Multiplicative induction and units for the ring of monomial representations

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Multiplicative induction and units for the ring of monomial representations

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Abstract

Let G be a finite group, and let A be a finite abelian G-group. For each subgroup H of G, $\Omega(H, A)$ denotes the ring of monomial representations of H with coefficients in A, which is a generalization of the Burnside ring $\Omega(H)$ of H. We research the multiplicative induction map $\Omega(H, A) \to \Omega(G, A)$ derived from the tensor induction map $\Omega(H) \to \Omega(G)$, and also research the unit group of $\Omega(G, A)$. The results are explained in terms of the first cohomology groups $H^1(K, A)$ for $K \leq G$. We see that tensor induction for 1-cocycles plays a crucial role in a description of multiplicative induction. The unit group of $\Omega(G, A)$ is identified as a finitely generated abelian group. We especially study the group of torsion units of $\Omega(G, A)$, and study the unit group of $\Omega(G)$ as well.

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1 Introduction

Let G be a finite group, and let A be a finite abelian group on which G acts via a homomorphism from G to the group of automorphisms of A. We are concerned with the ring $\Omega(G, A)$ of monomial representations of G with coefficients in A, which was introduced by Dress [12] and is called the monomial Burnside ring for short. This ring contains the ordinary Burnside ring $\Omega(G)$ as a subring, and is applicable to the representation theory of finite groups. There are some well-known facts about $\Omega(G, A)$ (see, *e.g.*, [2, 3, 12, 13, 22, 23]). Many properties of Burnside rings seem to be extended to monomial Burnside rings; for instance, the prime ideal spectrum of $\Omega(G, A)$ was studied in [12] (see also [10]). In this paper, among others, we focus our mind on the concept of multiplicative induction for monomial Burnside rings and the unit group of $\Omega(G, A)$. There are some specific characterizations of them which mean the algebraic peculiarities of $\Omega(G, A)$.

Following [12], we give the concept of (G, A)-sets and define simple (G, A)-sets $(G/K)_\nu$ for $K \leq G$ and 1-cocycles $\nu : K \to A$ in Section 2. The monomial Burnside ring $\Omega(G, A)$, which is defined to be the Grothendieck ring of the category of (G, A)-sets (see Definition 2.13), is the commutative unital ring consisting of all formal $\mathbb{Z}$-linear combinations of the symbols $[(G/K)_\nu]$ corresponding to the isomorphism classes of (G, A)-sets containing simple (G, A)-sets $(G/K)_\nu$ (see Proposition 2.14).

The concept of multiplicative induction for Burnside rings was introduced by tom Dieck [9] and Dress [11], and was developed by Yoshida [32]. In an attempt to introduce multiplicative induction for monomial Burnside rings, Barker [2] successfully defined the tenduction map ${}^{\mathbb{Z}}_{C}\mathrm{ten}^{G}_{H} : B(C, H) \to B(C, G)$ for each $H \leq G$, where C is a supercyclic group and $B(C, H)$ is the monomial Burnside ring for H with fibre group C, as a generalization of multiplicative induction for Burnside rings. (If C is a finite cyclic group on which G acts trivially, then $\Omega(G, C) \simeq B(C, G)$.)

In Section 3, we introduce the multiplicative induction map

$$\overline{\mathrm{Map}}_H(G, -) : \Omega(H, A) \to \Omega(G, A), \quad x \mapsto \overline{\mathrm{Map}}_H(G, x)$$

for each $H \leq G$. When A is a cyclic group on which G acts trivially, this map is associated with tensor induction for linear characters of G (cf. [8, §13A]). We have $\overline{\mathrm{Map}}_H(G, [(H/H)_\sigma]) = [\widehat{\mathrm{Map}}_H(G, (H/H)_\sigma)] = [(G/G)_{\sigma^{\otimes G}}]$ for all 1-cocycles $\sigma : H \to A$ (see Example 3.13), where 1-cocycles $\sigma^{\otimes G} : G \to A$ are obtained from $\sigma : H \to A$ by tensor induction. There is a nice formula of multiplicative (tensor) induction for Burnside rings (cf. [8, (80.49) Corollary]). The methods used in [8, §80C] enable us to establish that for any (H, A)-sets T_0 and T,

$$\overline{\mathrm{Map}}_H(G, [T_0] - [T]) = \sum_{i=0}^{n} (-1)^i [\widehat{\mathrm{Map}}_H(G, T_0, T_1, \ldots, T_i)], \tag{1.1}$$

where $n = |G : H|$ and $T = T_1 = \cdots = T_n$ (see Proposition 3.22).

The mark homomorphism ρ_G, which was introduced by Dress [12], is a ring monomorphism from $\Omega(G, A)$ to the set $\mho(G, A) := (\prod_{K \leq G} \mathbb{Z}H^1(K, A))^G$ of G-invariants in the direct product of integral group rings of the first cohomology groups $H^1(K, A)$ for $K \leq G$, where the action of G on $\prod_{K \leq G} \mathbb{Z}H^1(K, A)$ is given by the conjugation maps $\mathrm{con}_K^g : \mathbb{Z}H^1(K, A) \to \mathbb{Z}H^1({}^gK, A)$ for $K \leq G$ and $g \in G$. For each $U \leq G$, there is a ring homomorphism $-^{\otimes G} : \mathbb{Z}H^1(U, A) \to \mathbb{Z}H^1(G, A)$ derived from tensor induction which assigns to a 1-cocycle $\tau : U \to A$ the 1-cocycle $\tau^{\otimes G} : G \to A$. In Section 4, we describe $\overline{\mathrm{Map}}_H(G, x) \in \Omega(G, A)$ for each $x \in \Omega(H, A)$ via ρ_G as

$$\rho_G(\overline{\mathrm{Map}}_H(G, x)) = \left(\prod_{KgH \in K\backslash G/H} \mathrm{con}^g_{K^g \cap H}(x_{K^g \cap H})^{\otimes K} \right)_{K \leq G} \in \mho(G, A) \tag{1.2}$$

under the assumption that $\rho_H(x) = (x_L)_{L \leq H}$, where $\rho_H : \Omega(H, A) \to \mho(H, A)$ is the mark homomorphism (see Theorem 4.16). This fact is a generalization of [32, §3(b.3)]. We make use of Eq.(1.1) to prove Eq.(1.2).

The fundamental theorem of the Burnside ring $\Omega(G)$ (cf. [32, Lemma 2.1]) is a useful instrument for finding the idempotents of $\Omega(G)$ (cf. [33, 4.12 Theorem]), and is also essential to the Yoshida criterion (see Theorem 6.4) for the units of $\Omega(G)$. In Section 5, we insist on the existence of a short exact sequence

$$0 \longrightarrow \Omega(G, A) \xrightarrow{\varphi} \widetilde{\Omega}(G, A) \xrightarrow{\psi} \mathrm{Obs}\,(G, A) \longrightarrow 0$$

of additive groups (see Theorem 5.9) derived from the Cauchy-Frobenius lemma (see, *e.g.*, [33, 2.7 Lemma]), which generalizes the fundamental theorem of $\Omega(G)$.

Information of the primitive idempotents of the Burnside algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$ can help us to realize the units of $\Omega(G)$. Following [33, §4], we review the primitive idempotents of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$ and those of $\Omega(G)$; the latter are precisely the primitive idempotents of $\Omega(G, A)$ (see Theorem 5.18).

The unit group $\Omega(G)^\times$ of the Burnside ring $\Omega(G)$ is studied in many papers (see, *e.g.*, [6, 9, 11, 15, 18, 19, 20, 24, 30, 32]). Section 6 is devoted to a review of some well-known facts about $\Omega(G)^\times$. We also study a certain specific type of units (see Proposition 6.11), and present an additional fact about the structure of $\Omega(G)^\times$ for which the Yoshida criterion plays a crucial role (see Corollary 6.18).

The unit group $\Omega(G,A)^\times$ of the monomial Burnside ring $\Omega(G,A)$ was studied in [2, 22]. In Section 7, we show that $\Omega(G,A)^\times$ is a finitely generated abelian group (see Proposition 7.2). Consequently, the group $\Omega(G,A)^\omega$ of torsion units of $\Omega(G,A)$ is a finite abelian group. The basic structure of $\Omega(G,A)^\omega$ is analyzed on the basis of a generalization of the Yoshida criterion (see Theorem 7.3). We adapt the methods presented in [2, §8] for an analysis of $\Omega(G,A)^\omega$, and successfully elucidate the structure of $\Omega(G,A)^\omega$ in the sequel (see Corollary 7.4). Specifically, if G is nilpotent, then the universal result deduces that

$$\Omega(G,A)^\omega \simeq \Omega(G)^\times \times H^1(G,A)$$

(see Example 7.6). This fact is a generalization of [22, Proposition 5.1].

Notation Let G be a finite group. We denote by ϵ the identity of G, and denote by $\mathrm{S}(G)$ the set of subgroups of G. The subgroup generated by $g_1,\dots,g_k \in G$ is denoted by $\langle g_1,\dots,g_k\rangle$. We write $H \le G$ if H is a subgroup of G, and write $H < G$ if H is a proper subgroup of G. The Möbius function on the poset $(\mathrm{S}(G),\le)$ of all subgroups of G is denoted by μ (see, *e.g.*, [1]). We denote by $\mathrm{C}(G)$ a full set of non-conjugate subgroups of G. Let $H \le G$. We set ${}^gH = gHg^{-1}$ and $H^g = g^{-1}Hg$ for $g \in G$, and denote by (H) the set of conjugates of H in G. The normalizer of H in G is denoted by $N_G(H)$. We denote by $|G:H|$ the index of H in G, and denote by G/H the set of left cosets gH, $g \in G$, of H in G. Given $K, U \le G$, $K\backslash G/U$ denotes the set of (K,U)-double cosets KgU, $g \in G$, in G. The category of finite left G-sets and G-equivariant maps is denoted by G-**set**. For each finite set X, we denote by $|X|$ the cardinality of X. The natural numbers, the rational integers, the rational numbers, and the complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$, respectively. We set $[n] = \{1, 2, \dots, n\}$ for each $n \in \mathbb{N}$. The identity map on a set Σ is denoted by id_Σ. For each group V, we denote by $\mathrm{Hom}(V,\langle -1\rangle)$ the group consisting of all group homomorphisms from V to the unit group $\langle -1\rangle$ of $\mathbb{Z}$ with pointwise product.

2 Monomial Burnside rings

2A 1-cocycles

Throughout the paper, let G be a finite group, and let A be a finite G-group, that is, A is a finite group on which G acts via a homomorphism from G to the group of automorphisms of A (cf. [26, Chapter 1, Definition 8.1]). We start with the definition of (G,A)-sets introduced by Dress [12] (see also [27]). Given $g \in G$

and $a \in A$, the effect of g on a is denoted by ${}^{g}a$. A finite free right A-set Y is called a (G, A)-set if it is also a left G-set and if

$$g(ya) = (gy)\,{}^{g}a$$

for all $g \in G$, $a \in A$, and $y \in Y$. A map between (G, A)-sets is called a (G, A)-equivariant map if it is a morphism of both left G-sets and right A-sets. We now obtain the category of (G, A)-sets such that the empty set is an initial object, which is denoted by (G, A)-**set**. Under the assumption that A is abelian, the set of isomorphism classes of (G, A)-sets forms a commutative unital semiring, and the monomial Burnside ring $\Omega(G, A)$ is defined to be the associated Grothendieck ring (cf. [12]).

For a (G, A)-set Y, we denote by Y/A the set of A-orbits $yA := \{ya \mid a \in A\}$, $y \in Y$, on Y, which is considered as a left G-set with the action of G given by

$$g(yA) = gyA$$

for all $g \in G$ and $y \in Y$. A (G, A)-set Y is said to be simple if Y/A is a transitive left G-set. Given a pair of (G, A)-sets Y_1 and Y_2, their disjoint union $Y_1 \dot{\cup} Y_2$ is also a (G, A)-set. Every (G, A)-set is a disjoint union of simple (G, A)-sets. A subset of a (G, A)-set is said to be a (G, A)-subset if it is closed under the actions of G and A.

Let A° be the opposite group of A. For each $a \in A$, let a° denote the element of A° corresponding to a. By definition, $a^\circ b^\circ = (ba)^\circ$ for all $a, b \in A$. We view A° as a G-group with the action given by that of G on A, and denote by F the semidirect product $A^\circ \rtimes G$ of A° and G. Each (G, A)-set Y is viewed as a left F-set with the action of F given by

$$(a^\circ, g)y = (gy)a \tag{2.1}$$

for all $(a^\circ, g) \in F$ and $y \in Y$. A (G, A)-set is simple if and only if it is a transitive left F-set. A bijection between (G, A)-sets is an isomorphism of (G, A)-sets if and only if it is an isomorphism of left F-sets.

Let $H \le G$. By restriction of operators from G to H, we view A as an H-group. A map $\sigma : H \to A$ is called a 1-cocycle or a crossed homomorphism if

$$\sigma(h_1 h_2) = \sigma(h_1)\,{}^{h_1}\sigma(h_2)$$

for all $h_1, h_2 \in H$ (cf. [26, I, p. 243]). We define a 1-cocycle $1_H : H \to A$ by $1_H(h) = \epsilon_A$ for all $h \in H$, where ϵ_A is the identity of A.

Definition 2.1 For each $H \le G$, we denote by $Z^1(H, A)$ the set of 1-cocycles from H to A. Let $\mathcal{S}(G, A)$ be the set of pairs (H, σ) of $H \le G$ and $\sigma \in Z^1(H, A)$. Given $(H, \sigma) \in \mathcal{S}(G, A)$, we fix a complete set $\{g_1, g_2, \ldots, g_n\}$ with $g_1 = \epsilon$ of representatives of G/H, and define a (G, A)-set $(G/H)_\sigma$ to be the cartesian product $A \times (G/H)$ with the left action of G and the right action of A given by

$$g(a, g_j H) = ({}^{g_{j'}}\sigma(g_{j'}^{-1} g g_j)\,{}^{g}a, g_{j'} H) \quad \text{and} \quad (a, g_j H)b = (ab, g_j H),$$

where $gg_jH = g_{j'}H$, for all $g \in G$, a, $b \in A$, and $j \in [n]$, respectively.

Let $(H, \sigma) \in \mathcal{S}(G, A)$. Then $(G/H)_\sigma$ is a transitive left F-set. We define

$$F_{(H,\sigma)} := \{(\sigma(h)^{\circ -1}, h) \in F \mid h \in H\},$$

so that $F_{(H,\sigma)}$ is the stabilizer of $(\epsilon_A, H) \in (G/H)_\sigma$ in F (see [27, §2]), and make the set $F/F_{(H,\sigma)}$ of left cosets of $F_{(H,\sigma)}$ in F into a (G, A)-set by defining

$$g((a^\circ, r)F_{(H,\sigma)}) = ({}^g a^\circ, gr)F_{(H,\sigma)} \quad \text{and} \quad ((a^\circ, r)F_{(H,\sigma)})b = ((ab)^\circ, r)F_{(H,\sigma)} \qquad (2.2)$$

for all $g \in G$, $b \in A$, and $(a^\circ, r) \in F$.

Lemma 2.2 *Let $(H, \sigma) \in \mathcal{S}(G, A)$. Then $(G/H)_\sigma \simeq F/F_{(H,\sigma)}$ as (G, A)-sets. In particular, the isomorphism class of (G, A)-sets containing $(G/H)_\sigma$ is independent of the choice of $g_2, \ldots, g_n$ in Definition* 2.1.

Proof. There exists an isomorphism $F/F_{(H,\sigma)} \xrightarrow{\sim} (G/H)_\sigma$ of F-sets given by

$$(a^\circ, g)F_{(H,\sigma)} \mapsto (g(\epsilon_A, H))a$$

for all $(a^\circ, g) \in F$, because $F_{(H,\sigma)}$ is the stabilizer of $(\epsilon_A, H) \in (G/H)_\sigma$ in F. Thus we have $(G/H)_\sigma \simeq F/F_{(H,\sigma)}$ as (G, A)-sets, completing the proof. □

Remark 2.3 Given a simple (G, A)-set Y and $y \in Y$, the stabilizer F_y of y in F coincides with $F_{(H,\sigma)}$ for some $(H, \sigma) \in \mathcal{S}(G, A)$ (see the proof of [27, Lemma 2.1]), and hence $Y \simeq F/F_{(H,\sigma)}$ as (G, A)-sets. Under the notation of Definition 2.1, we may define $(G/H)_\sigma$ without assuming that $g_1 = \epsilon$. In such a case, $F_{(H,\sigma)}$ is the stabilizer of $(\sigma(g_1)^{-1}, H) \in (G/H)_\sigma$ in F, which yields $(G/H)_\sigma \simeq F/F_{(H,\sigma)}$.

2B Isomorphism classes

We give a complete set of representatives of isomorphism classes of (G, A)-sets.

Definition 2.4 Let $(H, \sigma) \in \mathcal{S}(G, A)$. Suppose that $g \in G$ and $a \in A$. We define two 1-cocycles $g\sigma : {}^gH \to A$ and $\sigma^a : H \to A$ by

$$(g\sigma)(ghg^{-1}) = {}^g\sigma(h) \quad \text{and} \quad \sigma^a(h) = a^{-1}\sigma(h)\,{}^h a$$

for all $h \in H$, respectively.

Let $H \leq G$, and let σ, $\tau \in Z^1(H, A)$. We write $\sigma =_A \tau$ if $\tau = \sigma^a$ for some $a \in A$.

Lemma 2.5 *Let $(H, \sigma) \in \mathcal{S}(G, A)$. Then $g\sigma =_A g(\sigma^a)$ for any $g \in G$ and $a \in A$.*

Proof. We have $g(\sigma^a) = (g\sigma)^{\,{}^g a}$ for any $g \in G$ and $a \in A$, completing the proof. □

The argument of the proof of Lemma 2.5 ensures that $\mathcal{S}(G, A)$ is a left F-set with the action of F given by

$$(a^\circ, g)(H, \sigma) = ({}^gH, (g\sigma)^a)$$

for all $(a^\circ, g) \in F$ and $(H, \sigma) \in \mathcal{S}(G, A)$.

By [27, Lemma 2.3], (H, σ) and (U, τ) are contained in the same F-orbit on $\mathcal{S}(G, A)$ if and only if $(G/H)_\sigma \simeq (G/U)_\tau$ as (G, A)-sets.

Lemma 2.6 *Let $H \leq G$, and let $\sigma \in Z^1(H, A)$. Then $h\sigma = \sigma^{\sigma(h)}$ for any $h \in H$. Moreover, given $\sigma_0 \in Z^1(H, A)$, $\sigma_0 =_A \sigma$ if and only if $(H/H)_{\sigma_0} \simeq (H/H)_\sigma$.*

Proof. The first assertion is shown in the proof of [27, Lemma 3.2]. Suppose that $\sigma_0 \in Z^1(H, A)$. By [27, Lemma 2.3], $(H/H)_{\sigma_0} \simeq (H/H)_\sigma$ if and only if there exist some $h \in H$ and $a \in A$ such that $\sigma_0 = (h\sigma)^a$. Hence the second assertion follows from the first one. This completes the proof. □

Definition 2.7 We define a subset $\mathcal{R}(G, A)$ of $\mathcal{S}(G, A)$ to be a complete set of representatives of F-orbits on $\mathcal{S}(G, A)$ such that $H \in \mathrm{C}(G)$ for any $(H, \sigma) \in \mathcal{R}(G, A)$.

The following proposition is [27, Proposition 2.4].

Proposition 2.8 *Let Y be a simple (G, A)-set. There exists a unique element (H, σ) of $\mathcal{R}(G, A)$ such that $Y \simeq (G/H)_\sigma$ as (G, A)-sets.*

Let $H \leq G$, and let $X \in H$-**set**. We define a left action of H on the cartesian product $G \times X$ of G and X by

$$h(g, x) = (gh^{-1}, hx)$$

for all $h \in H$ and $(g, x) \in G \times X$. Given $(g, x) \in G \times X$, let $g \otimes x$ denote the H-orbit containing (g, x). The left G-set $\mathrm{ind}_H^G(X)$ induced from X is the set of H-orbits on $G \times X$ with the action of G given by

$$g(r \otimes x) = gr \otimes x$$

for all g, $r \in G$ and $x \in X$ (cf. [11, §4]). Let $g \in G$, and set $g \otimes X = \{g \otimes x \mid x \in X\}$, which is a subset of $\mathrm{ind}_H^G(X)$. The left ${}^gH$-set $\mathrm{con}_H^g(X)$ conjugate to X is the set $g \otimes X$ with the action of ${}^gH$ given by

$$ghg^{-1}(g \otimes x) = g \otimes hx$$

for all $h \in H$ and $x \in X$, and is denoted simply by ${}^gX$.

Definition 2.9 Let $H \leq G$, and let T be an (H, A)-set. The (G, A)-set $\mathrm{ind}_H^G(T)$ induced from T is the left G-set $\mathrm{ind}_H^G(T)$ with the right action of A given by

$$(r \otimes t)a = r \otimes t\,{}^{r^{-1}}a$$

for all $r \in G$, $t \in T$, and $a \in A$ (cf. [27, Remark 6.2]). Let $g \in G$. The $({}^gH, A)$-set $\mathrm{con}_H^g(T)$ conjugate to T is the left ${}^gH$-set ${}^gT$ with the right action of A given by

$$(g \otimes t)a = g \otimes t\,{}^{g^{-1}}a$$

for all $t \in T$ and $a \in A$ (cf. [27, Remark 6.4]), and is denoted simply by ${}^gT$.

Lemma 2.10 *If $U \leq H \leq G$ and $\tau \in Z^1(U, A)$, then $\mathrm{ind}_H^G((H/U)_\tau) \simeq (G/U)_\tau$, ${}^g((H/U)_\tau) \simeq ({}^gH/{}^gU)_{g\tau}$ for each $g \in G$, and ${}^h((H/U)_\tau) \simeq (H/U)_\tau$ for all $h \in H$.*

Proof. The proof is straightforward. Note that the last assertion follows from the second one and [27, Lemma 2.3]. □

Let $(H, \sigma) \in \mathcal{S}(G, A)$, and let T be an (H, A)-set. For each $K \leq H$, we define a 1-cocycle $\sigma|_K : K \to A$, the restriction of σ, to be the map obtained by restriction of $\sigma : H \to A$ from H to K, and define a (K, A)-set $\mathrm{res}_K^H(T)$, the restriction of T, to be the (K, A)-set T obtained by restriction of operators from H to K.

We show a Mackey decomposition formula for (G, A)-sets (cf. [27, Lemma 6.5]).

Lemma 2.11 *Let $H \leq G$, and let $(U, \tau) \in \mathcal{S}(G, A)$. Then*

$$\mathrm{res}_H^G((G/U)_\tau) \simeq \dot{\bigcup_{HgU \in H\backslash G/U}} (H/(H \cap {}^gU))_{(g\tau)|_{H \cap {}^gU}},$$

where the disjoint union is taken over all (H, U)-double cosets HgU, $g \in G$, in G.

Proof. Let $\{g_1, g_2, \ldots, g_m\}$ be a complete set of representatives of $H\backslash G/U$. For each $i \in [m]$, let $\{h_{i1}, h_{i2}, \ldots, h_{i\ell_i}\}$ be a complete set of representatives of $H/(H \cap {}^{g_i}U)$. Then $\{h_{ij}g_i \mid i \in [m] \text{ and } j \in [\ell_i]\}$ is a complete set of representatives of G/U. We define a map $\Gamma : \mathrm{res}_H^G((G/U)_\tau) \to \dot{\bigcup}_{i \in [m]} (H/(H \cap {}^{g_i}U))_{(g_i\tau)|_{H \cap {}^{g_i}U}}$ by

$$(a, h_{ij}g_iU) \mapsto (a, h_{ij}(H \cap {}^{g_i}U))$$

for all $i \in [m]$, $j \in [\ell_i]$, and $a \in A$. Obviously, this map is bijective and A-equivariant. Given $h \in H$, $i \in [m]$, and $j \in [\ell_i]$, if $hh_{ij} = h_{ij'}h' \in h_{ij'}(H \cap {}^{g_i}U)$ with $h' \in H \cap {}^{g_i}U$, then we have $h(h_{ij}g_i) = h_{ij'}g_i(g_i^{-1}h'g_i) \in h_{ij'}g_iU$ and

$${}^{h_{ij'}g_i}\tau((h_{ij'}g_i)^{-1}h(h_{ij}g_i)) = {}^{h_{ij'}g_i}\tau(g_i^{-1}h'g_i) = {}^{h_{ij'}}(g_i\tau)(h_{i'j'}^{-1}hh_{ij}).$$

Thus Γ is H-equivariant. (See also Lemma 2.2.) This completes the proof. □

2C Tensor product

From now on, we assume that A is abelian. Hence $A = A^\circ$. Following [12], we define the tensor product $Y_1 \otimes Y_2$ of (G, A)-sets Y_1 and Y_2. The cartesian product $Y_1 \times Y_2$ is viewed as a free right A-set with the action of A given by

$$(y_1, y_2)a = (y_1 a^{-1}, y_2 a)$$

for all $a \in A$ and $(y_1, y_2) \in Y_1 \times Y_2$. For each $(y_1, y_2) \in Y_1 \times Y_2$, let $y_1 \otimes y_2$ be the A-orbit containing (y_1, y_2). We set

$$Y_1 \otimes Y_2 = \{y_1 \otimes y_2 \mid (y_1, y_2) \in Y_1 \times Y_2\},$$

and make it into a (G, A)-set by defining

$$g(y_1 \otimes y_2) = gy_1 \otimes gy_2 \quad \text{and} \quad (y_1 \otimes y_2)a = y_1 \otimes y_2 a$$

for all $g \in G$, $a \in A$, and $(y_1, y_2) \in Y_1 \times Y_2$. These actions are well-defined, because

$$g((y_1 b^{-1} \otimes y_2 b)a) = g(y_1 b^{-1}) \otimes g(y_2 ba) = (gy_1)\, {}^g b^{-1} \otimes g(y_2 a)\, {}^g b = g((y_1 \otimes y_2)a)$$

for all $g \in G$, a, $b \in A$, and $(y_1, y_2) \in Y_1 \times Y_2$. Obviously, $Y_1 \otimes Y_2 \simeq Y_2 \otimes Y_1$.

Lemma 2.12 *Let $K \le H \le G$, and let $g \in G$. For any (H, A)-sets T_1 and T_2,*

$$\mathrm{res}^H_K(T_1 \otimes T_2) \simeq \mathrm{res}^H_K(T_1) \otimes \mathrm{res}^H_K(T_2) \quad \textit{and} \quad {}^g(T_1 \otimes T_2) \simeq {}^g T_1 \otimes {}^g T_2.$$

Proof. The proof is straightforward. □

Let $\mathbf{F}(G, A)$ be the free abelian group on the set of isomorphism classes of (G, A)-sets. For each (G, A)-set Y, we denote by $\overline{Y}$ the isomorphism class of (G, A)-sets containing Y. Let $\mathbf{F}(G, A)_0$ be the subgroup of $\mathbf{F}(G, A)$ generated by the elements

$$\overline{Y_1 \dot\cup Y_2} - \overline{Y_1} - \overline{Y_2}$$

for (G, A)-sets Y_1 and Y_2. We define multiplication on the generators of $\mathbf{F}(G, A)$ by

$$\overline{Y_1} \cdot \overline{Y_2} = \overline{Y_1 \otimes Y_2}$$

for all (G, A)-sets Y_1 and Y_2, and extend it to $\mathbf{F}(G, A)$ by linearity. Then $\mathbf{F}(G, A)$ is a commutative unital ring; moreover, $\mathbf{F}(G, A)_0$ is an ideal of $\mathbf{F}(G, A)$.

Definition 2.13 We define a commutative unital ring $\Omega(G, A)$ to be the quotient $\mathbf{F}(G, A)/\mathbf{F}(G, A)_0$, which is the ring of monomial representations of G with coefficients in A introduced by Dress [12] (see also [2]).

When $A = \{\epsilon_A\}$, which is the group consisting of only the identity, $\Omega(G, A)$ is isomorphic to the Burnside ring $\Omega(G)$ (see §5C).

For each (G, A)-set Y, we denote by $[Y]$ the coset $\overline{Y} + \mathbf{F}(G, A)_0$ of $\mathbf{F}(G, A)_0$ in $\mathbf{F}(G, A)$. By [12, Proposition 1(b)] (or [27, Lemma 2.6]), $[Y_1] = [Y_2]$ if and only if $\overline{Y_1} = \overline{Y_2}$. Multiplication on the generators of $\Omega(G, A)$ is given by

$$[Y_1] \cdot [Y_2] = [Y_1 \otimes Y_2]$$

for all (G, A)-sets Y_1 and Y_2. The identity of $\Omega(G, A)$ is $[(G/G)_{1_G}]$.

A $\mathbb{Z}$-lattice is a finitely generated $\mathbb{Z}$-free $\mathbb{Z}$-module. Obviously, $\Omega(G, A)$ is a $\mathbb{Z}$-lattice. The statement of the following proposition is given in [12, Proposition 1(a)] (see also [2, Remark 2.2] and [27, Proposition 2.7]).

Proposition 2.14 *The elements $[(G/H)_\sigma]$ for $(H, \sigma) \in \mathcal{R}(G, A)$ form a free $\mathbb{Z}$-basis of the $\mathbb{Z}$-lattice $\Omega(G, A)$.*

Proof. The assertion follows from Proposition 2.8. □

We obtain a product formula of simple (G, A)-sets (see also [2, Remark 2.3]).

Lemma 2.15 *Let (H, σ), $(U, \tau) \in \mathcal{S}(G, A)$. Then*

$$(G/H)_\sigma \otimes (G/U)_\tau \simeq \dot{\bigcup_{HgU \in H\backslash G/U}} (G/(H \cap {}^gU))_{\sigma\cdot({}^g\tau)},$$

where $\sigma \cdot ({}^g\tau) : H \cap {}^gU \to A$ is the pointwise product of $\sigma|_{H\cap {}^gU}$ and $({}^g\tau)|_{H\cap {}^gU}$.

Proof. We view the tensor product $(F/F_{(H,\sigma)}) \otimes (F/F_{(U,\tau)})$ of (G, A)-sets $F/F_{(H,\sigma)}$ and $F/F_{(U,\tau)}$ as a left F-set. The left F-set $(F/F_{(H,\sigma)}) \otimes (F/F_{(U,\tau)})$ is expressed as a disjoint union of F-orbits. We identify each $g \in G$ with $(\epsilon_A, g) \in F$ for shortness' sake. For any (a, g), $(b, r) \in F$,

$$(a, g)^{-1}((a, g)F_{(H,\sigma)} \otimes (b, r)F_{(U,\tau)}) = ({}^{g^{-1}}b, \epsilon)(F_{(H,\sigma)} \otimes g^{-1}rF_{(U,\tau)})$$

(see Eqs. (2.1) and (2.2)), which means that there exists an F-orbit containing both $(a, g)F_{(H,\sigma)} \otimes (b, r)F_{(U,\tau)}$ and $F_{(H,\sigma)} \otimes g^{-1}rF_{(U,\tau)}$. Let g, $r \in G$. Suppose that

$$F_{(H,\sigma)} \otimes rF_{(U,\tau)} = (a, h)(F_{(H,\sigma)} \otimes gF_{(U,\tau)}) = hF_{(H,\sigma)} \otimes (a, hg)F_{(U,\tau)}$$

with $(a, h) \in F$. Then $h \in H$ and $r^{-1}hg \in U$, which yields $g \in HrU$. Conversely, if $g \in HrU$ and $r^{-1}hg \in U$ with $h \in H$, then we have

$$F_{(H,\sigma)} \otimes rF_{(U,\tau)} = ({}^{hg}\tau(g^{-1}h^{-1}r)\sigma(h)^{-1}, h)(F_{(H,\sigma)} \otimes gF_{(U,\tau)}).$$

Consequently, both $F_{(H,\sigma)} \otimes rF_{(U,\tau)}$ and $F_{(H,\sigma)} \otimes gF_{(U,\tau)}$ are contained in the same F-orbit if and only if $g \in HrU$. Suppose that

$$F_{(H,\sigma)} \otimes gF_{(U,\tau)} = (a, h)(F_{(H,\sigma)} \otimes gF_{(U,\tau)}) = (a, h)F_{(H,\sigma)} \otimes hgF_{(U,\tau)}$$

with $(a,h) \in F$. Then there exists some $b \in A$ such that

$$(F_{(H,\sigma)}, gF_{(U,\tau)}) = (((a,h)F_{(H,\sigma)})b^{-1}, (hgF_{(U,\tau)})b) = ((b^{-1}a,h)F_{(H,\sigma)}, (b,hg)F_{(U,\tau)}),$$

which yields $h \in H \cap {}^gU$ and

$$(\sigma \cdot (g\tau))(h) = \sigma(h)\, {}^g\tau(g^{-1}hg) = (a^{-1}b)\, {}^g({}^{g^{-1}}b^{-1}) = a^{-1}.$$

Hence $(a,h) \in F_{(H\cap {}^gU, \sigma\cdot(g\tau))}$. Moreover, it is easily verified that $F_{(H\cap {}^gU, \sigma\cdot(g\tau))}$ is the stabilizer of $F_{(H,\sigma)} \otimes gF_{(U,\tau)}$. Thus it turns out that

$$(F/F_{(H,\sigma)}) \otimes (F/F_{(U,\tau)}) \simeq \dot{\bigcup_{HgU \in H\backslash G/U}} F/F_{(H\cap {}^gU, \sigma\cdot(g\tau))}$$

as left F-sets. The lemma now follows from Lemma 2.2. This completes the proof. □

For each $K \leq H \leq G$ and $g \in G$, there are additive maps

$$\begin{array}{lll} \mathrm{con}^g_H : \Omega(H,A) \to \Omega({}^gH,A), & \sum_T \ell_T[T] \mapsto \sum_T \ell_T[\mathrm{con}^g_H(T)], & \\ \mathrm{res}^H_K : \Omega(H,A) \to \Omega(K,A), & \sum_T \ell_T[T] \mapsto \sum_T \ell_T[\mathrm{res}^H_K(T)], & \text{and} \\ \mathrm{ind}^H_K : \Omega(K,A) \to \Omega(H,A), & \sum_S k_S[S] \mapsto \sum_S k_S[\mathrm{ind}^H_K(S)], & \end{array}$$

where $S \in (K,A)$-**set**, $T \in (H,A)$-**set**, and k_S, $\ell_T \in \mathbb{Z}$; these maps are called the conjugation map, the restriction map, and the induction map, respectively. By Lemma 2.12, conjugation maps and restriction maps are ring homomorphisms.

Proposition 2.16 *The family of $\mathbb{Z}$-algebras $\Omega(H,A)$ for $H \leq G$, together with conjugation, restriction, and induction maps, defines a Green functor on G.*

Proof. The axioms of Green functor follow from Lemmas 2.10, 2.11, and 2.15 (cf. [4, 1.1. Definition]). As for the Frobenius axiom, we have

$$\mathrm{res}^G_K((G/H)_\sigma) \otimes (K/U)_\tau \simeq \dot{\bigcup_{KgH \in K\backslash G/H}} \; \dot{\bigcup_{L_geU \in L_g\backslash K/U}} (K/({}^gH \cap {}^eU))_{(g\sigma)|_{L_g}\cdot(e\tau)},$$

where $L_g = K \cap {}^gH$, and

$$\mathrm{ind}^G_K(\mathrm{res}^G_K((G/H)_\sigma) \otimes (K/U)_\tau) \simeq (G/H)_\sigma \otimes \mathrm{ind}^G_K((K/U)_\tau)$$

for all $K \leq G$, $(H,\sigma) \in \mathcal{S}(G,A)$, and $(U,\tau) \in \mathcal{S}(K,A)$, completing the proof. □

3 Multiplicative induction

3A Tensor induction

To begin with, we review the multiplicative induction $\mathrm{Jnd}_H^G : X \mapsto \mathrm{Map}_H(G, X)$, where $H \leq G$ and $X \in H$-**set**, given in [32, §3(a.3)] (see also [11, §4]).

Definition 3.1 Let $H \leq G$, and let T be an (H, A)-set. We define a left G-set $\mathrm{Map}_H(G, T)$ to be the set of maps $f : G \to T$ such that $f(hg) = hf(g)$ for all $h \in H$ and $g \in G$ with the action of G given by

$$(gf)(r) = f(rg)$$

for all g, $r \in G$ and $f \in \mathrm{Map}_H(G, T)$.

Remark 3.2 Under the notation of Definition 3.1, the left G-set $\mathrm{Map}_H(G, T)$ is viewed as a (G, A)-set with the right action of A given by

$$(fa)(r) = f(r)\,{}^{r}a$$

for all $r \in G$, $a \in A$, and $f \in \mathrm{Map}_H(G, T)$. However, we need hardly recall such a right action of A on $\mathrm{Map}_H(G, T)$ (see Definitions 3.3 and 3.5) in relation to multiplicative induction for monomial Burnside rings (see Proposition 3.20).

Let $H \leq G$, and let T be an (H, A)-set. The tensor induced G-set $T^{\otimes G}$ obtained from T (see [8, §80C]) is isomorphic to $\mathrm{Map}_H(G, T)$ and is related to tensor induction of modules. By modifying $\mathrm{Map}_H(G, T)$, we define tensor induction for (H, A)-sets, and then define multiplicative induction for monomial Burnside rings in §3C.

Let Hg, $g \in G$, be the right coset of H in G containing g. Given g, $r \in G$ with $Hg \neq Hr$ and $a \in A$, we define a relation $\sim_{(g,r,a)}$ on $\mathrm{Map}_H(G, T)$ by

$$f \sim_{(g,r,a)} f' \;:\Longleftrightarrow\; \begin{array}{l} f(hg)\,{}^{hg}a = f'(hg) \text{ and } f(hr) = f'(hr)\,{}^{hr}a \text{ for all } h \in H, \\ \text{and } f(g') = f'(g') \text{ for all } g' \in G - Hg\dot{\cup}Hr. \end{array}$$

Let $\sim_A$ be the equivalence relation on $\mathrm{Map}_H(G, T)$ generated by the relations $\sim_{(g,r,a)}$ for g, $r \in G$ and $a \in A$. For each $f \in \mathrm{Map}_H(G, T)$, we denote by $\widehat{f}$ the equivalence class containing f with respect to the equivalence relation $\sim_A$.

Definition 3.3 Let $H \leq G$, and let T be an (H, A)-set. We define

$$\widehat{\mathrm{Map}}_H(G, T) := \{\widehat{f} \mid f \in \mathrm{Map}_H(G, T)\},$$

and make it into a free right A-set by defining

$$\widehat{f}a = \widehat{f_a} \quad \text{with} \quad f_a : G \to T, \quad r \mapsto f_a(r) = \begin{cases} f(r)\,{}^{r}a & \text{if} \quad r \in H, \\ f(r) & \text{if} \quad r \in G - H \end{cases} \tag{3.1}$$

for all $a \in A$ and $f \in \mathrm{Map}_H(G, T)$.

The following lemma tells us of a suitable left action of G on the free right A-set $\widehat{\mathrm{Map}}_H(G,T)$ defined above for an extension of the multiplicative induction $\mathrm{Jnd}_H^G : X \mapsto \mathrm{Map}_H(G,X)$ where $X \in H$-**set**.

Lemma 3.4 *Let $H \le G$, and let T be an (H,A)-set. Then $\widehat{gf_a} = \widehat{gf}\,{}^g a$, where f_a is given in Eq.*(3.1)*, for all $g \in G$, $a \in A$, and $f \in \mathrm{Map}_H(G,T)$.*

Proof. Suppose that $g \in G$, $a \in A$, and $f \in \mathrm{Map}_H(G,T)$. By definition,

$$(gf_a)(r) = \begin{cases} f(rg)\,{}^{rg}a & \text{if} \quad rg \in H, \\ f(rg) & \text{if} \quad rg \in G-H, \end{cases}$$

and

$$(gf)_{{}^g a}(r) = \begin{cases} f(rg)\,{}^{rg}a & \text{if} \quad r \in H, \\ f(rg) & \text{if} \quad r \in G-H. \end{cases}$$

Hence we may assume that $g \notin H$. Observe that for any $h \in H$,

$$(gf_a)(h)\,{}^h({}^g a) = (gf)_{{}^g a}(h) \quad \text{and} \quad (gf_a)(hg^{-1}) = (gf)_{{}^g a}(hg^{-1})\,{}^{hg^{-1}}({}^g a).$$

Moreover, $(gf_a)(r) = (gf)_{{}^g a}(r)$ for all $r \in G - H\dot{\cup}Hg^{-1}$. Thus $gf_a \sim_{(\epsilon, g^{-1}, {}^g a)} (gf)_{{}^g a}$. We now obtain $\widehat{gf_a} = \widehat{gf}\,{}^g a$, completing the proof. □

Definition 3.5 (Tensor induction) Let $H \le G$, and let T be an (H,A)-set. We make the free right A-set $\widehat{\mathrm{Map}}_H(G,T)$ into a left G-set by defining

$$g\widehat{f} = \widehat{gf}$$

for all $g \in G$ and $f \in \mathrm{Map}_H(G,T)$, so that $\widehat{\mathrm{Map}}_H(G,T)$ is a (G,A)-set. The operation which assigns to T the (G,A)-set $\widehat{\mathrm{Map}}_H(G,T)$ is called tensor induction (cf. [8, §80C]), and is related to tensor induction for 1-cocycles (see §3B).

Remark 3.6 Keep the notation of Definition 3.5, and assume further that G acts trivially on A. Then the (G,A)-sets are considered as the A-fibred G-sets defined by Barker [2, §2], and the (G,A)-set $\widehat{\mathrm{Map}}_H(G,T)$ obtained from T by tensor induction is identified with the A-fibred G-set $\mathrm{Ten}_H^G(T)$ defined by Barker [2, §9].

We present a fundamental lemma which is essential to the investigation of multiplicative induction for monomial Burnside rings.

Lemma 3.7 *Let $H \le G$, and let T be an (H,A)-set. Suppose that $\{g_1, g_2, \ldots, g_n\}$ with $g_1 = \epsilon$ is a complete set of representatives of G/H. Let $f \in \mathrm{Map}_H(G,T)$, and define $f^{(0)} \in \mathrm{Map}_H(G,T)$ by $f^{(0)}(hg_j^{-1}) = f(hg_j^{-1})\,{}^h a_j$ with $a_j \in A$ for all $h \in H$ and $j \in [n]$. Then $f^{(0)} \sim_A f_a$, where $a = {}^{g_1}a_1\,{}^{g_2}a_2 \cdots {}^{g_n}a_n$, and hence $\widehat{f^{(0)}} = \widehat{f}a$.*

Proof. For each integer k with $1 \leq k \leq n$, we define $f^{(k)} \in \mathrm{Map}_H(G,T)$ by

$$f^{(k)}(hg_j^{-1}) = \begin{cases} f(hg_j^{-1}) & \text{if } j \in [k], \\ f^{(0)}(hg_j^{-1}) & \text{if } j = k+1,\, k+2, \dots,\, n \end{cases}$$

for all $h \in H$. In particular, $f^{(n)} = f$. Obviously, $f^{(1)} = f^{(0)}_{g_1 a_1^{-1}}$. Let k be an integer with $2 \leq k \leq n$. Then

$$f^{(k)}(hg_k^{-1}) = f^{(k-1)}(hg_k^{-1})\,{}^{h}a_k^{-1} \quad \text{and} \quad f^{(k)}(hg_j^{-1}) = f^{(k-1)}(hg_j^{-1})$$

for all $h \in H$ and $j \in [n]$ with $j \neq k$, and

$$f^{(k-1)}_{g_k a_k^{-1}}(hg_1^{-1}) = f^{(k-1)}(hg_1^{-1})\,{}^{hg_k}a_k^{-1} \quad \text{and} \quad f^{(k-1)}_{g_k a_k^{-1}}(hg_j^{-1}) = f^{(k-1)}(hg_j^{-1})$$

for all $h \in H$ and $j = 2,\, 3, \dots,\, n$. This shows that $f^{(k-1)}_{g_k a_k^{-1}} \sim_{(g_k^{-1}, g_1^{-1}, {}^{g_k}a_k^{-1})} f^{(k)}$. Hence we have $f^{(0)} \sim_A f_a$, completing the proof. □

Remark 3.8 Let $H \leq G$, and let T be an (H,A)-set. By Lemma 3.7, we have

$$|\widehat{\mathrm{Map}}_H(G,T)/A| = |\mathrm{Map}_H(G,T/A)|,$$

whence

$$|\widehat{\mathrm{Map}}_H(G,T)| = |T/A|^{|G/H|} \cdot |A|.$$

The following proposition, which is a generalization of [32, §3(a.13)], describes a Mackey decomposition formula (see also [2, Lemma 9.1]).

Proposition 3.9 *Let H, $K \leq G$. For each (H,A)-set T,*

$$\mathrm{res}^G_K(\widehat{\mathrm{Map}}_H(G,T)) \simeq \bigotimes_{KgH \in K\backslash G/H} \widehat{\mathrm{Map}}_{K \cap {}^gH}(K, \mathrm{res}^{{}^gH}_{K \cap {}^gH}({}^gT)).$$

Proof. Let $\{g_1,\, g_2, \dots,\, g_m\}$ with $g_1 = \epsilon$ be a complete set of representatives of $K\backslash G/H$. For each $i \in [m]$, let $\{r_{i1},\, r_{i2}, \dots,\, r_{i\ell_i}\}$ be a complete set of representatives of $K/(K \cap {}^{g_i}H)$. Then $\{r_{ij}g_i \mid i \in [m] \text{ and } j \in [\ell_i]\}$ is a complete set of representatives of G/H. Let $i \in [m]$. There is a map

$$\Phi_i : \mathrm{res}^G_K(\mathrm{Map}_H(G,T)) \to \mathrm{Map}_{K \cap {}^{g_i}H}(K, \mathrm{res}^{{}^{g_i}H}_{K \cap {}^{g_i}H}({}^{g_i}T))$$

given by

$$\Phi_i(f)({}^{g_i}hr_{ij}^{-1}) = g_i \otimes f(h(r_{ij}g_i)^{-1}) (= {}^{g_i}h(g_i \otimes f((r_{ij}g_i)^{-1}))) \in {}^{g_i}T$$

for all $h \in {}^{g_i^{-1}}K \cap H$, $j \in [\ell_i]$, and $f \in \mathrm{Map}_H(G,T)$. Given $j \in [\ell_i]$, we have

$$\Phi_i(rf)(r_{ij}^{-1}) = g_i \otimes f((r_{ij}g_i)^{-1}r) = g_i \otimes f(g_i^{-1}(r_{ij}^{-1}rr_{ij'})g_i(r_{ij'}g_i)^{-1})$$

and

$$(r\Phi_i(f))(r_{ij}^{-1}) = \Phi_i(f)(r_{ij}^{-1}r) = \Phi_i(f)((r_{ij}^{-1}rr_{ij'})r_{ij'}^{-1}),$$

where $r_{ij}(K \cap {}^{g_i}H) = rr_{ij'}(K \cap {}^{g_i}H)$, for all $r \in K$ and $f \in \mathrm{Map}_H(G,T)$. Thus Φ_i is a K-equivariant map. We now define a K-equivariant map

$$\widehat{\Phi} : \mathrm{res}^G_K(\widehat{\mathrm{Map}}_H(G,T)) \to \bigotimes_{i=1}^{m} \widehat{\mathrm{Map}}_{K\cap {}^{g_i}H}(K, \mathrm{res}^{{}^{g_i}H}_{K\cap {}^{g_i}H}({}^{g_i}T))$$

by

$$\widehat{f} \mapsto \widehat{\Phi_1(f)} \otimes \widehat{\Phi_2(f)} \otimes \cdots \otimes \widehat{\Phi_m(f)}$$

for all $f \in \mathrm{Map}_H(G,T)$. (Of course this map is well-defined; see Remark 3.10.) The map $\widehat{\Phi}$ is also a (K,A)-equivariant map, because

$$(\widehat{\Phi_1(f)} \otimes \widehat{\Phi_2(f)} \otimes \cdots \otimes \widehat{\Phi_m(f)})a = \widehat{\Phi_1(f)_a} \otimes \widehat{\Phi_2(f)} \otimes \cdots \otimes \widehat{\Phi_m(f)}$$

and

$$\Phi_i(f_a)(r_{ij}^{-1}) = \begin{cases} \epsilon \otimes f(r_{1j}^{-1})^{\,r_{1j}^{-1}}a & = \Phi_1(f)_a(r_{1j}^{-1}) \quad \text{if } i=1 \text{ and } r_{1j} \in H, \\ \epsilon \otimes f(r_{1j}^{-1}) & = \Phi_1(f)_a(r_{1j}^{-1}) \quad \text{if } i=1 \text{ and } r_{1j} \notin H, \\ g_i \otimes f((r_{ij}g_i)^{-1}) & = \Phi_i(f)(r_{ij}^{-1}) \quad \text{if } i \neq 1 \end{cases}$$

for all $i \in [m]$, $j \in [\ell_i]$, $a \in A$, and $f \in \mathrm{Map}_H(G,T)$. Thus it only remains for us to show that $\widehat{\Phi}$ is bijective. For each $i \in [m]$, choose $f_i \in \mathrm{Map}_{K\cap {}^{g_i}H}(K, \mathrm{res}^{{}^{g_i}H}_{K\cap {}^{g_i}H}({}^{g_i}T))$. Given $i \in [m]$ and $j \in [\ell_i]$, we suppose that $f_i(r_{ij}^{-1}) = g_i \otimes t_{ij} \in {}^{g_i}T$ with $t_{ij} \in T$. Now define $f \in \mathrm{Map}_H(G,T)$ by

$$f(h(r_{ij}g_i)^{-1}) = ht_{ij} \in T$$

for all $h \in H$, $i \in [m]$, and $j \in [\ell_i]$. Then $\widehat{\Phi}(\widehat{f}) = \widehat{f_1} \otimes \widehat{f_2} \otimes \cdots \otimes \widehat{f_m}$. Thus $\widehat{\Phi}$ is surjective, which means that it is also injective, because

$$|\widehat{\mathrm{Map}}_H(G,T)| = |T/A|^{\sum_{i=1}^m \ell_i} \cdot |A| = \left| \bigotimes_{i=1}^{m} \widehat{\mathrm{Map}}_{K\cap {}^{g_i}H}(K, \mathrm{res}^{{}^{g_i}H}_{K\cap {}^{g_i}H}({}^{g_i}T)) \right|$$

by Remark 3.8. We now conclude that $\widehat{\Phi}$ is bijective, completing the proof. □

Remark 3.10 In the proof of Proposition 3.9, assume that $f \sim_{((r_{ij}g_i)^{-1},(r_{i'j'}g_{i'})^{-1},a)} f'$ with f, $f' \in \mathrm{Map}_H(G,T)$ and $a \in A$. Let $u \in [m]$ and $v \in [\ell_u]$. Then we have

$$\Phi_u(f)({}^{g_u}hr_{uv}^{-1}) = \begin{cases} \Phi_i(f')({}^{g_i}hr_{ij}^{-1})^{\,{}^{g_i}hr_{ij}^{-1}}a^{-1} & \text{if } (u,v) = (i,j), \\ \Phi_u(f')({}^{g_{i'}}hr_{i'j'}^{-1})^{\,{}^{g_{i'}}hr_{i'j'}^{-1}}a & \text{if } (u,v) = (i',j'), \\ \Phi_u(f')({}^{g_u}hr_{uv}^{-1}) & \text{otherwise} \end{cases}$$

for all $h \in {}^{g_u^{-1}}K \cap H$. This, combined with Lemma 3.7, shows that

$$\widehat{\Phi_u(f)} = \widehat{\Phi_u(f')} \quad \text{if} \quad u \neq i,\, i', \quad \widehat{\Phi_i(f)} = \widehat{\Phi_i(f')}a^{-1}, \quad \text{and} \quad \widehat{\Phi_{i'}(f)} = \widehat{\Phi_{i'}(f')}a.$$

Hence we have $\widehat{\Phi}(\widehat{f}) = \widehat{\Phi}(\widehat{f'})$. Consequently, the map $\widehat{\Phi}$ is well-defined.

3B Tensor induction for 1-cocycles

We introduce tensor induction for 1-cocycles, and see that it is closely allied to tensor induction for (H, A)-sets with $H \leq G$.

Definition 3.11 Let $(H, \sigma) \in \mathcal{S}(G, A)$. We fix a complete set $\{g_1, g_2, \dots, g_n\}$ with $g_1 = \epsilon$ of representatives of G/H, and define a 1-cocycle $\sigma^{\otimes G} : G \to A$ by

$$\sigma^{\otimes G}(g) = \prod_{j=1}^{n} {}^{g_{j'}}\sigma(g_{j'}^{-1} g g_j),$$

where $gg_jH = g_{j'}H$, for all $g \in G$. The operation which assigns to σ the 1-cocycle $\sigma^{\otimes G} : G \to A$ is called tensor induction (cf. [8, §13A]).

Remark 3.12 Keep the notation of Definition 3.11, and let $h_1, h_2, \dots, h_n \in H$. Then

$$(\sigma^{\otimes G})^a(g) = \prod_{j=1}^{n} {}^{g_{j'}h_{j'}}\sigma(h_{j'}^{-1} g_{j'}^{-1} g g_j h_j) \quad \text{with} \quad a = \prod_{j=1}^{n} {}^{g_j}\sigma(h_j)$$

for all $g \in G$ (see Definition 2.4), because

$$\begin{aligned} {}^{g_{j'}h_{j'}}\sigma(h_{j'}^{-1} g_{j'}^{-1} g g_j h_j) &= {}^{g_{j'}h_{j'}}\sigma(h_{j'}^{-1})\, {}^{g_{j'}}\sigma(g_{j'}^{-1} g g_j h_j) \\ &= {}^{g_{j'}}\sigma(h_{j'})^{-1}\, {}^{g_{j'}}\sigma(g_{j'}^{-1} g g_j)\, {}^{gg_j}\sigma(h_j) \end{aligned}$$

for all $j \in [n]$. Hence the subset $\{(\sigma^{\otimes G})^a \mid a \in A\}$ of $Z^1(G, A)$ is independent of the choice of a complete set of representatives of G/H. Likewise, if $b \in A$, then

$$(\sigma^{\otimes G})^c = (\sigma^b)^{\otimes G} \quad \text{with} \quad c = \prod_{j=1}^{n} {}^{g_j}b.$$

Example 3.13 Let $(H, \sigma) \in \mathcal{S}(G, A)$. Obviously, A is a free right A-set with the action given by the product operation on A. We make it into an (H, A)-set $A^{(\sigma)}$ isomorphic to $(H/H)_\sigma$ by defining

$$ha = \sigma(h)\, {}^h a$$

for all $h \in H$ and $a \in A^{(\sigma)}$. For any $K \leq H$, $\mathrm{res}^H_K(A^{(\sigma)}) = A^{(\sigma|_K)}$ (see Lemma 2.11). Keep the notation of Definition 3.11, and identify $(H/H)_\sigma$ with $A^{(\sigma)}$. We define an element $\tilde{\sigma}$ of $\mathrm{Map}_H(G, A^{(\sigma)})$ by

$$\tilde{\sigma}(hg_i^{-1}) = \sigma(h)$$

for all $h \in H$ and $i \in [n]$. Let $f \in \mathrm{Map}_H(G, A^{(\sigma)})$. For each $j \in [n]$, we set $a_j = f(g_j^{-1}) \in A^{(\sigma)}$. Since $f(hg_j^{-1}) = \tilde{\sigma}(hg_j^{-1})\,{}^h a_j$ for all $h \in H$ and $j \in [n]$, it follows from Lemma 3.7 with $f^{(0)} = f$ that $\widehat{f} = \widehat{\tilde{\sigma}}a$ where $a = {}^{g_1}a_1\,{}^{g_2}a_2 \cdots {}^{g_n}a_n$. Hence $\widehat{\mathrm{Map}}_H(G, A^{(\sigma)}) = \{\widehat{\tilde{\sigma}}a \mid a \in A\}$. Let $g \in G$. We have

$$(g\tilde{\sigma})(hg_j^{-1}) = \tilde{\sigma}(hg_j^{-1}g) = \sigma(hg_j^{-1}gg_{j'}) = \sigma(h)\,{}^h\sigma(g_j^{-1}gg_{j'}) = \tilde{\sigma}(hg_j^{-1})\,{}^h\sigma(g_j^{-1}gg_{j'}),$$

where $g_jH = gg_{j'}H$, for all $h \in H$ and $j \in [n]$. Thus it follows from Lemma 3.7 that $g\widehat{\tilde{\sigma}} = \widehat{\tilde{\sigma}}\sigma^{\otimes G}(g)$. Moreover, there exists an isomorphism $\widehat{\mathrm{Map}}_H(G, A^{(\sigma)}) \xrightarrow{\sim} A^{(\sigma^{\otimes G})}$ of (G, A)-sets given by

$$\widehat{\tilde{\sigma}}a \mapsto a$$

for all $a \in A$. Thus $\widehat{\mathrm{Map}}_H(G, (H/H)_\sigma) \simeq (G/G)_{\sigma^{\otimes G}}$.

The following proposition describes a Mackey decomposition formula.

Proposition 3.14 *Let H, $K \leq G$. For each $\sigma \in Z^1(H, A)$,*

$$\sigma^{\otimes G}|_K =_A \prod_{KgH \in K\backslash G/H} (g\sigma)|_{K \cap {}^gH}{}^{\otimes K}.$$

Proof. By Lemma 2.10, Proposition 3.9, and Example 3.13, we have

$$(K/K)_{\sigma^{\otimes G}|_K} \simeq \bigotimes_{KgH \in K\backslash G/H} (K/K)_{(g\sigma)|_{K\cap {}^gH}{}^{\otimes K}},$$

which, combined with Lemma 2.15, implies that

$$(K/K)_{\sigma^{\otimes G}|_K} \simeq (K/K)_{\prod_{KgH\in K\backslash G/H} (g\sigma)|_{K\cap {}^gH}{}^{\otimes K}}.$$

The assertion follows from this fact and Lemma 2.6. This completes the proof. □

The following lemma states basic properties of tensor induction for 1-cocycles.

Lemma 3.15 *Let $U \leq K \leq H$, and let $g \in G$. Then*

$$g\nu^{\otimes H} =_A (g\nu)^{\otimes {}^gH} \quad \textit{and} \quad (\tau^{\otimes K})^{\otimes H} =_A \tau^{\otimes H}$$

for all $\nu \in Z^1(K, A)$ and $\tau \in Z^1(U, A)$.

Proof. Fix a complete set $\{h_1, h_2, \ldots, h_m\}$ with $h_1 = \epsilon$ of representatives of H/K and a complete set $\{r_1, r_2, \ldots, r_k\}$ with $r_1 = \epsilon$ of representatives of K/U. Given $\nu \in Z^1(K, A)$ and $\tau \in Z^1(U, A)$, we have

$$\begin{aligned}(g\nu^{\otimes H})({}^{g}h) &= \prod_{j=1}^{m} {}^{gh_{j'}}\nu(h_{j'}^{-1}hh_j) \\ &= \prod_{j=1}^{m} {}^{gh_{j'}g}\nu((h_{j'}^{-1}g^{-1})\,{}^{g}h(gh_j)) \\ &= \prod_{j=1}^{m} {}^{gh_{j'}g}\nu(g^{-1}({}^{g}h_{j'}^{-1}\,{}^{g}h\,{}^{g}h_j)g) \\ &= \prod_{j=1}^{m} {}^{gh_{j'}}(g\nu)({}^{g}h_{j'}^{-1}\,{}^{g}h\,{}^{g}h_j),\end{aligned}$$

where $hh_jK = h_{j'}K$, and

$$\begin{aligned}(\tau^{\otimes K})^{\otimes H}(h) &= \prod_{j=1}^{m} {}^{h_{j'}}\tau^{\otimes K}(h_{j'}^{-1}hh_j) \\ &= \prod_{j=1}^{m}\prod_{i=1}^{k} {}^{h_{j'}r_{i'}}\tau((h_{j'}r_{i'})^{-1}h(h_jr_i)),\end{aligned}$$

where $(h_{j'}^{-1}hh_j)r_iU = r_{i'}U$, for all $h \in H$. Consequently, the assertions follow from Remark 3.12. This completes the proof. □

3C Algebraic maps

We define a subset $\Omega(G, A)^+$ of $\Omega(G, A)$ to be the set consisting of all elements $\sum_{(U,\tau)\in\mathcal{R}(G,A)} \ell_{(U,\tau)}[(G/U)_\tau]$ with $\ell_{(U,\tau)} \geq 0$, which is an additive semigroup. By Lemma 2.15, $\Omega(G, A)^+$ is closed under multiplication. For each $H \leq G$, there is a map (tensor induction) $\widehat{\mathrm{Map}}_H(G, -) : \Omega(H, A)^+ \to \Omega(G, A)$ given by

$$[T] \mapsto [\widehat{\mathrm{Map}}_H(G, T)]$$

for all (H, A)-sets T (cf. [8, (80.42)]). This map is multiplicative (see Lemma 3.19).

We review the concept of algebraic maps which is due to Dress [11]. Let B be an additive semigroup with zero element, and let E be an additive group. Given $c \in B$ and a map $f : B \to E$, we define a map $D_cf : B \to E$ by

$$d \mapsto f(c + d) - f(d)$$

for all $d \in B$. A map $f : B \to E$ is said to be algebraic of degree n if n is the least integer such that

$$D_{c_1}D_{c_2}\cdots D_{c_{n+1}}f = 0$$

for all $c_1, c_2, \ldots, c_{n+1} \in B$ (cf. [8, §80C]). Let $f : B \to E$ be an algebraic map of degree n, and let $\overline{B}$ be the additive group generated by the elements of B. According to Dress [11, Proposition 1.1], there is a unique map $\overline{f} : \overline{B} \to E$ extending f, and $\overline{f}$ is also algebraic of degree n (see also [8, (80.44) Theorem (*Dress*)]). Assume further that $\overline{B}$ and E are commutative rings and B is closed under multiplication. If $f : B \to E$ is multiplicative, then the unique extension $\overline{f} : \overline{B} \to E$ of f to $\overline{B}$ is also multiplicative (cf. [8, (80.47) Theorem]).

Definition 3.16 Let $H \leq G$, and let $T_0, T_1, \ldots, T_i$ be (H, A)-sets, where i is an integer with $0 \leq i \leq |G : H| + 1$. We define a (G, A)-set $\widehat{\mathrm{Map}}_H(G, T_0, T_1, \ldots, T_i)$ to be the set consisting of all elements $\widehat{f}$ of $\widehat{\mathrm{Map}}_H(G, T_0 \dot{\cup} T_1 \dot{\cup} \cdots \dot{\cup} T_i)$ containing $f \in \mathrm{Map}_H(G, T_0 \dot{\cup} T_1 \dot{\cup} \cdots \dot{\cup} T_i)$ such that $|\mathrm{Im}\, f \cap T_\ell| \neq 0$ whenever $\ell \neq 0$ with the left action of G and the right action of A given by

$$g\widehat{f} = \widehat{gf} \quad \text{and} \quad \widehat{f}a = \widehat{f_a}$$

for all $g \in G$, $a \in A$, and $\widehat{f} \in \widehat{\mathrm{Map}}_H(G, T_0, T_1, \ldots, T_i)$.

Under the notation of Definition 3.16, we have

$$\widehat{\mathrm{Map}}_H(G, T_0, T_1, \ldots, T_i) = \begin{cases} \widehat{\mathrm{Map}}_H(G, T_0) & \text{if } i = 0, \\ \widehat{\mathrm{Map}}_H(G, T_1) & \text{if } T_0 = \emptyset \text{ and } i = 1, \\ \emptyset & \text{if } i = |G : H| + 1. \end{cases}$$

Proposition 3.17 *For each $H \leq G$, the map $\widehat{\mathrm{Map}}_H(G, -) : \Omega(H, A)^+ \to \Omega(G, A)$ is algebraic of degree $|G : H|$.*

This proposition is analogous to [8, (80.43) Proposition (*Dress*)], and is an immediate consequence of the following lemma.

Lemma 3.18 *Keep the notation of Definition* 3.16*, and assume further that $i \geq 1$. Set $\Theta_i = D_{[T_i]} \cdots D_{[T_1]} \widehat{\mathrm{Map}}_H(G, -)$. Then*

$$\Theta_i([T_0]) = [\widehat{\mathrm{Map}}_H(G, T_0, T_1, \ldots, T_i)].$$

Proof. The assertion is proved by an argument analogous to that in the proof of [8, (80.43) Proposition (*Dress*)]. □

Tensor induction is multiplicative.

Lemma 3.19 *For each $H \leq G$,*

$$\widehat{\mathrm{Map}}_H(G, T_1 \otimes T_2) \simeq \widehat{\mathrm{Map}}_H(G, T_1) \otimes \widehat{\mathrm{Map}}_H(G, T_2)$$

for all (H, A)-sets T_1 and T_2.

Proof. If $f \in \mathrm{Map}_H(G, T_1 \otimes T_2)$, then by Lemma 3.7, there exists a unique element $\widehat{\Psi_1(f)} \otimes \widehat{\Psi_2(f)}$ of $\widehat{\mathrm{Map}}_H(G, T_1) \otimes \widehat{\mathrm{Map}}_H(G, T_2)$, where $\Psi_i(f) \in \mathrm{Map}_H(G, T_i)$ with $i = 1, 2$, such that

$$f(g) = \Psi_1(f)(g) \otimes \Psi_2(f)(g)$$

for all $g \in G$. Obviously, $\widehat{\Psi_1(f)} \otimes \widehat{\Psi_2(f)} = \widehat{\Psi_1(f')} \otimes \widehat{\Psi_2(f')}$ whenever $f \sim_A f'$. We now define a map $\widehat{\Psi} : \widehat{\mathrm{Map}}_H(G, T_1 \otimes T_2) \to \widehat{\mathrm{Map}}_H(G, T_1) \otimes \widehat{\mathrm{Map}}_H(G, T_2)$ by

$$\widehat{f} \mapsto \widehat{\Psi_1(f)} \otimes \widehat{\Psi_2(f)}$$

for all $f \in \mathrm{Map}_H(G, T_1 \otimes T_2)$. Observe that this map is (G, A)-equivariant and surjective. Moreover, by Remark 3.8,

$$|\widehat{\mathrm{Map}}_H(G, T_1 \otimes T_2)| = (|T_1/A| \cdot |T_2/A|)^{|G/H|} \cdot |A| = |\widehat{\mathrm{Map}}_H(G, T_1) \otimes \widehat{\mathrm{Map}}_H(G, T_2)|.$$

Hence $\widehat{\Psi}$ is an isomorphism of (G, A)-sets. This completes the proof. □

Combining Proposition 3.17 and Lemma 3.19 with [8, (80.47) Theorem], we obtain a result analogous to [8, (80.48) Theorem (*Dress*)].

Proposition 3.20 *For any $H \leq G$, there is a unique multiplicative map*

$$\overline{\mathrm{Map}}_H(G, -) : \Omega(H, A) \to \Omega(G, A), \quad x \mapsto \overline{\mathrm{Map}}_H(G, x)$$

extending $\widehat{\mathrm{Map}}_H(G, -)$, called multiplicative induction or tensor induction, and this map is algebraic of degree $|G : H|$.

Remark 3.21 The multiplicative induction map $\overline{\mathrm{Map}}_H(G, -) : \Omega(H, A) \to \Omega(G, A)$ with $A = \{\epsilon_A\}$ is introduced by Dress [11, §4].

Our concern is an explicit description of each element of $\mathrm{Im}\,\overline{\mathrm{Map}}_H(G, -)$ with $H \leq G$, and is to prove Eq.(1.1) (see also [8, (80.49) Corollary]).

Proposition 3.22 *Let $H \leq G$. For any (H, A)-sets T_0 and T,*

$$\overline{\mathrm{Map}}_H(G, [T_0] - [T]) = \sum_{i=0}^{n} (-1)^i [\widehat{\mathrm{Map}}_H(G, T_0, T_1, \ldots, T_i)],$$

where $n = |G : H|$ and $T = T_1 = \cdots = T_n$.

Proof. We set $D^0_{[T]}\Theta = \Theta = \overline{\mathrm{Map}}_H(G, -) : \Omega(H, A) \to \Omega(G, A)$, and define inductively $D^i_{[T]}\Theta : \Omega(H, A) \to \Omega(G, A)$, $i = 1, 2, \cdots$, by $D^i_{[T]}\Theta = D_{[T]}(D^{i-1}_{[T]}\Theta)$. From [8, (80.45)], we know that $\Theta([T_0] - [T]) = \sum_{i=0}^{\infty} (-1)^i D^i_{[T]}\Theta([T_0])$. Hence the assertion follows from Proposition 3.17 and Lemma 3.18. This completes the proof. □

Remark 3.23 Let $(H,\sigma) \in \mathcal{S}(G,A)$. By Lemma 3.7 and Proposition 3.22, we can describe the structure of $\overline{\mathrm{Map}}_H(G, -[(H/H)_\sigma])$. For each $X \in G$-**set**, let $\widetilde{\Lambda}_{P(X)}$ be the reduced Lefschetz invariant of the poset $P(X)$ consisting of non-empty and proper subsets of X, which is an element of the Burnside ring $\Omega(G)$ (cf. [5, 29]). When $A = \{\epsilon_A\}$, $\overline{\mathrm{Map}}_H(G, -[(H/H)_{1_H}])$ is identified with $\widetilde{\Lambda}_{P(G/H)}$.

There is a Mackey decomposition formula which generalizes [32, §3(G.5)] (see also [2, Proposition 9.5]).

Proposition 3.24 *Let H, $K \le G$. For each $x \in \Omega(H,A)$,*

$$\mathrm{res}^G_K(\overline{\mathrm{Map}}_H(G,x)) = \prod_{KgH \in K\backslash G/H} \overline{\mathrm{Map}}_{K\cap {}^gH}(K, \mathrm{res}^{{}^gH}_{K\cap {}^gH} \circ \mathrm{con}^g_H(x)).$$

Proof. By [8, (80.44) Theorem (*Dress*)], $\mathrm{res}^G_K \circ \overline{\mathrm{Map}}_H(G,-) : \Omega(H,A) \to \Omega(K,A)$ is the unique map extending the algebraic map

$$\mathrm{res}^G_K \circ \widehat{\mathrm{Map}}_H(G,-) = \prod_{KgH \in K\backslash G/H} \widehat{\mathrm{Map}}_{K\cap {}^gH}(K,-) \circ \mathrm{res}^{{}^gH}_{K\cap {}^gH} \circ \mathrm{con}^g_H$$
$$: \Omega(H,A)^+ \to \Omega(K,A),$$

and so is $\prod_{KgH \in K\backslash G/H} \overline{\mathrm{Map}}_{K\cap {}^gH}(K,-) \circ \mathrm{res}^{{}^gH}_{K\cap {}^gH} \circ \mathrm{con}^g_H : \Omega(H,A) \to \Omega(K,A)$ (see [11, Proposition 1.2] and Propositions 3.9, 3.17, and 3.20). Thus the assertion holds. □

4 The mark homomorphism

4A The first cohomology group

Following [12, §2], we provide preliminaries of the mark homomorphism for $\Omega(G,A)$ which is given in §4B.

Let $H \le G$. The set $Z^1(H,A)$ is a right A-set with the action of A given in Definition 2.4, and is an abelian group with the product operation given by

$$\sigma \cdot \tau(h) = \sigma(h)\tau(h)$$

for all σ, $\tau \in Z^1(H,A)$ and $h \in H$. Obviously, the identity of $Z^1(H,A)$ is 1_H.

For each $\sigma \in Z^1(H,A)$, we denote by $\overline{\sigma}$ the A-orbit $\{\sigma^a \mid a \in A\}$ containing σ. Given σ, $\tau \in Z^1(H,A)$ and a, $b \in A$, it is easily seen that $\overline{\sigma^a \cdot \tau^b} = \overline{(\sigma\cdot\tau)^{ab}} = \overline{\sigma\cdot\tau}$.

Definition 4.1 For each $H \le G$, we define

$$H^1(H,A) := \{\overline{\sigma} \mid \sigma \in Z^1(H,A)\},$$

the set of A-orbits on $Z^1(H,A)$, and make it into an abelian group by defining

$$\overline{\sigma}\cdot\overline{\tau}=\overline{\sigma\cdot\tau}$$

for all σ, $\tau\in Z^1(H,A)$. (This product operation is well-defined.)

Let $H\le G$. We denote by $\mathbb{Z}H^1(H,A)$ the group ring of $H^1(H,A)$ over $\mathbb{Z}$. Given $K\le H$ and $g\in G$, there are ring homomorphisms

$$\begin{aligned}
&\mathrm{con}_H^g:\mathbb{Z}H^1(H,A)\to\mathbb{Z}H^1({}^gH,A), &&\sum_{\overline{\sigma}\in H^1(H,A)}\ell_{\overline{\sigma}}\,\overline{\sigma}\mapsto\sum_{\overline{\sigma}\in H^1(H,A)}\ell_{\overline{\sigma}}\,\overline{g\sigma}\quad\text{and}\\
&\mathrm{res}_K^H:\mathbb{Z}H^1(H,A)\to\mathbb{Z}H^1(K,A), &&\sum_{\overline{\sigma}\in H^1(H,A)}\ell_{\overline{\sigma}}\,\overline{\sigma}\mapsto\sum_{\overline{\sigma}\in H^1(H,A)}\ell_{\overline{\sigma}}\,\overline{\sigma|_K},
\end{aligned}$$

where $\ell_{\overline{\sigma}}\in\mathbb{Z}$ with $\sigma\in Z^1(H,A)$ (see §2B), which are called the conjugation map and the restriction map, respectively (cf. [12, §2.2]). Obviously, the restriction map is well-defined. Let $\sigma\in Z^1(H,A)$. By Lemma 2.5, we have $\overline{g(\sigma^a)}=\overline{g\sigma}$ for any $a\in A$. Thus the conjugation map is well-defined.

Let Y be a (G,A)-set. The set of A-orbits yA, $y\in Y$, on Y is a left G-set. For each $y\in Y$, we denote by G_{yA} the stabilizer of the A-orbit yA in G, that is,

$$G_{yA}=\{g\in G\mid gy=ya\text{ for some }a\in A\},$$

and define a 1-cocycle $\sigma_y:G_{yA}\to A$ by

$$gy=y\sigma_y(g)$$

for all $g\in G_{yA}$. Obviously, $G_{(ya)A}=G_{yA}$ and $\sigma_{ya}={\sigma_y}^a$ for any $y\in Y$ and $a\in A$.

Definition 4.2 Let Y be a (G,A)-set, and let $H\le G$. We define

$$\mathrm{inv}_H^A(Y):=\{y\in Y\mid H\le G_{yA}\},$$

which is viewed as an (H,A)-subset of $\mathrm{res}_H^G(Y)$, and define

$$[Y]_H:=\frac{1}{|A|}\sum_{y\in\mathrm{inv}_H^A(Y)}\mathrm{res}_H^{G_{yA}}(\overline{\sigma_y})=\sum_{yA\in\mathrm{inv}_H^A(Y)/A}\mathrm{res}_H^{G_{yA}}(\overline{\sigma_y})\in\mathbb{Z}H^1(H,A).$$

Let Y_1 and Y_2 be (G,A)-sets, and let $H\le G$. Obviously,

$$[Y_1\dot{\cup}Y_2]_H=[Y_1]_H+[Y_2]_H.$$

Let $(y_1,y_2)\in Y_1\times Y_2$. Given $g\in G$ and $a\in A$, $g(y_1\otimes y_2)=(y_1\otimes y_2)a$ if and only if $(gy_1b^{-1},gy_2b)=(y_1,y_2a)$ for some $b\in A$. Hence we have

$$G_{(y_1\otimes y_2)A}=G_{y_1A}\cap G_{y_2A}\quad\text{and}\quad\sigma_{y_1\otimes y_2}=\sigma_{y_1}|_{G_{(y_1\otimes y_2)A}}\cdot\sigma_{y_2}|_{G_{(y_1\otimes y_2)A}}$$

(cf. [12, §2.3]). Moreover, $y_1 \otimes y_2 \in \mathrm{inv}_H^A(Y_1 \otimes Y_2)$ if and only if $y_1 \in \mathrm{inv}_H^A(Y_1)$ and $y_2 \in \mathrm{inv}_H^A(Y_2)$. This means that

$$\begin{aligned}[Y_1]_H \cdot [Y_2]_H &= \frac{1}{|A|}\left(\sum_{y \in \mathrm{inv}_H^A(Y_1)} \mathrm{res}_H^{G_{yA}}(\overline{\sigma_y})\right)\left(\sum_{yA \in \mathrm{inv}_H^A(Y_2)/A} \mathrm{res}_H^{G_{yA}}(\overline{\sigma_y})\right)\\ &= \frac{1}{|A|}\sum_{y_1\otimes y_2 \in \mathrm{inv}_H^A(Y_1\otimes Y_2)} \mathrm{res}_H^{G_{(y_1\otimes y_2)A}}(\overline{\sigma_{y_1\otimes y_2}})\\ &= [Y_1 \otimes Y_2]_H.\end{aligned}$$

Given $H \le G$, we define a ring homomorphism $\rho_G^H : \Omega(G,A) \to \mathbb{Z}H^1(H,A)$ by

$$[Y] \mapsto [Y]_H$$

for all (G,A)-sets Y (cf. [12, §2.4]).

The ring homomorphisms $\rho_G^H : \Omega(G,A) \to \mathbb{Z}H^1(H,A)$ for $H \le G$ form the map

$$\prod_{H\le G} \rho_G^H : \Omega(G,A) \to \prod_{H \le G} \mathbb{Z}H^1(H,A), \quad x \mapsto (\rho_G^H(x))_{H\le G}$$

(cf. [12, §2.5]), which is injective (cf. [12, Theorem 1]).

4B The ghost ring

We continue reviewing part of [12, §2.4, §2.5], and define a ring monomorphism $\rho_G : \Omega(G,A) \to \mho(G,A)$, $x \mapsto \prod_{H\le G} \rho_G^H(x)$ (see Eq.(4.2)).

Definition 4.3 Let Y be a (G,A)-set, and let $(H,\sigma) \in \mathcal{S}(G,A)$. We define a subset $\mathrm{inv}_{(H,\sigma)}(Y)$ of Y to be the set of $F_{(H,\sigma)}$-invariants in Y, so that

$$\mathrm{inv}_{(H,\sigma)}(Y) = \{y \in Y \mid hy = y\sigma(h) \text{ for all } h \in H\} = \{y \in \mathrm{inv}_H^A(Y) \mid \sigma_y|_H = \sigma\},$$

and denote by A_σ the stabilizer $\{a \in A \mid \sigma = \sigma^a\}$ of $\sigma \in Z^1(H,A)$ in A.

Under the notation of Definition 4.3, the set $\mathrm{inv}_{(H,\sigma)}(Y)$ is a free right A_σ-set with the action inherited from that of A on Y. For each $(H,\sigma) \in \mathcal{S}(G,A)$, we denote by $\mathrm{inv}_{(H,\sigma)}(Y)/A_\sigma$ the set of A_σ-orbits on $\mathrm{inv}_{(H,\sigma)}(Y)$.

Lemma 4.4 *Let Y be a (G,A)-set, and let $H \le G$. Then*

$$[Y]_H = \sum_{\overline{\sigma} \in H^1(H,A)} |\mathrm{inv}_{(H,\sigma)}(Y)/A_\sigma| \cdot \overline{\sigma}.$$

Moreover, $|\mathrm{inv}_{({}^gH,{}^g\sigma)}(Y)/A_{{}^g\sigma}| = |\mathrm{inv}_{(H,\sigma)}(Y)/A_\sigma|$ for any $\sigma \in Z^1(H,A)$ and $g \in G$.

Proof. The second statement is clear. To prove the first statement, we set

$$(Y/A)_{(H,\sigma)} = \{yA \in \mathrm{inv}_H^A(Y)/A \mid \mathrm{res}_H^{G_{yA}}(\overline{\sigma_y}) = \overline{\sigma}\}$$

for each $\sigma \in Z^1(H, A)$, so that

$$[Y]_H = \sum_{\overline{\sigma} \in H^1(H,A)} |(Y/A)_{(H,\sigma)}| \cdot \overline{\sigma}.$$

Hence it suffices to verify that $|(Y/A)_{(H,\sigma)}| = |\mathrm{inv}_{(H,\sigma)}(Y)/A_\sigma|$ for any $\sigma \in Z^1(H, A)$. Let $\sigma \in Z^1(H, A)$. We make the set $\mathrm{inv}_H^A(Y)$ into a free right A_σ-set by restriction of operators from A to A_σ. By definition,

$$\mathrm{inv}_{(H,\sigma)}(Y)/A_\sigma = \{yA_\sigma \in \mathrm{inv}_H^A(Y)/A_\sigma \mid \sigma_y|_H = \sigma\},$$

where $\mathrm{inv}_H^A(Y)/A_\sigma$ is the set of A_σ-orbits $yA_\sigma := \{ya \mid a \in A_\sigma\}$, $y \in \mathrm{inv}_H^A(Y)$, on $\mathrm{inv}_H^A(Y)$. Let $y \in \mathrm{inv}_H^A(Y)$, and suppose that $\sigma_y|_H = \sigma^a = \sigma^b$ for some $a, b \in A$. Then $ab^{-1} \in A_\sigma$ and $ya^{-1}A_\sigma = yb^{-1}A_\sigma \in \mathrm{inv}_{(H,\sigma)}(Y)/A_\sigma$. (Note that $\sigma_{(yc)}|_H = \sigma^{ac}$ for any $c \in A$.) Hence there is a bijection $(Y/A)_{(H,\sigma)} \to \mathrm{inv}_{(H,\sigma)}(Y)/A_\sigma$ given by

$$yA \mapsto ya^{-1}A_\sigma,$$

where $\sigma_y|_H = \sigma^a$ with $a \in A$, for all $yA \in (Y/A)_{(H,\sigma)}$. This completes the proof. □

The following lemma is [27, Lemma 3.3].

Lemma 4.5 *Let (H, σ), $(U, \tau) \in \mathcal{S}(G, A)$. Then*

$$|\mathrm{inv}_{(H,\sigma)}((G/U)_\tau)/A_\sigma| = |\{gU \in G/U \mid H \le {}^gU \text{ and } (g\tau)|_H =_A \sigma\}|.$$

Let H, $U \le G$, and consider G/U to be a left G-set with the action of G given by the product operation on G. Following [8, (80.5) Proposition], we define

$$\mathrm{inv}_H(G/U) := \{gU \in G/U \mid H \le {}^gU\}. \tag{4.1}$$

Lemma 4.6 (a) *Let $H \le G$, and let $(U, \tau) \in \mathcal{S}(G, A)$. Then*

$$[(G/U)_\tau]_H = \sum_{gU \in \mathrm{inv}_H(G/U)} \mathrm{res}_H^{gU} \circ \mathrm{con}_U^g(\overline{\tau}).$$

(b) *Let $K \le H \le G$, and let $(U, \tau) \in \mathcal{S}(H, A)$. Then for any $r \in G$,*

$$[{}^r((H/U)_\tau)]_{{}^rK} = \mathrm{con}_K^r([(H/U)_\tau]_K).$$

If $H = G$, then for any $r \in G$,

$$[(G/U)_\tau]_{{}^rK} = \mathrm{con}_K^r([(G/U)_\tau]_K).$$

Proof. (a) Although the assertion follows from Lemmas 4.4 and 4.5, we directly prove it. In the proof of Lemma 4.4, if $Y = (G/U)_\tau$, then by Lemmas 2.5 and 2.6,

$$(Y/A)_{(H,\sigma)} = \{(\epsilon_A, gU)A \in \mathrm{inv}_H^A(Y)/A \mid \mathrm{res}_H^{gU} \circ \mathrm{con}_U^g(\overline{\tau}) = \overline{\sigma}\},$$

whence

$$[(G/U)_\tau]_H = \sum_{\overline{\sigma} \in H^1(H,A)} |(Y/A)_{(H,\sigma)}| \cdot \overline{\sigma} = \sum_{gU \in \mathrm{inv}_H(G/U)} \mathrm{res}_H^{gU} \circ \mathrm{con}_U^g(\overline{\tau}).$$

(b) By Lemma 2.10, it suffices to prove the first assertion. We have

$$\begin{aligned}[({}^rH/{}^rU)_{r\tau}]_{{}^rK} &= \sum_{h'\,{}^rU \in \mathrm{inv}_{{}^rK}({}^rH/{}^rU)} \mathrm{res}_{{}^rK}^{h'\,{}^rU} \circ \mathrm{con}_{{}^rU}^{h'}(\overline{r\tau}) \\ &= \sum_{h'\,{}^rU \in \mathrm{inv}_{{}^rK}({}^rH/{}^rU)} \mathrm{con}_K^r \circ \mathrm{res}_K^{r^{-1}h'rU} \circ \mathrm{con}_U^{r^{-1}h'r}(\overline{\tau}) \\ &= \mathrm{con}_K^r([(H/U)_\tau]_K).\end{aligned}$$

Hence the first assertion follows from Lemma 2.10. This completes the proof. □

Definition 4.7 We define

$$\mho(G,A) := \left\{ (x_H)_{H \le G} \in \prod_{H \le G} \mathbb{Z}H^1(H,A) \,\middle|\, \mathrm{con}_H^g(x_H) = x_{{}^gH} \text{ for all } g \in G \right\},$$

the ghost ring of $\Omega(G,A)$, which is a subring of $\prod_{H \le G} \mathbb{Z}H^1(H,A)$.

Remark 4.8 The family of $\mathbb{Z}$-algebras $\mathbb{Z}H^1(H,A)$ for $H \le G$, together with conjugation maps and restriction maps, defines a $\mathbb{Z}$-algebra restriction functor $\mathbb{Z}H^1(-,A)$ defined in [4, 1.1. Definition]. The rings $\Omega(G,A)$ and $\mho(G,A)$ are identified with $\mathbb{Z}H^1(G,A)_+$ and $\mathbb{Z}H^1(G,A)^+$, respectively, which are obtained by the plus constructions $\mathbb{Z}H^1(-,A) \mapsto \mathbb{Z}H^1(-,A)_+$ and $\mathbb{Z}H^1(-,A) \mapsto \mathbb{Z}H^1(-,A)^+$; moreover, the Green functor given in Proposition 2.16 is identified with $\mathbb{Z}H^1(-,A)_+$ (see [4]).

From Proposition 2.14 and Lemma 4.6, we know that there is an additive map $\rho_G : \Omega(G,A) \to \mho(G,A)$ given by

$$[(G/U)_\tau] \mapsto \left(\sum_{gU \in \mathrm{inv}_H(G/U)} \mathrm{res}_H^{gU} \circ \mathrm{con}_U^g(\overline{\tau}) \right)_{H \le G}$$

for all $(U,\tau) \in \mathcal{R}(G,A)$ (cf. [4, 2.3.]), which is called the mark homomorphism. Since

$$\rho_G([Y]) = ([Y]_H)_{H \le G}$$

for all (G, A)-sets Y, the mark homomorphism is a ring homomorphism defined by

$$\rho_G(x) = (\rho_G^H(x))_{H \le G} \tag{4.2}$$

for all $x \in \Omega(G, A)$ (cf. [12, §2.5]). We write $\rho = \rho_G$ for shortness' sake.

According to [4, (2.3a)], there is a map $\eta : \mho(G, A) \to \Omega(G, A)$ given by

$$\left(\sum_{\overline{\sigma} \in H^1(H,A)} \ell_{(H,\overline{\sigma})} \overline{\sigma} \right)_{H \le G} \mapsto \sum_{H \le G} \sum_{U \le H} |U| \mu(U, H) \sum_{\overline{\sigma} \in H^1(H,A)} \ell_{(H,\overline{\sigma})} [(G/U)_{\sigma|_U}]$$

for all $\ell_{(H,\overline{\sigma})} \in \mathbb{Z}$ with $H \le G$ and $\sigma \in Z^1(H, A)$.

We quote concise versions of [4, 2.4. Proposition] and [12, Theorem 1].

Proposition 4.9 (a) $\eta \circ \rho = |G| \mathrm{id}_{\Omega(G,A)}$. (b) $\rho \circ \eta = |G| \mathrm{id}_{\mho(G,A)}$.

Corollary 4.10 *The mark homomorphism ρ is injective.*

4C Invariant of tensor induction

Let $H \le G$. By Example 3.13 and Proposition 3.14, we have

$$\rho([\widehat{\mathrm{Map}}_H(G, (H/H)_\sigma)]) = (\overline{\sigma^{\otimes G}|_K})_{K \le G} = \left(\prod_{KgH \in K \backslash G/H} \overline{(g\sigma)|_{K \cap {}^gH}}^{\otimes K} \right)_{K \le G} \tag{4.3}$$

for all $\sigma \in Z^1(H, A)$. Let T be an (H, A)-set. We are interested in the description of $\rho([\widehat{\mathrm{Map}}_H(G, T)])$, which naturally extends Eq.(4.3) (see Proposition 4.14). For each $K \le G$, the K-component $[\widehat{\mathrm{Map}}_H(G, T)]_K$ of $\rho([\widehat{\mathrm{Map}}_H(G, T)])$ is also associated with a Mackey decomposition formula (see Proposition 3.9).

Let Y be a (G, A)-set, and let $K \le G$. By Definition 4.2,

$$[Y]_K = \frac{1}{|A|} \sum_{y \in \mathrm{inv}_K^A(Y)} \mathrm{res}_K^{G_y A}(\overline{\sigma_y}) = \sum_{yA \in \mathrm{inv}_K^A(Y)/A} \mathrm{res}_K^{G_y A}(\overline{\sigma_y}).$$

Concerning this formula, we have

$$[Y]_K = \frac{1}{|A|} \sum_{y \in \mathrm{inv}_K^A(Y)} \mathrm{res}_K^{G_y A}(\overline{\sigma_y}) = \frac{1}{|A|} \sum_{y \in \mathrm{inv}_K^A(\mathrm{res}_K^G(Y))} \overline{\sigma_y} = [\mathrm{res}_K^G(Y)]_K. \tag{4.4}$$

Obviously, this fact implies that $\rho_G^K(x) = \rho_K^K(\mathrm{res}_K^G(x))$ for any $x \in \Omega(G, A)$ which is applied to the following lemma.

Lemma 4.11 *Let H, $K \leq G$. For any $x \in \Omega(H, A)$,*

$$\rho_G^K(\overline{\mathrm{Map}}_H(G, x)) = \prod_{KgH \in K\backslash G/H} \rho_K^K(\overline{\mathrm{Map}}_{K\cap {}^gH}(K, \mathrm{res}^{gH}_{K\cap {}^gH} \circ \mathrm{con}^g_H(x))).$$

Proof. Since $\rho_G^K(\overline{\mathrm{Map}}_H(G, x)) = \rho_K^K(\mathrm{res}^G_K(\overline{\mathrm{Map}}_H(G, x)))$ for any $x \in \Omega(H, A)$, the assertion follows from Proposition 3.24. This completes the proof. □

Definition 4.12 Let $H \leq G$. We define a map $-^{\otimes G} : \mathbb{Z}H^1(H, A) \to \mathbb{Z}H^1(G, A)$ by

$$\sum_{\overline{\sigma}\in H^1(H,A)} \ell_{\overline{\sigma}}\overline{\sigma} \mapsto \left(\sum_{\overline{\sigma}\in H^1(H,A)} \ell_{\overline{\sigma}}\overline{\sigma}\right)^{\otimes G} := \sum_{\overline{\sigma}\in H^1(H,A)} \ell_{\overline{\sigma}}\overline{\sigma^{\otimes G}}$$

for all $\ell_{\overline{\sigma}} \in \mathbb{Z}$ with $\sigma \in Z^1(H, A)$. (This map is well-defined; see Remark 3.12.)

Lemma 4.13 *Let $H \leq G$, and let T be an (H, A)-set. Then*

$$[\widehat{\mathrm{Map}}_H(G, T)]_G = \frac{1}{|A|} \sum_{\widehat{f}\in \mathrm{inv}^A_G(\widehat{\mathrm{Map}}_H(G,T))} \overline{\sigma_{\widehat{f}}} = \frac{1}{|A|} \sum_{t\in \mathrm{inv}^A_H(T)} \overline{\sigma_t{}^{\otimes G}} = [T]_H{}^{\otimes G}.$$

Proof. Fix a complete set $\{g_1, g_2, \ldots, g_n\}$ with $g_1 = \epsilon$ of representatives of G/H. Let $t \in \mathrm{inv}^A_H(T)$. We define an element $f_{(t)}$ of $\mathrm{Map}_H(G, T)$ by

$$f_{(t)}(g_j^{-1}) = t$$

for all $j \in [n]$. For any $g \in G$ and $j \in [n]$, if $g_jH = gg_{j'}H$, then

$$(gf_{(t)})(g_j^{-1}) = (g_j^{-1}gg_{j'})f_{(t)}(g_{j'}^{-1}) = (g_j^{-1}gg_{j'})t = f_{(t)}(g_j^{-1})\sigma_t(g_j^{-1}gg_{j'}).$$

This, combined with Lemma 3.7, shows that $g\widehat{f_{(t)}} = \widehat{f_{(t)}}(\sigma_t{}^{\otimes G})(g)$ for all $g \in G$ (see Definition 3.11). Hence $G = G_{\widehat{f_{(t)}}A}$, $\widehat{f_{(t)}} \in \mathrm{inv}^A_G(\widehat{\mathrm{Map}}_H(G, T))$, and $\sigma_{\widehat{f_{(t)}}} = \sigma_t{}^{\otimes G}$.

We now define a map $\widehat{\Gamma} : \mathrm{inv}^A_H(T)/A \to \mathrm{inv}^A_G(\widehat{\mathrm{Map}}_H(G, T))/A$ by

$$\widehat{\Gamma}(tA) = \widehat{f_{(t)}}A$$

for all $t \in \mathrm{inv}^A_H(T)$. This map is well-defined, because, by Lemma 3.7, $\widehat{f_{(ta)}} = \widehat{f_{(t)}}b$ with $b = {}^{g_1}a\,{}^{g_2}a \cdots {}^{g_n}a$ for any $a \in A$. If $\widehat{\Gamma}(t_1A) = \widehat{\Gamma}(t_2A)$ with t_1, $t_2 \in \mathrm{inv}^A_H(T)$, then $\widehat{f_{(t_1)}} = \widehat{f_{(t_2)}}a$ for some $a \in A$, and hence $t_1 = t_2b$ for some $b \in A$. Thus $\widehat{\Gamma}$ is injective. Let $\widehat{f} \in \mathrm{inv}^A_G(\widehat{\mathrm{Map}}_H(G, T))$ with $f \in \mathrm{Map}_H(G, T)$, and let $g \in G$. Given $j \in [n]$, we have $(gf)(g_j^{-1}) = f(g_j^{-1})a_j(g)$ for some $a_j(g) \in A$. Set $t = f(\epsilon)$. Then

$$ht = hf(\epsilon) = f(h) = (hf)(\epsilon) = ta_1(h)$$

for all $h \in H$, which yields $t \in \mathrm{inv}_H^A(T)$. Observe now that for any $j \in [n]$,

$$f(g_j^{-1})a_j(g_j) = (g_j f)(g_j^{-1}) = f(\epsilon) = t = f_{(t)}(g_j^{-1}).$$

By Lemma 3.7, we have $\widehat{f} = \widehat{f_{(t)}}a$, where $a = ({}^{g_1}a_1(g_1)\,{}^{g_2}a_2(g_2)\cdots {}^{g_n}a_n(g_n))^{-1}$, so that $\widehat{\Gamma}(tA) = \widehat{f}A$. Thus $\widehat{\Gamma}$ is bijective. The assertion now follows from the fact that $\sigma_{\widehat{f_{(t)}}} = {\sigma_t}^{\otimes G}$ for all $t \in \mathrm{inv}_H^A(T)$. This completes the proof. □

The following proposition generalizes the equation in [32, p. 39] (see also [2, Lemma 9.2], [9, p. 149], and [30, p. 111, Eq.(2)]).

Proposition 4.14 *Let H, $K \leq G$. For each (H, A)-set T,*

$$[\widehat{\mathrm{Map}}_H(G, T)]_K = \prod_{KgH \in K\backslash G/H} [{}^gT]_{K\cap {}^gH}{}^{\otimes K}.$$

Proof. Combining Lemma 4.13 with Lemma 4.11, we have

$$\begin{aligned}[\widehat{\mathrm{Map}}_H(G, T)]_K &= \prod_{KgH \in K\backslash G/H} [\widehat{\mathrm{Map}}_{K\cap {}^gH}(K, \mathrm{res}^{{}^gH}_{K\cap {}^gH}({}^gT))]_K \\ &= \prod_{KgH \in K\backslash G/H} [\mathrm{res}^{{}^gH}_{K\cap {}^gH}({}^gT)]_{K\cap {}^gH}{}^{\otimes K}.\end{aligned}$$

Hence the assertion follows from Eq.(4.4). This completes the proof. □

How about the description of $\rho(\overline{\mathrm{Map}}_H(G, x))$ for any $H \leq G$ and $x \in \Omega(H, A)$? By using Eq.(1.1), we are successful in proving Eq.(1.2) (see Theorem 4.16).

Lemma 4.15 *Let $H \leq G$. For any (H, A)-sets T_0 and T,*

$$\rho_G^G(\overline{\mathrm{Map}}_H(G, [T_0] - [T])) = [\widehat{\mathrm{Map}}_H(G, T_0)]_G - [\widehat{\mathrm{Map}}_H(G, T)]_G.$$

Proof. We may assume that $H < G$. By Proposition 3.22,

$$\overline{\mathrm{Map}}_H(G, [T_0] - [T]) = [\widehat{\mathrm{Map}}_H(G, T_0)] + \sum_{i=1}^{n} (-1)^i [\widehat{\mathrm{Map}}_H(G, T_0, T_1, \ldots, T_i)],$$

where $n = |G : H|$ and $T = T_1 = \cdots = T_n$. If $i \in [n]$ and $i \geq 2$, then obviously, $[\widehat{\mathrm{Map}}_H(G, T_0, T_1, \ldots, T_i)]_G = 0$. Moreover, we have

$$\mathrm{inv}_G^A(\widehat{\mathrm{Map}}_H(G, T_0, T_1)) = \mathrm{inv}_G^A(\widehat{\mathrm{Map}}_H(G, \emptyset, T_1)) = \mathrm{inv}_G^A(\widehat{\mathrm{Map}}_H(G, T_1)),$$

completing the proof. □

The following theorem, which is equivalent to Eq.(1.2), is an extension of Proposition 4.14 and is a generalization of [32, §3(b.3)].

Theorem 4.16 *Let $H \leq G$, and define a map* $\mathrm{jnd}_H^G : \mho(H,A) \to \mho(G,A)$ *by*

$$(x_L)_{L\leq H} \mapsto \left(\prod_{KgH \in K\backslash G/H} \mathrm{con}^g_{K^g\cap H}(x_{K^g\cap H})^{\otimes K} \right)_{K\leq G}$$

for all $(x_L)_{L\leq H} \in \mho(H,A)$*. Then the diagram*

$$\begin{array}{ccc} \Omega(G,A) & \xrightarrow{\rho} & \mho(G,A) \\ {\scriptstyle \overline{\mathrm{Map}}_H(G,-)}\uparrow & & \uparrow{\scriptstyle \mathrm{jnd}_H^G} \\ \Omega(H,A) & \xrightarrow[\rho_H]{} & \mho(H,A) \end{array}$$

is commutative, where $\rho_H : \Omega(H,A) \to \mho(H,A)$ *is the mark homomorphism.*

Proof. We prove Eq.(1.2). Let $x \in \Omega(H,A)$. We may assume that $x = [T_0] - [T]$ for some (H,A)-sets T_0 and T. Let $K \leq G$. Then by Lemmas 4.11 and 4.15, we have

$$\begin{aligned} &\rho_G^K(\overline{\mathrm{Map}}_H(G,[T_0]-[T])) \\ &\quad = \prod_{KgH\in K\backslash G/H} \rho_K^K(\overline{\mathrm{Map}}_{K\cap {}^gH}(K,[\mathrm{res}^{{}^gH}_{K\cap {}^gH}({}^gT_0)] - [\mathrm{res}^{{}^gH}_{K\cap {}^gH}({}^gT)])) \\ &\quad = \prod_{KgH\in K\backslash G/H} \Big\{ [\widehat{\mathrm{Map}}_{K\cap {}^gH}(K,\mathrm{res}^{{}^gH}_{K\cap {}^gH}({}^gT_0))]_K \\ &\qquad\qquad - [\widehat{\mathrm{Map}}_{K\cap {}^gH}(K,\mathrm{res}^{{}^gH}_{K\cap {}^gH}({}^gT))]_K \Big\}. \end{aligned}$$

Moreover, it follows from Eq.(4.4) and Lemma 4.13 that

$$\rho_G^K(\overline{\mathrm{Map}}_H(G,[T_0]-[T])) = \prod_{KgH\in K\backslash G/H} \Big\{ [{}^gT_0]_{K\cap {}^gH}{}^{\otimes K} - [{}^gT]_{K\cap {}^gH}{}^{\otimes K} \Big\}. \quad (4.5)$$

By Lemma 4.6(b), $[{}^gT_1]_{K\cap {}^gH} = \mathrm{con}^g_{K^g\cap H}([T_1]_{K^g\cap H})$, where $T_1 = T_0$ or $T_1 = T$, for all $g \in G$. Hence Eq.(1.2) follows from Eq.(4.2). This completes the proof. □

Remark 4.17 Given $(H,\sigma) \in \mathcal{S}(G,A)$, it follows from Lemma 4.6 and Eq.(4.5) that

$$\rho(\overline{\mathrm{Map}}_H(G,-[(H/H)_\sigma])) = \left((-1)^{|K\backslash G/H|} \prod_{KgH\in K\backslash G/H} \overline{(g\sigma)|_{K\cap {}^gH}}^{\otimes K} \right)_{K\leq G}$$

(see also Eq.(4.3)). Here we return to Remark 3.23. Deducing this fact directly from Lemma 3.7 and Proposition 3.22 requires the use of [25, (24d)] which provides a combinatorial explanation. Let H, $K \leq G$. When $A = \{\epsilon_A\}$, the K-component of $\rho(\widetilde{\Lambda}_{P(G/H)})$ is $(-1)^{|K\backslash G/H|}$ (see [29, Proposition 5.1] and [32, Lemma 3.6]).

For each $H \leq G$, we denote by $\Omega(H, A)^{\times}$ the unit group of $\Omega(H, A)$, and consider this abelian group as a $\mathbb{Z}$-module. Note that the $\mathbb{Z}$-module structure of $\Omega(H, A)^{\times}$ is different from that of $\Omega(H, A)$.

There is a fact relative to [2, Theorem 9.6] and [32, Lemma 3.1].

Theorem 4.18 *The family of $\mathbb{Z}$-modules $\Omega(H, A)^{\times}$ for $H \leq G$, together with conjugation, restriction, and multiplicative induction maps inherited from those on the family of $\mathbb{Z}$-algebras $\Omega(H, A)$ for $H \leq G$ defines a Mackey functor on G.*

Proof. Let $\mathrm{jnd}_K^H : \Omega(K, A)^{\times} \to \Omega(H, A)^{\times}$ with $K \leq H \leq G$ be the map inherited from $\overline{\mathrm{Map}}_K(H, -) : \Omega(K, A) \to \Omega(H, A)$. By [4, 1.1. Definition], Lemma 2.10, and Proposition 3.24, it suffices to verify that for any $U \leq V \leq H \leq G$ and $g \in G$,

$$\mathrm{con}_H^g \circ \mathrm{jnd}_U^H = \mathrm{jnd}_{{}^gU}^{{}^gH} \circ \mathrm{con}_U^g \quad \text{and} \quad \mathrm{jnd}_V^H \circ \mathrm{jnd}_U^V = \mathrm{jnd}_U^H. \tag{4.6}$$

Given $H \leq G$ and $g \in G$, we define a map $\mathrm{con}_H^g : \mho(H, A) \to \mho({}^gH, A)$ by

$$(x_K)_{K \leq H} \mapsto (\mathrm{con}_K^g(x_K))_{{}^gK \leq {}^gH}$$

for all $(x_K)_{K \leq H} \in \mho(H, A)$. Let $U \leq V \leq H \leq G$, and let $g \in G$. Given $K \leq H$ and $(x_L)_{L \leq U} \in \mho(U, A)$, we have

$$\mathrm{con}_K^g(\mathrm{con}_{K^h \cap U}^h(x_{K^h \cap U})^{\otimes K}) = (\mathrm{con}^{{}^gh}_{({}^gK)^{{}^gh} \cap {}^gU} \circ \mathrm{con}^g_{K^h \cap U}(x_{K^h \cap U}))^{\otimes {}^gK}$$

for all $h \in H$ and

$$\prod_{KhV \in K \backslash H/V} \mathrm{con}_{K_h}^h \left(\prod_{K_h r U \in K_h \backslash V/U} \mathrm{con}^r_{K_h{}^r \cap U}(x_{K_h{}^r \cap U})^{\otimes K_h} \right)^{\otimes K} = \prod_{KhU \in K \backslash H/U} \mathrm{con}^h_{K^h \cap U}(x_{K^h \cap U})^{\otimes K},$$

where $K_h = K^h \cap V$ (see Lemma 3.15). Relative to 'jnd' defined in Theorem 4.16, these equations enable us to obtain the equations

$$\mathrm{con}_H^g \circ \mathrm{jnd}_U^H = \mathrm{jnd}_{{}^gU}^{{}^gH} \circ \mathrm{con}_U^g \quad \text{and} \quad \mathrm{jnd}_V^H \circ \mathrm{jnd}_U^V = \mathrm{jnd}_U^H.$$

By Lemma 4.6(b), $\mathrm{con}_H^g \circ \rho_H = \rho_{{}^gH} \circ \mathrm{con}_H^g$ and $\mathrm{con}_U^g \circ \rho_U = \rho_{{}^gU} \circ \mathrm{con}_U^g$. Hence Eq.(4.6) follows from Corollary 4.10 and Theorem 4.16. This completes the proof. □

5 Fundamentals of monomial Burnside rings

5A The Burnside homomorphism

The discussion in this section is a special case of [28, §9] (see also [27, §3, §4]).

For each $(U,\tau) \in \mathcal{S}(G,A)$, we set

$$N_G(U,\tau) = \{g \in G \mid {}^gU = U \text{ and } \mathrm{con}_U^g(\overline{\tau}) = \overline{\tau}\}.$$

By definition, the elements $(x_H^{(U,\tau)})_{H \le G}$ for $(U,\tau) \in \mathcal{R}(G,A)$, where

$$x_H^{(U,\tau)} = \begin{cases} \displaystyle\sum_{gN_G(U,\tau) \in N_G(U)/N_G(U,\tau)} \mathrm{con}_U^{rg}(\overline{\tau}) & \text{if } H = {}^rU \text{ with } r \in G, \\ 0 & \text{otherwise,} \end{cases}$$

form a free $\mathbb{Z}$-basis of the ghost ring $\mho(G,A)$. We define

$$\widetilde{\Omega}(G,A) := \coprod_{(K,\nu) \in \mathcal{R}(G,A)} \mathbb{Z},$$

so that there exists an isomorphism $\kappa : \widetilde{\Omega}(G,A) \xrightarrow{\sim} \mho(G,A)$ of $\mathbb{Z}$-lattices given by

$$(\delta_{(U,\tau)\,(K,\nu)})_{(K,\nu) \in \mathcal{R}(G,A)} \mapsto (x_H^{(U,\tau)})_{H \le G}$$

for all $(U,\tau) \in \mathcal{R}(G,A)$, where δ is the Kronecker delta.

Definition 5.1 We define an additive map $\varphi : \Omega(G,A) \to \widetilde{\Omega}(G,A)$ by

$$\varphi([(G/U)_\tau]) = (|\mathrm{inv}_{(K,\nu)}((G/U)_\tau)/A_\nu|)_{(K,\nu) \in \mathcal{R}(G,A)}$$

for all $(U,\tau) \in \mathcal{R}(G,A)$ (see Lemma 4.5), and call it the Burnside homomorphism.

Proposition 5.2 *The diagram*

$$\begin{array}{ccc} \Omega(G,A) & \xrightarrow{\varphi} & \widetilde{\Omega}(G,A) \\ & \underset{\rho}{\searrow} & \downarrow \kappa \\ & & \mho(G,A) \end{array}$$

is commutative. In particular, the Burnside homomorphism φ is injective.

Proof. The assertion follows from Lemma 4.4 and Corollary 4.10. □

Let $(U,\tau) \in \mathcal{R}(G,A)$. By Lemma 2.6, $N_G(U,\tau)$ contains U. Observe that for any $(K,\nu) \in \mathcal{R}(G,A)$, the (K,ν)-component of $\varphi([(G/U)_\tau])$ is divisible by $|N_G(U,\tau)/U|$ (see Lemma 4.5). We define

$$y^{(U,\tau)} := \frac{1}{|N_G(U,\tau)/U|}\varphi([(G/U)_\tau]) = \left(\frac{|\mathrm{inv}_{(K,\nu)}((G/U)_\tau)/A_\nu|}{|N_G(U,\tau)/U|}\right)_{(K,\nu) \in \mathcal{R}(G,A)}.$$

Proposition 5.3 *The elements $y^{(U,\tau)}$ for $(U,\tau) \in \mathcal{R}(G,A)$ form a free $\mathbb{Z}$-basis of the $\mathbb{Z}$-lattice $\widetilde{\Omega}(G,A)$.*

Proof. The proof is completely analogous to that of [8, (80.15) Proposition]. □

5B The Cauchy-Frobenius homomorphism

We aim to state a fundamental theorem for the monomial Burnside ring $\Omega(G,A)$ (see Theorem 5.9).

Definition 5.4 For each $(U,\tau) \in \mathcal{S}(G,A)$, let $W_G(U,\tau)$ denote the factor group $N_G(U,\tau)/U$. We define

$$\mathrm{Obs}\,(G,A) := \coprod_{(U,\tau)\in\mathcal{R}(G,A)} \mathbb{Z}/|W_G(U,\tau)|\mathbb{Z},$$

the obstruction group of $\Omega(G,A)$.

The following fact is a corollary to Proposition 5.3.

Corollary 5.5 $\widetilde{\Omega}(G,A)/\mathrm{Im}\varphi \simeq \mathrm{Obs}\,(G,A)$.

Proof. The proof is completely analogous to that of [27, Corollary 3.8]. □

Let p be a prime, and let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at p. For each $\mathbb{Z}$-module M, we set $M_{(p)} = \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} M$ and $M_{(\infty)} = M$. Let $(U,\tau) \in \mathcal{S}(G,A)$. We denote by $W_G(U,\tau)_p$ a Sylow p-subgroup of $W_G(U,\tau)$, and set $W_G(U,\tau)_\infty = W_G(U,\tau)$.

Let p be a prime or the symbol ∞ hereafter. By Proposition 2.14, the elements $[(G/H)_\sigma]$ for $(H,\sigma) \in \mathcal{R}(G,A)$ form a free $\mathbb{Z}_{(p)}$-basis of the $\mathbb{Z}_{(p)}$-lattice $\Omega(G,A)_{(p)}$. We identify $\widetilde{\Omega}(G,A)_{(p)}$ and $\mathrm{Obs}\,(G,A)_{(p)}$ with

$$\coprod_{(K,\nu)\in\mathcal{R}(G,A)} \mathbb{Z}_{(p)} \quad \text{and} \quad \coprod_{(U,\tau)\in\mathcal{R}(G,A)} \mathbb{Z}_{(p)}/|W_G(U,\tau)_p|\mathbb{Z}_{(p)},$$

respectively. Let $\varphi^{(p)}$ denote the monomorphism from $\Omega(G,A)_{(p)}$ to $\widetilde{\Omega}(G,A)_{(p)}$ determined by φ. (So $\varphi^{(\infty)} = \varphi$.) Then by Corollary 5.5, we have

$$\widetilde{\Omega}(G,A)_{(p)}/\mathrm{Im}\varphi^{(p)} \simeq \mathrm{Obs}\,(G,A)_{(p)}. \tag{5.1}$$

The expression 'x mod ℓ' with $x, \ell \in \mathbb{Z}_{(p)}$ denotes the coset $x + \ell\mathbb{Z}_{(p)}$ of $\ell\mathbb{Z}_{(p)}$ in $\mathbb{Z}_{(p)}$ containing x. Let $(U,\tau) \in \mathcal{S}(G,A)$. Given $(y_{(H,\sigma)})_{(H,\sigma)\in\mathcal{R}(G,A)} \in \widetilde{\Omega}(G,A)_{(p)}$, $y_{(U,\tau)}$ denotes $y_{(H,\sigma)}$ for a representative $(H,\sigma) \in \mathcal{R}(G,A)$ of the F-orbit on $\mathcal{S}(G,A)$ containing (U,τ). For each $g \in N_G(U,\tau)$, we set

$$H^1_\tau(\langle g\rangle U, A) = \{\overline{\nu} \in H^1(\langle g\rangle U, A) \mid \mathrm{res}^{\langle g\rangle U}_U(\overline{\nu}) = \overline{\tau}\}.$$

Definition 5.6 We define an additive map $\psi^{(p)} : \widetilde{\Omega}(G,A)_{(p)} \to \mathrm{Obs}\,(G,A)_{(p)}$ by

$$(y_{(K,\nu)})_{(K,\nu)\in\mathcal{R}(G,A)} \mapsto \left(\sum_{\substack{gU\in W_G(U,\tau)_p,\\ \overline{\nu}\in H^1_\tau(\langle g\rangle U,A)}} y_{(\langle g\rangle U,\nu)} \bmod |W_G(U,\tau)_p| \right)_{(U,\tau)\in\mathcal{R}(G,A)}$$

for all $(y_{(K,\nu)})_{(K,\nu)\in\mathcal{R}(G,A)} \in \widetilde{\Omega}(G,A)_{(p)}$, and call it the Cauchy-Frobenius homomorphism.

Remark 5.7 (1) When p is a prime, $\psi^{(p)}$ is independent of the choice of a Sylow p-subgroup $W_G(U,\tau)_p$ of $W_G(U,\tau)$ (cf. [28, §9]). (2) When $p=\infty$, we write $\psi=\psi^{(\infty)}$.

For each $(H,\sigma)\in\mathcal{R}(G,A)$, it follows from Lemma 4.5 that

$$\psi^{(p)}\circ\varphi^{(p)}([(G/H)_\sigma]) = \left(\sum_{gU\in W_G(U,\tau)_p} |I^{(H,\sigma)}_{gU,\tau}| \bmod |W_G(U,\tau)_p|\right)_{(U,\tau)\in\mathcal{R}(G,A)}, \quad (5.2)$$

where

$$I^{(H,\sigma)}_{gU,\tau} = \{rH\in G/H \mid \langle g\rangle U \le {}^rH \text{ and } (r\sigma)|_U =_A \tau\}.$$

The following lemma, which is a special case of [28, Lemma 9.2], is a consequence of the Cauchy-Frobenius lemma (see, *e.g.*, [33, 2.7 Lemma]).

Lemma 5.8 *Let (H,σ), $(U,\tau)\in\mathcal{R}(G,A)$. For any $V\le N_G(U,\tau)$ with $U\le V$,*

$$\sum_{gU\in V/U} |I^{(H,\sigma)}_{gU,\tau}| \equiv 0 \pmod{|V/U|}.$$

Proof. The proof is analogous to that of [28, Lemma 9.2], and is also analogous to part of the proof of [27, Lemma 4.1]. □

We are now in a position to show a special case of [28, Theorem 9.4], which is a generalization of [9, Proposition 1.3.5] and [32, Lemma 2.1].

Theorem 5.9 (Fundamental theorem) *The sequence*

$$0 \longrightarrow \Omega(G,A)_{(p)} \xrightarrow{\varphi^{(p)}} \widetilde{\Omega}(G,A)_{(p)} \xrightarrow{\psi^{(p)}} \mathrm{Obs}\,(G,A)_{(p)} \longrightarrow 0$$

of additive groups is exact.

Proof. By Proposition 5.2, $\varphi^{(p)}$ is injective. Moreover, it is easily verified that $\psi^{(p)}$ is surjective (see, *e.g.*, the proof of [27, Lemma 4.3]). Using Eqs.(5.1) and (5.2) and Lemma 5.8, we have $\mathrm{Im}\,\varphi^{(p)} = \mathrm{Ker}\,\psi^{(p)}$, completing the proof. □

5C Idempotents of Burnside rings

The Burnside ring $\Omega(G)$ of G, which is defined to be the Grothendieck ring of G-**set**, is the commutative unital ring consisting of all formal $\mathbb{Z}$-linear combinations of the symbols $[G/H]$ for $H\in\mathrm{C}(G)$ with multiplication given by

$$[G/H]\cdot[G/U] = \sum_{HgU\in H\backslash G/U} [G/(H\cap {}^gU)] \quad (5.3)$$

for all $H, U \in \mathrm{C}(G)$, where $[G/(H \cap {}^gU)] = [G/K]$ for a conjugate $K \in \mathrm{C}(G)$ of $H \cap {}^gU$ in G (see, *e.g.*, [33, 2.1]). The identity of $\Omega(G)$ is $[G/G]$.

We regard $\Omega(G)$ as $\Omega(G, A)$ with $A = \{\epsilon_A\}$. For each $X \in G$-**set**, the symbol $[X]$ denotes an element $\sum_{i=1}^{n}[G/H_i]$ of $\Omega(G)$ if $X \simeq \dot{\cup}_{i\in[n]} G/H_i$ with $H_i \in \mathrm{C}(G)$.

Remark 5.10 The product $X_1 \times X_2$ of $X_1, X_2 \in G$-**set** is their cartesian product with the componentwise action of G (cf. [8, §80A]). Let $H, U \leq G$, and let $\overline{H\backslash G/U}$ be a complete set of representatives of $H\backslash G/U$. Then there exists an isomorphism

$$(G/H) \times (G/U) \xrightarrow{\sim} \dot{\bigcup_{g\in\overline{H\backslash G/U}}} G/(H \cap {}^gU), \quad (g_1H, g_2U) \mapsto g_1h(H \cap {}^gU)$$

of G-sets, where $g_2U = g_1hgU$ with $h \in H$ and $g \in \overline{H\backslash G/U}$ (see Lemma 2.15). Hence Eq.(5.3) means that $[X_1] \cdot [X_2] = [X_1 \times X_2]$ for all $X_1, X_2 \in G$-**set**.

Definition 5.11 We define a ring homomorphism $\alpha : \Omega(G, A) \to \Omega(G)$ by

$$[(G/U)_\tau] \mapsto [G/U]$$

for all $(U, \tau) \in \mathcal{R}(G, A)$ and define a ring homomorphism $\iota : \Omega(G) \to \Omega(G, A)$ by

$$[G/U] \mapsto [(G/U)_{1_U}]$$

for all $U \in \mathrm{C}(G)$.

Since $\alpha \circ \iota = \mathrm{id}_{\Omega(G)}$, the Burnside ring $\Omega(G)$ is identified with $\mathrm{Im}\,\iota$. We define

$$\mho(G) := \prod_{H\in\mathrm{C}(G)} \mathbb{Z}.$$

There exists a ring monomorphism $\phi : \Omega(G) \to \mho(G)$ given by

$$[G/U] \mapsto (|\mathrm{inv}_H(G/U)|)_{H\in\mathrm{C}(G)}$$

for all $U \in \mathrm{C}(G)$ (cf. [8, (80.12) Proposition]), where $\mathrm{inv}_H(G/U)$ is given by Eq.(4.1).

The ring homomorphism $\varepsilon : \mathbb{Z}H^1(H, A) \to \mathbb{Z}$ with $H \leq G$ given by

$$\sum_{\overline{\sigma}\in H^1(H,A)} \ell_{\overline{\sigma}}\overline{\sigma} \mapsto \sum_{\overline{\sigma}\in H^1(H,A)} \ell_{\overline{\sigma}}$$

for all $\ell_{\overline{\sigma}} \in \mathbb{Z}$ with $\sigma \in Z^1(H, A)$ is called the augmentation map of $\mathbb{Z}H^1(H, A)$ (cf. [21, Definition 3.2.9]).

Definition 5.12 We define a ring homomorphism $\widetilde{\alpha} : \mho(G, A) \to \mho(G)$ by

$$(x_H)_{H \leq G} \mapsto (\varepsilon(x_H))_{H \in \mathrm{C}(G)}$$

for all $(x_H)_{H \leq G} \in \mho(G, A)$ and define a ring homomorphism $\widetilde{\iota} : \mho(G) \to \mho(G, A)$ by

$$(y_H)_{H \in \mathrm{C}(G)} \mapsto (\widetilde{y}_H)_{H \leq G},$$

where $\widetilde{y}_H = y_K$ for a conjugate $K \in \mathrm{C}(G)$ of H in G, for all $(y_H)_{H \in \mathrm{C}(G)} \in \mho(G)$.

Obviously, $\widetilde{\alpha} \circ \widetilde{\iota} = \mathrm{id}_{\mho(G)}$. We provide the following two lemmas.

Lemma 5.13 (a) *The diagrams*

$$\begin{array}{ccc} \Omega(G,A) & \xrightarrow{\ \rho\ } & \mho(G,A) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \widetilde{\alpha}} \\ \Omega(G) & \xrightarrow[\ \phi\ ]{} & \mho(G) \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \Omega(G,A) & \xrightarrow{\ \rho\ } & \mho(G,A) \\ {\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \widetilde{\iota}} \\ \Omega(G) & \xrightarrow[\ \phi\ ]{} & \mho(G) \end{array}$$

are commutative.

(b) *Let $x \in \Omega(G, A)$. If $\rho(x) = \widetilde{\iota}(y)$ for some $y \in \mho(G)$, then $\iota \circ \alpha(x) = x$.*

Proof. The statement (a) is clear. We prove the statement (b). Since $\widetilde{\alpha} \circ \widetilde{\iota} = \mathrm{id}_{\mho(G)}$, it follows from the statement (a) that

$$\rho \circ \iota \circ \alpha(x) = \widetilde{\iota} \circ \phi \circ \alpha(x) = \widetilde{\iota} \circ \widetilde{\alpha} \circ \rho(x) = \widetilde{\iota} \circ \widetilde{\alpha} \circ \widetilde{\iota}(y) = \widetilde{\iota}(y) = \rho(x).$$

This, combined with Corollary 4.10, shows that $\iota \circ \alpha(x) = x$, completing the proof. □

Lemma 5.14 (a) $\alpha \circ \eta \circ \widetilde{\iota} \circ \phi = |G|\mathrm{id}_{\Omega(G)}$. (b) $\phi \circ \alpha \circ \eta \circ \widetilde{\iota} = |G|\mathrm{id}_{\mho(G)}$.

Proof. The lemma follows from Proposition 4.9 and Lemma 5.13(a). □

The rest of this section is devoted to the idempotents of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$.

Definition 5.15 Given $U \leq G$, we define $W_G(U)$ to be the factor group $N_G(U)/U$.

Let p be a prime or the symbol ∞. For each $U \leq G$, we denote by $W_G(U)_p$ a Sylow p-subgroup of $W_G(U)$ provided p is a prime, and set $W_G(U)_\infty = W_G(U)$.

The elements $[G/H]$ for $H \in \mathrm{C}(G)$ form a free $\mathbb{Z}_{(p)}$-basis of the $\mathbb{Z}_{(p)}$-lattice $\Omega(G)_{(p)}$. We identify $\mho(G)_{(p)}$ with $\coprod_{H \in \mathrm{C}(G)} \mathbb{Z}_{(p)}$. Let $\phi^{(p)}$ denote the ring monomorphism from $\Omega(G)_{(p)}$ to $\mho(G)_{(p)}$ determined by ϕ.

We quote [9, Proposition 1.3.5] (see also [32, Lemma 2.1]).

Proposition 5.16 *Let* $\widetilde{x} = (x_H)_{H\in \mathrm{C}(G)} \in \mho(G)_{(p)}$. *Then* $\widetilde{x} \in \mathrm{Im}\phi^{(p)}$ *if and only if*

$$\sum_{gU\in W_G(U)_p} x_{\langle g\rangle U} \equiv 0 \pmod{|W_G(U)_p|},$$

where $x_{\langle g\rangle U} = x_K$ *for a conjugate* $K \in \mathrm{C}(G)$ *of* $\langle g\rangle U$ *in* G, *for all* $U \in \mathrm{C}(G)$.

Proof. The assertion follows from Theorem 5.9 and Lemma 5.13(a). □

By Lemma 5.14, the primitive idempotents of $\mathbb{Q}\otimes_{\mathbb{Z}}\Omega(G)$ are the elements

$$e_H := \frac{1}{|G|}\alpha\circ\eta\circ\widetilde{\iota}((\delta_{H\,K})_{K\in\mathrm{C}(G)}) = \frac{1}{|N_G(H)|}\sum_{U\le H}|U|\mu(U,H)[G/U] \qquad (5.4)$$

for $H \in \mathrm{C}(G)$. This fact was shown by Gluck [14] and independently by Yoshida [31]. Obviously, $e_He_K = \delta_{H\,K}e_H$ for all H, $K \in \mathrm{C}(G)$, and $[G/G] = \sum_{H\in\mathrm{C}(G)} e_H$.

Following [33], we present the primitive idempotents of $\Omega(G)$. Let $\sim_p$ be the equivalence relation on the set $\{(H) \mid H \le G\}$, where (H) is the set of conjugates of H in G, generated by

$$(\langle g\rangle U) \sim_p (U)$$

for $U \le G$ and $gU \in W_G(U)_p$ with $g \in N_G(U)$. We define an equivalence relation $\sim_p$ on the set $\mathrm{S}(G)$ of subgroups of G by

$$H \sim_p K \;:\Longleftrightarrow\quad (H) \sim_p (K).$$

Let $H \le G$. When p is a prime, we denote by $O^p(H)$ the smallest normal subgroup of H such that $H/O^p(H)$ is a p-group (cf. [31]). Suppose that

$$H = H^{(0)} \ge H^{(1)} \ge H^{(2)} \ge \cdots \ge H^{(i)} \ge \cdots$$

is the derived series of H (cf. [26, Chapter 2, Definition 3.11]). Then we define $O^\infty(H) := \cap_{i=0}^{\infty}H^{(i)}$. The following lemma is well-known (cf. [33, p. 535]).

Lemma 5.17 *Let* $H, U \le G$. *Then* $H \sim_p U$ *if and only if* $(O^p(H)) = (O^p(U))$.

Proof. The 'if' part follows from [26, Chapter 2, Theorem 1.6]. To prove the 'only if' part, we may assume that $H = \langle g\rangle U$ for some $gU \in W_G(U)_p$ with $g \in N_G(U)$. If p is a prime, then $U \ge O^p(U) \ge O^p(H)$, and hence $O^p(U) = O^p(H)$. Suppose that $p = \infty$. We have $U^{(i-1)} \ge H^{(i)} \ge U^{(i)}$ for any $i \ge 1$. If $U^{(i-1)} = U^{(i)}$ for some i, then $U^{(i-1)} = H^{(i)} = U^{(i)}$. Thus we have $O^\infty(H) = O^\infty(U)$, completing the proof. □

A subgroup H of G is said to be p-perfect if $H = O^p(H)$. For each $K \le G$, $K \sim_p O^p(K)$ by Lemma 5.17, and $O^p(K)$ is p-perfect. Let $\mathrm{C}^{(p)}(G)$ be a full set of non-conjugate p-perfect subgroups of G. For each $H \in \mathrm{C}^{(p)}(G)$, we define

$$e_H^{(p)} := \sum_{H\sim_p K\in\mathrm{C}(G)} e_K,$$

where the sum is taken over all $K \in \mathrm{C}(G)$ such that $H \sim_p K$.

The following theorem concerns [2, Theorem 7.3] and [33, 4.12 Theorem] (see also [14, Lemma 2] and [31, Theorem 3.1]).

Theorem 5.18 *The elements $e_H^{(p)}$ for $H \in \mathrm{C}^{(p)}(G)$ are the primitive idempotents of $\Omega(G)_{(p)}$, and the elements $e_H^{(\infty)}$ for $H \in \mathrm{C}^{(\infty)}(G)$ are also those of $\Omega(G, A)$.*

Proof. For any idempotent $(x_H)_{H\in \mathrm{C}(G)}$ of $\mho(G)_{(p)}$, it follows from Proposition 5.16 that $(x_H)_{H\in \mathrm{C}(G)} \in \mathrm{Im}\phi^{(p)}$ if and only if $x_K = x_U \in \{0, 1\}$ for all pairs (K, U) of K, $U \in \mathrm{C}(G)$ with $K \sim_p U$. Hence the elements $e_H^{(p)}$ for $H \in \mathrm{C}^{(p)}(G)$ are the primitive idempotents of $\Omega(G)_{(p)}$. Let x be an idempotent of $\Omega(G, A)$. According to [21, Corollary 7.2.4], $\mathbb{Z}H^1(H, A)$ with $H \leq G$ contains only trivial idempotents, whence $\rho(x) = \widetilde{\iota}(y)$ for some $y \in \mho(G)$. This, combined with Lemma 5.13(b), shows that $\iota \circ \alpha(x) = x$. By this fact, we may identify x with $\alpha(x) \in \Omega(G)$. Since the map $\alpha : \Omega(G, A) \to \Omega(G)$ is a ring homomorphism, it follows that $\alpha(x)$ is an idempotent of $\Omega(G)$. Consequently, the idempotents of $\Omega(G, A)$ are those of $\Omega(G)$. This completes the proof. □

There is an immediate consequence of Theorem 4.18 (see [4, 1.5. Proposition]).

Proposition 5.19 *The $\mathbb{Z}$-module $\Omega(G, A)^\times$ has a structure of an $\Omega(G)$-module, namely,*

$$\Omega(G) \otimes_{\mathbb{Z}} \Omega(G, A)^\times \to \Omega(G, A)^\times, \quad [G/H] \otimes_{\mathbb{Z}} x \mapsto \overline{\mathrm{Map}}_H(G, \mathrm{res}^G_H(x)).$$

Moreover,

$$\Omega(G, A)^\times = \prod_{H\in \mathrm{C}^{(\infty)}(G)} \{e_H^{(\infty)} x \mid x \in \Omega(G, A)^\times\},$$

where $e_H^{(\infty)} x$ denotes the effect of $e_H^{(\infty)}$ on x.

6 Units of Burnside rings

6A The Yoshida criterion for the units of Burnside rings

We turn to the unit group $\Omega(G)^\times$ of $\Omega(G)$. Let $\mho(G)^\times$ be the unit group of $\mho(G)$, and let $\phi^\times : \Omega(G)^\times \to \mho(G)^\times$ be the map obtained by restriction of $\phi : \Omega(G) \to \mho(G)$ from $\Omega(G)$ to $\Omega(G)^\times$. Obviously, $\mho(G)^\times = \prod_{H\in \mathrm{C}(G)} \langle -1\rangle$, where $\langle -1\rangle = \{\pm 1\}$, and hence $\Omega(G)^\times$ is embedded in $\prod_{H\in \mathrm{C}(G)} \langle -1\rangle$. In particular, $\Omega(G)^\times$ is an elementary abelian 2-group with identity $[G/G]$ (cf. [11, Proposition 3.1]). Thus $\Omega(G)^\times$ consists of all $x \in \Omega(G)$ such that $([G/G] \pm x)/2$ are idempotents of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$.

Example 6.1 Suppose that $K \leq G$ and $|G : K| = 2$. Then $[G/K] \cdot [G/K] = 2[G/K]$, and hence $[G/G] - [G/K] \in \Omega(G)^\times$. We have $\phi^\times([G/G] - [G/K]) = ((-1)^{\zeta(H,K)})_{H \in \mathrm{C}(G)}$, where $\zeta(H, K) = 1$ if $H \leq K$, and $\zeta(H, K) = 0$ otherwise.

Remark 6.2 According to Dress [10], G is solvable if and only if 0 and $[G/G]$ are the only idempotents of $\Omega(G)$ (see also Lemma 5.17 and Theorem 5.18). Suppose that G is of odd order. Then by Eq.(5.4), $\Omega(G)^\times$ consists of all $x \in \Omega(G)$ such that $([G/G] \pm x)/2$ are idempotents of $\Omega(G)$, whence $|\Omega(G)^\times|$ is the number of idempotents of $\Omega(G)$. Consequently, we have $\Omega(G)^\times = \langle -[G/G] \rangle$ because, by Feit-Thompson's theorem, G is solvable (cf. [9, Proposition 1.5.1]).

Definition 6.3 Given $\widetilde{x} = (x_H)_{H \in \mathrm{C}(G)} \in \mho(G)^\times$ and $U \leq G$, we define a class function $\gamma_U^{\widetilde{x}} : W_G(U) \to \langle -1 \rangle$ by

$$gU \mapsto x_U x_{\langle g \rangle U}$$

for all $g \in N_G(U)$, where $x_{\langle g \rangle U} = x_K$ for a conjugate $K \in \mathrm{C}(G)$ of $\langle g \rangle U$ in G.

We quote [32, Proposition 6.5] which is due to Yoshida.

Theorem 6.4 (The Yoshida criterion) *The subgroup* $\mathrm{Im}\phi^\times$ *of* $\mho(G)^\times$ *consists of all* $\widetilde{x} = (x_H)_{H \in \mathrm{C}(G)} \in \mho(G)^\times$ *such that* $\gamma_U^{\widetilde{x}} \in \mathrm{Hom}(W_G(U), \langle -1 \rangle)$ *for each* $U \leq G$.

Example 6.5 Let p be an odd prime, and suppose that G is a finite p-group. Let $\widetilde{x} = (x_H)_{H \in \mathrm{C}(G)} \in \mathrm{Im}\phi^\times$. If $G_r < \cdots < G_1 < G_0 = G$ is a sequence of subgroups of G with $|G_{i-1} : G_i| = p$ for all $i \in [r]$, then by Theorem 6.4, $x_{G_0} = x_{G_1} = \cdots = x_{G_r}$. Thus it follows from [26, Chapter 2, Theorem 1.9] that $\widetilde{x}$ is determined by x_G, and hence $\widetilde{x} \in \langle (-1, -1, \ldots, -1) \rangle$. Consequently, $\Omega(G)^\times = \langle -[G/G] \rangle$ (see Remark 6.2).

Definition 6.6 For each $\widetilde{x} \in \mathrm{Im}\phi$, $\phi^{-1}(\widetilde{x})$ denotes the unique element x of $\Omega(G)$ such that $\widetilde{x} = \phi(x)$. We define a subgroup $\Omega(G)_0^\times$ of $\Omega(G)^\times$ to be the product of the subgroups $\langle [G/K] - [G/G] \rangle$ for $K \leq G$ with $|G : K| = 2$, and define a subgroup $\Omega(G)_1^\times$ of $\Omega(G)^\times$ to be the group consisting of all $x = \phi^{-1}((x_H)_{H \in \mathrm{C}(G)})$ with $(x_H)_{H \in \mathrm{C}(G)} \in \mathrm{Im}\phi^\times$ such that $x_H = 1$ whenever H is cyclic.

The group $\mathrm{Hom}(G, \langle -1 \rangle)$ with pointwise product is isomorphic to the factor group G/G_2 where G_2 is the intersection of all subgroups of index 2 in G

Proposition 6.7 (a) $|\langle -[G/G] \rangle \times \Omega(G)_0^\times| = 2^{|\mathrm{Hom}(G, \langle -1 \rangle)|}$.

(b) $\Omega(G)^\times = \langle -[G/G] \rangle \times \Omega(G)_0^\times \Omega(G)_1^\times \simeq \langle -[G/G] \rangle \times \mathrm{Hom}(G, \langle -1 \rangle) \times \Omega(G)_1^\times$.

Proof. Obviously, $\Omega(G)_0^\times$ is the direct product of the subgroups $\langle [G/K] - [G/G] \rangle$ for $K \leq G$ with $|G : K| = 2$. Thus the assertion (a) holds. We prove the assertion (b). For each $K \leq G$ with $|G : K| = 2$, if $\phi([G/K] - [G/G]) = \widetilde{x} = (x_H)_{H \in \mathrm{C}(G)}$, then by Example 6.1 and Theorem 6.4, $\gamma_{(K)} := \gamma^{\widetilde{x}}_{\{\epsilon\}} \in \mathrm{Hom}(G, \langle -1\rangle)$, $\mathrm{Ker}\,\gamma_{(K)} = K$, and $\gamma_{(K)}(g) = x_{\langle g\rangle}$ for all $g \in G$. Let $y \in \Omega(G)^\times$, and suppose that the $\{\epsilon\}$-component of $\phi(y)$ is 1. If $\phi^\times(y) = \widetilde{y} = (y_H)_{H\in \mathrm{C}(G)}$ with $y_{\{\epsilon\}} = 1$, then $\gamma^{\widetilde{y}}_{\{\epsilon\}} \in \mathrm{Hom}(G, \langle -1\rangle)$ by Theorem 6.4, and $\gamma^{\widetilde{y}}_{\{\epsilon\}}(g) = y_{\langle g\rangle}$ for all $g \in G$. This, combined with the preceding argument, shows that $([G/K]-[G/G])\cdot y \in \Omega(G)^\times_1$ with $K = \mathrm{Ker}\,\gamma^{\widetilde{y}}_{\{\epsilon\}}$, which yields $y \in \Omega(G)^\times_0\Omega(G)^\times_1$. Hence $\Omega(G)^\times_0\Omega(G)^\times_1$ consists of all $x \in \Omega(G)^\times$ such that the $\{\epsilon\}$-component of $\phi(x)$ is 1. We now obtain

$$\Omega(G)^\times = \langle -[G/G]\rangle \times \Omega(G)^\times_0\Omega(G)^\times_1.$$

Let $K_1, K_2, \ldots, K_n$ be the subgroups of index 2 in G. Then $\Omega(G)^\times_0$ is the direct product of the subgroups $\langle [G/K_i] - [G/G]\rangle$ for $i \in [n]$ and $\mathrm{Hom}(G, \langle -1\rangle)$ is the group consisting of 1_G and the linear $\mathbb{C}$-characters $\gamma_{(K_i)}$ for $i \in [n]$. Define a group epimorphism $\gamma : \Omega(G)^\times_0 \to \mathrm{Hom}(G, \langle -1\rangle)$ by

$$\prod_{j=1}^{m}([G/K_{i_j}] - [G/G]) \mapsto \prod_{j=1}^{m}\gamma_{(K_{i_j})}$$

for all sequences $(i_1, i_2, \ldots, i_m)$ with $1 \le i_1 < i_2 < \cdots < i_m \le n$ of natural numbers. Then it is obvious that $\mathrm{Ker}\,\gamma = \Omega(G)^\times_0 \cap \Omega(G)^\times_1$. Consequently, we have

$$\Omega(G)^\times_0\Omega(G)^\times_1 \simeq \mathrm{Hom}(G, \langle -1\rangle) \times \Omega(G)^\times_1,$$

completing the proof. □

Proposition 6.8 *Let $\widehat{\mathrm{C}}(G)$ be the set of all $U \in \mathrm{C}(G)$ such that $|N_G(U) : U| \le 2$. For any $\widetilde{x} = (x_H)_{H\in \mathrm{C}(G)} \in \mathrm{Im}\phi^\times$, the values x_H for $H \in \mathrm{C}(G)$ are determined by the values x_U for $U \in \widehat{\mathrm{C}}(G)$. In particular, $|\Omega(G)^\times| \le 2^{|\widehat{\mathrm{C}}(G)|}$.*

Proof. Let $\widetilde{x} = (x_H)_{H\in \mathrm{C}(G)} \in \mathrm{Im}\phi^\times$, and let $H \le G$. By Theorem 6.4, we have

$$x_{\langle g_1\rangle H}x_{\langle g_2\rangle H}x_H = x_{\langle g_1 g_2\rangle H}$$

for all $g_1, g_2 \in N_G(H)$. Hence, if $|N_G(H) : H| > 2$, then the value x_H is determined by the values x_K with $H < K \le N_G(H)$ (cf. [7, p. 904]). This completes the proof. □

Example 6.9 Assume that G is abelian. Then by Propositions 6.7 and 6.8, we have $|\Omega(G)^\times| = 2^{|\mathrm{Hom}(G,\langle -1\rangle)|}$, because $\widehat{\mathrm{C}}(G)$ is the set of all $K \le G$ such that $|G : K| \le 2$ (cf. [32, Lemma 7.1]). This fact is due to Matsuda (cf. [18, Example 4.5]).

6B Structure of the unit groups of Burnside rings

We continue to discuss the structure of $\Omega(G)^\times$.

Definition 6.10 We define a subset $\overline{\mathrm{C}}(G)$ of $\mathrm{C}(G)$ to be the set consisting of all subgroups U which satisfy the following conditions.

(i) $|N_G(U) : U| \le 2$.

(ii) If L is a normal subgroup of U and if U/L is a non-trivial cyclic group, then U/L is a cyclic 2-group and there exists a subgroup K of index 2 in $N_G(L)$ containing L such that

$$\{\langle g\rangle L \mid g \in N_G(L) - K\} = \{\langle g\rangle L \mid g \in N_G(L) \text{ and } (\langle g\rangle L) = (U)\}.$$

Proposition 6.11 *Let $U \in \mathrm{C}(G)$, and set $\widetilde{x} = ((-1)^{\delta_{U\,H}})_{H\in\mathrm{C}(G)} \in \mho(G)^\times$. Then $\widetilde{x} \in \mathrm{Im}\,\phi^\times$ if and only if $U \in \overline{\mathrm{C}}(G)$. In particular, if $U \in \overline{\mathrm{C}}(G)$, then $2e_U \in \Omega(G)$, or equivalently, $[G/G] - 2e_U = \phi^{-1}(\widetilde{x}) \in \Omega(G)^\times$.*

Proof. Assume that $\widetilde{x} \in \mathrm{Im}\,\phi^\times$. For any $L \le G$, it follows from Theorem 6.4 that the map $\gamma_L^{\widetilde{x}} : W_G(L) \to \langle -1\rangle$ is a linear $\mathbb{C}$-character of $W_G(L)$. Moreover, by assumption, $\gamma_U^{\widetilde{x}}(gU) = -1$ for any $g \in N_G(U) - U$. This means that $\mathrm{Ker}\,\gamma_U^{\widetilde{x}} = U/U$. Consequently, $|N_G(U) : U| \le 2$. Let L be a normal subgroup of U, and suppose that U/L is non-trivial cyclic. Set $U = \langle r\rangle L$ with $r \in N_G(L) - L$. Then for any $g \in N_G(L)$, $\gamma_L^{\widetilde{x}}(gL) = -1$ if and only if $\langle g\rangle L$ is a conjugate of $\langle r\rangle L$ in G. In particular, rL must be a 2-element of $W_G(L)$, whence U/L is a cyclic 2-group. Moreover, there exists a subgroup K of index 2 in $N_G(L)$ containing L such that $K/L = \mathrm{Ker}\,\gamma_L^{\widetilde{x}}$ and

$$\{\langle g\rangle L \mid g \in N_G(L) - K\} = \{\langle g\rangle L \mid g \in N_G(L) \text{ and } (\langle g\rangle L) = (U)\}.$$

Thus $U \in \overline{\mathrm{C}}(G)$, as required. Conversely, if $U \in \overline{\mathrm{C}}(G)$, then by Theorem 6.4, we have $\widetilde{x} \in \mathrm{Im}\,\phi^\times$, completing the proof. □

Remark 6.12 Under the hypotheses of Proposition 6.11, it follows from Eq.(5.4) that $\widetilde{x} \in \mathrm{Im}\,\phi^\times$ if and only if $[G/G] - 2e_U \in \Omega(G)^\times$.

Corollary 6.13 *Let $U \in \overline{\mathrm{C}}(G)$, and suppose that U is non-trivial cyclic. Then U is a Sylow 2-subgroup of G, and $N_G(U) = U$.*

Proof. Set $\widetilde{x} = ((-1)^{\delta_{U\,H}})_{H\in\mathrm{C}(G)} \in \mho(G)^\times$. By Theorem 6.4 and Proposition 6.11, the map $\gamma_{\{\epsilon\}}^{\widetilde{x}} : G \to \langle -1\rangle$ is a linear $\mathbb{C}$-character of G. Since U is non-trivial cyclic, it follows that $\gamma_{\{\epsilon\}}^{\widetilde{x}}$ is not the trivial character of G. If $K = \mathrm{Ker}\,\gamma_{\{\epsilon\}}^{\widetilde{x}}$, then any cyclic subgroup $\langle g\rangle$ with $g \in G - K$ is a conjugate of U in G and

$$\frac{|G|}{2} = |K| = |G - K| = |G : N_G(U)| \cdot \frac{|U|}{2} = \frac{|G|}{2|N_G(U) : U|}$$

because U is a 2-group. Thus we have $|N_G(U) : U| = 1$. The corollary is now a consequence of [26, Chapter 2, Theorem 1.6]. This completes the proof. □

Let $\lambda = (\lambda_1, \ldots, \lambda_j, \ldots, \lambda_m, \lambda_{m+1}, \ldots)$, where $\lambda_1 > \cdots > \lambda_j > \cdots > \lambda_m > 0$ and $\lambda_\ell = 0$ for $\ell = m+1, m+2, \cdots$, be a partition of $n \in \mathbb{N}$. Such a partition is said to be strict. We set $S_\lambda = S_{(\lambda_1)} \times \cdots \times S_{(\lambda_j)} \times \cdots \times S_{(\lambda_m)}$, where each $S_{(\lambda_j)}$ is the symmetric group on $\{\sum_{i \geq j+1} \lambda_i + 1, \ldots, \sum_{i \geq j} \lambda_i\}$. Let S_n be the symmetric group on $[n]$. Then S_λ is a Young subgroup of S_n associated with the strict partition λ.

Proposition 6.14 *For any strict partition λ of n, the set $\overline{\mathrm{C}}(S_n)$ contains a conjugate of the Young subgroup S_λ of S_n associated with λ.*

Proof. We may assume that $S_\lambda \in \mathrm{C}(S_n)$. Obviously, $N_{S_n}(S_\lambda) = S_\lambda$. We show that $S_\lambda \in \overline{\mathrm{C}}(S_n)$. Under the preceding notation, let $A_{(\lambda_j)}$ with $j \in [m]$ be the subgroup of $S_{(\lambda_j)}$ consisting of all even permutations. Then the commutator subgroup of S_λ is $A_{(\lambda_1)} \times \cdots \times A_{(\lambda_j)} \times \cdots \times A_{(\lambda_m)}$. Hence every normal subgroup L of S_λ such that S_λ/L is non-trivial cyclic is a subgroup of index 2 in S_λ. If $N_{S_n}(L) = S_\lambda$ for a subgroup L of index 2 in S_λ, then $\langle g \rangle L = S_\lambda$ for any $g \in N_{S_n}(L) - L$. Thus it suffices to verify that, if $N_{S_n}(L) \neq S_\lambda$ for a subgroup L of index 2 in S_λ, then

$$\{\langle g \rangle L \mid g \in N_{S_n}(L) - K\} = \{\langle g \rangle L \mid g \in N_{S_n}(L) \text{ and } (\langle g \rangle L) = (S_\lambda)\}$$

for a subgroup K of index 2 in $N_{S_n}(L)$ containing L. Let $L \leq S_\lambda$ with $|S_\lambda : L| = 2$ and $N_{S_n}(L) \neq S_\lambda$. Then $\lambda_{m-1} = 2$, $\lambda_m = 1$, and every permutation in L fixes both $2 \in [n]$ and $3 \in [n]$. (In this case, $S_{(\lambda_{m-1})}$ is the symmetric group on $\{2, 3\}$). Hence it turns out that $L = S_{(\lambda_1)} \times \cdots \times S_{(\lambda_j)} \times \cdots \times S_{(\lambda_{m-2})}$, $S_\lambda = L \times S_{(\lambda_{m-1})} \times S_{(\lambda_m)}$, and $N_{S_n}(L) = L \times S_3$. Consequently, $L \leq L \times A_3 \leq N_{S_n}(L)$, $|N_{S_n}(L) : L \times A_3| = 2$, $(\langle g \rangle L) \neq (S_\lambda)$ for any $g \in L \times A_3$, where A_3 is the alternating group on [3], and the set of conjugates of S_λ in S_n includes the set $\{\langle g \rangle L \mid g \in N_{S_n}(L) - (L \times A_3)\}$, as required. We now conclude that $S_\lambda \in \overline{\mathrm{C}}(S_n)$, completing the proof. □

Definition 6.15 For each $L \leq G$, we define a subset $\mathrm{S}(G; L)$ of $\mathrm{S}(G)$ to be the set consisting of all subgroups U of $N_G(L)$ which satisfy the following conditions.

(i) U/L is a non-trivial cyclic 2-group.

(ii) There exists a subgroup K of index 2 in $N_G(L)$ containing L such that

$$\{\langle g \rangle L \mid g \in N_G(L) - K\} = \{\langle g \rangle L \mid g \in N_G(L) \text{ and } (\langle g \rangle L) = (U)\}.$$

Let $\approx$ be the equivalence relation on the set $\{(H) \mid G \geq H \neq \{\epsilon\}\}$ generated by

$$(\langle g \rangle L) \approx (L)$$

for $L \in \mathrm{C}(G)$ and $g \in N_G(L)$ such that $\langle g \rangle L \notin \mathrm{S}(G; L)$. We set $\mathrm{C}(G)^\circ = \mathrm{C}(G) - \{\epsilon\}$, and define an equivalence relation $\approx$ on $\mathrm{C}(G)^\circ$ by

$$H \approx K \;:\Longleftrightarrow\; (H) \approx (K).$$

Proposition 6.16 *If $|G| > 2$, then each $U \in \overline{\mathrm{C}}(G)$ forms an equivalence class consisting of a single element with respect to the equivalence relation $\approx$ on $\mathrm{C}(G)^\circ$.*

Proof. Suppose that $|G| > 2$, and let $U \in \overline{\mathrm{C}}(G)$. Then $U \neq \{\epsilon\}$ and $|N_G(U) : U| \leq 2$. If $N_G(U) \neq U$, then $|N_G(U) : U| = 2$ and $N_G(U) \in \mathrm{S}(G; U)$. Moreover, if L is a normal subgroup of U and if U/L is a non-trivial cyclic group, then $U \in \mathrm{S}(G; L)$. Thus (U) is isolated with respect to $\approx$. This completes the proof. □

Proposition 6.17 *Suppose that $\widetilde{y} = (y_H)_{H \in \mathrm{C}(G)} \in \mathrm{Im}\phi^\times$ and $\phi^{-1}(\widetilde{y}) \in \Omega(G)_1^\times$. Let $U \in \mathrm{C}(G)^\circ$, and define $\widetilde{x} = (x_H)_{H \in \mathrm{C}(G)} \in \mho(G)^\times$ by*

$$x_H = \begin{cases} y_H & \text{if } H \approx U, \\ 1 & \text{if } H \not\approx U \text{ or } H = \{\epsilon\}. \end{cases}$$

Then $\widetilde{x} \in \mathrm{Im}\phi^\times$ and $\phi^{-1}(\widetilde{x}) \in \Omega(G)_1^\times$.

Proof. By the definition of $\widetilde{x}$, the map $\gamma_{\{\epsilon\}}^{\widetilde{x}} : G \to \langle -1 \rangle$ is the trivial character of G. Hence it suffices to verify that $\widetilde{x} \in \mathrm{Im}\phi^\times$. Let $L \in \mathrm{C}(G)^\circ$. We show that the map $\gamma_L^{\widetilde{x}} : W_G(L) \to \langle -1 \rangle$ is a linear $\mathbb{C}$-character of $W_G(L)$. By Theorem 6.4, the map $\gamma_L^{\widetilde{y}} : W_G(L) \to \langle -1 \rangle$ is a linear $\mathbb{C}$-character of $W_G(L)$. We may assume that $\gamma_L^{\widetilde{x}} \notin \{\gamma_L^{\widetilde{y}}, 1_{W_G(L)}\}$. (If $\langle g \rangle L \notin \mathrm{S}(G; L)$ for all $g \in N_G(L) - L$, then either $\gamma_L^{\widetilde{x}} = \gamma_L^{\widetilde{y}}$ or $\gamma_L^{\widetilde{x}} = 1_{W_G(L)}$.) Obviously, $\gamma_L^{\widetilde{x}}(L) = 1$. We analysis the values $\gamma_L^{\widetilde{x}}(\langle g \rangle L)$ for $g \in N_G(L) - L$ in each of the cases where $L \approx U$ and $L \not\approx U$. Let r be any element of $N_G(L) - L$ such that $\langle r \rangle L \in \mathrm{S}(G; L)$. Then there exist a subgroup K of index 2 in $N_G(L)$ containing L such that for each $g \in N_G(L)$, $g \in N_G(L) - K$ if and only if $\langle g \rangle L$ is a conjugate of $\langle r \rangle L$ in G. We define a map $\beta_r : W_G(L) \to \langle -1 \rangle$ to be the linear $\mathbb{C}$-character of $W_G(L)$ whose kernel is K/L.

Case 1. Assume that $L \approx U$. Let $\mathcal{X} = \{\langle r_i \rangle L \mid i \in [\ell]\}$ be a full set of non-conjugate subgroups of G chosen from among the subgroups $\langle g \rangle L$ for $g \in N_G(L) - L$ with $\gamma_L^{\widetilde{x}}(gL) \neq \gamma_L^{\widetilde{y}}(gL)$. Then we have $(\langle r_i \rangle L) \not\approx (L)$ and $\langle r_i \rangle L \in \mathrm{S}(G; L)$ for all $i \in [\ell]$. For any $g \in N_G(L) - L$,

$$\gamma_L^{\widetilde{x}}(gL) = -\gamma_L^{\widetilde{y}}(gL) = \gamma_L^{\widetilde{y}}(gL) \prod_{i=1}^{\ell} \beta_{r_i}(gL)$$

if $\langle g \rangle L$ is a conjugate of some $\langle r_j \rangle L$ with $j \in [\ell]$ in G, and

$$\gamma_L^{\widetilde{x}}(gL) = \gamma_L^{\widetilde{y}}(gL) = \gamma_L^{\widetilde{y}}(gL) \prod_{i=1}^{\ell} \beta_{r_i}(gL)$$

otherwise. Thus we have

$$\gamma_L^{\widetilde{x}} = \gamma_L^{\widetilde{y}} \prod_{i=1}^{\ell} \beta_{r_i}.$$

Case 2. Assume that $L \not\approx U$. Then $x_L = 1$. Let $\mathcal{Y} = \{\langle r_i\rangle L \mid i \in [\ell]\}$ be a full set of non-conjugate subgroups of G chosen from among the subgroups $\langle g\rangle L$ for $g \in N_G(L) - L$ with $\gamma_L^{\widetilde{x}}(gL) \neq 1$. Then $(\langle r_i\rangle L) \approx (U)$, whence $(\langle r_i\rangle L) \not\approx (L)$ and $\langle r_i\rangle L \in \mathrm{S}(G; L)$ for all $i \in [\ell]$. By an argument analogous to that in Case 1, we have

$$\gamma_L^{\widetilde{x}} = \prod_{i=1}^{\ell} \beta_{r_i}.$$

We now conclude that the map $\gamma_L^{\widetilde{x}} : W_G(L) \to \langle -1\rangle$ is a linear $\mathbb{C}$-character of $W_G(L)$ in either case. Consequently, $\gamma_L^{\widetilde{x}} \in \mathrm{Hom}(W_G(L), \langle -1\rangle)$ for any $L \leq G$. This, combined with Theorem 6.4, shows that $\widetilde{x} \in \mathrm{Im}\,\phi^\times$, completing the proof. □

Corollary 6.18 *Let* $\mathrm{C}(G)^\circ/\approx$ *be a complete set of representatives of equivalence classes with respect to the equivalence relation* $\approx$ *on* $\mathrm{C}(G)^\circ$*. Set*

$$\Omega(G)_U^\times = \{\phi^{-1}(\widetilde{x}) \mid \widetilde{x} = (x_H)_{H\in \mathrm{C}(G)} \in \mathrm{Im}\,\phi^\times \text{ and } x_H = 1 \text{ if } H \not\approx U \text{ or } H = \{\epsilon\}\}$$

for each $U \in \mathrm{C}(G)^\circ/\approx$*. Then*

$$\Omega(G)_1^\times = \prod_{U\in \mathrm{C}(G)^\circ/\approx} (\Omega(G)_U^\times \cap \Omega(G)_1^\times).$$

Moreover, if $U \in \overline{\mathrm{C}}(G) \cap \mathrm{C}(G)^\circ$*, then* $U \in \mathrm{C}(G)^\circ/\approx$ *and* $\Omega(G)_U^\times = \langle [G/G] - 2e_U\rangle$*.*

Proof. The assertion follows from Propositions 6.11, 6.16, and 6.17. □

7 Units of monomial Burnside rings

7A The unit groups of monomial Burnside rings

We continue assuming that A is abelian. Given a commutative unital ring R, we denote by $R^\times$ the unit group of R, and denote by R^ω the group of torsion units of R. For each $H \leq G$, since $H^1(H, A)$ is a finite abelian group, it follows from [21, Theorem 8.3.1] that $(\mathbb{Z}H^1(H, A))^\times$ is a finitely generated abelian group.

Lemma 7.1 *The group* $\mho(G, A)^\times$ *is a finitely generated abelian group.*

Proof. Observe that $\mho(G, A) \simeq \prod_{H\in \mathrm{C}(G)} (\mathbb{Z}H^1(H, A))^{N_G(H)}$, where

$$(\mathbb{Z}H^1(H, A))^{N_G(H)} = \{x_H \in \mathbb{Z}H^1(H, A) \mid \mathrm{con}_H^g(x_H) = x_H \text{ for all } g \in N_G(H)\}.$$

Then we have $\mho(G, A)^\times \simeq \prod_{H\in \mathrm{C}(G)} J_H$, where

$$J_H = (\mathbb{Z}H^1(H, A))^\times \cap (\mathbb{Z}H^1(H, A))^{N_G(H)}.$$

Hence it suffices to verify that the groups J_H for $H \leq G$ are finitely generated. Let $H \leq G$, and assume that $(\mathbb{Z}H^1(H,A))^\times$ is generated by $x_1, \ldots, x_k$. We set $y_i = \prod_{g \in N_G(H)} \mathrm{con}_H^g(x_i)$ for all i, and set $\widehat{J}_H = \langle y_1, \ldots, y_k \rangle$. Obviously, $\widehat{J}_H$ is a subgroup of J_H. We have

$$x^{|N_G(H)|} = \prod_{g \in N_G(H)} \mathrm{con}_H^g(x) \in \widehat{J}_H$$

for any $x \in J_H$, so that $J_H/\widehat{J}_H$ is a torsion subgroup of $(\mathbb{Z}H^1(H,A))^\times/\widehat{J}_H$. Since $(\mathbb{Z}H^1(H,A))^\times/\widehat{J}_H$ is finitely generated, it follows from the fundamental theorem of abelian groups (see, *e.g.*, [16, I, §10, Theorem 8]) that $J_H/\widehat{J}_H$ is a finite group. Thus J_H is finitely generated, as desired. This completes the proof. □

Proposition 7.2 *The group $\Omega(G,A)^\times$ is a finitely generated abelian group. In particular, $\Omega(G,A)^\times$ is the direct product of $\Omega(G,A)^\omega$ and a free abelian group of finite rank, and $\Omega(G,A)^\omega$ is a finite abelian group.*

Proof. By the fundamental theorem of abelian groups, it suffices to prove the first statement. Using Proposition 5.2 and Corollary 5.5, we have

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Im}\rho = \mathbb{Q} \otimes_{\mathbb{Z}} \mho(G,A).$$

This, combined with [21, Lemma 2.9.5], shows that $|\mho(G,A)^\times : (\mathrm{Im}\rho)^\times|$ is finite. Moreover, by Lemma 7.1, $\mho(G,A)^\times$ is finitely generated. Hence it follows from [17, Corollary 2.7.1] that $(\mathrm{Im}\rho)^\times$ is finitely generated. By Corollary 4.10, we have $\Omega(G,A)^\times \simeq (\mathrm{Im}\rho)^\times$, completing the proof. □

7B Torsion units of monomial Burnside rings

From Higman's theorem (cf. [21, Theorem 7.1.4]), we know that for any $H \leq G$,

$$(\mathbb{Z}H^1(H,A))^\omega = \langle -1 \rangle \times H^1(H,A) = \{\pm\overline{\sigma} \mid \sigma \in Z^1(H,A)\}. \tag{7.1}$$

Theorem 7.3 *The necessary and sufficient condition for an element $\widetilde{x} = (x_H)_{H \leq G}$ of $\mho(G,A)^\omega$ to be contained in $\mathrm{Im}\rho$ is that $\gamma_U^{\widetilde{\alpha}(\widetilde{x})} \in \mathrm{Hom}(W_G(U), \langle -1 \rangle)$ for all $U \leq G$ and $(\varepsilon(x_H)x_H)_{H \leq G} \in \Upsilon(G,A)$ (see Definitions 5.12 and 6.3),* where

$$\Upsilon(G,A) = \left\{ (\overline{\sigma_H})_{H \leq G} \in \mho(G,A)^\omega \;\middle|\; \begin{array}{r} \sigma_U \in Z^1(U,A) \text{ and } \overline{\sigma_U} = \mathrm{res}_U^{\langle g \rangle U}(\overline{\sigma_{\langle g \rangle U}}) \\ \text{for all } U \leq G \text{ and } g \in N_G(U) \end{array} \right\}.$$

Proof. Let $\widetilde{x} = (x_H)_{H \leq G} \in \mho(G,A)^\omega$. Suppose that for each $H \leq G$, $\overline{\sigma_H} = \varepsilon(x_H)x_H$ with $\sigma_H \in Z^1(H,A)$ (see Eq.(7.1)). We first prove 'sufficient' part. By assumption, $\mathrm{con}_U^g(\overline{\sigma_U}) = \overline{\sigma_U}$ and $\overline{\sigma_U} = \mathrm{res}_U^{\langle g \rangle U}(\overline{\sigma_{\langle g \rangle U}})$ for all $U \leq G$ and $g \in N_G(U)$, so that

$$\psi \circ \kappa^{-1}(\widetilde{x}) = (z_{(U,\tau)} \bmod |W_G(U,\tau)|)_{(U,\tau) \in \mathcal{R}(G,A)},$$

where

$$z_{(U,\tau)} = \begin{cases} \displaystyle\sum_{gU \in W_G(U)} \varepsilon(x_{\langle g \rangle U}) & \text{if } \overline{\tau} = \overline{\sigma_U}, \\ 0 & \text{otherwise.} \end{cases}$$

For any $U \leq G$, since $\gamma_U^{\widetilde{\alpha}(\widetilde{x})} \in \mathrm{Hom}(W_G(U), \langle -1 \rangle)$, we have

$$\frac{1}{|W_G(U)|} \sum_{gU \in W_G(U)} \varepsilon(x_U)\varepsilon(x_{\langle g \rangle U}) \in \{0,\, 1\}$$

by [8, (9.21) Proposition]. Hence either $z_{(U,\tau)} = \varepsilon(x_U)|W_G(U)|$ or $z_{(U,\tau)} = 0$ for all $(U, \tau) \in \mathcal{R}(G, A)$, which yields $\psi \circ \kappa^{-1}(\widetilde{x}) = 0 \in \mathrm{Obs}\,(G, A)$. This, combined with Proposition 5.2 and Theorem 5.9, shows that $\widetilde{x} \in \mathrm{Im}\,\rho$, as desired. We next prove 'necessary' part. Assume that $\rho(x) = \widetilde{x}$ with $x \in \Omega(G, A)^\omega$. Then $\alpha(x) \in \Omega(G)^\times$, because the map $\alpha : \Omega(G, A) \to \Omega(G)$ is a ring homomorphism. By Lemma 5.13(a) and Theorem 6.4, $\gamma_U^{\widetilde{\alpha}(\widetilde{x})} \in \mathrm{Hom}(W_G(U), \langle -1 \rangle)$ for all $U \leq G$, and

$$\rho(\iota \circ \alpha(x) \cdot x) = (\overline{\sigma_H})_{H \leq G} \in \mho(G, A)^\omega.$$

In particular, we have $\mathrm{con}_U^g(\overline{\sigma_U}) = \overline{\sigma_U}$ for all $U \leq G$ and $g \in N_G(U)$. For each $(U, \tau) \in \mathcal{R}(G, A)$ with $\overline{\tau} = \overline{\sigma_U}$, the (U, τ)-component of $\psi \circ \kappa^{-1}((\overline{\sigma_H})_{H \leq G})$ is

$$\sum_{gU \in W_G(U),\, \overline{\sigma_U} = \mathrm{res}_U^{\langle g \rangle U}(\overline{\sigma_{\langle g \rangle U}})} 1 \bmod |W_G(U)|,$$

where the sum is taken over all left cosets gU, $g \in N_G(U)$, of U in $N_G(U)$ such that $\overline{\sigma_U} = \mathrm{res}_U^{\langle g \rangle U}(\overline{\sigma_{\langle g \rangle U}})$. Since $(\overline{\sigma_H})_{H \leq G} \in \mathrm{Im}\,\rho$, it follows from Proposition 5.2 and Theorem 5.9 that $\overline{\sigma_U} = \mathrm{res}_U^{\langle g \rangle U}(\overline{\sigma_{\langle g \rangle U}})$ for all $U \leq G$ and $g \in N_G(U)$, as desired. This completes the proof. □

In §4A, the ring epimorphism $\rho_G^G : \Omega(G, A) \to \mathbb{Z}H^1(G, A)$ is given by

$$[(G/U)_\tau] \mapsto \begin{cases} \overline{\tau} & \text{if } G = U, \\ 0 & \text{otherwise} \end{cases}$$

for all $(U, \tau) \in \mathcal{R}(G, A)$ (see Lemma 4.6(a)). Following [2, §7], we define a ring monomorphism $\upsilon : \mathbb{Z}H^1(G, A) \to \Omega(G, A)$ by

$$\overline{\chi} \mapsto [(G/G)_\chi]$$

for all $\chi \in Z^1(G, A)$ (see Lemmas 2.6 and 2.15). There are group homomorphisms

$$\upsilon^\omega : (\mathbb{Z}H^1(G, A))^\omega \to \Omega(G, A)^\omega \quad \text{and} \quad \theta^\omega : \Omega(G, A)^\omega \to (\mathbb{Z}H^1(G, A))^\omega$$

inherited from υ and ρ_G^G, respectively (see Eq.(7.1)). Hence it turns out that

$$\Omega(G, A)^\omega = \mathrm{Im}\,\upsilon^\omega \times \mathrm{Ker}\,\theta^\omega \simeq \langle -1 \rangle \times H^1(G, A) \times \mathrm{Ker}\,\theta^\omega$$

(cf. [2, §8]), because $\theta^\omega \circ \upsilon^\omega = \mathrm{id}_{(\mathbb{Z}H^1(G,A))^\omega}$. We continue to describe $\Omega(G, A)^\omega$.

Corollary 7.4 *Identify the finite groups* $\Omega(G)^\times$ *and* $H^1(G,A)$ *with the subgroups* $\{\iota(u) \mid u \in \Omega(G)^\times\}$ *and* $\{[(G/G)_\chi] \mid \chi \in Z^1(G,A)\}$ *of* $\Omega(G,A)^\omega$*, respectively. Set*

$$\nabla(G,A) = \left\{ \frac{1}{|G|} \sum_{H \leq G} \sum_{U \leq H} |U| \mu(U,H) [(G/U)_{\sigma_H|_U}] \;\middle|\; \begin{array}{l} (\overline{\sigma_H})_{H \leq G} \in \Upsilon(G,A) \\ \text{with } \sigma_G = 1_G \end{array} \right\}.$$

Then

$$\Omega(G,A)^\omega = \Omega(G)^\times \times H^1(G,A) \times \nabla(G,A).$$

Proof. Let $x \in \Omega(G,A)^\omega$, and suppose that $\rho(x) = (x_H)_{H \leq G}$. By Theorem 7.3, $\rho(\iota \circ \alpha(x) \cdot x) = (\varepsilon(x_H)x_H)_{H \leq G} \in \Upsilon(G,A)$. Since the map $\alpha : \Omega(G,A) \to \Omega(G)$ is a ring epimorphism, it follows from Proposition 4.9 and Theorem 7.3 that

$$\Omega(G,A)^\omega = \Omega(G)^\times \times \left\{ \frac{1}{|G|} \eta((\overline{\sigma_H})_{H \leq G}) \;\middle|\; (\overline{\sigma_H})_{H \leq G} \in \Upsilon(G,A) \right\}.$$

Moreover, $\rho([(G/G)_\chi]) = (\mathrm{res}^G_H(\overline{\chi}))_{H \leq G} \in \Upsilon(G,A)$ for all $\chi \in Z^1(G,A)$, and hence

$$\Upsilon(G,A) = \{\rho([(G/G)_\chi]) \mid \chi \in Z^1(G,A)\} \times \{(\overline{\sigma_H})_{H \leq G} \in \Upsilon(G,A) \mid \sigma_G = 1_G\}.$$

The assertion now follows from Proposition 4.9. This completes the proof. □

Remark 7.5 Suppose that G is of odd order and that G acts trivially on A. Then by Remark 6.2 and Corollary 7.4, we have

$$\Omega(G,A)^\omega = \langle -[(G/G)_{1_G}] \rangle \times \Omega(G,A)^{\mathrm{odd}},$$

where $\Omega(G,A)^{\mathrm{odd}}$ is the Hall $2'$-subgroup of $\Omega(G,A)^\omega$ (cf. [2, Proposition 8.2]).

Example 7.6 Suppose that G is nilpotent. Then by [26, Chapter 4, Theorem 2.9], $\nabla(G,A) = \langle [(G/G)_{1_G}] \rangle$ in Corollary 7.4, and hence

$$\Omega(G,A)^\omega \simeq \Omega(G)^\times \times H^1(G,A).$$

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