{3} = -c_{10} \frac{2^5}{3}, & w_1(3,1,0) &= c_{10} 2^5, \\ w_1(4,0,0) &= c_6 2^3 \cdot \frac{2}{3} = \frac{1936}{135}, & w_1(4,1,0) &= -c_6 2^4 = -\frac{1936}{45}, \\ w_1(4,2,0) &= c_6 2^5 = \frac{3872}{45}, & w_1(4,2,1) &= c_6 2^2 = \frac{484}{45}, \\ w_1(5,0,0) &= -c_2 2^2 \cdot \frac{7}{15} = -\frac{56}{15}, & w_1(5,1,0) &= c_2 2^3 \cdot \frac{2}{3} = \frac{32}{3}, \\ w_1(5,2,0) &= -c_2 2^4 = -32, & w_1(5,2,1) &= -c_2 2 = -4, \end{aligned}$$

and

$$w_5 2^7 = -4 - 32 + \frac{32}{3} - \frac{56}{15} + \frac{484}{45} + \frac{3872}{45} - \frac{1936}{45} + \frac{1936}{135} + a = \frac{5272}{135} + a = 8 \cdot \frac{659}{135} + a,$$

where $a = c_{10}2^6/3 - c_{14}2^5 - c_{18}2^6 + c_{22}2^7$. Thus we have

$$w_1 2 + w_5 2^7 X = 8 \cdot \frac{19}{45} + 8 \cdot \frac{659}{135} X + aX.$$

Since $T_2(X) \in 1 + 3 \cdot 2^3 X + 2^2 X^2 \mathbb{Z}_2\langle X\rangle$, it follows from Lemma 2.6 and Eq.(4.11) that

$$g_{1,2}(X) \in \left(8 \cdot \frac{19}{45} + 8 \cdot \frac{659}{135} X + aX\right)(1 - 3 \cdot 2^3 X) + \sum_{i=2}^{\ell} a_i X^{\underline{i}} + 2^6 X \mathbb{Z}_2\langle X\rangle$$

for some $a_2, a_3, \dots, a_\ell \in 2^2\mathbb{Z}_2$. Likewise, by using Eqs.(4.8) and (4.9) with $u = v = 1$ and $q = r = 2$, we have

$$g_{2,2}(X) \in \left(-4 \cdot \frac{83}{45} + c_{10}2^2 - 32 \cdot \frac{2243}{675} X + bX\right)(1 - 3 \cdot 2^3 X) + \sum_{i=2}^{m} b_i X^{\underline{i}} + 2^5 X \mathbb{Z}_2\langle X\rangle,$$

where $b = c_{10}(2^3 + 2^7/3) + c_{14}2^7/3 - c_{18}2^6 - c_{22}2^7 + c_{26}2^8$, for some $b_2, b_3, \dots, b_m \in 2^2\mathbb{Z}_2$. According to Mathematica,

$$c_{10} = \frac{33991}{14175} \quad \text{and} \quad c_{14} = \frac{69934958}{42567525}.$$

In particular, $\mathrm{ord}_2(a) \geq 6$ and $\mathrm{ord}_2(b) = 3$. Consequently, we have

$$\begin{aligned}
g_{1,2}(0) &= 8 \cdot \frac{19}{45}, \qquad g_{1,2}(1) \equiv 8 \cdot \frac{19}{45} + 8 \cdot \frac{659}{135} = 32 \cdot \frac{179}{135} \pmod{2^6},\\
g_{2,2}(0) &= -4 \cdot \frac{83}{45} + 4 \cdot \frac{33991}{14175} = 8 \cdot \frac{3923}{14175},\\
g_{2,2}(1) &\equiv 8 \cdot \frac{3923}{14175} - 32 \cdot \frac{2243}{675} + c_{10}2^3 = -16 \cdot \frac{929}{175} \pmod{2^5},
\end{aligned}$$

which implies that $e_6(P) = 3$, $e_{10}(P) = 3$, $e_{22}(P) = 5$, and $e_{26}(P) = 4$. If $y \geq 2$, then $2^2 y(y-1) \equiv 0 \pmod{2^3}$, whence $e_{16y+6}(P) \geq 3$ and $e_{16y+10}(P) \geq 3$.

5 Abelian p-groups

Recall that G is a finite abelian p-group of order p^s and of type $\lambda = (\lambda_1, \lambda_2, \dots)$,

$$u = \max\left\{\lambda_1, \left[\frac{s+1}{2}\right]\right\}, \quad \text{and} \quad v = s - u = \min\left\{s - \lambda_1, \left[\frac{s}{2}\right]\right\}.$$

In this section we study the p-adic properties of $h_n(G)$ and extend the theorems on a p-adic estimate of $h_n(P)$. The facts about the number of subgroups of G and the formula of $E_G(X)$ given in [20] play an important role in the study of p-adic properties of $h_n(G)$.

For each nonnegative integer i, we denote by $\alpha_\lambda(i;p)$ the number of subgroups of order p^i in a finite abelian p-group of type λ. By the Wohlfahrt formula (1.1),

$$E_G(X) = \exp\left(\sum_{k=0}^{\infty} \frac{\alpha_\lambda(s-k;p)}{p^k} X^{p^k}\right).$$

There exist nonnegative integers $a_{i,j}$ for $i, j \in \mathbb{Z}$ such that $\alpha_\lambda(i;p) = \sum_j a_{i,j}p^j$ which depend only on λ and i (see, *e.g.*, [6]). By the duality of finite abelian p-groups,

$$\alpha_\lambda(i;p) = \alpha_\lambda(s-i;p),$$

and hence $a_{i,j} = a_{s-i,j}$. Obviously, $\alpha_\lambda(0;p) = \alpha_\lambda(s;p) = 1$. If $\lambda_r \geq 1$ and $\lambda_{r+1} = 0$, then $\alpha_\lambda(1;p) = \alpha_\lambda(s-1;p) = 1 + p + \cdots + p^{r-1}$.

For each pair $(\ell, m) \in \mathbb{Z} \times \mathbb{Z}$ of nonnegative integers ℓ and m with $m \leq s$, we set $b_{\ell,m} = a_{\ell,m} - a_{\ell-1,m-1}$ and

$$t_{\ell,m} = \begin{cases} b_{\ell,m} - b_{\ell-1,m} & \text{if } 0 \leq \ell \leq s-m \text{ and } 0 \leq m \leq v, \\ a_{\ell,m} - a_{\ell-1,m} & \text{if } 0 \leq \ell \leq s-m \text{ and } v < m \leq s, \\ a_{m,\ell} & \text{if } \ell > s-m. \end{cases}$$

The following theorem is [20, Theorem 4.1].

Theorem 5.1 *The formal power series $E_G(X)$ is expressed in the form*

$$E_G(X) = \sum_{n=0}^{\infty} \frac{h_n(G)}{n!} X^n = \Phi_\lambda(X) \prod_{m=0}^{s} \prod_{\ell=s-m+1}^{\infty} \exp(p^{\ell+m-s} X^{p^{s-m}})^{t_{\ell,m}},$$

where

$$\Phi_\lambda(X) = \prod_{m=0}^{v} \prod_{\ell=0}^{m} E_{C_{p^{u-\ell}} \times C_{p^{v-m}}}(X^{p^m})^{t_{\ell,m}} \prod_{m=v+1}^{s} \prod_{\ell=0}^{s-m} E_{C_{p^{s-\ell-m}}}(X^{p^m})^{t_{\ell,m}}.$$

We quote [20, Proposition 4.1, Lemma 4.1, Lemma 4.2], namely,

Lemma 5.2 (1) *The integer $t_{\ell,m}$ is nonnegative for any nonnegative integers ℓ and m with $m \leq s$. In particular, $t_{0,0} = 1$, and $t_{0,m} = 0$ if $m \geq 1$.*

(2) *Suppose that $0 \leq m \leq v$. Unless $\lambda_3 \geq 1$ and $(\ell, m) = (1, 2)$, $t_{\ell,m} = 0$ for any positive integer ℓ with $m - 1 \leq \ell \leq s - m$. If $\lambda_3 \geq 1$ and $m = 2$, then $t_{1,2} = 1$.*

(3) *Suppose that $v < m \leq u$. If $u = \lambda_1$, then $t_{\ell,m} = 0$ for any integer ℓ with $v \leq \ell \leq s-m$. If $u > \lambda_1$, then $u = v + 1$ and $t_{v,v+1} = 1$.*

The following lemma is deduced from Lemma 5.2(3).

Lemma 5.3 *Suppose that $v < m \leq s$ and $v \leq \ell \leq s - m$. If $u = \lambda_1$, then $t_{\ell,m} = 0$. If $u > \lambda_1$, then $\ell = v$, $m = u = v + 1$, and $t_{\ell,m} = 1$.*

Proof. By the hypotheses, we have $v < m \leq u$ and $v \leq \ell \leq s - m < u$. Hence the assertion follows from Lemma 5.2(3). □

We provide one more lemma which states the value of $t_{v-1,v+1}$.

Lemma 5.4 *The following statements hold.*

(1) *Suppose that* $\lambda_3 \geq 1$ *and* $\lambda_4 = 0$. *Then*

$$t_{v-1,v+1} = \begin{cases} 0 & \textit{if either } \lambda_3 = 1 \textit{ or } \lambda_3 \geq 2 \textit{ and } u = \lambda_1, \\ 1 & \textit{if } \lambda_3 \geq 2 \textit{ and } u > \lambda_1. \end{cases}$$

(2) *Suppose that* $\lambda_4 \geq 1$. *Then*

$$t_{v-1,v+1} = \begin{cases} 1 & \textit{if either } u = s-2 \textit{ or } u = \lambda_1, \\ 2 & \textit{if } \lambda_1 < u < s-2. \end{cases}$$

Proof. There exist nonnegative integers $\widehat{a}_{ij}$ for i, $j \in \mathbb{Z}$ such that $\alpha_{\widehat{\lambda}}(i;p) = \sum_j \widehat{a}_{ij} p^j$ with $\widehat{\lambda} = (\lambda_2, \lambda_3, \dots)$ which depend only on $\widehat{\lambda}$ and i. By [20, Lemma 3.1], we have

$$t_{v-1,v+1} = a_{v-1,v+1} - a_{v-2,v+1} = \widehat{a}_{v-1,2}.$$

Hence the assertion is a consequence of [19, Proposition 5.3] ([20, Proposition 3.2]) and [20, Proposition 3.1(2),(4),(5)]. □

Under the notation of Theorem 5.1, it follows from Lemmas 5.2(1),(2) and 5.3 that

$$\Phi_\lambda(X) = E_{C_{p^u} \times C_{p^v}}(X) \prod_{m=2}^{v} \prod_{\ell=1}^{m-1} E_{C_{p^{u-\ell}} \times C_{p^{v-m}}}(X^{p^m})^{t_{\ell,m}} \prod_{\ell=1}^{v} \prod_{m=v+1}^{s-\ell} E_{C_{p^{s-\ell-m}}}(X^{p^m})^{t_{\ell,m}}.$$

To extend Eqs.(3.1) and (3.2), we define a formal power series $\sum_{n=0}^{\infty} \dot{c}_n^{(v)} X^n$ by

$$\sum_{n=0}^{\infty} \dot{c}_n^{(v)} X^n = E_p^{(v)}(X) \prod_{m=2}^{v} \prod_{\ell=1}^{m-1} E_p^{(v-m)}(X^{p^m})^{t_{\ell,m}} \prod_{\ell=1}^{v} \prod_{m=v+1}^{s-\ell} E_p^{(0)}(X^{p^m})^{t_{\ell,m}} \times \prod_{m=0}^{s} \prod_{\ell=s-m+1}^{\infty} \exp(p^{\ell+m-s} X^{p^{s-m}})^{t_{\ell,m}},$$

so that

$$\sum_{n=1}^{\infty} \dot{c}_n^{(v)} X^n \in X\mathbb{Z}_p[[X]]$$

(see Proposition 2.3 and Lemma 2.4). Note that the p-adic integers $\dot{c}_n^{(v)}$ for $n \in \mathbb{N}$ depend on λ. We next define formal power series $D_p^{(\ell,m)}(X)$ with $(\ell, m) \in \mathbb{Z} \times \mathbb{Z}$ and $D_p(X)$ by

$$D_p^{(\ell,m)}(X) = \begin{cases} \exp\left(-\sum_{i=0}^{\infty} \frac{[v-m+1]_p - [v-m-i]_p}{p^{u-\ell+i+1}} (-p^{u-v+1}X)^{p^{i+m-\ell}} \right) \\ \qquad \text{if } 2 \leq m \leq v \text{ and } 1 \leq \ell \leq m-1, \\ \exp\left(-\sum_{i=0}^{\infty} \frac{1}{p^{s-\ell-m+i+1}} (-p^{u-v+1}X)^{p^{i+v-\ell}} \right) \\ \qquad \text{if } 1 \leq \ell \leq v < m \leq s-\ell \end{cases}$$

(cf. Eqs.(2.1) and (2.2)) and

$$D_p(X) = F_p^{(u,v)}(X) \prod_{m=2}^{v} \prod_{\ell=1}^{m-1} D_p^{(\ell,m)}(X)^{t_{\ell,m}} \prod_{\ell=1}^{v} \prod_{m=v+1}^{s-\ell} D_p^{(\ell,m)}(X)^{t_{\ell,m}}.$$

Let r be a nonnegative integer less that p^{u+1}. By Theorem 5.1 and an argument analogous to that in the case where $G = P$, we have

$$\sum_{y=0}^{\infty} \frac{h_{p^{u+1}y+r}(G)}{(p^{u+1}y+r)!} X^{p^{u+1}y} = \left(\sum_{j=0}^{\infty} \dot{c}^{(v)}_{p^{u+1}j+r} X^{p^{u+1}j} \right) \times \exp\left(-\frac{1}{p^{u-v+1}} X^{p^{u+1}} \right) D_p\left(-\frac{1}{p^{u-v+1}} X^{p^{u+1}} \right) \quad (5.1)$$

and

$$\sum_{y=0}^{\infty} \frac{h_{p^{u+1}y+r}(G)}{(p^{u+1}y+r)!} (-p^{u-v+1}X)^y = \exp(X) \left(\sum_{j=0}^{\infty} \dot{c}^{(v)}_{p^{u+1}j+r} (-p^{u-v+1}X)^j \right) D_p(X), \quad (5.2)$$

which are the extensions of Eqs.(3.1) and (3.2), respectively.

The p-adic properties of $h_n(G)$ are explored on the basis of Eqs.(5.1) and (5.2).

Lemma 5.5 *Suppose that $2 \le m \le v$ and $1 \le \ell \le m-1$. Let i be a nonnegative integer.*

(1) *If $p \ge 3$, then*

$$\mathrm{ord}_p\left(\frac{[v-m+1]_p - [v-m-i]_p}{p^{u-\ell+i+1}} \, p^{(u-v+1)p^{i+m-\ell}} \right) \ge (p-2)(u-v+1+i) + u - v.$$

(2) *If $p = 2$, then*

$$\mathrm{ord}_2\left(\frac{[v-m+1]_2 - [v-m-i]_2}{2^{u-\ell+i+1}} \, 2^{(u-v+1)2^{i+m-\ell}} \right) \ge (u-v)(2i+m-\ell).$$

Proof. Since $m - \ell \ge 1$ and $p^{i+m-\ell} \ge (i+m-\ell)p$, it follows that

$$\begin{aligned} &\mathrm{ord}_p\left(\frac{[v-m+1]_p - [v-m-i]_p}{p^{u-\ell+i+1}} \, p^{(u-v+1)p^{i+m-\ell}} \right) \\ &\quad \ge v-m-i+(u-v+1)p^{i+m-\ell} - (u-\ell+i+1) \\ &\quad \ge (u-v+1)(i+m-\ell)p - 2i - (u-v+1) - (m-\ell) \\ &\quad \ge (p-2)(u-v+1) + (u-v+1)(pi+2m-2\ell-1) - 2i - (m-\ell) \\ &\quad \ge (p-2)(u-v+1+i) + (u-v)(pi+2m-2\ell-1) + m - \ell - 1. \end{aligned}$$

The assertions are consequences of this fact. □

Lemma 5.6 *Suppose that $1 \le \ell < v < m \le s - \ell$. Let i be a nonnegative integer.*

(1) *If* $p \geq 3$*, then*

$$\mathrm{ord}_p\left(\frac{1}{p^{s-\ell-m+i+1}}\, p^{(u-v+1)p^{i+v-\ell}}\right) \geq u-v+2.$$

(2) *If* $p = 2$*, then*

$$\mathrm{ord}_2\left(\frac{1}{2^{s-\ell-m+i+1}}\, 2^{(u-v+1)2^{i+v-\ell}}\right) \geq (u-v)(v-\ell)+m-v+i.$$

Proof. Since $v > \ell$ and $p^{i+v-\ell} \geq (i+v-\ell)p$, it follows that

$$\begin{aligned}\mathrm{ord}_p&\left(\frac{1}{p^{s-\ell-m+i+1}}\, p^{(u-v+1)p^{i+v-\ell}}\right)\\ &= (u-v+1)p^{i+v-\ell} - (s-\ell-m+i+1)\\ &\geq (u-v+1)(i+v-\ell)p - (s-\ell-m+i+1)\\ &\geq (p-1)(u-v+1+i) + (u-v+1)(v-\ell) - (s-\ell-m+1)\\ &= (p-2)(u-v+1) + (u-v)(v-\ell) + m - v + i.\end{aligned}$$

The assertions are consequences of this fact. □

The following lemma is an extension of Lemma 3.4.

Lemma 5.7 *The following statements hold.*

(1) *If* $p = 3$*,* $u = \lambda_1$*, and* $u + \delta_{v0} = v+1$*, then*

$$D_3(X) \in 1 + 3^{u-v+1}X^3 + 3^{u-v+2}X\mathbb{Z}_3\langle X\rangle.$$

(2) *If* $p = 3$ *and* $u + \delta_{v0} > v+1$ *or if* $p > 3$ *and* $u = \max\{\lambda_1, v\}$*, then*

$$D_p(X) \in 1 + p^{u-v+2}X\mathbb{Z}_p\langle X\rangle.$$

(3) *If* $p \geq 3$ *and* $u = v+1 > \lambda_1$*, then*

$$D_p(X) \in 1 + pX + \frac{p^2}{2}X^2 + p^3X\mathbb{Z}_p\langle X\rangle.$$

Proof. If $u = \lambda_1$, then by Lemma 5.2(3), $t_{v,m} = 0$ for any integer m with $v < m \leq u$. On the other hand, if $u = v+1 > \lambda_1$, then by Lemmas 2.5 and 5.2(3), $t_{v,v+1} = 1$ and

$$\begin{aligned}D_p^{(v,v+1)}(X) &= \exp\left(\sum_{i=0}^{\infty} p^{2p^i-i-1}\, X^{p^i}\right)\\ &\in \begin{cases} 1 + 3X + \dfrac{3^2}{2}X^2 + \dfrac{3^2}{2}X^3 + 3^3X\mathbb{Z}_3\langle X\rangle & \text{if } p = 3,\\ 1 + pX + \dfrac{p^2}{2}X^2 + p^3X\mathbb{Z}_p\langle X\rangle & \text{if } p > 3.\end{cases}\end{aligned}$$

Hence the assertion follows from Lemmas 2.5, 3.4, 5.5, and 5.6. □

Combining Eqs.(5.1) and (5.2) and Lemmas 5.4–5.7 with arguments in the preceding sections, we are successful in extending Theorems 3.6, 3.9, 3.11, 4.1, and 4.2 as follows.

Theorem 5.8 *Let r be a nonnegative integer less than p^{u+1}.*

(1) *Suppose that either $p \geq 3$, $u = \lambda_1$, and $u + \delta_{v0} \geq v + 1$ or $p > 3$ and $u = v \geq \lambda_1$. Then there exists a p-adic analytic function $g_r(X)$ contained in $\mathbb{Z}_p\langle X\rangle$ such that*

$$g_r(y) = \frac{h_{p^{u+1}y+r}(G)}{(p^{u+1}y+r)!}(-p^{u-v+1})^y y!$$

for any nonnegative integer y and

$$g_r(X) \in \widehat{g}_r(X) + p^{u-v+2}X\mathbb{Z}_p\langle X\rangle,$$

where

$$\widehat{g}_r(X) = \begin{cases} \dot{c}_r^{(0)}(1+3X^{\underline{3}}) - \dot{c}_{3+r}^{(0)}3X & \text{if } p=3 \text{ and } u=v=0, \\ \dot{c}_r^{(v)}(1+3^2X^{\underline{3}}) - \dot{c}_{3^{u+1}+r}^{(v)}3^2X & \text{if } p=3 \text{ and } u=v+1 \geq 2, \\ \dot{c}_r^{(v)} - \dot{c}_{p^{u+1}+r}^{(v)}p^{u-v+1}X & \text{otherwise.} \end{cases}$$

(2) *Suppose that $p \geq 3$ and $u = v + 1 > \lambda_1$. Then there exists a p-adic analytic function $g_r(X)$ contained in $\mathbb{Z}_p\langle X\rangle$ such that*

$$g_r(y) = \frac{h_{p^{u+1}y+r}(G)}{(p^{u+1}y+r)!}(-p^2)^y y!$$

for any nonnegative integer y and

$$g_r(X) \in \dot{c}_r^{(v)}\left(1 + pX + \frac{p^2}{2}X^{\underline{2}}\right) - \dot{c}_{p^{u+1}+r}^{(v)}p^2X + p^3X\mathbb{Z}_p\langle X\rangle.$$

Proof. The theorem follows from Lemmas 2.6 and 5.7 and Eq.(5.2). □

Theorem 5.9 *Suppose that $p = 2$ and $u + \delta_{v0} \geq v + 3$. Let r be a nonnegative integer less than 2^{u+1}. Then there exists a 2-adic analytic function $g_r(X)$ contained in $\mathbb{Z}_2\langle X\rangle$ such that*

$$g_r(y) = \frac{h_{2^{u+1}y+r}(G)}{(2^{u+1}y+r)!}(-2^{u-v+1})^y y!$$

for any nonnegative integer y and

$$g_r(X) \in \dot{g}_r(X) - \dot{c}_{2^{u+1}+r}^{(v)}2^{u-v+1}X + 2^{u-v+2}X\mathbb{Z}_2\langle X\rangle,$$

where

$$\dot{g}_r(X) = \begin{cases} \dot{c}_r^{(v)}(1 - 2^uX^{\underline{2}} + 2^{2u-1}X^{\underline{4}}) & \text{if } \lambda_2 = 0, \\ \dot{c}_r^{(v)}(1 - 3\cdot 2^{u-v-1}X^{\underline{2}} + 9\cdot 2^{2u-2v-3}X^{\underline{4}}) & \text{if } \lambda_2 \geq 1 \text{ and } \lambda_3 = 0, \\ \dot{c}_r^{(v)}(1 - 5\cdot 2^{u-v-1}X^{\underline{2}} + 9\cdot 2^{2u-2v-3}X^{\underline{4}}) & \text{if } \lambda_3 \geq 1 \text{ and } \lambda_4 = 0, \\ \dot{c}_r^{(v)}(1 - 9\cdot 2^{u-v-1}X^{\underline{2}} + 9\cdot 2^{2u-2v-3}X^{\underline{4}}) & \text{if } \lambda_4 \geq 1. \end{cases}$$

Proof. If $\lambda_3 = 0$, then the assertion is given in Theorem 3.9. Assume that $\lambda_3 \geq 1$. Then $v \geq 2$, because $u = \lambda_1 \geq v + 3$. Hence it follows from Lemma 5.2(2) that $t_{1,2} = 1$ and $t_{m-1,m} = 0$ for any integer m with $2 < m \leq v$. By Lemmas 2.5 and 5.5, we have

$$\begin{aligned} D_2^{(1,2)}(X)^{t_{1,2}} &= \exp\left(-\sum_{i=0}^{\infty} \frac{[v-1]_2 - [v-i-2]_2}{2^{u+i}}(-2^{u-v+1}X)^{2^{i+1}}\right) \\ &\in 1 - 2^{u-v}X^2 + 2^{u-v+2}X\mathbb{Z}_2\langle X\rangle. \end{aligned}$$

Since $\lambda_3 \geq 1$ and $u = \lambda_1$, Lemma 5.4 asserts that

$$t_{v-1,v+1} = \begin{cases} 0 & \text{if } \lambda_4 = 0, \\ 1 & \text{if } \lambda_4 \geq 1. \end{cases}$$

If $\lambda_4 \geq 1$, then by Lemmas 2.5 and 5.6, we have

$$\begin{aligned} D_2^{(v-1,v+1)}(X)^{t_{v-1,v+1}} &= \exp\left(-\sum_{i=0}^{\infty} \frac{1}{2^{u-v+i+1}}(-2^{u-v+1}X)^{2^{i+1}}\right) \\ &\in 1 - 2^{u-v+1}X^2 + 2^{u-v+2}X\mathbb{Z}_2\langle X\rangle. \end{aligned}$$

Moreover, it follows from Lemma 5.2(3) that $t_{v,m} = 0$ for any integer m with $v < m \leq u$. Combining these facts with Lemmas 2.5, 3.8, 5.5, and 5.6, we have

$$D_2(X) \in \begin{cases} 1 - 5\cdot 2^{u-v-1}X^2 + 9\cdot 2^{2u-2v-3}X^4 + 2^{u-v+2}X\mathbb{Z}_2\langle X\rangle & \text{if } \lambda_4 = 0, \\ 1 - 9\cdot 2^{u-v-1}X^2 + 9\cdot 2^{2u-2v-3}X^4 + 2^{u-v+2}X\mathbb{Z}_2\langle X\rangle & \text{if } \lambda_4 \geq 1. \end{cases}$$

Hence the assertion is a consequence of Lemma 2.6 and Eq.(5.2). □

Theorem 5.10 *Suppose that $p = 2$ and $u + \delta_{v0} = v + 2$. Let r be a nonnegative integer less than 2^{u+1}. Then there exists a 2-adic analytic function $g_r(X)$ contained in $\mathbb{Z}_2\langle X\rangle$ such that*

$$g_r(y) = \frac{h_{2^{u+1}y+r}(G)}{(2^{u+1}y+r)!}\,2^{(u-v+1)y}y!$$

for any nonnegative integer y and

$$g_r(X) \in \dot{g}_r(X) + \dot{c}^{(v)}_{2^{u+1}+r}2^{u-v+1}X + 2^{u-v+2}X\mathbb{Z}_2\langle X\rangle,$$

where

$$\dot{g}_r(X) = \begin{cases} \dot{c}_r^{(0)}(1 - 2X - 4X^{\underline{4}}) & \text{if } \lambda_2 = 0, \\ \dot{c}_r^{(v)}(1 - 2X - 4X^{\underline{2}} - 4X^{\underline{4}} + 8X^{\underline{5}} + 8X^{\underline{8}}) & \text{if } \lambda_2 \geq 1 \text{ and } \lambda_3 = 0, \\ \dot{c}_r^{(v)}(1 - 2X - 8X^{\underline{2}} + 8X^{\underline{3}} + 4X^{\underline{4}} + 8X^{\underline{5}} + 8X^{\underline{8}}) & \text{if } \lambda_3 \geq 1 \text{ and } \lambda_4 = 0, \\ \dot{c}_r^{(v)}(1 - 2X + 8X^{\underline{3}} + 4X^{\underline{4}} + 8X^{\underline{5}} + 8X^{\underline{8}}) & \text{if } \lambda_4 \geq 1. \end{cases}$$

Proof. If $\lambda_3 = 0$, then the assertion is given in Theorem 3.11. Assume that $\lambda_3 \geq 1$. Then $v \geq 2$, because $u = \lambda_1 = v + 2$. By Lemmas 2.5, 3.10, 5.2, and 5.4–5.6 and an argument analogous to that in the proof Theorem 5.9, we have $D_2(-X) = D_2(X)$ and

$$\exp(-2X)D_2(X) \in \overline{D}_2(X) + 16X\mathbb{Z}_2\langle X\rangle,$$

where

$$\overline{D}_2(X) = \begin{cases} 1 - 2X - 8X^2 + 8X^3 + 4X^4 + 8X^5 + 8X^8 & \text{if } \lambda_4 = 0, \\ 1 - 2X + 8X^3 + 4X^4 + 8X^5 + 8X^8 & \text{if } \lambda_4 \geq 1. \end{cases}$$

Substitute $-X$ for X in Eq.(5.2). Then it turns out that

$$\sum_{y=0}^{\infty} \frac{h_{2^{u+1}y+r}(G)}{(2^{u+1}y+r)!}(2^{u-v+1}X)^y = \exp(X)\exp(-2X)\left(\sum_{j=0}^{\infty} \dot{c}^{(v)}_{2^{u+1}j+r}(2^{u-v+1}X)^j\right) D_2(X).$$

Hence the assertion is a consequence of Lemma 2.6. □

Theorem 5.11 *Suppose that $p = 3$ and $u = v \geq 1$. Let r be a nonnegative integer less than 3^{u+1} and q a nonnegative integer less than 3. Then there exists a 3-adic analytic function $g_{q,r}(X)$ contained in $\mathbb{Z}_3\langle X\rangle$ such that*

$$g_{q,r}(y) = \frac{h_{3^{u+1}(3y+q)+r}(G)}{(3^{u+1}(3y+q)+r)!}3^{4y+q}y!$$

for any nonnegative integer y and

$$g_{q,r}(X) \in \widehat{g}_{q,r}(X) + 3^2X\mathbb{Z}_3\langle X\rangle,$$

where

$$\widehat{g}_{q,r}(X) = \begin{cases} \dot{c}^{(v)}_r\left(1 - 3^3\dfrac{h_9(C_3)}{9!}X^{\underline{3}}\right) & \text{if } q = 0, \\ \dot{c}^{(v)}_r\left(-1 + 3X + 3^3\dfrac{h_{10}(C_3)}{10!}X^{\underline{3}}\right) + 3\dot{c}^{(v)}_{3^{u+1}+r} & \text{if } q = 1, \\ \dot{c}^{(v)}_r\left(\dfrac{1}{2} - 3^3\dfrac{h_{11}(C_3)}{11!}X^{\underline{3}}\right) - 3\dot{c}^{(v)}_{3^{u+1}+r} + 3^2\dot{c}^{(v)}_{2\cdot 3^{u+1}+r} & \text{if } q = 2. \end{cases}$$

Proof. If $\lambda_3 = 0$, then the assertion is given in Theorem 4.1. Assume that $\lambda_3 \geq 1$, and set

$$\begin{aligned} H_3(X) &= \exp\left(-3^2X - \sum_{i=2}^{\infty}\frac{[v+1]_3 - [v-i]_3}{3^{u+i+1}}(3^4X)^{3^{i-1}}\right) \\ &\quad \times \prod_{m=2}^{v}\prod_{\ell=1}^{m-1}\exp\left(-\sum_{i=0}^{\infty}\frac{[v-m+1]_3 - [v-m-i]_3}{3^{u-\ell+i+1}}(3^4X)^{3^{i+m-\ell-1}}\right)^{t_{\ell,m}} \\ &\quad \times \prod_{\ell=1}^{v-1}\prod_{m=v+1}^{s-\ell}\exp\left(-\sum_{i=0}^{\infty}\frac{1}{3^{s-\ell-m+i+1}}(3^4X)^{3^{i+v-\ell-1}}\right)^{t_{\ell,m}}. \end{aligned}$$

Then by Lemmas 2.5, 3.3, 5.5, and 5.6, $H_3(X) \in 1 + 3^2X\mathbb{Z}_3\langle X\rangle$. Hence we can show the assertion in a similar fashion as in the proof of Theorem 4.1, completing the proof. □

Theorem 5.12 *Suppose that $p = 2$ and $u = v + 1 \geq 2$. Let r be a nonnegative integer less than 2^{u+1} and q a nonnegative integer less than 2. Define a polynomial $g_{q,r}(X)$ by*

$$g_{q,r}(X) = \begin{cases} \dot{c}^{(v)}_r(1 - X - X^2) & \text{if } q = 0, \\ (-1)^{\delta_{u\lambda_1}}\dot{c}^{(v)}_r(1 + X - X^2) & \text{if } q = 1. \end{cases}$$

Then for any nonnegative integer y*,*

$$\frac{h_{2^{u+1}(2y+q)+r}(G)}{(2^{u+1}(2y+q)+r)!}\,2^{5y+2q}y! \equiv g_{q,r}(y) \pmod{2^2}.$$

Proof. If $\lambda_3 = 0$, then the assertion is given in Theorem 4.2. Assume that $\lambda_3 \geq 1$. By Lemma 5.2(3), $t_{v,u} = 0$ if $u = \lambda_1$, and $t_{v,u} = 1$ otherwise. We set

$$\begin{aligned} H_2(X) = \exp&\left(-2^3X - \sum_{i=2}^{\infty}\frac{[v+1]_2-[v-i]_2}{2^{u+i+1}}(2^5X)^{2^{i-1}}\right)\\ &\times\prod_{m=2}^{v}\prod_{\ell=1}^{m-1}\exp\left(-\sum_{i=0}^{\infty}\frac{[v-m+1]_2-[v-m-i]_2}{2^{u-\ell+i+1}}(2^5X)^{2^{i+m-\ell-1}}\right)^{t_{\ell,m}}\\ &\times\prod_{\ell=1}^{v-1}\prod_{m=v+1}^{s-\ell}\exp\left(-\sum_{i=0}^{\infty}\frac{1}{2^{s-\ell-m+i+1}}(2^5X)^{2^{i+v-\ell-1}}\right)^{t_{\ell,m}}\\ &\times\exp\left(-2^3X-\sum_{i=1}^{\infty}\frac{1}{2^{i+1}}(2^5X)^{2^{i-1}}\right)^{(1-\delta_{u\lambda_1})}. \end{aligned}$$

For any positive integer i,

$$\mathrm{ord}_2\left(\frac{1}{2^{i+1}}\,2^{5\cdot 2^{i-1}}\right) \geq 5\cdot 2^{i-1}-i-1 \geq 4i-1 \geq 3.$$

Hence it follows from Lemmas 2.5, 3.7, 5.5, and 5.6 that $H_2(X) \in 1+2^2X\mathbb{Z}_2\langle X\rangle$. We define a formal power series $\sum_{j=0}^{\infty} w_jX^{2^{u+1}j}$ by

$$\sum_{j=0}^{\infty} w_jX^{2^{u+1}j} = \left(\sum_{j=0}^{\infty}\dot{c}^{(v)}_{2^{u+1}j+r}X^{2^{u+1}j}\right)\exp\left(-\frac{3-2\delta_{u\lambda_1}}{2^2}\,X^{2^{u+1}}+\frac{(3-2\delta_{u\lambda_1})^2}{2^5}\,X^{2^{u+2}}\right),$$

so that by Eq.(5.1),

$$\sum_{y=0}^{\infty}\frac{h_{2^{u+1}y+r}(G)}{(2^{u+1}y+r)!}\,X^{2^{u+1}y} = \left(\sum_{j=0}^{\infty}w_jX^{2^{u+1}j}\right)\exp\left(\frac{1}{2^5}\,X^{2^{u+2}}\right)H_2\left(\frac{1}{2^5}\,X^{2^{u+2}}\right)$$

(see Section 4B). Given a nonnegative integer q less than 2, these equations imply that

$$\sum_{y=0}^{\infty}\frac{h_{2^{u+1}(2y+q)+r}(G)}{(2^{u+1}(2y+q)+r)!}\,2^{5y+2q}X^y = \exp(X)\left(\sum_{j=0}^{\infty}w_{2j+q}2^{5j+2q}X^j\right)H_2(X)$$

and

$$w_{2j+q}2^{5j+2q} = \sum_{i=0}^{2j+q}(2\delta_{u\lambda_1}-3)^i\dot{c}^{(v)}_{2^{u+1}(2j+q-i)+r}2^{5j+2(q-i)}\frac{h_i(C_2)}{i!}$$

for any nonnegative integer j (cf. Eqs.(4.5) and (4.6)). Moreover, we can show the assertion in a similar fashion as in the proof of Theorem 4.2, completing the proof. □

The statements of Corollary 3.13 and Theorems 4.3 and 5.8–5.12, together with the following theorem, imply [20, Theorem 6.1, Theorem 7.1] in the sequel.

When $p = 2$ and $u = v \geq 2$, no 2-adic analytic function completely controls $h_n(G)$.

Theorem 5.13 *Suppose that $p = 2$, $\lambda_3 \geq 1$, and $u = v \geq 2$. Let r be a nonnegative integer less than 2^{u+1} and q a nonnegative integer less than 4. Then for any nonnegative integer y,*

$$\frac{h_{2^{u+1}(4y+q)+r}(G)}{(2^{u+1}(4y+q)+r)!}(-1)^y 2^{6y+q} y! \equiv \begin{cases} \dot{c}_r^{(v)} \pmod{2^2} & \text{if } q=0, \\ -\dot{c}_r^{(v)} + 2\dot{c}_{2^{u+1}+r}^{(v)} \pmod{2^2} & \text{if } q=1, \\ -2(t_{v-1,v+1}+1)\dot{c}_r^{(v)} - 2\dot{c}_{2^{u+1}+r}^{(v)} \pmod{2^2} & \\ & \text{if } q=2, \\ \dfrac{1+6(t_{v-1,v+1}+1)}{3}\dot{c}_r^{(v)} \pmod{2^2} & \text{if } q=3. \end{cases}$$

Proof. By Lemma 5.2(2), $t_{1,2} = 1$ and $t_{m-1,m} = 0$ for any integer m with $3 \leq m \leq v$. Set

$$\begin{aligned} T_2(X) &= \exp\left(3\cdot 2^3 X - 2^2 X^2 - \sum_{i=3}^{\infty} \frac{[v+1]_2 - [v-i]_2}{2^{u+i+1}}(-2^6 X)^{2^{i-2}}\right) \\ &\times \exp\left(-\sum_{i=1}^{\infty} \frac{[v-1]_2 - [v-i-2]_2}{2^{u+i}}(-2^6 X)^{2^{i-1}}\right) \\ &\times \prod_{m=3}^{v}\prod_{\ell=1}^{m-2} \exp\left(-\sum_{i=0}^{\infty} \frac{[v-m+1]_2 - [v-m-i]_2}{2^{u-\ell+i+1}}(-2^6 X)^{2^{i+m-\ell-2}}\right)^{t_{\ell,m}} \\ &\times \prod_{\ell=1}^{v-2}\prod_{m=v+1}^{s-\ell} \exp\left(-\sum_{i=0}^{\infty} \frac{1}{2^{s-\ell-m+i+1}}(-2^6 X)^{2^{i+v-\ell-2}}\right)^{t_{\ell,m}} \\ &\times \exp\left(-\sum_{i=1}^{\infty} \frac{1}{2^{i+1}}(-2^6 X)^{2^{i-1}}\right)^{t_{v-1,v+1}}. \end{aligned}$$

We define a formal power series $\sum_{j=0}^{\infty} w_j X^{2^{u+1}j}$ by

$$\begin{aligned} \sum_{j=0}^{\infty} w_j X^{2^{u+1}j} &= \left(\sum_{j=0}^{\infty} \dot{c}_{2^{u+1}j+r}^{(v)} X^{2^{u+1}j}\right) \\ &\quad \times \exp\left(-\frac{1}{2}X^{2^{u+1}} - \frac{5+4\varepsilon}{2^3}X^{2^{u+2}} + \frac{11}{2^6}X^{2^{u+3}} + \frac{1}{2^{10}}X^{2^{u+4}}\right), \end{aligned}$$

where $\varepsilon = t_{v-1,v+1}$, so that by Eq.(5.1),

$$\sum_{y=0}^{\infty} \frac{h_{2^{u+1}y+r}(G)}{(2^{u+1}y+r)!} X^{2^{u+1}y} = \left(\sum_{j=0}^{\infty} w_j X^{2^{u+1}j}\right) \exp\left(-\frac{1}{2^6}X^{2^{u+3}}\right) T_2\left(-\frac{1}{2^6}X^{2^{u+3}}\right)$$

(see Section 4C). Let q be a nonnegative integer less than 4. The above formula implies that

$$\sum_{y=0}^{\infty} \frac{h_{2^{u+1}(4y+q)+r}(G)}{(2^{u+1}(4y+q)+r)!} 2^{6y+q}(-X)^y = \exp(X)\left(\sum_{j=0}^{\infty} w_{4j+q} 2^{6j+q}(-X)^j\right) T_2(X) \qquad (5.3)$$

(cf. Eq.(4.8)). By an argument analogous to that in the proof of Theorem 4.3, we have

$$\sum_{j=0}^{\infty} w_j 2^{3j/2} X^j = \left(\sum_{j=0}^{\infty} \dot{c}^{(v)}_{2^{u+1}j+r} 2^{3j/2} X^j \right) \left(\sum_{n=0}^{\infty} \frac{h_n(C_4)}{n!} (-2^{1/2}X)^n \right) \times \left(\sum_{n=0}^{\infty} \frac{h_n(C_4)}{n!} (-2X^2)^n \right) \left(\sum_{n=0}^{\infty} \frac{h_n(C_2)}{n!} (-2^2X^2)^n \right) \exp\left(-4\varepsilon X^2\right),$$

because

$$\begin{aligned} &\exp(-2^{1/2}X - 5X^2 + 11X^4 + 4X^8) \\ &\quad = \exp(-2^{1/2}X + X^2 + X^4) \exp\left(-2X^2 + 2X^4 + 4X^8\right) \exp\left(-4X^2 + 8X^4\right) \\ &\quad = \left(\sum_{n=0}^{\infty} \frac{h_n(C_4)}{n!} (-2^{1/2}X)^n \right) \left(\sum_{n=0}^{\infty} \frac{h_n(C_4)}{n!} (-2X^2)^n \right) \left(\sum_{n=0}^{\infty} \frac{h_n(C_2)}{n!} (-2^2X^2)^n \right) \end{aligned}$$

by Eq.(1.1). Let j be a nonnegative integer. By the above equation, we have

$$w_{4j+q} 2^{6j+q} 2^{q/2} = \sum_{i=0}^{4j+q} \dot{c}^{(v)}_{2^{u+1}(4j+q-i)+r} 2^{6j+q-i} 2^{(q-i)/2} \sum_{k=0}^{[i/2]} \frac{h_{i-2k}(C_4)}{(i-2k)!} (-2^{1/2})^{i-2k} \times \sum_{z=0}^{k} \frac{h_{k-z}(C_4)}{(k-z)!} (-2)^{k-z} \sum_{m=0}^{z} \frac{h_{z-m}(C_2)}{(z-m)!} (-2^2)^{z-m} \frac{(-4\varepsilon)^m}{m!},$$

and thus

$$w_{4j+q} 2^{6j+q} = \sum_{i=0}^{4j+q} \sum_{k=0}^{[i/2]} \sum_{z=0}^{k} \sum_{m=0}^{z} w_j(i,k,z,m), \tag{5.4}$$

where

$$w_j(i,k,z,m) = (-1)^{i-k} \dot{c}^{(v)}_{2^{u+1}(4j+q-i)+r} 2^{6j+q-i+z} \frac{h_{i-2k}(C_4)}{(i-2k)!} \cdot \frac{h_{k-z}(C_4)}{(k-z)!} \cdot \frac{h_{z-m}(C_2)}{(z-m)!} \cdot \frac{\varepsilon^m}{m!}.$$

Let i, k, z, and m be nonnegative integers, and suppose that $i \le 4j+q$, $k \le [i/2]$, $z \le k$, and $m \le z$. Then it follows from Lemma 2.4 and Corollary 3.13 that

$$\begin{aligned} \mathrm{ord}_2\left(\frac{h_{i-2k}(C_4)}{(i-2k)!} \right) &\ge \sum_{\ell=1}^{2} \left[\frac{i-2k}{2^\ell} \right] - 2\left[\frac{i-2k}{2^3} \right] - (i - 2k - s_2(i-2k)) \\ &\ge -\left[\frac{i+1}{2} \right] + \left[\frac{i-2k}{2^2} \right] - 2\left[\frac{i-2k}{2^3} \right] + k + s_2(i-2k) \end{aligned} \tag{5.5}$$

and

$$\mathrm{ord}_2(w_j(i,k,z,m)) \ge 6j + q - i - \left[\frac{i+1}{2} \right] + s_2(i-2k) + s_2(k-z) + s_2(z-m) + s_2(m).$$

We explore $\mathrm{ord}_2(w_j(i,k,z,m))$ in each of the following cases **(a)**–**(e)**. Assume that $j \ge 1$.

(a) If $i \le 3j-1$, then

$$\mathrm{ord}_2(w_j(i,k,z,m)) \ge 6j + q - i - \left[\frac{i+1}{2} \right] \ge \left[\frac{3j+1}{2} \right] + 1.$$

(b) If $3j \leq i \leq 4j \leq 4j+q-1$, then

$$\mathrm{ord}_2(w_j(i,k,z,m)) \geq 6j+q-i-\left[\frac{i+1}{2}\right]+1 \geq 2.$$

(c) If $4j+1 \leq i \leq 4j+q-1$, then either $i=4j+1$ or $i=4j+2$, and hence

$$\mathrm{ord}_2(w_j(i,k,z,m)) \geq 6j+q-i-\left[\frac{i+1}{2}\right]+2 \geq 2.$$

(d) If $i=4j+q$, then

$$\mathrm{ord}_2(w_j(i,k,z,m)) \geq 6j+q-i-\left[\frac{i+1}{2}\right]+1+\delta_{q1}+\delta_{q2}+2\delta_{q3} \geq 1.$$

(e) If $j=1$ and $i=4+q$, then by Lemma 2.4 and Corollary 3.13,

$$\mathrm{ord}_2(w_1(i,k,z,m)) \geq 6+q-i+z-m+s_2(m) \geq 2.$$

After all, $\mathrm{ord}_2(w_1(i,k,z,m)) \geq 2$ and $\mathrm{ord}_2(w_j(i,k,z,m)) \geq 1$ if $j \geq 2$. We now define

$$\sum_{j=0}^{\infty} \widetilde{w}_j X^j := \left(\sum_{j=0}^{\infty} w_{4j+q} 2^{6j+q}(-X)^j\right) T_2(X).$$

By Lemmas 2.5, 3.7, 5.5, and 5.6, $T_2(X) \in 1+2^2X\mathbb{Z}_2\langle X\rangle$. Hence it follows from Eq.(5.4) and the above fact that $\mathrm{ord}_2(\widetilde{w}_1) \geq 2$ and $\mathrm{ord}_2(\widetilde{w}_j) \geq 1$ if $j \geq 2$. Consequently, by Eq.(5.3),

$$\frac{h_{2^{u+1}(4y+q)+r}(G)}{(2^{u+1}(4y+q)+r)!}(-1)^y 2^{6y+q} y! = \sum_{j=0}^{y} \frac{y!}{(y-j)!}\widetilde{w}_j \equiv \widetilde{w}_0 \pmod{2^2}$$

for any nonnegative integer y. Moreover, Eq.(5.4) implies that

$$\widetilde{w}_0 = \sum_{i=0}^{q}\sum_{k=0}^{[i/2]}\sum_{z=0}^{k}\sum_{m=0}^{z} w_0(i,k,z,m)$$

and

$$\begin{aligned}\widetilde{w}_0 &= \sum_{i=0}^{q}\sum_{k=0}^{[i/2]}\sum_{z=0}^{k}\sum_{m=0}^{z}(-1)^{i-k}\dot{c}^{(v)}_{2^{u+1}(q-i)+r}2^{q-i+z}\frac{h_{i-2k}(C_4)}{(i-2k)!}\cdot\frac{h_{k-z}(C_4)}{(k-z)!}\cdot\frac{h_{z-m}(C_2)}{(z-m)!}\cdot\frac{\varepsilon^m}{m!}\\ &= \begin{cases} \dot{c}^{(v)}_r & \text{if } q=0,\\ -\dot{c}^{(v)}_r+2\dot{c}^{(v)}_{2^{u+1}+r} & \text{if } q=1,\\ -2(\varepsilon+1)\dot{c}^{(v)}_r-2\dot{c}^{(v)}_{2^{u+1}+r}+2^2\dot{c}^{(v)}_{2^{u+2}+r} & \text{if } q=2,\\ \dfrac{1+6(\varepsilon+1)}{3}\dot{c}^{(v)}_r-2^2(\varepsilon+1)\dot{c}^{(v)}_{2^{u+1}+r}-2^2\dot{c}^{(v)}_{2^{u+2}+r}+2^3\dot{c}^{(v)}_{2^{u+2}+2^{u+1}+r} & \text{if } q=3.\end{cases}\end{aligned}$$

This completes the proof. □

6 Conclusions

In this section we prove Theorems 1.1 and 1.2. In the proof, some technical computations are required. Let k be a positive integer. We define

$$\eta_k(X) := \frac{1}{p^{\frac{p^k-1}{p-1}-k}} \prod_{i=1}^{p^k-1} (p^k X - i),$$

so that for any nonnegative integer y,

$$p^{\frac{p^k-1}{p-1}y} \prod_{j=1}^{y} \eta_k(j) = \prod_{j=1}^{y} \frac{p^k(p^k j-1)!}{(p^k(j-1))!} = \frac{1}{y!} \prod_{j=1}^{y} \frac{(p^k j)!}{(p^k(j-1))!} = \frac{(p^k y)!}{y!}. \tag{6.1}$$

Since $\mathrm{ord}_p((p^k-1)!) = 1 + p + \cdots + p^{k-1} - k$ by Lemma 2.4, it follows that $\eta_k(X)$ is a polynomial of degree $p^k - 1$ with integer coefficients and

$$\eta_k(j) = \frac{1}{p^{\mathrm{ord}_p((p^k-1)!)}} \prod_{i=1}^{p^k-1} (p^k j - i) \equiv (-1)^{p^k-1}((p-1)!)^{\frac{p^k-1}{p-1}} \not\equiv 0 \pmod{p} \tag{6.2}$$

for any integer j. Assume that $k = \kappa_p(u,v)$. We define

$$\eta(X) := \begin{cases} \eta_{\kappa_p(u,v)}(X) & \text{if } p = 2 \text{ and either } u = v = 1 \text{ or } 1 \leq u - v + \delta_{v0} \leq 2, \\ & \text{or if } p = 3 \text{ and } u = v \geq 1, \\ -\eta_{\kappa_p(u,v)}(X) & \text{otherwise,} \end{cases}$$

which is the polynomial '$\eta(X)$' appearing in a statement of Theorem 1.1.

Let r be a nonnegative integer less than $p^{\kappa_p(u,v)}$. We set $q = [r/p^{u+1}]$ and $r' = r - p^{u+1}q$. If $\kappa_p(u,v) = u+1$, then it is obvious that $r = r'$. Let $g_r(X)$ and $g_{q,r'}(X)$ denote the polynomials and formal power series given in Theorems 4.3 and 5.8–5.12. For the sake of convenience, we set $g_r(X) = c_r^{(0)}$ if $p = 2$ and $u = v = 0$. Also, $g_{q,r'}(X)$ denotes a p-adic integer on the right side of the congruence in Theorem 5.13 if $p = 2$, $\lambda_3 \geq 1$, and $u = v \geq 2$. To prove Theorems 1.1 and 1.2, we define

$$f_r(X) := \begin{cases} \dfrac{g_{q,r'}(X)}{p^q} \displaystyle\prod_{i=1}^{r} (p^{\kappa_p(u,v)} X + i) & \text{if } p \leq 3 \text{ and } u = v \geq 1, \\ \dfrac{g_{q,r'}(X)}{2^{2q}} \displaystyle\prod_{i=1}^{r} (2^{\kappa_2(u,v)} X + i) & \text{if } p = 2 \text{ and } u = v + 1 \geq 2, \\ g_r(X) \displaystyle\prod_{i=1}^{r} (p^{\kappa_p(u,v)} X + i) & \text{otherwise.} \end{cases}$$

Proof of Theorem 1.1. By Lemma 2.4, $\mathrm{ord}_p(f_r(y)) \geq \tau_p^{(u,v)}(r)$ and $f_r(X) \in \mathbb{Z}_p\langle X \rangle$. Let y be a nonnegative integer. Except for the case where $p = 2$ and either $u + \delta_{v0} = v + 1$ or $\lambda_3 \geq 1$ and $u = v \geq 2$, it follows from Theorems 4.3 and 5.8–5.11 and Eq.(6.1) that

$$h_{p^{\kappa_p(u,v)}y+r}(G) = p^{\tau_p^{(u,v)}(p^{\kappa_p(u,v)}y)} f_r(y) \prod_{j=1}^{y} \eta(j).$$

If $p = 2$ and either $u + \delta_{v0} = v + 1$ or $\lambda_3 \geq 1$ and $u = v \geq 2$, then by Corollary 3.13, Theorems 5.12 and 5.13, and Eq.(6.1), we have

$$h_{2^{\kappa_2(u,v)}y+r}(G) \equiv 2^{\tau_2^{(u,v)}(2^{\kappa_2(u,v)}y)} f_r(y) \prod_{j=1}^{y} \eta(j) \pmod{p^{\tau_2^{(u,v)}(2^{\kappa_2(u,v)}y+r)+2-\delta_{v0}}}.$$

Since $\mathrm{ord}_p(\prod_{j=1}^{y} \eta(j)) = 0$ by Eq.(6.2), it turns out that

$$\mathrm{ord}_p(h_{p^{\kappa_p(u,v)}y+r}(G)) = \tau_p^{(u,v)}(p^{\kappa_p(u,v)}y) + \mathrm{ord}_p(f_r(y)) \geq \tau_p^{(u,v)}(p^{\kappa_p(u,v)}y + r) \tag{6.3}$$

in any case. Hence the theorem holds. □

Proof of Theorem 1.2. Let y be a nonnegative integer. By Theorems 4.3 and 5.8–5.13,

$$\mathrm{ord}_p(f_r(y)) = \tau_p^{(u,v)}(r) + \mathrm{ord}_p(\dot{c}_{r'}^{(v)})$$

except for the case where $p = 2$ and either $\lambda_3 = 0$, $u = v \geq 1$, and $q = 3$ or $\lambda_3 \geq 1$, $u = v \geq 2$, and $q = 2$. Obviously, $\dot{c}_0^{(v)} = 1$. Hence the theorem is a consequence of Eq.(6.3). □

7 Wreath products

Under the assumption that $u \geq 1$, we explore a p-adic estimate of $h_n(P; C_p)$, where $P = C_{p^u} \times C_{p^v}$ and $h_n(P; C_p)$ is the number of homomorphisms from P to $C_p \wr S_n$, and give some p-adic properties of $h_n(P; C_p)$ which are analogous to those of $h_n(P)$ in Sections 2–4.

By [19, Lemma 5.4], the number of cyclic subgroups of order p^k in P is $p^{k-1} + p^k$ if $0 < k \leq v$, and is p^v if $v < k \leq u$. Equivalently, the number of cyclic subgroups of index p^k in P is $p^{u+v-k-1} + p^{u+v-k}$ if $u + v > k \geq u$, and is p^v if $u > k \geq v$. This, combined with [19, Proposition 5.3] ([20, Proposition 3.2]) and Eq.(1.3), shows that

$$\sum_{n=0}^{\infty} \frac{h_n(P; C_p)}{p^n n!} X^n = \exp\left(\sum_{k=0}^{v-1} \frac{p + \cdots + p^{k+1}}{p^k} X^{p^k} + \sum_{k=v}^{u-1} \frac{p + \cdots + p^v}{p^k} X^{p^k} + \sum_{k=v}^{u-1} \frac{p^v}{p^k} X^{p^k} \right.$$
$$\left. + \sum_{k=u}^{u+v-1} \frac{p + \cdots + p^{u+v-k}}{p^k} X^{p^k} + \sum_{k=u}^{u+v} \frac{p^{u+v-k-1}}{p^k} X^{p^k} \right),$$

and thus

$$\sum_{n=0}^{\infty} \frac{h_n(P; C_p)}{p^n n!} X^n = \exp\left(\sum_{k=0}^{v-1} \frac{p(p^{k+1} - 1)}{p^k(p-1)} X^{p^k} + \sum_{k=v}^{u-1} \frac{p^v(2p-1) - p}{p^k(p-1)} X^{p^k} \right.$$
$$\left. + \sum_{k=u}^{u+v-1} \frac{p(p^{u+v-k} - 1)}{p^k(p-1)} X^{p^k} + \sum_{k=u}^{u+v} \frac{p^{u+v}}{p^{2k+1}} X^{p^k} \right)$$

(cf. [19, Proposition 5.5]). We define a formal power series $\widetilde{E}_p^{(v)}(X) = \sum_{n=0}^{\infty} \tilde{c}_n^{(v)} X^n$ by

$$\widetilde{E}_p^{(v)}(X) = \sum_{n=0}^{\infty} \tilde{c}_n^{(v)} X^n = \exp\left(\sum_{k=0}^{v-1} \frac{p(p^{k+1} - 1)}{p^k(p-1)} X^{p^k} + \sum_{k=v}^{\infty} \frac{p^v(2p-1) - p}{p^k(p-1)} X^{p^k} \right), \tag{7.1}$$

and use a consequence of [3, Proposition 1] without mention (cf. Proposition 2.3):

Proposition 7.1 *For any nonnegative integer n,* $\mathrm{ord}_p(\tilde{c}_n^{(v)}) \geq 0$*. Equivalently,*

$$\widetilde{E}_p^{(v)}(X) - 1 \in X\mathbb{Z}_p[[X]].$$

Set $\iota_{u,v}^{(p)} = (p^{u-v+1}/(p-1))^{1/p^u}$. Substituting $\iota_{u,v}^{(p)}X$ for X in Eq.(7.1), we have

$$\begin{aligned}
\widetilde{E}_p^{(v)}\left(\iota_{u,v}^{(p)}X\right) &= \exp\left(\sum_{k=0}^{v-1}\frac{p(p^{k+1}-1)}{p^k(p-1)}(\iota_{u,v}^{(p)}X)^{p^k} + \sum_{k=v}^{\infty}\frac{p^v(2p-1)-p}{p^k(p-1)}(\iota_{u,v}^{(p)}X)^{p^k}\right)\\
&= \left\{\sum_{n=0}^{\infty}\frac{h_n(P;C_p)}{p^n n!}(\iota_{u,v}^{(p)}X)^n\right\}\\
&\quad \times \exp\left(X^{p^u} + \sum_{k=u+1}^{u+v}\left(\frac{p^v(2p-1-p^{u-k+1})}{p^k(p-1)} - \frac{p^{u+v}}{p^{2k+1}}\right)(\iota_{u,v}^{(p)}X)^{p^k}\right.\\
&\qquad\qquad \left. + \sum_{i=0}^{\infty}\frac{p^v(2p-1)-p}{p^{u+v+i+1}(p-1)}(\iota_{u,v}^{(p)}X)^{p^{u+v+i+1}}\right).
\end{aligned}$$

Let r be a nonnegative integer less than p^u. The above formula implies that

$$\begin{aligned}
\sum_{y=0}^{\infty}\frac{h_{p^u y+r}(P;C_p)}{p^{p^u y+r}(p^u y+r)!}(\iota_{u,v}^{(p)}X)^{p^u y} &= \left(\sum_{j=0}^{\infty}\tilde{c}_{p^u j+r}^{(v)}(\iota_{u,v}^{(p)}X)^{p^u j}\right)\\
&\quad \times \exp\left(-X^{p^u} - \sum_{k=u+1}^{u+v}\left(\frac{p^v(2p-1-p^{u-k+1})}{p^k(p-1)} - \frac{p^{u+v}}{p^{2k+1}}\right)(\iota_{u,v}^{(p)}X)^{p^k}\right.\\
&\qquad\qquad \left. - \sum_{i=0}^{\infty}\frac{p^v(2p-1)-p}{p^{u+v+i+1}(p-1)}(\iota_{u,v}^{(p)}X)^{p^{u+v+i+1}}\right).
\end{aligned} \tag{7.2}$$

Substituting $-X$ for X^{p^u}, we have

$$\begin{aligned}
&\sum_{y=0}^{\infty}\frac{h_{p^u y+r}(P;C_p)}{p^{p^u y+r}(p^u y+r)!}\left(-\frac{p^{u-v+1}}{p-1}X\right)^y\\
&\qquad = \exp(X)\left(\sum_{j=0}^{\infty}\tilde{c}_{p^u j+r}^{(v)}\left(-\frac{p^{u-v+1}}{p-1}X\right)^j\right)F_p^{(u,v)}(X),
\end{aligned} \tag{7.3}$$

where

$$\begin{aligned}
F_p^{(u,v)}(X) &= \exp\left(-\sum_{k=u+1}^{u+v}\left(\frac{p^v(2p-1-p^{u-k+1})}{p^k(p-1)} - \frac{p^{u+v}}{p^{2k+1}}\right)\left(-\frac{p^{u-v+1}}{p-1}X\right)^{p^{k-u}}\right.\\
&\qquad\qquad \left. - \sum_{i=0}^{\infty}\frac{p^v(2p-1)-p}{p^{u+v+i+1}(p-1)}\left(-\frac{p^{u-v+1}}{p-1}X\right)^{p^{v+i+1}}\right).
\end{aligned}$$

The p-adic properties of $h_n(P;C_p)$ are explored on the basis of Eqs.(7.2) and (7.3), including Theorem 1.3. For each integer k with $u+1 \leq k \leq u+v$, we set

$$R(k) = \left(\frac{p^v(2p-1-p^{u-k+1})}{p^k(p-1)} - \frac{p^{u+v}}{p^{2k+1}}\right)\left(-\frac{p^{u-v+1}}{p-1}\right)^{p^{k-u}}, \tag{7.4}$$

so that

$$F_p^{(u,v)}(X) = \exp\left(-\sum_{k=u+1}^{u+v} R(k) X^{p^{k-u}}\right) \overline{F}_p^{(u,v)}(X), \tag{7.5}$$

where

$$\overline{F}_p^{(u,v)}(X) = \exp\left(-\sum_{i=0}^{\infty} \frac{p^v(2p-1)-p}{p^{u+v+i+1}(p-1)} \left(-\frac{p^{u-v+1}}{p-1} X\right)^{p^{v+i+1}}\right).$$

Lemma 7.2 *For any integer k with $u+2 \leq k \leq u+v$, $\mathrm{ord}_p(R(k)) \geq u-v+4$ unless $p=2$ and $k \leq u+3$. If $p=2$, then $\mathrm{ord}_2(R(u+2)) = 3(u-v)-1$ and $\mathrm{ord}_2(R(u+3)) = 7(u-v)+1$.*

Proof. Let k be an integer greater than $u+1$. We have

$$\frac{p^v(2p-1-p^{u-k+1})}{p^k(p-1)} = \frac{2p^{k-u}-p^{k-u-1}-1}{p^{2k-u-v-1}(p-1)} = \frac{2}{p^{k-v}} + \frac{1+p+\cdots+p^{k-u-2}}{p^{2k-u-v-1}}.$$

This, combined with Eq.(7.4), shows that

$$\mathrm{ord}_p(R(u+2)) = p^2(u-v+1)-(u-v+5) = (p^2-1)(u-v)+p^2-5$$

and

$$\mathrm{ord}_p(R(u+3)) = p^3(u-v+1)-(u-v+7) = (p^3-1)(u-v)+p^3-7.$$

Moreover, if $k \geq u+4$, then $p^{k-u} > 3(k-u+1)$, and hence

$$\begin{aligned}\mathrm{ord}_p(R(k)) &= p^{k-u}(u-v+1)+u+v-2k-1 \\ &> 3(u-v+1)(k-u+1)+u+v-2k-1 \\ &= 3(u-v+1)+(u-v)(3k-3u-1)+k-u-1.\end{aligned}$$

The proof is now complete. □

The following lemma is an analogue of Lemma 3.3.

Lemma 7.3 *Let i be a nonnegative integer. Then*

$$\mathrm{ord}_p\left(\frac{p^v(2p-1)-p}{p^{u+v+i+1}(p-1)}\left(-\frac{p^{u-v+1}}{p-1}\right)^{p^{v+i+1}}\right) \geq (p-1)(u-v+1)+i+1-\delta_{v0}.$$

In particular, $\overline{F}_p^{(u,v)}(X) \in 1+p^{u-v+2}X^p\mathbb{Z}_p\langle X\rangle$ unless $p=2$ and $v=0$.

Proof. Since $p^{v+i+1} \geq (v+i+1)p$, it follows that

$$\begin{aligned}\mathrm{ord}_p\left(\frac{p^v(2p-1)-p}{p^{u+v+i+1}(p-1)}\left(-\frac{p^{u-v+1}}{p-1}\right)^{p^{v+i+1}}\right) &\geq (u-v+1)p^{v+i+1}-(u+v+i+\delta_{v0}) \\ &\geq (u-v+1)(v+i+1)p-(u+v+i+\delta_{v0}) \\ &\geq p(u-v+1)+2(v+i)-(u+v+i+\delta_{v0}) \\ &\geq (p-1)(u-v+1)+i+1-\delta_{v0},\end{aligned}$$

which is the first assertion. The second assertion follows from the first one and Lemma 2.5. □

The following lemma is part of [12, Theorem 7.2].

Lemma 7.4 *Suppose that $v = 0$. Let r be a nonnegative integer less than p^u. Then there exists a p-adic analytic function $g_r(X)$ contained in $\mathbb{Z}_p\langle X\rangle$ such that*

$$g_r(y) = \frac{h_{p^u y+r}(P; C_p)}{p^{p^u y+r}(p^u y + r)!} \left(-\frac{p^{u+1}}{p-1}\right)^y y!$$

for any nonnegative integer y and

$$g_r(X) \in \widehat{g}_r(X) + p^{u+2} X\mathbb{Z}_p\langle X\rangle,$$

where

$$\widehat{g}_r(X) = \begin{cases} \tilde{c}_r^{(0)}(1 - 2^{u+1} X^{\underline{2}}) - \tilde{c}_{2^u+r}^{(0)} 2^{u+1} X & \text{if } p = 2, \\ \tilde{c}_r^{(0)} - \tilde{c}_{p^u+r}^{(0)} \dfrac{p^{u+1}}{p-1} X & \text{otherwise.} \end{cases}$$

Proof. By Lemmas 2.5 and 7.3, $\overline{F}_2^{(u,0)}(X) \in 1 - 2^{u+1}X^2 + 2^{u+2}X^2\mathbb{Z}_2\langle X\rangle$. Hence the assertion follows from Lemma 2.6 and Eqs.(7.3) and (7.5). □

We are now in the position to present a series of theorems on a p-adic estimate of $h_n(P; C_p)$, which, combined with Lemma 2.4, proves Theorems 1.3 and 8.1. The following theorems are analogues of Theorems 3.6, 3.9, 3.11, and 4.1–4.3; no 2-adic analytic function completely controls $h_n(P; C_p)$ under the assumption that $p = 2$ and $1 \leq v \leq u \leq v + 1$, however.

Theorem 7.5 *Suppose that either $p \geq 3$ and $u \geq v + 1 \geq 1$ or $p > 3$ and $u = v \geq 1$. Let r be a nonnegative integer less than p^u. Then there exists a p-adic analytic function $g_r(X)$ contained in $\mathbb{Z}_p\langle X\rangle$ such that*

$$g_r(y) = \frac{h_{p^u y+r}(P; C_p)}{p^{p^u y+r}(p^u y + r)!} \left(-\frac{p^{u-v+1}}{p-1}\right)^y y!$$

for any nonnegative integer y and

$$g_r(X) \in \widehat{g}_r(X) + p^{u-v+2} X\mathbb{Z}_p\langle X\rangle,$$

where

$$\widehat{g}_r(X) = \begin{cases} \tilde{c}_r^{(v)} \left(1 + \dfrac{3^2}{4} X^{\underline{3}}\right) - \tilde{c}_{3^u+r}^{(v)} \dfrac{3^2}{2} X & \text{if } p = 3 \text{ and } u = v + 1 \geq 2, \\ \tilde{c}_r^{(v)} - \tilde{c}_{p^u+r}^{(v)} \dfrac{p^{u-v+1}}{p-1} X & \text{otherwise.} \end{cases}$$

Proof. If $p > 3$ and $v \geq 1$ or if $u > v + 1 \geq 2$, then $\mathrm{ord}_p(R(u + 1)) \geq u - v + 2$, because

$$R(u+1) = \left(\frac{p^v(2p-2)}{p^{u+1}(p-1)} - \frac{p^{u+v}}{p^{2u+3}}\right)\left(-\frac{p^{u-v+1}}{p-1}\right)^p = \left(\frac{2}{p^{u-v+1}} - \frac{1}{p^{u-v+3}}\right)\left(-\frac{p^{u-v+1}}{p-1}\right)^p$$

and

$$\mathrm{ord}_p(R(u + 1)) = p(u - v + 1) - (u - v + 3) = (p - 2)(u - v + 1) + u - v - 1.$$

Moreover, if $p = 3$ and $u = v + 1 \geq 2$, then $R(u + 1) = -153/8$. Hence the assertion follows from Lemmas 2.5, 2.6, and 7.2–7.4 and Eqs.(7.3)–(7.5). □

Theorem 7.6 *Suppose that $p = 2$ and $u + \delta_{v0} \geq v + 3$. Let r be a nonnegative integer less than 2^u. Then there exists a 2-adic analytic function $g_r(X)$ contained in $\mathbb{Z}_2\langle X\rangle$ such that*

$$g_r(y) = \frac{h_{2^u y+r}(P; C_2)}{2^{2^u y+r}(2^u y + r)!}(-2^{u-v+1})^y y!$$

for any nonnegative integer y and

$$g_r(X) \in \widehat{g}_r(X) + 2^{u-v+2}X\mathbb{Z}_2\langle X\rangle,$$

where

$$\widehat{g}_r(X) = \begin{cases} \tilde{c}_r^{(0)}(1 - 2^{u+1}X^{\underline{2}}) - \tilde{c}_{2^u+r}^{(0)}2^{u+1}X & \text{if } v = 0, \\ \tilde{c}_r^{(v)}(1 - 7 \cdot 2^{u-v-1}X^{\underline{2}} + 49 \cdot 2^{2u-2v-3}X^{\underline{4}}) - \tilde{c}_{2^u+r}^{(v)}2^{u-v+1}X & \text{if } v \geq 1. \end{cases}$$

Proof. If $v \geq 1$, then we have $R(u + 1) = 7 \cdot 2^{u-v-1}$. Hence the assertion follows from Lemmas 2.5, 2.6, and 7.2–7.4 and Eq.(7.3)–(7.5). □

Theorem 7.7 *Suppose that $p = 2$ and $u + \delta_{v0} = v + 2$. Let r be a nonnegative integer less than 2^u. Then there exists a 2-adic analytic function $g_r(X)$ contained in $\mathbb{Z}_p\langle X\rangle$ such that*

$$g_r(y) = \frac{h_{2^u y+r}(P; C_2)}{2^{2^u y+r}(2^u y + r)!}((-1)^{\delta_{v0}}2^{u-v+1})^y y!$$

for any nonnegative integer y and

$$g_r(X) \in \widehat{g}_r(X) + 2^{u-v+2}X\mathbb{Z}_2\langle X\rangle,$$

where

$$\widehat{g}_r(X) = \begin{cases} \tilde{c}_r^{(0)}(1 - 4X^{\underline{2}}) - \tilde{c}_{2^u+r}^{(0)}4X & \text{if } v = 0, \\ \tilde{c}_r^{(v)}(1 - 2X + 4X^{\underline{2}} - 4X^{\underline{4}} + 8X^{\underline{5}} + 8X^{\underline{8}}) + \tilde{c}_{2^u+r}^{(v)}8X & \text{if } v \geq 1. \end{cases}$$

Proof. If $v = 0$, then the assertion is given in Lemma 7.4. Assume that $v \geq 1$. Then $u = v + 2 \geq 3$ and it follows from Eqs.(7.3)–(7.5) that

$$\sum_{y=0}^{\infty} \frac{h_{2^u y+r}(P; C_2)}{2^{2^u y+r}(2^u y + r)!}(-2^3X)^y$$
$$= \exp(X)\left(\sum_{j=0}^{\infty} \tilde{c}_{2^u j+r}^{(v)}(-2^3X)^j\right)\exp(-14X^2)\widehat{F}_2^{(u,v)}(X)\overline{F}_2^{(u,v)}(X),$$

where

$$\widehat{F}_2^{(u,v)}(X) = \exp\left(-\sum_{k=u+2}^{u+v} R(k)X^{2^{k-u}}\right).$$

Substituting $-X$ for X, we have

$$\sum_{y=0}^{\infty} \frac{h_{2^u y+r}(P; C_2)}{2^{2^u y+r}(2^u y + r)!}2^{3y}X^y = \exp(X)\exp(-2X + 2X^2 + 4X^4)\exp(-16X^2 - 4X^4)$$
$$\times \left(\sum_{j=0}^{\infty} \tilde{c}_{2^u j+r}^{(v)}2^{3j}X^j\right)\widehat{F}_2^{(u,v)}(X)\overline{F}_2^{(u,v)}(X).$$

Moreover, it follows from Eq.(1.1), Lemma 2.4, and Corollary 3.13 that

$$\exp(-2X+2X^2+4X^4)=\sum_{n=0}^{\infty}\frac{h_n(C_4)}{n!}(-2X)^n\in 1-2X+4X^2+16X\mathbb{Z}_2\langle X\rangle$$

(see the proof of [12, Theorem 4.4]). By Lemmas 2.5, 7.2, and 7.3, we have

$$\begin{aligned}\exp(-2X+2X^2+4X^4)\exp(-16X^2-4X^4)\left(\sum_{j=0}^{\infty}\tilde{c}^{(v)}_{2^uj+r}2^{3j}X^j\right)\widehat{F}_2^{(u,v)}(X)\overline{F}_2^{(u,v)}(X)\\ \in \tilde{c}^{(v)}_r(1-2X+4X^2)(1-4X^4+8X^8)+\tilde{c}^{(v)}_{2^u+r}8X+16X\mathbb{Z}_2\langle X\rangle.\end{aligned}$$

Hence the assertion is a consequence of Lemma 2.6. □

Theorem 7.8 *Suppose that $p=3$ and $u=v\geq 1$. Let r be a nonnegative integer less than 3^u and q a nonnegative integer less than 3. Then there exists a 3-adic analytic function $g_{q,r}(X)$ contained in $\mathbb{Z}_3\langle X\rangle$ such that*

$$g_{q,r}(y)=\sum_{y=0}^{\infty}\frac{h_{3^u(3y+q)+r}(P;C_3)}{3^{3^u(3y+q)+r}(3^u(3y+q)+r)!}\cdot\frac{3^{4y+q}}{2^{3y+q}}y!$$

for any nonnegative integer y and

$$g_{q,r}(X)\in\widehat{g}_{q,r}(X)+3^2X\mathbb{Z}_3\langle X\rangle,$$

where

$$\widehat{g}_{q,r}(X)=\begin{cases}\tilde{c}^{(v)}_r\left(1-3^3\dfrac{h_9(C_3)}{9!}X^{\frac{3}{}}\right) & \text{if } q=0,\\ \tilde{c}^{(v)}_r\left(-1+3X+3^3\dfrac{h_{10}(C_3)}{10!}X^{\frac{3}{}}\right)+\dfrac{3}{2}\tilde{c}^{(v)}_{3^u+r} & \text{if } q=1,\\ \tilde{c}^{(v)}_r\left(\dfrac{1}{2}-3^3\dfrac{h_{11}(C_3)}{11!}X^{\frac{3}{}}\right)-\dfrac{3}{2}\tilde{c}^{(v)}_{3^u+r}+\left(\dfrac{3}{2}\right)^2\tilde{c}^{(v)}_{2\cdot 3^u+r} & \text{if } q=2.\end{cases}$$

Proof. We follow the argument in Section 4A. By Eq.(7.2),

$$\begin{aligned}&\sum_{y=0}^{\infty}\frac{h_{3^uy+r}(P;C_3)}{3^{3^uy+r}(3^uy+r)!}\left(\frac{3}{2}\right)^y X^{3^uy}\\ &=\left(\sum_{j=0}^{\infty}\tilde{c}^{(v)}_{3^uj+r}\left(\frac{3}{2}\right)^j X^{3^uj}\right)\exp\left(-X^{3^u}-\frac{1}{3}X^{3^{u+1}}\right)\exp\left(\frac{1}{3}X^{3^{u+1}}\right)\\ &\quad\times\exp\left(-\frac{17}{2^3}X^{3^{u+1}}-\sum_{k=u+2}^{u+v}\left(\frac{3^v(5-3^{u-k+1})}{2\cdot 3^k}-\frac{3^{u+v}}{3^{2k+1}}\right)\left(\frac{3}{2}\right)^{3^{k-u}}X^{3^k}\right.\\ &\qquad\left.-\sum_{i=0}^{\infty}\frac{5\cdot 3^v-3}{2\cdot 3^{u+v+i+1}}\left(\frac{3}{2}\right)^{3^{v+i+1}}X^{3^{u+v+i+1}}\right).\end{aligned}$$

To show Eq.(7.7), we define

$$H_3(X) := \exp\left(-\frac{51}{2^3}X - \sum_{k=u+2}^{u+v}\left(\frac{3^v(5-3^{u-k+1})}{2\cdot 3^k} - \frac{3^{u+v}}{3^{2k+1}}\right)\left(\frac{3}{2}\right)^{3^{k-u}}(3X)^{3^{k-u-1}} - \sum_{i=0}^{\infty}\frac{5\cdot 3^v - 3}{2\cdot 3^{u+v+i+1}}\left(\frac{3}{2}\right)^{3^{v+i+1}}(3X)^{3^{v+i}}\right)$$

and

$$\sum_{j=0}^{\infty} w_j X^{3^u j} := \left(\sum_{j=0}^{\infty}\tilde{c}^{(v)}_{3^u j+r}\left(\frac{3}{2}\right)^j X^{3^u j}\right)\exp\left(-X^{3^u} - \frac{1}{3}X^{3^{u+1}}\right), \tag{7.6}$$

so that

$$\sum_{y=0}^{\infty}\frac{h_{3^u y+r}(P;C_3)}{3^{3^u y+r}(3^u y+r)!}\left(\frac{3}{2}\right)^y X^{3^u y} = \left(\sum_{j=0}^{\infty} w_j X^{3^u j}\right)\exp\left(\frac{1}{3}X^{3^{u+1}}\right)H_3\left(\frac{1}{3}X^{3^{u+1}}\right).$$

Let q be a nonnegative integer less than 3. The above formula implies that

$$\sum_{y=0}^{\infty}\frac{h_{3^u(3y+q)+r}(P;C_3)}{3^{3^u(3y+q)+r}(3^u(3y+q)+r)!}\left(\frac{3}{2}\right)^{(3y+q)} X^{3^u(3y+q)} = \left(\sum_{j=0}^{\infty} w_{3j+q} X^{3^u(3j+q)}\right)\exp\left(\frac{1}{3}X^{3^{u+1}}\right)H_3\left(\frac{1}{3}X^{3^{u+1}}\right).$$

Substituting $3^{1/3}X$ for X^{3^u}, we have

$$\sum_{y=0}^{\infty}\frac{h_{3^u(3y+q)+r}(P;C_3)}{3^{3^u(3y+q)+r}(3^u(3y+q)+r)!}\left(\frac{3}{2}\right)^{(3y+q)} 3^y X^{3y}(3^{1/3}X)^q = \left(\sum_{j=0}^{\infty} w_{3j+q}3^j X^{3j}(3^{1/3}X)^q\right)\exp(X^3)H_3(X^3).$$

Now omit $(3^{1/3}X)^q$ and substitute X for X^3. Then

$$\sum_{y=0}^{\infty}\frac{h_{3^u(3y+q)+r}(P;C_3)}{3^{3^u(3y+q)+r}(3^u(3y+q)+r)!}\left(\frac{3}{2}\right)^{(3y+q)} 3^y X^y = \exp(X)\left(\sum_{j=0}^{\infty} w_{3j+q}3^j X^j\right)H_3(X). \tag{7.7}$$

Moreover, $H_3(X) \in 1 - 3X + 3^2X\mathbb{Z}_3\langle X\rangle$ by Lemmas 2.5, 7.2, and 7.3.

Substitute $3^{1/3}X$ for X^{3^u} in Eq.(7.6). Then it follows from Eq.(1.1) that

$$\sum_{j=0}^{\infty} w_j 3^{j/3}X^j = \left(\sum_{j=0}^{\infty}\tilde{c}^{(v)}_{3^u j+r}\left(\frac{3}{2}\right)^j 3^{j/3}X^j\right)\left(\sum_{n=0}^{\infty}\frac{h_n(C_3)}{n!}(-3^{1/3}X)^n\right).$$

By an argument analogous to that in the proof of Theorem 4.1, we have

$$\sum_{j=0}^{\infty} w_{3j+q}3^jX^j = \sum_{j=0}^{\infty}\left(\sum_{i=0}^{3j+q}(-1)^i\tilde{c}^{(v)}_{3^u(3j+q-i)+r}\frac{3^{4j+q-i}}{2^{3j+q-i}}\cdot\frac{h_i(C_3)}{i!}\right)X^j,$$

$$\sum_{j=1}^{\infty} w_{3j+q}3^jX^j \in (-1)^{3+q}\tilde{c}^{(v)}_r 3\frac{h_{3+q}(C_3)}{(3+q)!}X + (-1)^{9+q}\tilde{c}^{(v)}_r 3^3\frac{h_{9+q}(C_3)}{(9+q)!}X^3 + 3^2X\mathbb{Z}_3\langle X\rangle,$$

and

$$\begin{aligned}\sum_{j=0}^{\infty} w_{3j+q}3^jX^j &= \sum_{i=0}^{q}(-1)^i\tilde{c}^{(v)}_{3^u(q-i)+r}\left(\frac{3}{2}\right)^{q-i}\frac{h_i(C_3)}{i!} + \sum_{j=1}^{\infty} w_{3j+q}3^jX^j\\ &\in \widetilde{g}_{q,r}(X) + 3^2X\mathbb{Z}_3\langle X\rangle,\end{aligned}$$

where

$$\widetilde{g}_{q,r}(X) = \begin{cases}\tilde{c}^{(v)}_r\left(1-\frac{3}{2}X - 3^3\frac{h_9(C_3)}{9!}X^3\right) & \text{if } q=0,\\ \tilde{c}^{(v)}_r\left(-1+3^3\frac{h_{10}(C_3)}{10!}X^3\right)+\frac{3}{2}\tilde{c}^{(v)}_{3^u+r} & \text{if } q=1,\\ \tilde{c}^{(v)}_r\left(\frac{1}{2}-\frac{21}{40}X - 3^3\frac{h_{11}(C_3)}{11!}X^3\right)-\frac{3}{2}\tilde{c}^{(v)}_{3^u+r}+\left(\frac{3}{2}\right)^2\tilde{c}^{(v)}_{2\cdot 3^u+r} & \text{if } q=2.\end{cases}$$

Hence the assertion is a consequence of Lemma 2.6 and Eq.(7.7). This completes the proof. □

Theorem 7.9 *Suppose that* $p=2$ *and* $u=v+1\geq 2$*. Let* r *be a nonnegative integer less than* 2^u *and* q *a nonnegative integer less than* 2*. Define a polynomial* $g_{q,r}(X)$ *by*

$$g_{q,r}(X) = \begin{cases}\tilde{c}^{(v)}_r(1-X-X^2) & \text{if } q=0,\\ -\tilde{c}^{(v)}_r(1+X-X^2) & \text{if } q=1.\end{cases}$$

Then for any nonnegative integer y,

$$\frac{h_{2^u(2y+q)+r}(P;C_2)}{2^{2^u(2y+q)+r}(2^u(2y+q)+r)!}2^{5y+2q}y! \equiv g_{q,r}(y) \pmod{2^2}.$$

Proof. We follow the argument in Section 4B. By Eq.(7.2),

$$\begin{aligned}&\sum_{y=0}^{\infty}\frac{h_{2^uy+r}(P;C_2)}{2^{2^uy+r}(2^uy+r)!}2^{2y}X^{2^uy}\\ &\quad=\left(\sum_{j=0}^{\infty}\tilde{c}^{(v)}_{2^uj+r}2^{2j}X^{2^uj}\right)\exp\left(-X^{2^u}+\frac{1}{2}X^{2^{u+1}}\right)\exp\left(\frac{1}{2}X^{2^{u+1}}\right)\\ &\quad\quad\times\exp\left(-2^3X^{2^{u+1}}-\sum_{k=u+2}^{u+v}\left(\frac{2^v(3-2^{u-k+1})}{2^k}-\frac{2^{u+v}}{2^{2k+1}}\right)2^{2^{k-u+1}}X^{2^k}\right.\\ &\quad\quad\quad\left.-\sum_{i=0}^{\infty}\frac{3\cdot 2^v-2}{2^{u+v+i+1}}2^{2^{v+i+2}}X^{2^{u+v+i+1}}\right).\end{aligned}$$

To show Eq.(7.9), we define

$$H_2(X) := \exp\left(-2^4X - \sum_{k=u+2}^{u+v}\left(\frac{2^v(3-2^{u-k+1})}{2^k} - \frac{2^{u+v}}{2^{2k+1}}\right)2^{2^{k-u+1}}(2X)^{2^{k-u-1}} - \sum_{i=0}^{\infty}\frac{3\cdot 2^v-2}{2^{u+v+i+1}}2^{2^{v+i+2}}(2X)^{2^{v+i}}\right)$$

and

$$\sum_{j=0}^{\infty} w_j X^{2^u j} := \exp\left(\sum_{j=0}^{\infty}\tilde{c}^{(v)}_{2^u j+r}2^{2j}X^{2^u j}\right)\exp\left(-X^{2^u}+\frac{1}{2}X^{2^{u+1}}\right), \tag{7.8}$$

so that

$$\sum_{y=0}^{\infty}\frac{h_{2^u y+r}(P;C_2)}{2^{2^u y+r}(2^u y+r)!}\,2^{2y}X^{2^u y} = \left(\sum_{j=0}^{\infty}w_j X^{2^u j}\right)\exp\left(\frac{1}{2}X^{2^{u+1}}\right)H_2\left(\frac{1}{2}X^{2^{u+1}}\right).$$

Let q be a nonnegative integer less than 2. The above formula implies that

$$\sum_{y=0}^{\infty}\frac{h_{2^u(2y+q)+r}(P;C_2)}{2^{2^u(2y+q)+r}(2^u(2y+q)+r)!}\,2^{2(2y+q)}X^{2^u(2y+q)} = \left(\sum_{j=0}^{\infty}w_{2j+q}X^{2^u(2j+q)}\right)\exp\left(\frac{1}{2}X^{2^{u+1}}\right)H_2\left(\frac{1}{2}X^{2^{u+1}}\right).$$

Substituting $2^{1/2}X$ for X^{2^u}, we have

$$\sum_{y=0}^{\infty}\frac{h_{2^u(2y+q)+r}(P;C_2)}{2^{2^u(2y+q)+r}(2^u(2y+q)+r)!}\,2^{5y+2q}X^{2y}(2^{1/2}X)^q = \left(\sum_{j=0}^{\infty}w_{2j+q}2^jX^{2j}(2^{1/2}X)^q\right)\exp(X^2)H_2(X^2).$$

Now omit $(2^{1/2}X)^q$ and substitute X for X^2. Then

$$\sum_{y=0}^{\infty}\frac{h_{2^u(2y+q)+r}(P;C_2)}{2^{2^u(2y+q)+r}(2^u(2y+q)+r)!}\,2^{5y+2q}X^{y} = \exp(X)\left(\sum_{j=0}^{\infty}w_{2j+q}2^jX^j\right)H_2(X). \tag{7.9}$$

Moreover, $H_2(X) \in 1 + 2^2X\mathbb{Z}_2\langle X\rangle$ by Lemmas 2.5, 7.2, and 7.3.

Substitute $2^{1/2}X$ for X^{2^u} in Eq.(7.8). Then it follows from Eq.(1.1) that

$$\sum_{j=0}^{\infty}w_j2^{j/2}X^j = \exp\left(\sum_{j=0}^{\infty}\tilde{c}^{(v)}_{2^u j+r}2^{5j/2}X^j\right)\left(\sum_{n=0}^{\infty}\frac{h_n(C_2)}{n!}(-2^{1/2}X)^n\right)$$

and

$$w_{2j+q}2^j = \sum_{i=0}^{2j+q}(-1)^i\tilde{c}^{(v)}_{2^u(2j+q-i)+r}2^{5j+2(q-i)}\frac{h_i(C_2)}{i!}$$

for any nonnegative integer j (cf. Eq.(4.6)). Moreover, by Eq.(7.9) and an argument analogous to that in the proof of Theorem 4.2, we have

$$\frac{h_{2^u(2y+q)+r}(P;C_2)}{2^{2^u(2y+q)+r}(2^u(2y+q)+r)!}\,2^{5y+2q}y! \equiv \begin{cases} \tilde{c}_r^{(v)}(1-y-y^2) \pmod{2^2} & \text{if } q=0,\\ -\tilde{c}_r^{(v)}(1+y-y^2) \pmod{2^2} & \text{if } q=1 \end{cases}$$

for any nonnegative integer y. This completes the proof. □

Theorem 7.10 *Suppose that $p=2$ and $u=v\geq 1$. Let r be a nonnegative integer less than 2^u and q a nonnegative integer less than 4. Then for any nonnegative integer y,*

$$\frac{h_{2^u(4y+q)+r}(P;C_2)}{2^{2^u(4y+q)+r}(2^u(4y+q)+r)!}(-1)^{\delta_{u1}y}2^{6y+q}y! \equiv \begin{cases} \tilde{c}_r^{(v)} \pmod{2^2} & \textit{if } q=0,\\ -\tilde{c}_r^{(v)}+2\tilde{c}_{2^u+r}^{(v)} \pmod{2^2} & \textit{if } q=1,\\ -3\tilde{c}_r^{(v)}-2\tilde{c}_{2^u+r}^{(v)} \pmod{2^2} & \textit{if } q=2,\\ \dfrac{10}{3}\tilde{c}_r^{(v)}-6\tilde{c}_{2^u+r}^{(v)} \pmod{2^2} & \textit{if } q=3. \end{cases}$$

Proof. We follow the argument in Section 4C. By Eq.(7.2),

$$\begin{aligned}\sum_{y=0}^{\infty}&\frac{h_{2^u y+r}(P;C_2)}{2^{2^u y+r}(2^u y+r)!}\,2^y X^{2^u y}\\ &=\left(\sum_{j=0}^{\infty}\tilde{c}_{2^u j+r}^{(v)}2^j X^{2^u j}\right)\exp\left(-X^{2^u}-\frac{7}{2}X^{2^{u+1}}+\frac{33}{4}X^{2^{u+2}}\right)\exp\left((-1)^{\delta_{u1}}\frac{1}{4}X^{2^{u+2}}\right)\\ &\quad\times\exp\left((10\delta_{u1}-18)X^{2^{u+2}}-\sum_{k=u+3}^{u+v}\left(\frac{2^v(3-2^{u-k+1})}{2^k}-\frac{2^{u+v}}{2^{2k+1}}\right)2^{2^{k-u}}X^{2^k}\right.\\ &\qquad\left.-\sum_{i=0}^{\infty}\frac{3\cdot 2^v-2}{2^{u+v+i+1}}\,2^{2^{v+i+1}}X^{2^{u+v+i+1}}\right).\end{aligned}$$

To show Eq.(7.11), we define

$$\begin{aligned}T_2(X):=\exp&\left((40\delta_{u1}-72)X-\sum_{k=u+3}^{u+v}\left(\frac{2^v(3-2^{u-k+1})}{2^k}-\frac{2^{u+v}}{2^{2k+1}}\right)2^{2^{k-u}}(4X)^{2^{k-u-2}}\right.\\ &\qquad\left.-\sum_{i=0}^{\infty}\frac{3\cdot 2^v-2}{2^{u+v+i+1}}\,2^{2^{v+i+1}}(4X)^{2^{v+i-1}}\right)\end{aligned}$$

and

$$\sum_{j=0}^{\infty}w_j X^{2^u j}:=\left(\sum_{j=0}^{\infty}\tilde{c}_{2^u j+r}^{(v)}2^j X^{2^u j}\right)\exp\left(-X^{2^u}-\frac{7}{2}X^{2^{u+1}}+\frac{33}{4}X^{2^{u+2}}\right), \tag{7.10}$$

so that

$$\sum_{y=0}^{\infty}\frac{h_{2^u y+r}(P;C_2)}{2^{2^u y+r}(2^u y+r)!}\,2^y X^{2^u y}=\left(\sum_{j=0}^{\infty}w_j X^{2^u j}\right)\exp\left((-1)^{\delta_{u1}}\frac{1}{4}X^{2^{u+2}}\right)T_2\left(\frac{1}{4}X^{2^{u+2}}\right).$$

Let q be a nonnegative integer less than 4. The above formula implies that

$$\sum_{y=0}^{\infty} \frac{h_{2^u(4y+q)+r}(P;C_2)}{2^{2^u(4y+q)+r}(2^u(4y+q)+r)!}\, 2^{4y+q} X^{2^u(4y+q)} = \left(\sum_{j=0}^{\infty} w_{4j+q} X^{2^u(4j+q)}\right) \exp\left((-1)^{\delta_{u1}} \frac{1}{4}\, X^{2^{u+2}}\right) T_2\left(\frac{1}{4}\, X^{2^{u+2}}\right).$$

Substituting $2^{1/2}X$ for X^{2^u}, we have

$$\sum_{y=0}^{\infty} \frac{h_{2^u(4y+q)+r}(P;C_2)}{2^{2^u(4y+q)+r}(2^u(4y+q)+r)!}\, 2^{6y+q} X^{4y}(2^{1/2}X)^q = \left(\sum_{j=0}^{\infty} w_{4j+q} 2^{2j} X^{4j}(2^{1/2}X)^q\right) \exp((-1)^{\delta_{u1}} X^4) T_2(X^4).$$

Now omit $(2^{1/2}X)^q$ and substitute $(-1)^{\delta_{u1}}X$ for X^4. Then

$$\sum_{y=0}^{\infty} \frac{h_{2^u(4y+q)+r}(P;C_2)}{2^{2^u(4y+q)+r}(2^u(4y+q)+r)!}\, 2^{6y+q}((-1)^{\delta_{u1}}X)^y = \exp(X) \left(\sum_{j=0}^{\infty} w_{4j+q} 2^{2j}((-1)^{\delta_{u1}}X)^j\right) T_2((-1)^{\delta_{u1}}X). \tag{7.11}$$

Moreover, $T_2((-1)^{\delta_{u1}}X) \in 1 + 2^2 X\mathbb{Z}_2\langle X\rangle$ by Lemmas 2.5, 7.2, and 7.3.

Substitute $2^{1/2}X$ for X^{2^u} in Eq.(7.10). Then it follows from Eq.(1.1) that

$$\sum_{j=0}^{\infty} w_j 2^{j/2} X^j = \left(\sum_{j=0}^{\infty} \tilde{c}^{(v)}_{2^u j+r} 2^{3j/2} X^j\right) \left(\sum_{n=0}^{\infty} \frac{h_n(C_4)}{n!}(-2^{1/2}X)^n\right) \times \left(\sum_{n=0}^{\infty} \frac{h_n(C_2)}{n!}(-2^3 X^2)^n\right).$$

Let j be a nonnegative integer. By the above equation, we have

$$w_{4j+q} 2^{2j} 2^{q/2} = \sum_{i=0}^{4j+q} \tilde{c}^{(v)}_{2^u(4j+q-i)+r} 2^{6j+q-i} 2^{(q-i)/2} \sum_{k=0}^{[i/2]} \frac{h_{i-2k}(C_4)}{(i-2k)!}(-2^{1/2})^{i-2k} \frac{h_k(C_2)}{k!}(-2^3)^k,$$

and thus

$$w_{4j+q} 2^{2j} = \sum_{i=0}^{4j+q} \sum_{k=0}^{[i/2]} w_j(i,k),$$

where

$$w_j(i,k) = (-1)^{i-k} \tilde{c}^{(v)}_{2^u(4j+q-i)+r} 2^{6j+q-i+2k} \frac{h_{i-2k}(C_4)}{(i-2k)!} \cdot \frac{h_k(C_2)}{k!}.$$

Let i and k be nonnegative integers, and suppose that $i \leq 4j + q$ and $k \leq [i/2]$. Then by Lemma 2.4, Corollary 3.13, and Eq.(5.5), we have

$$\mathrm{ord}_2(w_j(i,k)) \geq 6j + q - i - \left[\frac{i+1}{2}\right] + \left[\frac{i-2k}{2^2}\right] - 2\left[\frac{i-2k}{2^3}\right] + 2k + s_2(i-2k) + s_2(k).$$

Hence it is easily seen that $\mathrm{ord}_2(w_1(i,k)) \geq 2$ and $\mathrm{ord}_2(w_j(i,k)) \geq 1$ if $j \geq 2$. Consequently, by Eq.(7.11) and an argument analogous to that in the proof of Theorem 5.13, we have

$$\frac{h_{2^u(4y+q)+r}(P;C_2)}{2^{2^u(4y+q)+r}(2^u(4y+q)+r)!}(-1)^{\delta_{u1}y}2^{6y+q}y! \equiv \sum_{i=0}^{q}\sum_{k=0}^{[i/2]} w_0(i,k) \pmod{2^2}$$

for any nonnegative integer y. Moreover, it turns out that

$$\begin{aligned}\sum_{i=0}^{q}\sum_{k=0}^{[i/2]} w_0(i,k) &= \sum_{i=0}^{q}\sum_{k=0}^{[i/2]}(-1)^{i-k}\tilde{c}^{(v)}_{2^u(q-i)+r}2^{q-i+2k}\frac{h_{i-2k}(C_4)}{(i-2k)!}\cdot\frac{h_k(C_2)}{k!}\\ &= \begin{cases} \tilde{c}^{(v)}_r & \text{if } q=0,\\ -\tilde{c}^{(v)}_r + 2\tilde{c}^{(v)}_{2^u+r} & \text{if } q=1,\\ -3\tilde{c}^{(v)}_r - 2\tilde{c}^{(v)}_{2^u+r} + 2^2\tilde{c}^{(v)}_{2^{u+1}+r} & \text{if } q=2,\\ \dfrac{10}{3}\tilde{c}^{(v)}_r - 6\tilde{c}^{(v)}_{2^u+r} - 2^2\tilde{c}^{(v)}_{2^{u+1}+r} + 2^3\tilde{c}^{(v)}_{2^{u+1}+2^u+r} & \text{if } q=3.\end{cases}\end{aligned}$$

This completes the proof. □

8 Additional results

In order to state an analogue of Theorem 1.1, we set

$$\overline{\tau}^{(u,v)}_p(n) = \begin{cases} \displaystyle\sum_{j=0}^{u-1}\left[\frac{n}{2^j}\right] + \left[\frac{n}{2^{u+1}}\right] - \left[\frac{n}{2^{u+2}}\right] & \text{if } p=2 \text{ and } u=v\geq 1,\\ \displaystyle\sum_{j=0}^{u-1}\left[\frac{n}{p^j}\right] - (u-v)\left[\frac{n}{p^u}\right] & \begin{array}{l}\text{if } p\geq 2 \text{ and } u\geq v+1\geq 1,\\ \text{or if } p>2 \text{ and } u=v\geq 1\end{array}\end{cases}$$

for each nonnegative integer n, and set

$$\overline{\kappa}_p(u,v) = \begin{cases} u+2 & \text{if } p=2 \text{ and } u=v\geq 1,\\ u+1 & \text{if } p=2 \text{ and } u=v+1\geq 2, \text{ or if } p=3 \text{ and } u=v\geq 1,\\ u & \text{otherwise.}\end{cases}$$

The following theorem, which includes Theorem 1.3, is inspired by Theorems 1.1 and 1.2 and is a consequence of Lemma 2.4 of Theorems 7.5–7.10.

Theorem 8.1 *Suppose that $|P| > 1$. The following statements hold.*

(1) *Except for the case where $p = 2$ and $1 \le v \le u \le v+1$, there exist p-adic analytic functions $f_r(X)$ for $r = 0, 1, \ldots, p^{\overline{\kappa}_p(u,v)} - 1$ contained in $\mathbb{Z}_p\langle X\rangle$ and a polynomial $\eta(X)$ of degree $p^{\overline{\kappa}_p(u,v)} - 1$ with integer coefficients such that for any nonnegative integer y,*

$$h_{p^{\overline{\kappa}_p(u,v)}y+r}(P; C_p) = p^{\overline{\tau}_p^{(u,v)}(p^{\overline{\kappa}_p(u,v)}y)} f_r(y) \prod_{j=1}^{y} \eta(j)$$

and

$$\operatorname{ord}_p(h_{p^{\overline{\kappa}_p(u,v)}y+r}(P; C_p)) = \overline{\tau}_p^{(u,v)}(p^{\overline{\kappa}_p(u,v)}y) + \operatorname{ord}_p(f_r(y)).$$

Moreover, $\operatorname{ord}_p(h_n(P; C_p)) \ge \overline{\tau}_p^{(u,v)}(n)$ *for any nonnegative integer n.*

(2) *Except for the case where $p = 2$ and $u = v \ge 1$,* $\operatorname{ord}_p(h_n(P; C_p)) = \overline{\tau}_p^{(u,v)}(n)$ *for each nonnegative integer n such that $n \equiv 0 \pmod{p^u}$.*

(3) *Assume that $p = 2$ and that $u = v \ge 1$. Then* $\operatorname{ord}_2(h_n(P; C_2)) = \overline{\tau}_2^{(u,v)}(n)$ *for each nonnegative integer n such that $n \equiv 0$, 2^u, or $2^{u+1} \pmod{2^{u+2}}$.*

Proof. The proof is analogous to that of Theorems 1.1 and 1.2. □

We turn to the disks of convergence for the p-adic power series $E_G(X)$ and $E_P(X; C_p)$.

Theorem 8.2 *The following statements hold.*

(1) *The p-adic power series $E_G(X)$ converges only in the open disc of radius p^a, where*

$$a = \begin{cases} -\dfrac{7}{2^{u+3}} & \text{if } p = 2 \text{ and } u = v \ge 1, \\ -\dfrac{1}{p^u(p-1)} - \dfrac{u-v}{p^{u+1}} & \text{otherwise.} \end{cases}$$

(2) *The p-adic power series $E_P(X; C_p)$ converges only in the open disc of radius p^a, where*

$$a = \begin{cases} -\dfrac{7}{2^{u+2}} & \text{if } p = 2 \text{ and } u = v \ge 1, \\ -\dfrac{1}{p^{u-1}(p-1)} - \dfrac{u-v}{p^u} & \text{if } p \ge 2 \text{ and } u \ge v+1 \ge 1, \\ & \text{or if } p > 2 \text{ and } u = v \ge 1. \end{cases}$$

Proof. The theorem follows from Theorems 1.1–1.3. The statement (1) is proved in [20]. The proof of the statement (2) is completely analogous to that of the statement (1). □

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