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AIP | Conference Proceedings

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Citation: AIP Conf. Proc. **839**, 453 (2006); doi: 10.1063/1.2216655
View online: http://dx.doi.org/10.1063/1.2216655
View Table of Contents: http://proceedings.aip.org/dbt/dbt.jsp?KEY=APCPCS&Volume=839&Issue=1
Published by the American Institute of Physics.

# A Note on Granular Reasoning and Semantics of Four-Valued Logics

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**Abstract.** Zooming reasoning systems is a mechanism for reasoning using granular computing. The key concept of the zooming reasoning system is focus, which represents sentences we use in the current step of reasoning. Murai et al. has proposed a three-valued valuation based on focus. On the other hand, the authors have proposed another concept of granularity, called visibility, and constructed a four-valued truth valuation based on visibility and focus. However, our formulation of the four-valued valuation causes some difficulties to extend to all non-modal sentences. In this paper, we explore and refine connections between granular reasoning and semantics of four-valued logic. In particular, we refine the four-valued semantics based on visibility and focus, and demonstrate some properties of the four-valued semantics.

**Keywords:** Granular reasoning, four-valued logic, zooming reasoning system, focus, visibility
**PACS:** 02.10.Ab Logic and set theory

## 1 INTRODUCTION

*Granular computing* based on rough set theory (Pawlak [13, 14]) has been widely studied as a new paradigm of computing (for example, see [6, 15]). In particular, Murai et al. has proposed *granular reasoning* as a mechanism for reasoning using granular computing [7], and developed a framework of granular reasoning, called a zooming reasoning system [8, 9, 10]. The key concept of the zooming reasoning system is *focus*, which represents sentences we use in the current step of reasoning. The focus provides "granularized" possible worlds, and a three-valued valuation that assigns the truth value "true" or "false" to atomic sentences that appear in the focus, and assigns the truth value "unknown" to other atomic sentences. Murai et al. have also provided methods of control of the *degree of granularity*, and illustrated that such control of the degree of granularity represents reasoning steps. Moreover, deduction, non-monotonic reasoning, and abduction are also illustrated by control of the degree of granularity [11, 12].

On the other hand, the authors have proposed another concept of granularity, called *visibility* [5]. Visibility is an analogy of the term about vision that means the range of vision, which introduces the concept of "range" of sentences we consider. Visibility separates all atomic sentences into "visible" atomic sentences, that is, atomic sentences we consider, and "invisible" atomic sentences which are out of consideration. Combining visibility and focus, the authors have constructed a four-valued valuation with the following four values: true, false, unknown and undefined. Using the four-valued valuation, all atomic sentences are separated in the following three groups: *invisible* sentences, that is, atomic sentences with the truth value "undefined", *obscurely visible* sentences with the truth value "unknown", and *clearly visible* sentences with the truth value "true" or "false". However, our formulation had some difficulties to extend the four-valued valuation to all non-modal sentences.

In this paper, to overcome the difficulties of extending four-valued valuations, we refine connections between granular reasoning and semantics of four-valued logic. In particular, we refine the formulation of visibility and focus by further granularization to granularized possible worlds. We also reconstruct the four-valued valuation, which is extended to all non-modal sentences. Moreover, we discuss semantic characterization of visibility and focus.

# 2 BACKGROUNDS

## 2.1 Kripke-Style Models

Let $\mathscr{P}$ be a set of (at most countably infinite) atomic sentences. We construct a language $\mathscr{L}_{\text{ML}}(\mathscr{P})$ for modal logic from $\mathscr{P}$ using logical symbols $\top$ (the truth constant), $\bot$ (the falsity constant), $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (material implication), and two modal operators $\Box$ (necessity) and $\Diamond$ (possibility) by the following rules (1) $\mathsf{p} \in \mathscr{P} \Rightarrow \mathsf{p} \in \mathscr{L}_{\text{ML}}(\mathscr{P})$, (2) $p \in \mathscr{L}_{\text{ML}}(\mathscr{P}) \Rightarrow \neg p \in \mathscr{L}_{\text{ML}}(\mathscr{P})$, (3) $p\ q \in \mathscr{L}_{\text{ML}}(\mathscr{P}) \Rightarrow (p \wedge q)\ (p \vee q)\ (p \rightarrow q) \in \mathscr{L}_{\text{ML}}(\mathscr{P})$, (4) $p \in \mathscr{L}_{\text{ML}}(\mathscr{P}) \Rightarrow \Box p\ \Diamond p \in \mathscr{L}_{\text{ML}}(\mathscr{P})$. A sentence is called *non-modal* if the sentence does not contain any modal operators.

A *Kripke model* is a triple $\mathscr{M} = \langle W\ R\ v \rangle$, where $W$ is a non-empty set of possible worlds, $R$ is a binary relation on $W$, and $v$ is a valuation that assigns either the truth value $\mathbf{t}$ (true) or $\mathbf{f}$ (false) to every atomic sentence $\mathsf{p} \in \mathscr{P}$ at every world $w \in W$. We define $\mathscr{M}\ w \models \mathsf{p} \iff v(\mathsf{p}\ w) = \mathbf{t}$. The relation $\models$ is naturally extended to every sentence $p \in \mathscr{L}_{\text{ML}}(\mathscr{P})$ by the usual way. For any sentence $p \in \mathscr{L}_{\text{ML}}(\mathscr{P})$, we define the *truth set* of $p$ in $\mathscr{M}$ as $\|p\| = \{w \in W \mid \mathscr{M}\ w \models p\}$.

## 2.2 Zooming Reasoning Systems

*Zooming reasoning systems* provide reasoning processes using reconstruction of models by generating equivalent classes of possible worlds [9, 10]. Such construction operations are called *zooming in & out* [8].

Zooming reasoning systems are formalized as follows: Let $\mathscr{M} = \langle W\ R\ v \rangle$ be a Kripke model, and $\mathscr{L}(\mathscr{P})$ be a propositional language generated from $\mathscr{P}$ by usual way similar to constructing $\mathscr{L}_{\text{ML}}(\mathscr{P})$. Suppose we consider a set $\Gamma$ of non-modal sentences that illustrates the set of sentences we need to use the current reasoning step. The set $\Gamma$ is called a *focal point* or a *focus*. We define the set $\mathscr{P}_\Gamma$ of atomic sentences that appear in the current reasoning step by $\mathscr{P}_\Gamma = \mathscr{P} \cap \text{Sub}(\Gamma)$, where $\text{Sub}(\Gamma)$ is the union of the sets of subsentences of each sentence in $\Gamma$. Using $\mathscr{P}_\Gamma$, an equivalence relation $R_\Gamma$ over $W$, called an *agreement relation*, is defined by

$$xR_\Gamma y \overset{\text{def}}{\iff} v(\mathsf{p}\ x) = v(\mathsf{p}\ y) \text{ for all } \mathsf{p} \in \mathscr{P}_\Gamma \tag{1}$$

The agreement relation $R_\Gamma$ induces the quotient set $\tilde{W}_\Gamma \overset{\text{def}}{=} W\ \ R_\Gamma$. Each element $[x]_{R_\Gamma} \in \tilde{W}_\Gamma$ is a granule of possible worlds under $\Gamma$, and called a *granularized possible world*. Hereafter, we denote a granularized world $[x]_{R_\Gamma}$ by $\tilde{x}$. We also construct a truth valuation $\tilde{v}_\Gamma$ for granularized possible worlds. The valuation $\tilde{v}_\Gamma$ becomes the following three-valued one:

$$\tilde{v}_\Gamma : \mathscr{P} \times \tilde{W}_\Gamma \longrightarrow 2^{\{\mathbf{t}\ \mathbf{f}\}} \setminus \{\emptyset\} \tag{2}$$

The three-valued valuation $\tilde{v}_\Gamma$ is defined by

$$\tilde{v}_\Gamma(\mathsf{p}\ \tilde{x}) = \begin{cases} \{\mathbf{t}\} & \text{if } v(\mathsf{p}\ w) = \mathbf{t} \text{ for all } w \in \tilde{x} \\ \{\mathbf{f}\} & \text{if } v(\mathsf{p}\ w) = \mathbf{f} \text{ for all } w \in \tilde{x} \\ \{\mathbf{t}\ \mathbf{f}\} & \text{otherwise} \end{cases} \tag{3}$$

Now we have a *granularized model*

$$\tilde{\mathscr{M}}_\Gamma \overset{\text{def}}{=} \langle \tilde{W}_\Gamma\ \cdots\ \tilde{v}_\Gamma \rangle \tag{4}$$

of $\mathscr{M}$ with respect to $\Gamma$. The three-valued semantic consequence relation $\models_3$ is partially defined: $\tilde{M}_\Gamma\ \tilde{w} \models_3 \mathsf{p} \overset{\text{def}}{\iff} \tilde{v}_\Gamma(\mathsf{p}\ \tilde{x}) = \{\mathbf{t}\}$, and extended by the usual way.

When we move to the next step in some reasoning process, we need to reconstruct the granularized possible worlds and the granularized model. Let $\Gamma$ be the current focus, and $\Delta$ be the focus in the next step.

1. When $\mathscr{P}_\Gamma \supset \mathscr{P}_\Delta$, we need further granularization, which is represented by a mapping

$$\mathscr{O}_\Delta^\Gamma : W_\Gamma \longrightarrow W_\Delta \tag{5}$$

$$\mathscr{O}_\Delta^\Gamma(\tilde{x}) \overset{\text{def}}{=} \{w \in W \mid v(\mathsf{p}\ w) = v(\mathsf{p}\ x) \text{ for all } \mathsf{p} \in \mathscr{P}_\Delta \text{ and } x \in \tilde{x}\} \tag{6}$$

and called a *zooming out from* $\Gamma$ *to* $\Delta$.

2. When $\mathscr{P}_\Gamma \subset \mathscr{P}_\Delta$, we need the inverse operation of granularization. we represent this operation by a mapping

$$\mathscr{I}_\Delta^\Gamma : W_\Gamma \longrightarrow 2^{W_\Delta} \tag{7}$$

$$\mathscr{I}_\Delta^\Gamma(\tilde{x}) \stackrel{\text{def}}{=} \{\tilde{y} \in W_\Delta \mid v(\mathsf{p}\ x) = v(\mathsf{p}\ y) \text{ for all } \mathsf{p} \in \mathscr{P}_\Gamma\ \ x \in \tilde{x} \text{ and } y \in \tilde{y}\} \tag{8}$$

and called a *zooming in from* $\Gamma$ *to* $\Delta$.

3. If $\mathscr{P}_\Gamma$ and $\mathscr{P}_\Delta$ are not nested each other, the movement from $\Gamma$ to $\Delta$ is represented by combination of "zooming in & out", that is, a zooming in from $\Gamma$ to $\Gamma \cup \Delta$ first, and next, a zooming out from $\Gamma \cup \Delta$ to $\Delta$.

## 2.3 Visibility: Another Concept of Granularization

*Visibility* is a term about vision that means the range of vision. Visibility divides objects we can see into two types primitively: objects inside of the range of vision, that is, currently visible objects, and outside objects, that is, currently invisible objects. Moreover, combining the visibility and the focus, visible objects are further divided into two types. If an object is in the range of vision but out of focus, it looks obscurely, and we can look the object clearly only if it is in the focal point.

The authors have introduced the concept of visibility to granular reasoning as an another concept of granularization, and have proposed a four-valued valuation based on the visibility and focus [5]. Let $\Gamma$ be a set of non-modal sentences considered in the current step of reasoning. Using $\Gamma$, we define the *visibility* relative to $\Gamma$. Moreover, we redefine the the concept of the focus, and proposed the *focus* relative to $\Gamma$. The definitions of the visibility $Vs(\Gamma)$ and focus $Fc(\Gamma)$ relative to $\Gamma$ are as follows:

$$Vs(\Gamma) \stackrel{\text{def}}{=} \mathscr{P} \cap \mathrm{Sub}(\Gamma) \ = \ \mathscr{P}_\Gamma \tag{9}$$

$$Fc(\Gamma) \stackrel{\text{def}}{=} \{\mathsf{p} \in \mathscr{P} \mid \text{either } \Gamma \vdash \mathsf{p} \text{ or } \Gamma \vdash \neg\mathsf{p}\} \tag{10}$$

Note that we have $Fc(\Gamma) \subseteq Vs(\Gamma)$ for any $\Gamma$.

To characterize the semantic meaning of visibility and focus, we also construct a granularized model $\tilde{\mathscr{M}}_{Fc(\Gamma)}$ based on the focus $Fc(\Gamma)$ relative to $\Gamma$. First, if we have $Fc(\Gamma) \neq \emptyset$, we define the agreement relation $R_{Fc(\Gamma)}$ by (1), and construct the set of granularized possible worlds $\tilde{W}_{Fc(\Gamma)}$. On the other hand, if $Fc(\Gamma) = \emptyset$, we define $\tilde{W}_{Fc(\Gamma)} \stackrel{\text{def}}{=} \{W\}$.

Next, we construct a valuation $\tilde{v}_{Fc(\Gamma)}$ in the granularized model $\tilde{\mathscr{M}}_{Fc(\Gamma)}$ as the following four-valued valuation:

$$\tilde{v}_{Fc(\Gamma)} : \mathscr{P} \times \tilde{W}_{Fc(\Gamma)} \longrightarrow 2^{\{\mathbf{t}\ \mathbf{f}\}} \tag{11}$$

Actually, the four-valued valuation $\tilde{v}_{Fc(\Gamma)}$ is defined by

$$\tilde{v}_{Fc(\Gamma)}(\mathsf{p}\ \tilde{w}) \stackrel{\text{def}}{=} \begin{cases} \{\mathbf{t}\} & \text{if } \mathsf{p} \in Vs(\Gamma) \text{ and } v(\mathsf{p}\ x) = \mathbf{t} \text{ for all } x \in \tilde{w} \\ \{\mathbf{f}\} & \text{if } \mathsf{p} \in Vs(\Gamma) \text{ and } v(\mathsf{p}\ x) = \mathbf{f} \text{ for all } x \in \tilde{w} \\ \{\mathbf{t}\ \mathbf{f}\} & \begin{array}{l}\text{if } \mathsf{p} \in Vs(\Gamma) \text{ but } v(\mathsf{p}\ x) = \mathbf{t} \text{ for some } x \in \tilde{w} \\ \text{and } v(\mathsf{p}\ y) = \mathbf{f} \text{ for some } y \in \tilde{w}\end{array} \\ \emptyset & \text{if } \mathsf{p} \notin Vs(\Gamma) \end{cases} \tag{12}$$

An atomic sentence $\mathsf{p}$ is *clearly visible* at the granularized possible world $\tilde{w}$ if and only if either $\tilde{v}_{Vs(\Gamma)}(\mathsf{p}\ \tilde{w}) = \{\mathbf{t}\}$ or $\tilde{v}_{Vs(\Gamma)}(\mathsf{p}\ \tilde{w}) = \{\mathbf{f}\}$. On the other hand, $\mathsf{p}$ is *obscurely visible* if and only if $\tilde{v}_{Vs(\Gamma)}(\mathsf{p}\ \tilde{w}) = \{\mathbf{t}\ \mathbf{f}\}$. Moreover, $\mathsf{p}$ is *invisible* if and only if $\tilde{v}_{Vs(\Gamma)}(\mathsf{p}\ \tilde{w}) = \emptyset$.

# 3 VISIBILITY AND FOCUS: REVISED

In this section, we revise our formulation of visibility and focus. We have proposed visibility as another concept of granular reasoning, and formulated the four-valued valuation $\tilde{v}_{Fc(\Gamma)}$ based on visibility and focus [5], however, our formulation had some difficulties to extend $\tilde{v}_{Fc(\Gamma)}$ to all non-modal sentences. In the definition of $\tilde{v}_{Fc(\Gamma)}$ by (12), we have used the visibility $Vs(\Gamma)$ to check whether an atomic sentence $\mathsf{p}$ is visible. If we extend $\tilde{v}_{Fc(\Gamma)}$ to all non-modal sentences, we need to check, for any non-modal sentence $p$, whether $p$ is visible. However, it is determined by the truth value $\tilde{v}_{Fc(\Gamma)}(p, \tilde{w})$. Therefore, it becomes a circular argument.

Our main idea to overcome these difficulties is to construct *equivalence classes of granularized possible worlds* by the following two steps:

1. Using visibility, we construct granularized possible worlds to divide all non-modal sentences into "visible" ones and "invisible" ones.
2. Using focus, we construct equivalence classes of granularized possible worlds to divide all "visible "sentences into "clearly visible" ones and "obscurely visible" ones.

## 3.1 Part 1: Granularized Possible Worlds Based on Visibility

We formulate a set of granularized possible worlds based on visibility and a three-valued valuation to determine whether each non-modal sentence is visible.

Let $\Gamma$ be a set of non-modal sentences considered in the current step of reasoning. Using $\Gamma$, we define the visibility $Vs(\Gamma)$ and focus $Fc(\Gamma)$ relative to $\Gamma$ by (9) and (10), respectively. We construct the agreement relation $R_{Vs(\Gamma)}$ based on the visibility $Vs(\Gamma)$ as follows:

$$xR_{Vs(\Gamma)}y \overset{\text{def}}{\Longleftrightarrow} v(\mathsf{p}, x) = v(\mathsf{p}, y) \quad \forall \mathsf{p} \in Vs(\Gamma) \tag{13}$$

The agreement relation $R_{Vs(\Gamma)}$ induces the set of granularized possible worlds $\tilde{W} \overset{\text{def}}{=} W / R_{Vs(\Gamma)}$. We also construct a truth valuation $\tilde{v}_{Vs(\Gamma)}$ for granularized possible worlds $\tilde{x} \overset{\text{def}}{=} [x]_{R_{Vs(\Gamma)}} \in \tilde{W}$ . The valuation $\tilde{v}_{Vs(\Gamma)}$ becomes the following three-valued one:

$$\tilde{v}_{Vs(\Gamma)} : \mathscr{P} \times \tilde{W} \longrightarrow 2^{\{\mathbf{t},\mathbf{f}\}} \setminus \{\{\mathbf{t}, \mathbf{f}\}\} \tag{14}$$

The three-valued valuation $\tilde{v}_{Vs(\Gamma)}$ is defined by:

$$\tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{w}) \overset{\text{def}}{=} \begin{cases} \{\mathbf{t}\} & \text{if } v(\mathsf{p}, x) = \mathbf{t} \text{ for all } x \in \tilde{w} \\ \{\mathbf{f}\} & \text{if } v(\mathsf{p}, x) = \mathbf{f} \text{ for all } x \in \tilde{w} \\ \emptyset & \text{otherwise} \end{cases} \tag{15}$$

Hereafter, we use the following notations: $\mathbf{T} \overset{\text{def}}{=} \{\mathbf{t}\}$ and $\mathbf{F} \overset{\text{def}}{=} \{\mathbf{f}\}$, respectively. We call that an atomic sentence $\mathsf{p}$ is *visible* at $\tilde{w}$ if and only if either $\tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{w}) = \mathbf{T}$ or $\tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{w}) = \mathbf{F}$. Otherwise, we call that $\mathsf{p}$ is *invisible* at $\tilde{w}$. By this definition, for any $\mathsf{p} \in Vs(\Gamma)$, it is clear that $\mathsf{p}$ is visible at all $\tilde{w} \in \tilde{W}$.

The three-valued valuation $\tilde{v}_{Vs(\Gamma)}$ is extended to all non-modal sentences by truth assignments of connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication) illustrated in Table 1. We denote the extended three-valued valuation by the same notation $\tilde{v}_{Vs(\Gamma)}$. Similar to the case of atomic sentences, for any non-modal sentence $p$, we call $p$ is visible at $\tilde{w}$ if and only if either $\tilde{v}_{Vs(\Gamma)}(p, \tilde{w}) = \mathbf{T}$ or $\tilde{v}_{Vs(\Gamma)}(p, \tilde{w}) = \mathbf{F}$. Otherwise, we call that $p$ is invisible at $\tilde{w}$. Hence, if both $p$ and $q$ are visible, it is clear that $\neg p$, $p \wedge q$, $p \vee q$ and $p \rightarrow q$ are also visible.

These truth assignments are direct extensions of two-valued truth assignments by simply adding the third truth value $\emptyset$, and may look unnatural. By these truth assignments, we intend to extend the concept of visibility to all non-modal sentences. The visibility $Vs(\Gamma)$ relative to $\Gamma$ is the set of *all* atomic sentences that we consider at the current reasoning step, thus, the set of all non-modal sentences that we can consider at the current step becomes the sublanguage $\mathscr{L}(Vs(\Gamma))$, that is, the subset of $\mathscr{L}(\mathscr{P})$ that are generated from $Vs(\Gamma)$ by usual way. It is easy to check that, for any sentence $p \in \mathscr{L}(Vs(\Gamma))$, there is some $\tilde{w} \in \tilde{W}$ such that $p$ is visible at $\tilde{w}$. These facts indicate that the definition of $\tilde{v}_{Vs(\Gamma)}(p, \tilde{w})$ and truth assignments illustrated in Table 1 are well-defined, and they capture some of the important properties of visibility.

## 3.2 Part 2: Equivalence Classes of Granularized Possible Worlds Based on Focus

As we mentioned at the first of this section, using focus, we intend to divide all "visible "sentences into "clearly visible" ones and "obscurely visible" ones. To illustrate this intention, we formulate a set of equivalence classes of granularized possible worlds, and a four-valued valuation to determine whether each visible sentence is "clearly visible".

**TABLE 1.** Truth tables of the three-valued valuation

Negation $\neg p$

| $p$ | $\neg p$ |
|---|---|
| $\emptyset$ | $\emptyset$ |
| **F** | **T** |
| **T** | **F** |

Conjunction $p \wedge q$

| $p \backslash q$ | $\emptyset$ | **F** | **T** |
|---|---|---|---|
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| **F** | $\emptyset$ | **F** | **F** |
| **T** | $\emptyset$ | **F** | **T** |

Disjunction $p \vee q$

| $p \backslash q$ | $\emptyset$ | **F** | **T** |
|---|---|---|---|
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| **F** | $\emptyset$ | **F** | **T** |
| **T** | $\emptyset$ | **T** | **T** |

Implication $p \rightarrow q$

| $p \backslash q$ | $\emptyset$ | **F** | **T** |
|---|---|---|---|
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| **F** | $\emptyset$ | **T** | **T** |
| **T** | $\emptyset$ | **F** | **T** |

Using the focus $Fc(\Gamma)$ relative to $\Gamma$, we construct an agreement relation $R_{Fc(\Gamma)}$ over the *set of granularized possible worlds* $\tilde{W}$. If $Fc(\Gamma) \neq \emptyset$, we define the agreement relation $R_{Fc(\Gamma)}$ as follows:

$$\tilde{x}\, R_{Fc(\Gamma)}\, \tilde{y} \overset{\text{def}}{\Longleftrightarrow} \tilde{v}_{Vs(\Gamma)}(\mathsf{p}\ \tilde{x}) = \tilde{v}_{Vs(\Gamma)}(\mathsf{p}\ \tilde{y}) \quad \forall \mathsf{p} \in Fc(\Gamma) \tag{16}$$

Note that, because $Fc(\Gamma) \subseteq Vs(\Gamma)$, each $\mathsf{p} \in Fc(\Gamma)$ has either $\tilde{v}_{Vs(\Gamma)}(\mathsf{p}\ \tilde{w}) = \mathbf{T}$ or $\tilde{v}_{Vs(\Gamma)}(\mathsf{p}\ \tilde{w}) = \mathbf{F}$ at each $\tilde{w} \in \tilde{W}$. The agreement relation $R_{Fc(\Gamma)}$ over $\tilde{W}$ induce the quotient set of granularized possible worlds $\widehat{W} \overset{\text{def}}{=} \tilde{W}\ \ R_{Fc(\Gamma)}$. We treat each equivalence class $\widehat{w} \overset{\text{def}}{=} [\tilde{w}]_{R_{Fc(\Gamma)}}$ as a unit of consideration as if each $\widehat{w}$ were a "possible world". On the other hand, if $Fc(\Gamma) = \emptyset$, we can not construct the agreement relation. In this case, we define $\widehat{W} \overset{\text{def}}{=} \{\tilde{W}\}$.

We consider a valuation function $\widehat{v}_{Fc(\Gamma)}$ for equivalence classes of granularized possible worlds as the following four-valued one:

$$\widehat{v}_{Fc(\Gamma)} : \mathscr{P} \times \widehat{W} \longrightarrow 2^{\{\mathbf{T}\ \mathbf{F}\}} \tag{17}$$

The valuation $\widehat{v}_{Fc(\Gamma)}$ is defined by:

$$\widehat{v}_{Fc(\Gamma)}(\mathsf{p}\ \widehat{w}) \overset{\text{def}}{=} \begin{cases} \{\mathbf{T}\} & \tilde{v}_{Vs(\Gamma)}(\mathsf{p}\ \tilde{x}) = \mathbf{T} \text{ for all } \tilde{x} \in \widehat{w} \\ \{\mathbf{F}\} & \tilde{v}_{Vs(\Gamma)}(\mathsf{p}\ \tilde{x}) = \mathbf{F} \text{ for all } \tilde{x} \in \widehat{w} \\ \{\mathbf{T}\ \mathbf{F}\} & \begin{array}{l} \tilde{v}_{Vs(\Gamma)}(\mathsf{p}\ \tilde{x}) = \mathbf{T} \text{ for some } \tilde{x} \in \widehat{w} \\ \text{and } \tilde{v}_{Vs(\Gamma)}(\mathsf{p}\ \tilde{y}) = \mathbf{F} \text{ for some } \tilde{y} \in \widehat{w} \end{array} \\ \emptyset & \text{otherwise} \end{cases} \tag{18}$$

We call that an atomic sentence $\mathsf{p}$ is *clearly visible* at an equivalence class of granularized possible worlds $\widehat{w}$ if and only if either $\widehat{v}_{Fc(\Gamma)}(\mathsf{p}\ \widehat{w}) = \{\mathbf{T}\}$ or $\widehat{v}_{Fc(\Gamma)}(\mathsf{p}\ \widehat{w}) = \{\mathbf{F}\}$. On the other hand, $\mathsf{p}$ is *obscurely visible* at $\widehat{w}$ if and only if $\widehat{v}_{Fc(\Gamma)}(\mathsf{p}\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$. Otherwise, $\mathsf{p}$ is *invisible* at $\widehat{w}$.

Similar to the case of the three-valued valuation $\tilde{v}_{Vs(\Gamma)}$, the four-valued valuation $\widehat{v}_{Fc(\Gamma)}$ is extended to all non-modal sentences by truth assignments illustrated in Table 2. We denote the extended four-valued valuation by the same notation $\widehat{v}_{Fc(\Gamma)}$. Similar to the three-valued case, for any clearly visible sentences $p$ and $q$, it is obvious that $\neg p$, $p \wedge q$, $p \vee q$ and $p \rightarrow q$ are also clearly visible. Note that there is at least one equivalence class $\widehat{w} \in \widehat{W}$ such that $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \{\mathbf{T}\}$ for all $p \in \Gamma$. Moreover, it is easy to check that $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \emptyset$ at all $\widehat{w} \in \widehat{W} \iff \tilde{v}_{Fc(\Gamma)}(p\ \tilde{w}) = \emptyset$ at all $\tilde{w} \in \tilde{W}$, that is, $p$ is invisible at all $\widehat{w} \in \widehat{W}$ if and only if $p$ is invisible at all $\tilde{w} \in \tilde{W}$.

We denote $p \equiv q$ if and only if $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \widehat{v}_{Fc(\Gamma)}(q\ \widehat{w})$ at all $\widehat{w} \in \widehat{W}$. It is easy to check that $\tilde{v}_{Fc(\Gamma)}$ satisfies some two-valued tautologies.

**Proposition 1** For any non-modal sentences $p\ q\ r \in \mathscr{L}(\mathscr{P})$, the four-valued valuation $\tilde{v}_{Fc(\Gamma)}$ validates the following properties:

- All associative and commutative laws for $\wedge$ and $\vee$.
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (Distributive laws).
- $\neg\neg p \equiv p$, $p \rightarrow q \equiv \neg p \vee q$.
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$, $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (De Morgan's laws).
- $p \rightarrow q \equiv \neg q \rightarrow \neg p$ (Contraposition).
- $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$ (Exportation).

The proof of Proposition 1 is in Appendix.

Note that, however, not all two-valued tautologies are satisfied by $\tilde{v}_{Fc(\Gamma)}$. For example, for any invisible sentence $p$ and obscurely visible sentence $q$, exclusive middle is not satisfied: $\tilde{v}_{Fc(\Gamma)}(p \vee \neg p\ \widehat{w}) = \emptyset$ and $\tilde{v}_{Fc(\Gamma)}(q \vee \neg q\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ for all $\widehat{w} \in \widehat{W}$.

**TABLE 2.** Truth tables of the four-valued valuation

Negation $\neg p$

| $p$ | $\neg p$ |
|---|---|
| $\emptyset$ | $\emptyset$ |
| $\{\mathbf{F}\}$ | $\{\mathbf{T}\}$ |
| $\{\mathbf{T}\}$ | $\{\mathbf{F}\}$ |
| $\{\mathbf{T}\ \mathbf{F}\}$ | $\{\mathbf{T}\ \mathbf{F}\}$ |

Disjunction $p \vee q$

| $p \backslash q$ | $\emptyset$ | $\{\mathbf{F}\}$ | $\{\mathbf{T}\}$ | $\{\mathbf{T}\ \mathbf{F}\}$ |
|---|---|---|---|---|
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $\{\mathbf{F}\}$ | $\emptyset$ | $\{\mathbf{F}\}$ | $\{\mathbf{T}\}$ | $\{\mathbf{T}\ \mathbf{F}\}$ |
| $\{\mathbf{T}\}$ | $\emptyset$ | $\{\mathbf{T}\}$ | $\{\mathbf{T}\}$ | $\{\mathbf{T}\}$ |
| $\{\mathbf{T}\ \mathbf{F}\}$ | $\emptyset$ | $\{\mathbf{T}\ \mathbf{F}\}$ | $\{\mathbf{T}\}$ | $\{\mathbf{T}\ \mathbf{F}\}$ |

Conjunction $p \wedge q$

| $p \backslash q$ | $\emptyset$ | $\{\mathbf{F}\}$ | $\{\mathbf{T}\}$ | $\{\mathbf{T}\ \mathbf{F}\}$ |
|---|---|---|---|---|
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $\{\mathbf{F}\}$ | $\emptyset$ | $\{\mathbf{F}\}$ | $\{\mathbf{F}\}$ | $\{\mathbf{F}\}$ |
| $\{\mathbf{T}\}$ | $\emptyset$ | $\{\mathbf{F}\}$ | $\{\mathbf{T}\}$ | $\{\mathbf{T}\ \mathbf{F}\}$ |
| $\{\mathbf{T}\ \mathbf{F}\}$ | $\emptyset$ | $\{\mathbf{F}\}$ | $\{\mathbf{T}\ \mathbf{F}\}$ | $\{\mathbf{T}\ \mathbf{F}\}$ |

Implication $p \to q$

| $p \backslash q$ | $\emptyset$ | $\{\mathbf{F}\}$ | $\{\mathbf{T}\}$ | $\{\mathbf{T}\ \mathbf{F}\}$ |
|---|---|---|---|---|
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $\{\mathbf{F}\}$ | $\emptyset$ | $\{\mathbf{T}\}$ | $\{\mathbf{T}\}$ | $\{\mathbf{T}\}$ |
| $\{\mathbf{T}\}$ | $\emptyset$ | $\{\mathbf{F}\}$ | $\{\mathbf{T}\}$ | $\{\mathbf{T}\ \mathbf{F}\}$ |
| $\{\mathbf{T}\ \mathbf{F}\}$ | $\emptyset$ | $\{\mathbf{T}\ \mathbf{F}\}$ | $\{\mathbf{T}\}$ | $\{\mathbf{T}\ \mathbf{F}\}$ |

Now we have a *four-valued granularized model*

$$\widehat{\mathscr{M}_\Gamma} \stackrel{\text{def}}{=} \langle \widehat{W} \ \cdots \ \widehat{v}_{Fc(\Gamma)} \rangle \tag{19}$$

of $\mathscr{M}$ with respect to $Vs(\Gamma)$ and $Fc(\Gamma)$. Moreover, the four-valued semantic consequence relation $\models_4$ is partially defined and extended by the usual way as follows. $\widehat{M} \ \widehat{w} \models_4 p$ means that the sentence $p$ is true at the equivalence class $\widehat{w}$ in the model $\widehat{\mathscr{M}_\Gamma}$:

$$\begin{aligned}
\widehat{M} \ \widehat{w} \models_4 \mathsf{p} &\stackrel{\text{def}}{\iff} \widehat{v}_{Fc(\Gamma)}(\mathsf{p} \ \widehat{w}) = \{\mathbf{T}\} \ \ \forall \mathsf{p} \in \mathscr{P} \\
\widehat{M} \ \widehat{w} \models_4 \neg p &\iff \widehat{v}_{Fc(\Gamma)}(p \ \widehat{w}) = \{\mathbf{F}\} \\
\widehat{M} \ \widehat{w} \models_4 p \wedge q &\iff \widehat{v}_{Fc(\Gamma)}(p \ \widehat{w}) = \{\mathbf{T}\} \text{ and } \widehat{v}_{Fc(\Gamma)}(q \ \widehat{w}) = \{\mathbf{T}\} \\
\widehat{M} \ \widehat{w} \models_4 p \vee q &\iff \widehat{v}_{Fc(\Gamma)}(p \ \widehat{w}) = \{\mathbf{T}\} \text{ or } \widehat{v}_{Fc(\Gamma)}(q \ \widehat{w}) = \{\mathbf{T}\} \\
\widehat{M} \ \widehat{w} \models_4 p \to q &\iff \widehat{v}_{Fc(\Gamma)}(q \ \widehat{w}) = \{\mathbf{T}\} \text{ whenever } \widehat{v}_{Fc(\Gamma)}(p \ \widehat{w}) = \{\mathbf{T}\}
\end{aligned}$$

For any subset $\Delta$ of non-modal sentences, we denote $\widehat{\mathscr{M}} \ \widehat{w} \models_4 \Delta$ if $\widehat{\mathscr{M}} \ \widehat{w} \models_4 p$ for all $p \in \Delta$. We define the set of equivalence classes $\|\Delta\|_4 \stackrel{\text{def}}{=} \{\widehat{w} \in \widehat{W} \mid \widehat{\mathscr{M}} \ \widehat{w} \models_4 \Delta\}$, and called the *truth set* of $\Delta$. If $\widehat{\mathscr{M}} \ \widehat{w} \models_4 \Delta$ holds for all $\widehat{w} \in \widehat{W}$, we denote $\widehat{\mathscr{M}} \models_4 \Delta$. In the case $\Delta = \{p\}$, we abbreviate $\widehat{\mathscr{M}} \models_4 p$, and called that *$p$ is valid in $\widehat{\mathscr{M}}$*. Moreover, if $\widehat{\mathscr{M}} \ \widehat{w} \models_4 \Delta$ implies $\widehat{\mathscr{M}} \ \widehat{w} \models_4 p$ for all $\widehat{w} \in \widehat{W}$, we denote $\widehat{\mathscr{M}} \ \Delta \models_4 p$, and called that $\Delta$ *concludes $p$ in $\widehat{\mathscr{M}}$*. Similar to the classical propositional logic, the semantic version of the *deduction theorem* is satisfied.

**Proposition 2** If $p$ is clearly visible at all $\widehat{w} \in \widehat{W}$, then

$$\widehat{\mathscr{M}} \ \Delta \cup \{p\} \models_4 q \iff \widehat{\mathscr{M}} \ \Delta \models_4 p \to q$$

The proof of Proposition 2 is in Appendix.

**Example 1** Let $\mathscr{P} = \{\mathsf{p} \ \mathsf{q} \ \mathsf{r}\}$ be a set of atomic sentences, and $\mathscr{M} = \langle W \ R \ v \rangle$ be a Kripke model with the following eight possible worlds:

$$\begin{array}{llll}
w_1 = \{\mathsf{p} \ \mathsf{q} \ \mathsf{r}\}, & w_2 = \{\mathsf{p} \ \mathsf{q}\}, & w_3 = \{\mathsf{p} \ \mathsf{r}\}, & w_4 = \{\mathsf{p}\}, \\
w_5 = \{\mathsf{q} \ \mathsf{r}\}, & w_6 = \{\mathsf{q}\}, & w_7 = \{\mathsf{r}\}, & w_8 = \emptyset.
\end{array}$$

We define the truth value of each atomic sentence $p \in \mathscr{P}$ at each world $w \in W$ by $v(p \ w) = \mathbf{t} \iff p \in w$. By this truth assignment, for example, all atomic sentences are true at $w_1$. On the other hand, all atomic sentences are false at $w_8$.

Suppose we have the following set of non-modal sentences considered in the current step of reasoning: $\Gamma = \{\mathsf{q} \ \mathsf{p} \to \mathsf{q}\}$. Hence, we have the visibility $Vs(\Gamma) = \{\mathsf{p} \ \mathsf{q}\}$, and focus $Fc(\Gamma) = \{\mathsf{q}\}$ relative to $\Gamma$, respectively:

Constructing the agreement relation $R_{Vs(\Gamma)}$ by (13), we have the following four granularized possible worlds:

$$\tilde{w}_1 = \{w_1 \ w_2\}, \tilde{w}_3 = \{w_3 \ w_4\}, \tilde{w}_5 = \{w_5 \ w_6\}, \tilde{w}_7 = \{w_7 \ w_8\}.$$

Each atomic sentence has the following three-valued truth value:

$$\begin{array}{lll} \tilde{v}_{Fc(\Gamma)}(\mathsf{p}\ \tilde{w}_1) = \mathbf{T} & \tilde{v}_{Fc(\Gamma)}(\mathsf{q}\ \tilde{w}_1) = \mathbf{T} & \tilde{v}_{Fc(\Gamma)}(\mathsf{r}\ \tilde{w}_1) = \emptyset \\ \tilde{v}_{Fc(\Gamma)}(\mathsf{p}\ \tilde{w}_3) = \mathbf{T} & \tilde{v}_{Fc(\Gamma)}(\mathsf{q}\ \tilde{w}_3) = \mathbf{F} & \tilde{v}_{Fc(\Gamma)}(\mathsf{r}\ \tilde{w}_3) = \emptyset \\ \tilde{v}_{Fc(\Gamma)}(\mathsf{p}\ \tilde{w}_5) = \mathbf{F} & \tilde{v}_{Fc(\Gamma)}(\mathsf{q}\ \tilde{w}_5) = \mathbf{T} & \tilde{v}_{Fc(\Gamma)}(\mathsf{r}\ \tilde{w}_5) = \emptyset \\ \tilde{v}_{Fc(\Gamma)}(\mathsf{p}\ \tilde{w}_7) = \mathbf{F} & \tilde{v}_{Fc(\Gamma)}(\mathsf{q}\ \tilde{w}_7) = \mathbf{F} & \tilde{v}_{Fc(\Gamma)}(\mathsf{r}\ \tilde{w}_7) = \emptyset \end{array}$$

These truth values indicate that $\mathsf{p}$ and $\mathsf{q}$ are visible, while $\mathsf{r}$ is invisible.

Next, we construct the agreement relation $R_{Fc(\Gamma)}$ over $\tilde{W}$, and get the following two equivalence classes:

$$\widehat{w_1} = \{\tilde{w_1}\ \tilde{w_5}\} = \{\{w_1\ w_2\}\ \{w_5\ w_6\}\}, \widehat{w_3} = \{\tilde{w_3}\ \tilde{w_7}\} = \{\{w_3\ w_4\}\ \{w_7\ w_8\}\}.$$

By (18), each atomic sentence has the following four-valued truth value:

$$\begin{array}{lll} \widehat{v}_{Fc(\Gamma)}(\mathsf{p}\ \widehat{w}_1) = \{\mathbf{T}\ \mathbf{F}\} & \widehat{v}_{Fc(\Gamma)}(\mathsf{q}\ \widehat{w}_1) = \{\mathbf{T}\} & \widehat{v}_{Fc(\Gamma)}(\mathsf{r}\ \widehat{w}_1) = \emptyset \\ \widehat{v}_{Fc(\Gamma)}(\mathsf{p}\ \widehat{w}_3) = \{\mathbf{T}\ \mathbf{F}\} & \widehat{v}_{Fc(\Gamma)}(\mathsf{q}\ \widehat{w}_3) = \{\mathbf{F}\} & \widehat{v}_{Fc(\Gamma)}(\mathsf{r}\ \widehat{w}_3) = \emptyset \end{array}$$

This means that $\mathsf{q}$ is clearly visible, but $\mathsf{p}$ is obscurely visible. Similar to the three-valued case, $\mathsf{r}$ is invisible. Four-valued truth values of any non-modal sentences are calculated based on Table 2. For example, the truth value of $\mathsf{p} \to \mathsf{q}$ is: $\widehat{v}_{Fc(\Gamma)}(\mathsf{p} \to \mathsf{q}\ \widehat{w}_1) = \{\mathbf{T}\}$ and $\widehat{v}_{Fc(\Gamma)}(\mathsf{p} \to \mathsf{q}\ \widehat{w}_3) = \{\mathbf{T}\ \mathbf{F}\}$. Thus, all non-modal sentences in $\Gamma$ are true, that is, clearly visible, at $\widehat{w}_1$.

## 4 DISCUSSION

As illustrated in Example 1, our revised framework of four-valued valuation avoids the circular argument mentioned in Section 3, which overcome the difficulties of the previous formulation about extension of four-valued valuation. However, our four-valued valuation is quite different to Belnap's four-valued logics based on bilattices [1, 2]. Comparison our valuation and other four-valued semantics is a future work.

Murai et al.'s formulation of focus is included into our formulation as a special case. Their formulation corresponds to the case that, ignoring (9) and (10), we define $Vs(\Gamma) \stackrel{\text{def}}{=} \mathscr{P}$ and $Fc(\Gamma) \stackrel{\text{def}}{=} \mathscr{P}_\Gamma$, that is, all atomic sentences are visible, and all atomic sentences which appear in $\Gamma$ are clearly visible.

If we regard the set $\Gamma$ as the current *knowledge base* about the current world $w$, our intention about visibility and focus becomes more clear. We illustrate this by using example 1. Suppose we *believe* $\Gamma = \{\mathsf{q}\ \mathsf{p} \to \mathsf{q}\}$ that represents our beliefs about $w$. The meaning of "we believe $\Gamma$" is that we believe that all sentences in $\Gamma$ are true. Thus, constructing $\widehat{W}$ and $\widehat{v}_{Fc(\Gamma)}$ by the visibility $Vs(\Gamma)$ and focus $Fc(\Gamma)$, we have a "model" $\widehat{w}_1$ about $w$, which illustrates that $\mathsf{q}$ is "true", and $\mathsf{p}$ is "unknown". In $\Gamma$, we just consider $\mathsf{p}$ and $\mathsf{q}$, and do not know the existence an other atomic sentence, $\mathsf{r}$, therefore $\mathsf{r}$ should be "undefined" at the model.

Moreover, if we observe the world, and get new information about the world, it cause some changes of $\Gamma$, and reconstruction $\widehat{W}$ and $\widehat{v}_{Fc(\Gamma)}$. This connects to *belief change* (for example, see [3, 4]) and *zooming in & out* operations [8, 9, 10, 11], which is a future work.

## 5 CONCLUSION

In this paper, we refined our previous formulation of visibility and focus [5] by further granularization to granularized possible worlds. We also reformulated the four-valued valuation based on visibility and focus, and it was extended to all non-modal sentences. These results overcome the difficulties of the previous formulation about extension of four-valued valuation. More refinement of the proposed framework and explore of connections with, for example, belief change and zooming in & out operations discussed in the previous section are future works.


## ACKNOWLEDGMENT

The first author was partially supported by Grant-in-Aid No.17700222-1 for Young Scientists(B) of the Ministry of Education, Culture, Sports, Science and Technology, Japan.

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# APPENDIX

## Proofs

**Proposition 1** For any non-modal sentences $p\ q\ r \in \mathscr{L}(\mathscr{P})$, the four-valued valuation $\tilde{v}_{Fc(\Gamma)}$ validates the following properties:

- All associative and commutative laws for $\wedge$ and $\vee$.
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (Distributive laws).
- $\neg\neg p \equiv p$, $p \to q \equiv \neg p \vee q$.
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$, $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (De Morgan's laws).
- $p \to q \equiv \neg q \to \neg p$ (Contraposition).
- $(p \wedge q) \to r \equiv p \to (q \to r)$ (Exportation).

*Proof.*

We show that the distributive law $p \wedge (q \vee r) \equiv (p \vee q) \wedge (p \vee r)$ is satisfied by checking $\widehat{v}_{Fc(\Gamma)}(p \wedge (q \vee r)\ \widehat{w}) = \widehat{v}_{Fc(\Gamma)}((p \wedge q) \vee (p \wedge r)\ \widehat{w})$ for each case that $\widehat{v}_{Fc(\Gamma)}(p \wedge (q \vee r)\ \widehat{w}) = \mathbf{T}\ \ \mathbf{F}$, $\{\mathbf{T}\ \mathbf{F}\}$ and $\emptyset$, respectively. Other properties are proved similarly. By the truth tables of the four-valued valuations illustrated in TABLE 2, each case is proved as follows:

$$
\begin{aligned}
\widehat{v}_{Fc(\Gamma)}(p \wedge (q \vee r)\ \widehat{w}) = \mathbf{T} \quad &\Longleftrightarrow \quad \widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \mathbf{T}, \text{ and } \widehat{v}_{Fc(\Gamma)}(q \vee r\ \widehat{w}) = \mathbf{T} \\
&\Longleftrightarrow \quad \widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \mathbf{T}, \text{ and, either } \widehat{v}_{Fc(\Gamma)}(q\ \widehat{w}) = \mathbf{T} \text{ or } \widehat{v}_{Fc(\Gamma)}(r\ \widehat{w}) = \mathbf{T} \\
&\Longleftrightarrow \quad \begin{array}{l} \text{either } (\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \mathbf{T}, \text{ and } \widehat{v}_{Fc(\Gamma)}(q\ \widehat{w}) = \mathbf{T}\ ), \\ \qquad \text{or } (\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \mathbf{T}, \text{ and } \widehat{v}_{Fc(\Gamma)}(r\ \widehat{w}) = \mathbf{T}) \end{array} \\
&\Longleftrightarrow \quad \widehat{v}_{Fc(\Gamma)}((p \wedge q) \vee (p \wedge r)\ \widehat{w}) = \mathbf{T}.
\end{aligned}
$$

$\widehat{v}_{Fc(\Gamma)}(p \wedge (q \vee r)\ \widehat{w}) = \mathbf{F}$ $\iff$ either $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \mathbf{F}$ or $\widehat{v}_{Fc(\Gamma)}(q \vee r\ \widehat{w}) = \mathbf{F}$
$\iff$ either $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \mathbf{F}$, or, both $\widehat{v}_{Fc(\Gamma)}(q\ \widehat{w}) = \mathbf{F}$ and $\widehat{v}_{Fc(\Gamma)}(r\ \widehat{w}) = \mathbf{F}$
$\iff$ both $\widehat{v}_{Fc(\Gamma)}(p \wedge q\ \widehat{w}) = \mathbf{F}$ and $\widehat{v}_{Fc(\Gamma)}(p \wedge r\ \widehat{w}) = \mathbf{F}$
$\iff$ $\widehat{v}_{Fc(\Gamma)}((p \wedge q) \vee (p \wedge r)\ \widehat{w}) = \mathbf{F}$.

$\widehat{v}_{Fc(\Gamma)}(p \wedge (q \vee r)\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ $\iff$ either
1. $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \mathbf{T}$, and $\widehat{v}_{Fc(\Gamma)}(q \vee r\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$, or
2. $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$, and $\widehat{v}_{Fc(\Gamma)}(q \vee r\ \widehat{w}) = \mathbf{T}$, or
3. $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$, and $\widehat{v}_{Fc(\Gamma)}(q \vee r\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$

In the case of 1. $\iff$ $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \mathbf{T}$, and either
1-1. $\widehat{v}_{Fc(\Gamma)}(q\ \widehat{w}) = \mathbf{F}$ and $\widehat{v}_{Fc(\Gamma)}(r\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$, or
1-2. $\widehat{v}_{Fc(\Gamma)}(q\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ and $\widehat{v}_{Fc(\Gamma)}(r\ \widehat{w}) = \mathbf{F}$, or
1-3. $\widehat{v}_{Fc(\Gamma)}(q\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ and $\widehat{v}_{Fc(\Gamma)}(r\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$
$\iff$ 1-1. $\widehat{v}_{Fc(\Gamma)}(p \wedge q\ \widehat{w}) = \mathbf{F}$ and $\widehat{v}_{Fc(\Gamma)}(p \wedge r\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$, or
1-2. $\widehat{v}_{Fc(\Gamma)}(p \wedge q\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ and $\widehat{v}_{Fc(\Gamma)}(p \wedge r\ \widehat{w}) = \mathbf{F}$, or
1-3. $\widehat{v}_{Fc(\Gamma)}(p \wedge q\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ and $\widehat{v}_{Fc(\Gamma)}(p \wedge r\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$
$\iff$ $\widehat{v}_{Fc(\Gamma)}((p \wedge q) \vee (p \wedge r)\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$.

In the case of 2. $\iff$ $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$, and either $\widehat{v}_{Fc(\Gamma)}(q\ \widehat{w}) = \mathbf{T}$ or $\widehat{v}_{Fc(\Gamma)}(r\ \widehat{w}) = \mathbf{T}$
$\iff$ either $\widehat{v}_{Fc(\Gamma)}(p \wedge q\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ and $\widehat{v}_{Fc(\Gamma)}(p \wedge r\ \widehat{w}) = \mathbf{F}$, or
$\widehat{v}_{Fc(\Gamma)}(p \wedge q\ \widehat{w}) = \mathbf{F}$ and $\widehat{v}_{Fc(\Gamma)}(p \wedge r\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$, or
$\widehat{v}_{Fc(\Gamma)}(p \wedge q\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ and $\widehat{v}_{Fc(\Gamma)}(p \wedge r\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$
$\widehat{v}_{Fc(\Gamma)}((p \wedge q) \vee (p \wedge r)\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$.

In the case of 3. $\iff$ $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$, and either
3-1. $\widehat{v}_{Fc(\Gamma)}(q\ \widehat{w}) = \mathbf{F}$ and $\widehat{v}_{Fc(\Gamma)}(r\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$, or
3-2. $\widehat{v}_{Fc(\Gamma)}(q\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ and $\widehat{v}_{Fc(\Gamma)}(r\ \widehat{w}) = \mathbf{F}$, or
3-3. $\widehat{v}_{Fc(\Gamma)}(q\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ and $\widehat{v}_{Fc(\Gamma)}(r\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$
$\iff$ 3-1. $\widehat{v}_{Fc(\Gamma)}(p \wedge q\ \widehat{w}) = \mathbf{F}$ and $\widehat{v}_{Fc(\Gamma)}(p \wedge r\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$, or
3-2. $\widehat{v}_{Fc(\Gamma)}(p \wedge q\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ and $\widehat{v}_{Fc(\Gamma)}(p \wedge r\ \widehat{w}) = \mathbf{F}$, or
3-3. $\widehat{v}_{Fc(\Gamma)}(p \wedge q\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ and $\widehat{v}_{Fc(\Gamma)}(p \wedge r\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$
$\iff$ $\widehat{v}_{Fc(\Gamma)}((p \wedge q) \vee (p \wedge r)\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$.

Thus, we have $\widehat{v}_{Fc(\Gamma)}(p \vee (q \wedge r)\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$ if and only if $\widehat{v}_{Fc(\Gamma)}((p \wedge q) \vee (p \wedge r)\ \widehat{w}) = \{\mathbf{T}\ \mathbf{F}\}$. It is clear that $\widehat{v}_{Fc(\Gamma)}(p \vee (q \wedge r)\ \widehat{w}) = \emptyset$ if and only if $\widehat{v}_{Fc(\Gamma)}((p \wedge q) \vee (p \wedge r)\ \widehat{w}) = \emptyset$. Therefore, $\widehat{v}_{Fc(\Gamma)}(p \wedge (q \vee r)\ \widehat{w}) = \widehat{v}_{Fc(\Gamma)}((p \wedge q) \vee (p \wedge r)\ \widehat{w})$ holds.

**Proposition 2** If $p$ is clearly visible at all $\widehat{w} \in \widehat{W}$, then

$$\widehat{\mathscr{M}}\ \Delta \cup \{p\} \models_4 q \iff \widehat{\mathscr{M}}\ \Delta \models_4 p \to q$$

*Proof.*

($\Longrightarrow$) Assume that, for any $\widehat{w} \in \widehat{W}$, if $\widehat{\mathscr{M}}\ \widehat{w} \models_4 \Delta \cup \{p\}$ holds, then $\widehat{\mathscr{M}}\ \widehat{w} \models_4 q$ also holds. Because $p$ is clearly visible at all $\widehat{w} \in \widehat{W}$, we have either $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \mathbf{T}$ or $\widehat{v}_{Fc(\Gamma)}(p\ \widehat{w}) = \mathbf{F}$ for all $\widehat{w} \in \widehat{W}$. This means that the truth set $\|\Delta\|_4$ is represented by

$$\|\Delta\|_4 = (\|\Delta\|_4 \cap \|p\|_4) \cup (\|\Delta\|_4 \cap \|\neg p\|_4)$$

Hence, if $\widehat{w} \in (\|\Delta\|_4 \cap \|p\|_4)$, we have $\widehat{\mathscr{M}}\ \widehat{w} \models_4 q$ by the assumption, and therefore $\widehat{\mathscr{M}}\ \widehat{w} \models_4 p \to q$ holds. On the other hand, if $\widehat{w} \in (\|\Delta\|_4 \cap \|\neg p\|_4)$, it is clear that $\widehat{\mathscr{M}}\ \widehat{w} \models_4 p \to q$ because $\widehat{w} \in \|\neg p\|_4$. Therefore, in the both cases, we conclude that $\widehat{\mathscr{M}}\ \widehat{w} \models_4 p \to q$ for any $\widehat{w} \in \|\Delta\|_4$.

($\Longleftarrow$) Assume that, for any $\widehat{w} \in \widehat{W}$, if $\widehat{\mathscr{M}}\ \widehat{w} \models_4 \Delta$ holds, then $\widehat{\mathscr{M}}\ \widehat{w} \models_4 p \to q$ also holds. Because $\widehat{w} \in \|\Delta \cup \{p\}\|_4$ satisfies both $\widehat{w} \in \|\Delta\|_4$ and $\widehat{w} \in \|p\|_4$, we have both $\widehat{\mathscr{M}}\ \widehat{w} \models_4 p \to q$ and $\widehat{\mathscr{M}}\ \widehat{w} \models_4 p$ for such $\widehat{w} \in \|\Delta \cup \{p\}\|_4$, which means that $\widehat{\mathscr{M}}\ \widehat{w} \models_4 q$ holds for any $\widehat{w} \in \|\Delta \cup \{p\}\|_4$.