



## Rough-Set-Based Interrelationship Mining for Incomplete Decision Tables

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# Rough-set-based Interrelationship Mining for Incomplete Decision Tables

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**Abstract.** Rough-set-based interrelationship mining enables to extract characteristics by comparing the values of the same object between different attributes. To apply this interrelationship mining to incomplete decision tables with null values, in this study, we discuss the treatment of null values in interrelationships between attributes. We introduce three types of null values for interrelated condition attributes and formulate a similarity relation by such attributes with these null values.

**Keywords:** Interrelationship mining, Incomplete decision tables, Rough set, Similarity relation

## 1. Introduction

Rough set theory, originally proposed by Pawlak [12, 13], provides a mathematical basis for logical data analysis, and attribute reduction and decision rule extraction are currently useful tools for data mining [14]. Application of rough set theory to incomplete decision tables was first reported by Kryszkiewicz [11], who used null values to represent a lack of values of objects in some condition attributes and introduced similarity relations between objects rather than equivalence relations, i.e., null values weaken the discernibility of objects by comparing attribute values.

Rough-set-based interrelationship mining, proposed by previous studies [3–7, 10], enables to extract characteristics by interrelationships between attributes. We introduced the concept of interrelationships between attributes by comparing values of the same object between different attributes and formulated indiscernibility between objects by the interrelationship between attributes as the indiscernibility based on whether both objects support (or do not support) the interrelationship. We also introduced new attributes, called interrelated condition attributes, to explicitly represent the interrelationship between two condition attributes.

To apply rough-set-based interrelationship mining to

decision tables with null values, i.e., incomplete decision tables, we need to discuss how to treat null values in interrelationships between attributes. We introduced three types of null values for interrelated condition attributes and formulated the similarity relation by such attributes with these null values.

The reminder of this paper is organized as follows. Section 2 reviews rough set theory applied to incomplete decision tables, and Section 3 reviews rough-set-based interrelationship mining for complete decision tables. In Section 4, we introduce similarity relations by interrelationship between attributes in incomplete decision tables, and in Section 5, we describe interrelated condition attributes in such tables. Section 6 presents the conclusions of this study. Note that this paper is a revised and extended version of a previous paper [8].

## 2. Rough Sets for Imcomplete Decision Tables

In this section, we review rough sets for incomplete decision tables proposed by Kryszkiewicz [11].

### 2.1. Decision tables and similarity relations

The subjects of rough set data analysis are described by decision tables. According to the authors' previous study [5–7, 10], we use a general expression of decision tables used by Yao et al. [15], which can be expressed as

$$DT = (U, AT, \{V_a \mid a \in AT\}, \mathcal{R}_{AT}, \rho), \quad . \quad . \quad . \quad (1)$$

where  $U$  is a finite and nonempty set of objects;  $AT = C \cup \{d\}$  is a finite and nonempty set of attributes (where  $C$  is a set of condition attributes and  $d \notin C$  is a decision attribute);  $V_a$  is a nonempty set of values for  $a \in AT$ ;  $\mathcal{R}_{AT} = \{\{R_a\} \mid a \in AT\}$  is a set of sets  $\{R_a\}$  of binary relations defined on each  $V_a$ ; and  $\rho$  is an information function  $\rho : U \times AT \rightarrow V$  that presents a value  $\rho(x, a) \in V_a$  of each object  $x \in U$  at the attribute  $a \in AT$  (where  $V = \bigcup_{a \in AT} V_a$  is the set of values of all attributes in  $AT$ ).

The set  $\{R_a\}$  of binary relations for each attribute  $a \in AT$  can contain various binary relations: similarity, dissimilarity, the ordering relation on  $V_a$  and typical infor-

mation tables are implicitly assumed that the set  $\{R_a\}$  consists of only the equality relation  $=$  on  $V_a$  [15]. We also assume that the equality relation  $=$  is included in the set  $\{R_a\}$  for each attribute  $a \in AT$ .

A decision table  $DT$  is considered an incomplete decision table if there is at least one condition attribute  $a \in C$  for which  $V_a$  contains a null value [11]. We denote a null value by the  $*$  symbol and assume that the decision attribute  $d$  does not have null value, i.e.,  $*$   $\notin V_d$ . We also assume that, for any attribute  $a \in AT$  and any binary relation  $R \in \{R_a\}$  except for the equality relation  $=$ , if  $*$   $\in V_a$  holds, then the null value  $*$  does not relate to other values  $v \in V_a$  by the binary relation  $R$ , i.e., neither  $*Rv$  nor  $vR*$  holds.

A similarity relation  $SIM(A)$  between objects that are possibly indiscernible in terms of values of attributes  $A \subseteq AT$  is expressed as

$$SIM(A) = \left\{ (x, y) \mid \begin{array}{l} \forall a \in A, \rho(x, a) = \rho(y, a) \text{ or} \\ \rho(x, a) = * \text{ or } \rho(y, a) = * \end{array} \right\}. \quad (2)$$

If a pair  $(x, y)$  is in  $SIM(A)$ , then we say that object  $x$  is similar to object  $y$  in terms of the attribute values in  $A$ . It can be clearly observed that such similarity relation is a tolerance relation, i.e., it is reflexive and symmetric; however, it may not be transitive in general. Note that if all attributes  $a \in A$  do not have null value, the similarity relation  $SIM(A)$  is identical to the indiscernibility relation  $IND(A)$  [11]. Moreover, the following property holds for any similarity relation:

$$SIM(A) = \bigcap_{a \in A} SIM(\{a\}). \quad (3)$$

Let  $S_A(x) = \{y \in U \mid (x, y) \in SIM(A)\}$  be the set of similar objects with the object  $x$  by  $A \subseteq AT$ . The collection of sets  $S_A(x)$  for all  $x \in U$  comprises a covering of  $U$ , denoted  $U/SIM(A)$ . Each element in  $U/SIM(A)$  is called a tolerance class. In particular, the covering  $U/SIM(\{d\})$  generated by the decision attribute  $d$  becomes a partition of  $U$ , and each element in it is called a decision class.

For any set of objects  $X \subseteq U$ , the lower approximation  $\underline{A}(X)$  and upper approximation  $\overline{A}(X)$  of  $X$  by  $A \subseteq AT$  are defined in a manner similar to Pawlak's rough sets:

$$\underline{A}(X) = \{x \in U \mid S_A(x) \subseteq X\}, \quad (4)$$

$$\overline{A}(X) = \{x \in U \mid S_A(x) \cap X \neq \emptyset\}. \quad (5)$$

## 2.2. Relative reducts and decision rules

To describe relative reducts for incomplete decision tables, we introduce the generalized decision proposed by Kryszkiewicz [11].

Let  $\partial_A$ ,  $A \subseteq C$ , be a function  $\partial_A : U \rightarrow \mathcal{P}(V_d)$ , where  $\mathcal{P}(V_d)$  is the power set of  $V_d$ , defined as follows:

$$\partial_A(x) = \{v \in V_d \mid y \in S_A(x) \text{ and } \rho(y, d) = v\}. \quad (6)$$

The function  $\partial_A$  is called a generalized decision by  $A$ .

Relative reducts for incomplete decision tables are minimal sets of condition attributes that preserve the general-

ized decision  $\partial_C$  by the set of all condition attributes. Formally, a set  $A \subseteq C$  is called a relative reduct of  $DT$  if and only if  $A$  satisfies the following condition:

$$\partial_A = \partial_C \text{ and } \forall B \subset A, \partial_B \neq \partial_C. \quad (7)$$

We also introduce the relative reducts of  $DT$  for each object in  $U$ . A set  $A \subseteq C$  is called a relative reduct of  $DT$  for object  $x \in U$  if and only if  $A$  satisfies the following condition:

$$\partial_A(x) = \partial_C(x) \text{ and } \forall B \subset A, \partial_B(x) \neq \partial_C(x). \quad (8)$$

We denote a decision rule generated from a set  $A \subseteq C$  of condition attributes in the following form:

$$\bigwedge_{a \in A} (a, v) \rightarrow \bigvee (d, w), \quad (9)$$

where  $v \in V_a$  and  $w \in V_d$ . The form  $(a, v)$  is called a descriptor, and the forms  $\bigwedge_{a \in A} (a, v)$  and  $\bigvee (d, w)$  are called the condition and decision of the decision rule, respectively.

Let  $r$  be a decision rule defined by (9) with a set of condition attributes  $A \subseteq C$ ,  $X$  be a set of objects of property  $\bigwedge_{a \in A} (a, v)$ , and  $Y$  be a set of objects of property  $\bigvee (d, w)$ . The decision rule  $r$  is true in  $DT$  if and only if  $\overline{A}(X) \subseteq Y$  holds. Moreover,  $r$  is optimal in  $DT$  if and only if  $r$  is true in  $DT$  and no other rule  $r'$  constructed from the proper subset  $B \subset A$  is untrue [11]. Note that, similar to Pawlak's rough sets for complete decision tables, discernibility functions can also be used to compute relative reducts in incomplete decision tables [11].

*Example 1:* Table 1 shows an example incomplete decision table  $DT$ . The decision table consists of a set of eight users of sample products. Here  $U = \{u_1, \dots, u_8\}$  is the set of users, the set of attributes  $AT$  consists of the set of condition attributes  $C = \{\text{Gender}, Q.1, Q.2, Q.3\}$  that represents user gender and the answers to questions 1, 2, and 3 about the sample products, and the decision attribute Purchase that represents user answers to the question about purchase.

Each attribute  $a \in AT$  has the following range of values  $V_a$ . Note that each condition attribute in this example has null value  $*$  in the following range:

$$V_{\text{Gender}} = \{\text{female}, \text{male}, *\},$$

$$V_{Q.1} = \{\text{no}, \text{yes}, *\},$$

$$V_{Q.2} = V_{Q.3} = \{\text{v.b.}, \text{bad}, \text{normal}, \text{good}, \text{v.g.}, *\},$$

$$V_{\text{Purchase}} = \{\text{no}, \text{maybe}, \text{yes}\},$$

and each attribute  $a \in AT$  has the following set of binary relations on the set  $V_a$  of values:

$$\text{Gender} : \{=\}, Q.1 : \{=, \prec_{Q.1}, \preceq_{Q.1}\},$$

$$Q.2 : \{=, \prec_{Q.2}, \preceq_{Q.2}\}, Q.3 : \{=, \prec_{Q.3}, \preceq_{Q.3}\},$$

$$\text{Purchase} : \{=, \prec_{\text{Purchase}}, \preceq_{\text{Purchase}}\},$$

where each relation  $\prec_a$  is a preference relation that is defined as follows:

$$\prec_{Q.1} : \text{no} \prec \text{yes},$$

$$\prec_{Q.2} : \text{v.b.} \prec \text{bad} \prec \text{normal} \prec \text{good} \prec \text{v.g.},$$

$$\prec_{Q.3} : \text{v.b.} \prec \text{bad} \prec \text{normal} \prec \text{good} \prec \text{v.g.},$$

$$\prec_{\text{Purchase}} : \text{no} \prec \text{maybe} \prec \text{yes}.$$

The contents of Table 1 is described by the infor-

**Table 1.** Incomplete decision table

$U$	Gender	Q.1	Q.2	Q.3	Purchase
$u1$	female	yes	good	v. g.	yes
$u2$	male	no	good	v. g.	yes
$u3$	male	no	*	good	yes
$u4$	female	yes	normal	normal	yes
$u5$	female	*	*	*	maybe
$u6$	male	no	good	*	maybe
$u7$	male	no	*	bad	no
$u8$	*	yes	good	normal	no

mation function  $\rho$ ; e.g.,  $\rho(u1, \text{Gender}) = \text{female}$ , and  $\rho(u5, \text{Q.1}) = *$ .

Here, we consider the similarity relation and relative reduct of Table 1. The generalized decision  $\partial_C$  of Table 1 by the set of all condition attributes  $C \subseteq AT$  is defined as follows:

$$\partial_C(u1) = \partial_C(u2) = \partial_C(u3) = \partial_C(u4) \\ = \{\text{yes}, \text{maybe}\},$$

$$\partial_C(u5) = \partial_C(u6) \\ = \{\text{yes}, \text{maybe}, \text{no}\},$$

$$\partial_C(u7) = \partial_C(u8) \\ = \{\text{maybe}, \text{no}\}.$$

Let  $A = \{\text{Q.1}, \text{Q.2}, \text{Q.3}\}$  be a subset of condition attributes. For each object  $u \in U$ , the set of similar objects  $S_A(u)$  with respect to the similarity relation  $\text{SIM}(A)$  by (2) is as follows:

$$\begin{aligned} S_A(u1) &= \{u1, u5\}, & S_A(u2) &= \{u2, u5, u6\}, \\ S_A(u3) &= \{u3, u5, u6\}, & S_A(u4) &= \{u4, u5\}, \\ S_A(u5) &= U, & S_A(u6) &= \{u2, u3, u5, u6, u7\}, \\ S_A(u7) &= \{u5, u6, u7\}, & S_A(u8) &= \{u5, u8\}. \end{aligned}$$

For example,  $u5, u6 \in S_A(u3)$  means that the object  $u3$  is indiscernible from the objects  $u5$  and  $u6$  by the values of attributes in  $A$ , which confirms that the generalized decision  $\partial_A(u3)$  is  $\{\text{yes}, \text{maybe}\}$ . Consequently, it is clearly observed that  $A = \{\text{Q.1}, \text{Q.2}, \text{Q.3}\}$  is a relative reduct of Table 1, i.e.,  $\partial_A = \partial_C$  and  $\forall B \subset A, \partial_B \neq \partial_C$  holds.

Similarly, we can also consider a relative reduct of some objects and decision rules. Let  $X = \{\text{Q.3}\}$ . It can be clearly observed that  $S_X(u3) = \{u3, u5, u6\}$  and it provides  $\partial_X(u3) = \{\text{yes}, \text{maybe}\} = \partial_C(u3)$ , i.e.,  $X$  is a relative reduct of  $u3$  in Table 1. By using the relative reduct  $X$  and the generalized decision of  $u3$ , we obtain the following optimal decision rule:

$$(\text{Q.3}, \text{good}) \rightarrow (\text{Purchase}, \text{yes}) \vee (\text{Purchase}, \text{maybe}).$$

### 3. Interrelationship Mining for Complete Decision Tables

In this section, we review rough-set-based interrelationship mining for complete decision tables [3–7, 10].

#### 3.1. Observations and motivations

Comparison of attribute values between objects is a common basis of rough set data analysis. For example, in Pawlak's rough set, the equality of attribute values for each attribute  $a \in A$  is essential for an indiscernibility relation  $\text{IND}(A)$  by the subset of attributes  $A \subseteq AT$ . In the dominance-based rough set approach [1], the total pre-order relation among attribute values in each attribute provides a dominance relation between objects.

Generally, the domain of such comparison between attribute values is separated into each set of attribute values  $V_a, a \in AT$ . However, the following characteristics are difficult to describe without comparing the values between different attributes:

- the evaluation score of movie A is better than that of movie B,
- the color of sample A is similar to that of sample B,
- users prefer the design of car A to that of car B.

To treat the above characteristics in the framework of rough sets, we must extend the domain of binary relations  $R$  for comparison of attribute values from each range  $V_a$ , i.e.,  $R \subseteq V_a \times V_a$  to the Cartesian product of two ranges  $V_a$  and  $V_b$ , i.e.,  $R \subseteq V_a \times V_b$ . This extension enables us to describe the interrelationships between attributes by comparing the attribute values of different attributes in the framework of rough set theory [3].

#### 3.2. Interrelationships between attributes and indiscernibility of objects by interrelationships

Various relations between attribute values can be considered to compare values, e.g., equality, equivalence, order relations, and similarity. These situations between attributes in a given decision table can be formulated as interrelationships as follows.

Let  $DT$  be a decision table by (1),  $a, b \in C$  be two condition attributes with ranges  $V_a$  and  $V_b$ , respectively, and  $R \subseteq V_a \times V_b$  be a binary relation from  $V_a$  to  $V_b$ . The attributes  $a$  and  $b$  are interrelated by  $R$  if and only if there exists an object  $x \in U$  such that  $(\rho(x, a), \rho(x, b)) \in R$  holds [3].

We denote the set of objects wherein the values of attributes  $a$  and  $b$  satisfy the relation  $R$  as follows:

$$R(a, b) = \{x \in U \mid (\rho(x, a), \rho(x, b)) \in R\}, \quad . . . (10)$$

and we call the set  $R(a, b)$  the support set of the interrelationship between  $a$  and  $b$  by  $R$ .

An interrelationship between two attributes by a binary relation provides a way to compare attribute values between different attributes; however, for simplicity, we also allow the interrelationship between the same attributes.

Indiscernibility relations in a given decision table by interrelationships between attributes are introduced as follows [3]. Let  $a, b \in C$  be two condition attributes in a decision table  $DT$ . We assume that  $a, b \in C$  are interrelated by a binary relation  $R \subseteq V_a \times V_b$ , i.e.,  $R(a, b) \neq \emptyset$  holds.

The indiscernibility relation on  $U$  based on the interrelationship between  $a$  and  $b$  by  $R$  is defined by

$$IND(aRb) = \left\{ (x, y) \mid \begin{array}{l} x \in U, y \in U, \text{ and} \\ x \in R(a, b) \text{ iff } y \in R(a, b) \end{array} \right\}. \quad (11)$$

For any objects  $x$  and  $y$ ,  $(x, y) \in IND(aRb)$  means that  $x$  is not discernible from  $y$  relative to whether the interrelationship between the attributes  $a$  and  $b$  by relation  $R$  holds. Any binary relation  $IND(aRb)$  on  $U$  defined by (11) is an equivalence relation, and we can construct equivalence classes from an indiscernibility relation  $IND(aRb)$  [3].

### 3.3. Decision tables for interrelationship mining

The decision table  $DT$  by (1) is reconstructed to treat interrelationships between attributes explicitly by introducing the given binary relations to compare the values of different attributes.

**Definition 1**—[6, 10]: Let  $DT$  be a decision table by (1). The decision table  $DT_{int}$  for interrelationship mining with respect to  $DT$  is defined as follows:

$$DT_{int} = (U, AT_{int}, V \cup \{0, 1\}, \mathcal{R}_{int}, \rho_{int}), \quad \dots \quad (12)$$

where  $U$  and  $V = \bigcup_{a \in AT} V_a$  are identical to  $DT$ .

The set  $AT_{int}$  is defined by

$$AT_{int} = AT \cup \{aRb \mid \exists R \in \{R_{a \times b}\}, R(a, b) \neq \emptyset\}, \quad (13)$$

where  $AT = C \cup \{d\}$  is identical to  $DT$  and each expression  $aRb$  is a newly introduced condition attribute called an interrelated condition attribute. The set  $\{R_{a \times b}\}$  of binary relation(s) is defined below.

The set  $\mathcal{R}_{int}$  of sets of binary relations is defined as

$$\mathcal{R}_{int} = \mathcal{R}_{AT} \cup \left\{ \{R_{a_i \times b_i}\} \mid \begin{array}{l} R_{a_i \times b_i} \subseteq V_{a_i} \times V_{b_i}, \\ \exists a_i, b_i \in C \end{array} \right\} \cup \{\{=\}\} \quad \text{For each } aRb, \quad (14)$$

where each set  $\{R_{a_i \times b_i}\} = \{R_{a_i \times b_i}^1, \dots, R_{a_i \times b_i}^{n_i}\}$  consists of  $n_i$  ( $n_i \geq 0$ ) binary relation(s) defined on  $V_{a_i} \times V_{b_i}$ .

The information function  $\rho_{int}$  is defined by

$$\rho_{int}(x, c) = \begin{cases} \rho(x, c), & \text{if } c \in AT, \\ 1, & c = aRb \text{ and } x \in R(a, b), \\ 0, & c = aRb \text{ and } x \notin R(a, b). \end{cases} \quad (15)$$

Each interrelated condition attribute  $aRb$  represents whether each object  $x \in U$  supports the interrelationship between the attributes  $a, b \in C$  by the binary relation  $R \subseteq V_a \times V_b$  [3]. Therefore, interrelated condition attributes are nominal attributes and we only treat the equality relation to compare attribute values of different objects for each interrelated condition attribute.

Note that the equation (14) means that binary relations for comparing attribute values between different attributes are assumed to be given. In general, comparison of attribute values between different attributes is needed to treat carefully and we need to determine whether two attributes in a given decision table are comparable in some sense based on some background knowledge. However, comparability between attribute values depends on the “meaning of dataset,” in other words, semantics of interrelationship mining and the formulation of semantics

is one of the most important issues for rough-set-based interrelationship mining [10].

The following property guarantees the equivalence of representation ability between the interrelationship between attributes  $a$  and  $b$  by  $R$  and the corresponding interrelated condition attribute  $aRb$ .

**Proposition 1**—[3]: Let  $DT$  be a decision table by (1) and  $DT_{int}$  be a decision table by (12). The following equality holds.

$$IND_{DT}(aRb) = IND_{DT_{int}}(\{aRb\}), \quad \dots \quad (16)$$

where  $IND_{DT}(aRb)$  is the indiscernibility relation in  $DT$  with respect to the interrelationship between  $a$  and  $b$  by  $R$  defined by (11), and  $IND_{DT_{int}}(\{aRb\})$  is the indiscernibility relation in  $DT_{int}$  constructed from a singleton  $\{aRb\}$ .

The authors have discussed effectiveness of interrelated condition attributes in relative reducts of complete decision tables [9]. The following property guarantees that interrelated condition attributes in relative reducts enhance representation ability of decision rules generated from relative reducts.

**Proposition 2**—[9]: Suppose  $A \subseteq AT_{int}$  is a relative reduct of the decision table  $DT_{int}$  for interrelationship mining. For any two condition attributes  $a, b \in A$ , let a set  $A'$  be either  $A' = (A \setminus \{a\}) \cup \{aRb\}$  or  $A' = (A \setminus \{b\}) \cup \{aRb\}$  and  $A'$  is also a relative reduct of  $DT_{int}$ . The following inequality holds:

$$ACov(A) \leq ACov(A'). \quad \dots \quad (17)$$

Note that  $ACov(\cdot)$  is an evaluation criterion for relative reducts proposed by the authors [2] based on the roughness of partitions generated from the relative reducts. The criterion  $ACov(\cdot)$  is defined by

$$ACov(A) = \frac{|\mathcal{D}|}{\sum_{[x]_A \in U/IND(A)} |\{D_i \in \mathcal{D} \mid [x]_A \cap D_i \neq \emptyset\}|}, \quad \dots \quad (18)$$

where  $|X|$  is the cardinality of the set  $X$ , and  $\mathcal{D} = U/IND(\{d\})$  is the set of decision classes of the given decision table. For every relative reduct  $A \subseteq AT_{int}$ , the score  $ACov(A)$  represents the average of coverage scores of decision rules generated from  $A$ , the decision attribute  $d$ , and every object  $x \in U$  [2].

## 4. Interrelationships between Attributes in Incomplete Decision Tables

In this section, we revise the concept of interrelationships between condition attributes to treat null value in the interrelationships and introduce similarity relations based on the interrelationship between attributes.

### 4.1. Three cases in which interrelationships are not available by null value

Here, let  $a, b \in C$  be two condition attributes and  $R$  be a binary relation defined on  $V_a \times V_b$ . Similar to the case of binary relations  $R_a$  on  $V_a$ , we do not treat the relation between null value  $*$  and other values by  $R$ , which causes

the following three cases in which we cannot consider the interrelationship between  $a$  and  $b$  by  $R$ :

1. The value of  $a$  is null but that of  $b$  is not null:  $\rho(x, a) = *$  and  $\rho(x, b) \neq *$ ,
2. The value of  $b$  is null but that of  $a$  is not null:  $\rho(x, a) \neq *$  and  $\rho(x, b) = *$ ,
3. The values of both  $a$  and  $b$  are null:  $\rho(x, a) = \rho(x, b) = *$ .

Because the occurrences of null value are different among these three cases, we can consider the three cases separately to treat null value in the interrelationship between  $a$  and  $b$  by  $R$ . Moreover, this fact indicates that each object  $x \in U$  fits one of the following three situations:

- (a)  $x$  supports the interrelationship between  $a$  and  $b$  by  $R$ , i.e.,  $x \in R(a, b)$ ,
- (b)  $x$  does not support the interrelationship between  $a$  and  $b$  by  $R$ ,
- (c) The values of  $x$  at  $a$  and  $b$  are not comparable by one of the above three cases.

The situation (c) causes difficulty of interrelationship between attributes, and therefore, in the next section, we discuss an approach to treat the situation (c) and introduce similarity relations by interrelationship between attributes.

#### 4.2. Similarity relation by interrelationship between attributes

According to the discussion in Section 4.1, we introduce the following three sets of objects:

$$LN(a, b) = \{x \in U \mid \rho(x, a) = * \text{ and } \rho(x, b) \neq *\}, \quad (19)$$

$$RN(a, b) = \{x \in U \mid \rho(x, a) \neq * \text{ and } \rho(x, b) = *\}, \quad (20)$$

$$BN(a, b) = \{x \in U \mid \rho(x, a) = \rho(x, b) = *\}. \quad (21)$$

The sets  $LN(a, b)$ ,  $RN(a, b)$ , and  $BN(a, b)$  correspond to the cases 1, 2, and 3, respectively. We denote the union of  $LN(a, b)$ ,  $RN(a, b)$ , and  $BN(a, b)$  as  $NULL(a, b) \stackrel{\text{def}}{=} LN(a, b) \cup RN(a, b) \cup BN(a, b)$ . The set  $NULL(a, b)$  corresponds to the set of objects in the situation (c).

For every binary relation  $R$  defined on  $V_a \times V_b$ , the domain of  $R$  can be naturally extended to the case that  $V_a$  and  $V_b$  include the null value  $*$ . As we mentioned in Section 4.1, for any two condition attributes  $a, b \in C$  and any binary relation  $R$  defined on  $V_a \times V_b$ , we do not treat the relationship between the null value  $*$  and all values  $v \in V_a$  and  $w \in V_b$  by  $R$ , i.e., neither  $vR*$  nor  $*Rw$  holds. Note that this modification does not affect the definition of support set  $R(a, b)$  by (10).

We then represent the set of objects that does not support the interrelationship between  $a$  and  $b$  by  $R$  as follows:

$$R^c(a, b) = U - (R(a, b) \cup NULL(a, b)). \quad (22)$$

Note that the set  $R^c(a, b)$  defined by (22) corresponds to the support set of the interrelationship between  $a$  and  $b$  by the binary relation  $R^c \stackrel{\text{def}}{=} (U_a \times U_b) - R$ , where the set  $U_i \stackrel{\text{def}}{=} \{x \in U \mid \rho(x, i) \neq *\}$  is a set of objects in which the value of the attribute  $i$  is not null.

The following is obvious from the definitions of the support set  $R(a, b)$ , nonsupport set  $R^c(a, b)$ , and  $LN(a, b)$ ,  $RN(a, b)$ , and  $BN(a, b)$ .

*Lemma 1:* The collection  $\mathcal{S}$  of sets defined as

$$\mathcal{S} = \left\{ R(a, b), R^c(a, b), LN(a, b), RN(a, b), BN(a, b) \right\} - \{\emptyset\} \quad (23)$$

comprises a partition of  $U$ .

The proof of Lemma 1 can be found in Appendix A.

Here, we introduce a similarity relation by the interrelationship between attributes.

*Definition 2:* Let  $DT$  be a given decision table, and  $\mathcal{S}$  be a family of object sets by (23). Suppose that two condition attributes  $a, b \in C$  are interrelated by the binary relation  $R \subseteq V_a \times V_b$ , i.e.,  $R(a, b) \neq \emptyset$  holds.

The similarity relation on  $U$  by interrelationship between  $a$  and  $b$  by  $R$  is defined as follows:

$$SIM(aRb) = \left\{ (x, y) \mid \begin{array}{l} \exists S \in \mathcal{S}, x, y \in S, \text{ or} \\ x \in NULL(a, b) \text{ and } y \notin NULL(a, b), \text{ or} \\ x \notin NULL(a, b) \text{ and } y \in NULL(a, b) \end{array} \right\}. \quad (24)$$

It is easy to confirm that the binary relation  $SIM(aRb)$  by the interrelationship between  $a$  and  $b$  by  $R$  is a tolerance relation on  $U$ .

*Theorem 1:* Let  $a, b \in C$  be two condition attributes in a given decision table  $DT$  and  $R \subseteq V_a \times V_b$  be a binary relation. The binary relation  $SIM(aRb)$  defined by (24) satisfies reflexivity and symmetry.

The proof of Theorem 1 can be found in Appendix A.

If both attribute  $a$  and  $b$  do not have null value, the similarity relation  $SIM(aRb)$  is identical to the indiscernibility relation  $IND(aRb)$  by the interrelationship between  $a$  and  $b$  by a binary relation  $R$ , i.e.,  $IND(aRb)$  is a special case of  $SIM(aRb)$ .

*Corollary 1:* If both  $* \notin V_a$  and  $* \notin V_b$  hold, then  $SIM(aRb) = IND(aRb)$  holds, where  $IND(aRb)$  is the indiscernibility relation by the interrelationship between the attributes defined by (11).

*Example 2:* We introduce two interrelationships between two condition attributes Q.2 and Q.3 in Table 1 by comparing the values of these attributes by the following two binary relations defined on  $V_{Q.2} \times V_{Q.3}$ :

- $\prec_{Q.2 \times Q.3}$ : the answer to Q.3 is better than the answer to Q.2,
- $\preceq_{Q.2 \times Q.3}$ : The answer to Q.3 is equal to or better than the answer to Q.2.

The range of these two attributes is identical; therefore, we can consider the preference relation,  $\prec_{Q.2}$ , as the binary relation  $\prec_{Q.2 \times Q.3}$  defined on  $V_{Q.2} \times V_{Q.3}$  with the following order:

$$\prec_{Q.2 \times Q.3}: v.b. \prec \text{bad} \prec \text{normal} \prec \text{good} \prec v.g..$$

Here, we concentrate on the similarity relation between Q.2 and Q.3 by the binary relation  $\prec_{Q.2 \times Q.3}$ . Hereafter, we omit the indices of binary relations. We construct the support set  $\prec (Q.2, Q.3)$  of the interrelationship defined by (10) and the other four sets defined by (19)-(22) as follows:

$$\begin{aligned}\prec (Q.2, Q.3) &= \{u1, u2\}, \\ \prec^c (Q.2, Q.3) &= \{u4, u8\}, \\ LN(Q.2, Q.3) &= \{u3, u7\}, \\ RN(Q.2, Q.3) &= \{u6\}, \\ BN(Q.2, Q.3) &= \{u5\}.\end{aligned}$$

We then obtain the set  $NULL(Q.2, Q.3) = \{u3, u5, u6, u7\}$  and a partition  $\mathcal{S}$  by (23) as follows:

$$\mathcal{S} = \{\{u1, u2\}, \{u4, u8\}, \{u3, u7\}, \{u6\}, \{u5\}\}.$$

The support set  $\prec (Q.2, Q.3)$  is not empty, therefore, we can construct the similarity relation  $SIM(Q.2 \prec Q.3)$  based on the interrelationship between Q.2 and Q.3 by the binary relation  $\prec$  by (24). From the constructed similarity relation, we obtain the following results:

- $(u1, u2), (u4, u8) \in SIM(Q.2 \prec Q.3)$ :  $u1$  and  $u2$ , and  $u4$  and  $u8$  are indiscernible from each other, respectively; both  $u1$  and  $u2$  ( $u4$  and  $u8$ ) support (do not support) the interrelationship.
- $(u3, u4), (u4, u5) \in SIM(Q.2 \prec Q.3)$ :  $u3$  and  $u4$ , and  $u4$  and  $u5$  are indiscernible from each other, respectively; an object in the pair adopts one of the three cases that cannot treat the interrelationship between attributes and another does not adopt any of the three cases.
- $(u3, u7) \in SIM(Q.2 \prec Q.3)$ : both  $u3$  and  $u7$  adopt the same case that cannot treat the interrelationship between attributes.
- $(u3, u5) \notin SIM(Q.2 \prec Q.3)$ :  $u3$  and  $u5$  adopt different cases that cannot treat the interrelationship between attributes.

The results  $(u3, u4), (u4, u5) \in SIM(Q.2 \prec Q.3)$  but  $(u3, u5) \notin SIM(Q.2 \prec Q.3)$  demonstrate that the similarity relation by the interrelationship between attributes generally does not satisfy transitivity.

## 5. Interrelated Attributes for Incomplete Decision Tables

To describe the three cases discussed in Section 4.1, we introduce three null values  $*_l$ ,  $*_r$ , and  $*_b$  for interrelated condition attributes. These null values  $*_l$ ,  $*_r$ , and  $*_b$  correspond to cases 1, 2, and 3, respectively.

**Definition 3:** Let  $DT$  be a decision table by (1), and  $DT_{int}$  be a decision table for interrelationship mining with respect to the  $DT$  defined by (12). We redefine the information function  $\rho_{int}$  for interrelationship mining defined

by (15). The redefined information function  $\rho_{int}$  is expressed as follows:

$$\rho_{int} : U \times AT_{int} \rightarrow V \cup \{0, 1\} \cup \{*_l, *_r, *_b\} \quad . \quad (25)$$

such that

$$\rho_{int}(x, c) = \begin{cases} \rho(x, c), & \text{if } c \in AT, \\ 1, & c = aRb \text{ and } x \in R(a, b), \\ 0, & c = aRb \text{ and } x \in R^c(a, b), \\ *_l, & c = aRb \text{ and } x \in LN(a, b), \\ *_r, & c = aRb \text{ and } x \in RN(a, b), \\ *_b, & c = aRb \text{ and } x \in BN(a, b). \end{cases} \quad (26)$$

Note that there exists various interpretations of the null value  $*$ , e.g., missing value, nondeterministic value, and do not care. We interpret the newly introduced null value  $*_l$  ( $*_r$ ,  $*_b$ ) as "the value of the interrelated condition attribute is not determined by the null value of condition attribute(s) at left (right, both) side(s)." This interpretation does not depend on the interpretation of null value  $*$  that causes the occurrence of null values  $*_l$ ,  $*_r$ , and  $*_b$ .

Similar to the interrelationship mining for complete decision tables, we can represent the similarity relation based on an interrelationship between attributes using the corresponding interrelated condition attribute.

**Definition 4:** Let  $DT_{int}$  be a decision table for the interrelationship mining defined by (12) and assume that  $DT_{int}$  has a revised information function  $\rho_{int}$  defined by (26). For any interrelated condition attribute  $aRb$ , a similarity relation based on the singleton  $\{aRb\}$  is defined as follows:

$$SIM(\{aRb\}) = \left\{ (x, y) \mid \begin{array}{l} \rho_{int}(x, aRb) = \rho_{int}(y, aRb), \\ \text{or} \\ \rho_{int}(x, aRb) \in \{1, 0\} \text{ and} \\ \rho_{int}(y, aRb) \in \{*_l, *_r, *_b\}, \\ \text{or} \\ \rho_{int}(x, aRb) \in \{*_l, *_r, *_b\} \\ \text{and } \rho_{int}(y, aRb) \in \{1, 0\}. \end{array} \right\}. \quad (27)$$

It is easy to confirm that the binary relation  $SIM(\{aRb\})$  from an interrelated condition attribute  $aRb$  by (27) is a tolerance relation.

**Theorem 2:** Let  $DT_{int}$  be a decision table for the interrelationship mining defined by (12) and assume that  $DT_{int}$  has a revised information function  $\rho_{int}$  defined by (26). For any interrelated condition attribute  $aRb \in AT_{int}$ , the binary relation  $SIM(\{aRb\})$  defined by (27) satisfies reflexivity and symmetry.

The proof of Theorem 2 can be found in Appendix A.

The newly introduced null values  $*_l$ ,  $*_r$ , and  $*_b$  are mutually discernible.

**Corollary 2:** Let  $SIM(\{aRb\})$  be the similarity relation defined by (27), and  $i, j \in \{l, r, b\}$ . For any object  $x, y \in U$ , if  $\rho(x, aRb) = *_i$ ,  $\rho(y, aRb) = *_j$ , and  $i \neq j$  hold, then  $(x, y) \notin SIM(\{aRb\})$  holds.

Similar to the indiscernibility relations in complete decision tables by Proposition 1, the similarity relation by the interrelationship between attributes is perfectly representable by the similarity relation with respect to the corresponding interrelated condition attribute.

**Theorem 3:** Let  $DT$  be a decision table by (1) and  $DT_{int}$

be a decision table for interrelationship mining by (12) with the redefined information function by (26). The following equality holds.

$$SIM_{DT}(aRb) = SIM_{DT_{int}}(\{aRb\}), \quad . . . . . (28)$$

where  $SIM_{DT}(aRb)$  is the similarity relation in  $DT$  with respect to the interrelationship between  $a$  and  $b$  by  $R$  defined by (24), and  $SIM_{DT_{int}}(\{aRb\})$  is the similarity relation in  $DT_{int}$  constructed from the singleton  $\{aRb\}$  of the interrelated condition attribute  $aRb$  by (27).

The proof of Theorem 3 can be found in Appendix A.

Similar to the case of similarity relation by the interrelationship between attributes by Corollary 1, the similarity relation  $SIM(\{aRb\})$  by the interrelated condition attribute  $aRb$  can be considered as the indiscernibility relation  $IND(\{aRb\})$ , if attributes  $a$  and  $b$  do not have null value.

**Corollary 3:** If both  $* \notin V_a$  and  $* \notin V_b$  hold, then  $SIM_{DT_{int}}(\{aRb\}) = IND_{DT_{int}}(\{aRb\})$  holds.

By Theorem 3 and the tolerance of similarity relation by (3), we can combine the similarity relation  $SIM(\{aRb\})$  by the interrelated condition attribute  $aRb$  with the similarity relation  $SIM(A), A \subseteq C$ , defined by (2). Moreover, by Corollary 3 and the property of similarity relation [11], the combined similarity relation can be considered an indiscernibility relation if there is no null value in the ranges of the attributes in  $A \cup \{aRb\}$  used to construct the similarity relation.

Therefore, the results of this study enable us to flexibly treat the similarity between objects with respect to both characteristics by comparing the values of the same attributes and by the interrelationship between attributes.

**Example 3:** Here, we introduce two interrelated condition attributes  $Q.2 \prec Q.3$  and  $Q.2 \preceq Q.3$  based on binary relations  $\prec$  and  $\preceq$  defined on  $V_{Q.2} \times V_{Q.3}$ , respectively. We can use the results of Example 2 to construct the interrelated condition attribute  $Q.2 \prec Q.3$  and define the value for each object. For the interrelated condition attribute  $Q.2 \preceq Q.3$ , the support set  $\preceq(Q.2, Q.3)$  defined by (10) and the non-support set  $\preceq^c(Q.2, Q.3)$  defined by (22) are as follows, respectively:

$$\preceq(Q.2, Q.3) = \{u1, u2, u4\},$$

$$\preceq^c(Q.2, Q.3) = \{u8\}.$$

The sets  $LN(Q.2, Q.3)$ ,  $RN(Q.2, Q.3)$ , and  $BN(Q.2, Q.3)$  are identical to the case of relation  $\prec_{Q.2 \times Q.3}$  in Example 2.

We then construct the two interrelated condition attributes  $Q.2 \prec Q.3$  and  $Q.2 \preceq Q.3$  using the five sets for each binary relation  $\prec$  and  $\preceq$ , respectively. Table 2 is the incomplete decision table with the two interrelated condition attributes.

Similar to Example 1, we consider the similarity relation and relative reduct of Table 2. Let  $C_{int} = C \cup \{Q.2 \prec Q.3, Q.2 \preceq Q.3\}$ . The generalized decision  $\partial_{C_{int}}$  of Table 1 by the set of all condition attributes  $C \subseteq AT$  is defined as

follows:

$$\begin{aligned} \partial_{C_{int}}(u1) &= \partial_{C_{int}}(u2) = \partial_{C_{int}}(u4) = \partial_{C_{int}}(u6) \\ &= \{\text{yes, maybe}\}, \end{aligned}$$

$$\partial_{C_{int}}(u3) = \{\text{yes}\},$$

$$\partial_{C_{int}}(u5) = \{\text{yes, maybe, no}\},$$

$$\partial_{C_{int}}(u7) = \{\text{no}\},$$

$$\partial_{C_{int}}(u8) = \{\text{maybe, no}\}.$$

In Table 2, the object  $u3$  is discernible from  $u6$  by the difference in the null values in the interrelated condition attributes, i.e.,  $*_l$  at  $u3$  and  $*_r$  at  $u6$ . Similarly,  $u7$  is discernible from  $u6$  and these differences provide the above revision of the generalized decision, i.e.,  $\partial_{C_{int}}(u3) = \{\text{yes}\}$  and  $\partial_{C_{int}}(u7) = \{\text{no}\}$ , respectively.

We construct the similarity relation  $SIM(\{Q.2 \prec Q.3\})$  with respect to the singleton of the interrelated condition attribute by (27). By Theorem 3, the constructed similarity relation  $SIM(\{Q.2 \prec Q.3\})$  correctly reflects all relationships between objects by the similarity relation  $SIM_{DT}(Q.2 \prec Q.3)$  by interrelationship between  $Q.2$  and  $Q.3$  with the binary relation  $\prec$  in Example 2. For example, similar to the above discussion, the null values of  $u3$  and  $u5$  are different in Table 2, i.e.,  $*_l$  at  $u3$  and  $*_b$  at  $u5$ , which provides  $(u3, u4), (u4, u5) \in SIM(\{Q.2 \prec Q.3\})$  but  $(u3, u5) \notin SIM(\{Q.2 \prec Q.3\})$ .

Here, we consider the relative reduct of Table 2. Let  $A = \{Q.1, Q.3, Q.2 \preceq Q.3\}$ . For each object  $u \in U$ , the set of similar objects  $S_A(u)$  with respect to the similarity relation  $Sim(A)$  by (2) is as follows, respectively:

$$S_A(u1) = \{u1, u5\}, \quad S_A(u2) = \{u2, u5, u6\},$$

$$S_A(u3) = \{u3\}, \quad S_A(u4) = \{u4, u5\},$$

$$S_A(u5) = \left\{ \begin{matrix} u1, u2, u4, \\ u5, u8 \end{matrix} \right\}, \quad S_A(u6) = \{u2, u6\},$$

$$S_A(u7) = \{u7\}, \quad S_A(u8) = \{u5, u8\}.$$

If we remove some attribute from  $A$ , e.g.,  $Q.3$ , the subset  $A' = \{Q.1, Q.2 \preceq Q.3\}$  cannot keep the generalized decision by  $C_{int}$ , e.g.,  $S_{A'}(u7) = \{u2, u3, u7\}$  and  $\partial_{A'}(u7) = \{\text{yes, no}\} \neq \partial_{C_{int}}(u7)$ . Consequently, it is easy to observe that  $A = \{Q.1, Q.3, Q.2 \preceq Q.3\}$  is a relative reduct of Table 2.

Similarly, we can obtain relative reducts of an object, which are used to express optimal decision rules. Let  $X = \{Q.3, Q.2 \preceq Q.3\}$ . It can be clearly observed that  $S_X(u7) = \{u7\}$  and  $\partial_X(u7) = \{\text{no}\} = \partial_{C_{int}}(u7)$ , and any subsets cannot keep the generalized decision, e.g., if  $X' = \{Q.3\}$ , then  $S_{X'}(u7) = \{u5, u6, u7\}$  and  $\partial_{X'}(u7) = \{\text{maybe, no}\} \neq \partial_{C_{int}}(u7)$ . Therefore,  $X$  is a relative reduct of  $u7$ . We then obtain an optimal decision rule of  $u7$  as follows:

$$(Q.3, \text{bad}) \wedge (Q.2 \preceq Q.3, *_l) \rightarrow (\text{Purchase, no}).$$

## 6. Conclusion

In this study, we applied rough-set-based interrelationship mining to incomplete decision tables. First, we observed three cases in which the interrelationship between attributes unavailable by null values and introduced sim-



**Table 2.** Incomplete decision table for interrelationship mining

$U$	Gender	Q.1	Q.2	Q.3	$Q.2 \prec Q.3$	$Q.2 \preceq Q.3$	Purchase
$u1$	female	yes	good	v. g.	1	1	yes
$u2$	male	no	good	v. g.	1	1	yes
$u3$	male	no	*	good	$*_l$	$*_l$	yes
$u4$	female	yes	normal	normal	0	1	yes
$u5$	female	*	*	*	$*_b$	$*_b$	maybe
$u6$	male	no	good	*	$*_r$	$*_r$	maybe
$u7$	male	no	*	bad	$*_l$	$*_l$	no
$u8$	*	yes	good	normal	0	0	no

ilarity relations based on the interrelationship between attributes. Next, we introduced three types of null values that correspond to the above three cases and demonstrated that the similarity relation by the interrelationship between attributes is perfectly representable by the similarity relation with respect to the corresponding interrelated condition attribute. In future, we plan to refine the theoretical aspects of the proposed approach and apply it to real-life data analysis. In particular, how to determine the existence or non-existence of interrelationships between condition attributes and how to construct interrelated condition attributes from the scratch by generating binary relations between attribute values are very important issues for applying rough-set-based interrelationship mining to real-life data.

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## Appendix A. Proofs of Theoretical Results

**Lemma 1:** The collection  $\mathcal{S}$  of sets defined as

$$\mathcal{S} = \left\{ R(a, b), R^c(a, b), LN(a, b), \right. \\ \left. RN(a, b), BN(a, b) \right\} - \{\emptyset\}$$

comprises a partition of  $U$ .

**Proof:** From the definitions of support set  $R(a, b)$ , nonsupport set  $R^c(a, b)$ , and  $LN(a, b)$ ,  $RN(a, b)$ , and  $BN(a, b)$ , it is obvious that, for any  $S_i, S_j \in \mathcal{S}$ , if  $S_i \neq \emptyset$ ,  $S_j \neq \emptyset$ , and  $i \neq j$  hold, then  $S_i \cap S_j = \emptyset$  holds. Thus, it is sufficient to show  $\cup_{S \in \mathcal{S}} S = U$ . Moreover, every set  $S \in \mathcal{S}$  is obviously a subset of  $U$ , which implies  $\cup_{S \in \mathcal{S}} S \subseteq U$ .

Let  $x \in U$  be any object. If both  $v = \rho(x, a) \neq *$  and  $w = \rho(x, b) \neq *$  hold, either  $(v, w) \in R$  or  $(v, w) \notin R$  holds, which implies that either  $x \in R(a, b)$  or  $x \in R^c(a, b)$  holds. Otherwise, if  $v = *$  or  $w = *$  or both hold, either  $x \in LN(a, b)$  or  $x \in RN(a, b)$  or  $x \in BN(a, b)$  hold. These facts imply that  $\cup_{S \in \mathcal{S}} S \supseteq U$ ; thus,  $\cup_{S \in \mathcal{S}} S = U$ . ■

**Theorem 1:** Let  $a, b \in C$  be two condition attributes in a given decision table  $DT$  and  $R \subseteq V_a \times V_b$  be a binary relation. The binary relation  $SIM(aRb)$  defined by (24) satisfies reflexivity and symmetry.

**Proof:** First, we show that  $SIM(aRb)$  satisfies reflexivity, i.e.,  $(x, x) \in SIM(aRb)$  for all  $x \in U$ . Because  $\mathcal{S}$  is a partition by Lemma 1, for every  $x \in U$ , there exists a set  $S \in \mathcal{S}$  such that  $x \in S$ ; thus  $(x, x) \in SIM(aRb)$  for all  $x \in U$  by (24).

Next, we show symmetry, i.e., if  $(x, y) \in SIM(aRb)$  holds, then  $(y, x) \in SIM(aRb)$  holds. Suppose  $(x, y) \in SIM(aRb)$  holds. If there exists a set  $S \in \mathcal{S}$  such that  $x, y \in S$  holds, it is obvious that  $(y, x) \in SIM(aRb)$  also holds. We then assume that  $x \in NULL(a, b)$ , which implies that  $y \notin NULL(a, b)$  by (24). Therefore,  $(y, x) \in SIM(aRb)$  also holds by (24). The case of  $x \notin NULL(a, b)$  is shown similarly, which indicates that  $SIM(aRb)$  satisfies symmetry. ■

**Theorem 2:** Let  $DT_{int}$  be a decision table for interrelationship mining defined by (12) and assume that  $DT_{int}$  has a revised information function  $\rho_{int}$  defined by (26). For any interrelated condition attribute  $aRb \in AT_{int}$ , the binary relation  $SIM(\{aRb\})$  defined by (27) satisfies reflexivity and symmetry.

**Proof:** It is obvious that  $\rho(x, aRb) = \rho(x, aRb)$  for every  $x \in U$ ; therefore,  $(x, x) \in SIM(\{aRb\})$  holds for every  $x \in U$ , i.e., the binary relation  $SIM(\{aRb\})$  satisfies reflexivity.

We show that  $SIM(\{aRb\})$  satisfies symmetry. Suppose  $(x, y) \in SIM(\{aRb\})$  holds. If  $\rho(x, aRb) = \rho(y, aRb)$  holds, it is obvious that  $\rho(y, aRb) = \rho(x, aRb)$  also holds and  $(y, x) \in SIM(\{aRb\})$ . If  $\rho(x, aRb) \neq \rho(y, aRb)$  and  $\rho(x, aRb) \in \{1, 0\}$  hold, according to the definition of  $SIM(\{aRb\})$  by (27), it is implied that  $\rho(y, aRb) \in \{*_l, *_r, *_b\}$  holds. Thus,  $(y, x) \in SIM(\{aRb\})$ . We can also prove the case of  $\rho(x, aRb) \in \{*_l, *_r, *_b\}$  similarly. Thus,  $SIM(\{aRb\})$  satisfies symmetry. ■

**Theorem 3:** Let  $DT$  be a decision table by (1) and  $DT_{int}$  be a decision table for interrelationship mining by (12) with the redefined information function by (26). The following equality holds.

$$SIM_{DT}(aRb) = SIM_{DT_{int}}(\{aRb\}),$$

where  $SIM_{DT}(aRb)$  is the similarity relation in  $DT$  with respect to the interrelationship between  $a$  and  $b$  by  $R$  defined by (24), and  $SIM_{DT_{int}}(\{aRb\})$  is the similarity relation in  $DT_{int}$  constructed from the singleton  $\{aRb\}$  of the interrelated condition attribute  $aRb$ .

**Proof:** Suppose that  $(x, y) \in SIM_{DT}(aRb)$  holds. If there exists a set  $S \in \mathcal{S}$  such that  $x, y \in S$  holds,

it is clear that  $\rho_{int}(x, aRb) = \rho_{int}(y, aRb)$  holds by the definition of the redefined information function  $\rho_{int}$  by (26), which indicates that  $(x, y) \in SIM_{DT_{int}}(\{aRb\})$  by (27). Otherwise, if  $x \in NULL(a, b)$  and  $y \notin NULL(a, b)$ , it is implied that  $\rho_{int}(x, aRb) \in \{*_l, *_r, *_b\}$  and  $\rho_{int}(y, aRb) \in \{0, 1\}$  hold by (26); thus,  $(x, y) \in SIM_{DT_{int}}(\{aRb\})$  holds by (27). We can also show the case of  $x \in NULL(a, b)$  and  $y \notin NULL(a, b)$  similarly, thus,  $SIM_{DT}(aRb) \subseteq SIM_{DT_{int}}(\{aRb\})$  holds. The opposite inclusion  $SIM_{DT}(aRb) \supseteq SIM_{DT_{int}}(\{aRb\})$  is also provable in the same manner as the above proof, which completes the proof. ■

## References:

- [1] S. Greco, B. Matarazzo, and R. Słowiński, Rough set theory for multicriteria decision analysis, *European Journal of Operational Research*, Vol. 129, pp. 1–47, 2002.
- [2] Y. Kudo and T. Murai, An Evaluation method of Relative Reducts based on Roughness of partitions, *International Journal of Cognitive Informatics and Natural Intelligence*, Vol. 4, No. 2, pp. 50–62, 2010.
- [3] Y. Kudo and T. Murai, Indiscernibility Relations by Interrelationships between Attributes in Rough Set Data Analysis, *Proc. of IEEE GrC 2012*, pp. 264–269, 2012.
- [4] Y. Kudo and T. Murai, A Plan of Interrelationship Mining Using Rough Sets, *Proc. of the 29th Fuzzy System Symposium*, pp. 33–36, 2013 (in Japanese).
- [5] Y. Kudo and T. Murai, Decision Logic for Rough Set-based Interrelationship Mining, *Proc. of IEEE GrC 2013*, pp. 172–177, 2013.
- [6] Y. Kudo and T. Murai, Interrelationship Mining from a Viewpoint of Rough Sets on Two Universes, *Proc. of IEEE GrC 2014*, pp. 137–140, 2014.
- [7] Y. Kudo and T. Murai, Some Properties of Interrelated Attributes in Relative Reducts for Interrelationship Mining, *Proc. of SCIS&ISIS 2014*, SOFT, pp. 998–1001, 2014.
- [8] Y. Kudo and T. Murai, A Note on Application of Interrelationship Mining to Incomplete Information Systems, *Proc. of the 31st Fuzzy System Symposium*, pp. 777–778, 2015 (in Japanese).
- [9] Y. Kudo and T. Murai, On Representation Ability of Interrelated Attributes in Rough Set-based Interrelationship Mining, *Proc. of ISIS 2015*, pp. 1229–1237, 2015.
- [10] Y. Kudo, Y. Okada, and T. Murai, On a Possibility of Applying Interrelationship Mining to Gene Expression Data Analysis, *Brain and Health Informatics*, LNAI 8211, pp. 379–388, 2013.
- [11] M. Kryszkiewicz, Rough set approach to incomplete information systems, *Information Sciences*, Vol. 112, pp. 39–49, 1998.
- [12] Z. Pawlak, Rough Sets, *International Journal of Computer and Information Science*, Vol. 11, pp. 341–356, 1982.
- [13] Z. Pawlak, *Rough Sets: Theoretical Aspects of Reasoning about Data*, Kluwer Academic Publishers, 1991.
- [14] Z. Pawlak and A. Skowron, Rough Sets: Some Extensions, *Information Sciences*, Vol. 177, pp. 28–40, 2007.
- [15] Y. Y. Yao, B. Zhou, and Y. Chen, Interpreting Low and High Order Rules: A Granular Computing Approach, *Proc. of RSEISP 2007*, LNCS 4585, pp. 371–380, 2007.



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