Ikeda's conjecture on the period of the Duke-Imamoglu-Ikeda lift

著者	KATSURADA Hidenori, KAWAMURA Hisa-aki
journal or publication title	Proceedings of the London Mathematical Society. Ser. 3
volume	111
number	5
page range	445-483
year	2015
URL	http://hdl.handle.net/10258/3828

doi: info:doi/10.1112/plms/pdv011

Ikeda's conjecture on the period of the Duke-Imamoglu-Ikeda lift

著者	KATSURADA Hidenori, KAWAMURA Hisa-aki
journal or publication title	Proceedings of the London Mathematical Society. Ser. 3
volume	111
number	5
page range	445-483
year	2015
URL	http://hdl.handle.net/10258/3828

doi: info:doi/10.1112/plms/pdv011

Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

Ikeda's conjecture on the period of the Duke-Imamoḡlu-Ikeda lift

Hidenori Katsurada and Hisa-aki Kawamura

Abstract

Let k and n be positive even integers. For a cuspidal Hecke eigenform h in the Kohnen plus space of weight $k-n/2+1/2$ for $\Gamma_0(4)$, let $I_n(h)$ be the Duke-Imamoḡlu-Ikeda lift of h in the space of cusp forms of weight k for $Sp_n(\mathbf{Z})$, and f the primitive form of weight $2k-n$ for $SL_2(\mathbf{Z})$ corresponding to h under the Shimura correspondence. We then express the ratio $\langle I_n(h), I_n(h)\rangle/\langle h,h\rangle$ of the period of $I_n(h)$ to that of h in terms of special values of certain L-functions of f. This proves the conjecture proposed by Ikeda concerning the period of the Duke-Imamoḡlu-Ikeda lift.

1. *Introduction*

One of the fascinating problems in the theory of modular forms is to find the relation between the periods (or the Petersson products) of cuspidal Hecke eigenforms which are related with each other through their L-functions. In particular, there are several important results concerning the relation between the period of a cuspidal Hecke eigenform g for an elliptic modular group $\Gamma \subseteq SL_2(\mathbf{Z})$ and that of its lift $\widehat{g}$. Here, by a lift of g we mean a cuspidal Hecke eigenform for another modular group Γ' (e.g. the symplectic group, the orthogonal group, the unitary group) such that a certain automorphic L-function of $\widehat{g}$ can be expressed in terms of some L-functions related with g. Thus we propose the following problem:

Problem A. *Let $\langle \widehat{g}, \widehat{g}\rangle$ (resp. $\langle g, g\rangle$) be the period of $\widehat{g}$ (resp. g). Then express the ratio $\langle \widehat{g}, \widehat{g}\rangle/\langle g, g\rangle^e$ in terms of arithmetic invariants of g, for example, the special values of certain L-functions related with g for some integer e.*

For instance, Zagier [**39**] solved Problem A for the Doi-Naganuma lift $\widehat{f}$ of a primitive form f of integral weight. Murase and Sugano [**32**] also solved Problem A for the Kudla lift $\widehat{f}$ of a primitive form f of integral weight. In addition, Kohnen and Skoruppa [**30**] solved Problem A in the case where $\widehat{h}$ is the Saito-Kurokawa lift of a cuspidal Hecke eigenform h in the Kohnen plus space of half-integral weight.

We should also note that this type of period relation is not only interesting and important in its own right but also plays an important role in arithmetic theory of modular forms. For instance, by using the result of Kohnen and Skoruppa, Brown [**5**] and Katsurada [**19**] independently proved a modification of Harder's conjecture on congruences occurring between Saito-Kurokawa lifts and non-Saito-Kurokawa lifts under mild conditions. Furthermore, by using such congruences, Brown constructed a non-trivial element of the Bloch-Kato Selmer group attached to a modular two-dimensional Galois representation. A similar type of result

---

2000 *Mathematics Subject Classification* 11F67,11F46, 11F66.

The first named author was partly supported by the JSPS Grant-in-Aid for Scientific Research, Number 24540005 and the second named author was partly supported by the JSPS Grant-in-Aid for Scientific Research, Number 23224001.

can be found in [**3**]. We note that this type of congruence relation was conjectured by Doi, Hida and Ishii [**7**] in the case where $\widehat{f}$ is the Doi-Naganuma lift of f.

Now let us explain our main result briefly. Let k and n be positive even integers. Let h be a cuspidal Hecke eigenform in the Kohnen plus space of weight $k-n/2+1/2$ for $\Gamma_0(4)$, and f the primitive form of weight $2k-n$ for $SL_2(\mathbf{Z})$ corresponding to h under the Shimura correspondence. Then Ikeda [**14**] constructed a cuspidal Hecke eigenform $I_n(h)$ of weight k for $Sp_n(\mathbf{Z})$ whose standard L-function can be expressed as $\zeta(s)\prod_{i=1}^n L(s+k-i,f)$, where $\zeta(s)$ is Riemann's zeta function and $L(s,f)$ is Hecke's L-function of f. The existence of such a Hecke eigenform was conjectured by Duke and Imamoḡlu in their unpublished paper. We call $I_n(h)$ the Duke-Imamoḡlu-Ikeda lift of h (or of f). (See also the remark after Theorem 2.1.) We note that $I_2(h)$ is nothing but the Saito-Kurokawa lift of h. Then, as a generalization of the result of Kohnen and Skoruppa, Ikeda among others proposed the following remarkable conjecture in [**15**]:

The ratio $\langle I_n(h), I_n(h)\rangle/\langle h,h\rangle$ should be expressed, up to elementary factor, as

$$L(k,f)\zeta(n)\prod_{i=1}^{n/2-1} L(2i+1,f,\mathrm{Ad})\zeta(2i),$$

where $L(s,f,\mathrm{Ad})$ is the adjoint L-function of f (cf. Conjecture A).

The aim of this paper is to prove the above conjecture (cf. Theorem 2.2). We note that $I_n(h)$ is not likely to be realized as a theta lift except in the case $n=2$ (cf. Schulze-Pillot [**35**]). Therefore we cannot adapt a general method of Rallis [**34**] for our purpose. We also note that the conjecture cannot be explained within the framework of motives since there is no principle attaching motives to half-integral weight modular forms so far. Taking these remarks into account, we take an approach based on the classical Rankin-Selberg method to our problem. Namely, the method we use is to give an explicit formula for the Rankin-Selberg series of a certain half-integral weight Siegel modular form related with $I_n(h)$, and to compute its residue at a pole. We explain it more precisely.

First let $\phi_{I_n(h),1}$ be the first coefficient of the Fourier-Jacobi expansion of $I_n(h)$ and $\sigma_{n-1}(\phi_{I_n(h),1})$ the cusp form in the generalized Kohnen plus space of weight $k-1/2$ for $\Gamma_0^{(n-1)}(4)\subset Sp_{n-1}(\mathbf{Z})$ corresponding to $\phi_{I_n(h),1}$ under the Ibukiyama isomorphism σ_{n-1}. For the precise definition of the generalized Kohnen plus space and the Ibukiyama isomorphism, see Section 3. Then we have the following Fourier expansion of $\sigma_{n-1}(\phi_{I_n(h),1})$:

$$\sigma_{n-1}(\phi_{I_n(h),1})(Z)=\sum_A c(A)\exp(2\pi\sqrt{-1}\mathrm{tr}(AZ)),$$

where A runs over all positive definite half-integral matrices of degree $n-1$, and tr denotes the trace of a matrix. Then, in Section 3, we consider the following Rankin-Selberg series $R(s,\sigma_{n-1}(\phi_{I_n(h),1}))$ of $\sigma_{n-1}(\phi_{I_n(h),1})$:

$$R(s,\sigma_{n-1}(\phi_{I_n(h),1}))=\sum_A \frac{|c(A)|^2}{e(A)(\det A)^s},$$

where A runs over all the $SL_{n-1}(\mathbf{Z})$-equivalence classes of positive definite half-integral matrices of degree $n-1$ and $e(A)$ denotes the order of the unit group of A in $SL_{n-1}(\mathbf{Z})$. In the integral weight Siegel modular form case, the analytic properties of this type of Dirichlet series have been studied by many people (e.g. Kalinin [**17**]). Similarly to that case, we also get analytic properties of $R(s,\sigma_{n-1}(\phi_{I_n(h),1}))$. While such a Dirichlet series with no Euler product has never been regarded as significant as automorphic L-functions until now, it should be emphasized that it plays a very important role in the proof of our main result. Indeed, as one of the most significant properties, $R(s,\sigma_{n-1}(\phi_{I_n(h),1}))$ has a simple pole at $s=k-1/2$ with residue expressed in terms of the period of $\phi_{I_n(h),1}$ (cf. Corollary to Proposition 3.1).

Hence, by virtue of the main identity in [**23**], this enables us to rewrite Ikeda's conjecture in terms of the relation between the residue of $R(s, \sigma_{n-1}(\phi_{I_n(h),1}))$ at $s = k - 1/2$ and the period of h (cf. Theorem 3.2). In order to prove Theorem 3.2, we will get an explicit formula for $R(s, \sigma_{n-1}(\phi_{I_n(h),1}))$ in terms of $L(s, f, \mathrm{Ad})$ and $L(s, f)$. To get it, in Section 4, we reduce our computation to that of certain formal power series, which we call formal power series of Rankin-Selberg type, associated with local Siegel series similarly to [**11**, **12**] (cf. Theorem 4.2). Section 5 is devoted to the computation of them. This computation is similar to those in [**11**, **12**], but is more elaborate and longer than them. In particular we should be careful in dealing with the case $p = 2$. After overcoming such obstacles we can get explicit formulas for formal power series of Rankin-Selberg type (cf. Theorem 5.3.1). In Section 6, by using Theorem 5.3.1, we immediately get an explicit formula for $R(s, \sigma_{n-1}(\phi_{I_n(h),1}))$ (cf. Theorem 6.2), and by taking the residue of it at $k - 1/2$ we prove Theorem 3.2, and therefore prove Conjecture A (cf. Theorem 6.3).

We note that we can also give an explicit formula for the Rankin-Selberg series of $I_n(h)$. However, it seems difficult to prove Conjecture A directly from such a formula.

By Theorem 2.2, we can give a refined version of a result concerning the algebraicity of $\langle f, f\rangle^{n/2}/\langle I_n(h), I_n(h)\rangle$ due to Choie and Kohnen (cf. Theorem 2.3). Moreover we can apply this result to characterize prime ideals giving congruences between Duke-Imamoḡlu-Ikeda lifts and non-Duke-Imamoḡlu-Ikeda lifts. This will be discussed in [**21**].

Acknowledgement. The authors thank T. Ikeda, Y. Ishikawa, Y. Mizuno and S. Yamana for their valuable comments. The authors are also thankful to an anonymous referee, whose comments made our paper more readable.

Notation. Let R be a commutative ring. We denote by $R^\times$ and R^* the semigroup of non-zero elements of R and the unit group of R, respectively. We also put $S^\square = \{a^2 \mid a \in S\}$ for a subset S of R. We denote by $M_{mn}(R)$ the set of $m \times n$-matrices with entries in R. In particular put $M_n(R) = M_{nn}(R)$. Put $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, where $\det A$ denotes the determinant of a square matrix A. For an $m \times n$-matrix X and an $m \times m$-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Let $S_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, if R is an integral domain of characteristic different from 2, let $\mathcal{L}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathcal{L}_n(R)$ is the subset of symmetric matrices of degree n with entries in the field of fractions of R whose (i, j)-component belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. In particular, we put $\mathcal{L}_n = \mathcal{L}_n(\mathbf{Z})$, and $\mathcal{L}_{n,p} = \mathcal{L}_n(\mathbf{Z}_p)$ for a prime number p. For a subset S of $M_n(R)$ we denote by $S^\times$ the subset of S consisting of non-degenerate matrices. If S is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. The group $GL_n(R)$ acts on the set $S_n(R)$ in the following way:

$$GL_n(R) \times S_n(R) \ni (g, A) \longmapsto A[g] \in S_n(R).$$

Let G be a subgroup of $GL_n(R)$. For a G-stable subset $\mathcal{B}$ of $S_n(R)$ we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ under the action of G. We sometimes use the same symbol $\mathcal{B}/G$ to denote a complete set of representatives of $\mathcal{B}/G$. We abbreviate $\mathcal{B}/GL_n(R)$ as $\mathcal{B}/\sim$ if there is no fear of confusion. Let R' be a subring of R. Then two symmetric matrices A and A' with entries in R are said to be equivalent over R' with each other and write $A \sim_{R'} A'$ if there is an element X of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X \bot Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

For an integer $D \in \mathbf{Z}$ such that $D \equiv 0$ or $\equiv 1 \bmod 4$, let $\mathfrak{d}_D$ be the discriminant of $\mathbf{Q}(\sqrt{D})$, and put $\mathfrak{f}_D = \sqrt{\frac{D}{\mathfrak{d}_D}}$. We call an integer D a fundamental discriminant if it is the discriminant of some quadratic extension of $\mathbf{Q}$ or 1. For a fundamental discriminant D, let $\left(\frac{D}{*}\right)$ be the character corresponding to $\mathbf{Q}(\sqrt{D})/\mathbf{Q}$. Here we make the convention that $\left(\frac{D}{*}\right) = 1$ if $D = 1$.

We put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbf{C}$. For a prime number p we denote by $\nu_p(*)$ the additive valuation of $\mathbf{Q}_p$ normalized so that $\nu_p(p) = 1$, and by $\mathbf{e}_p(*)$ the continuous additive character of $\mathbf{Q}_p$ such that $\mathbf{e}_p(x) = \mathbf{e}(x)$ for $x \in \mathbf{Z}[p^{-1}]$.

For a non-negative integer r we define a polynomial $\phi_r(x)$ in x by $\phi_r(x) = \prod_{i=1}^{r}(1 - x^i)$. Here we understand that $\phi_0(x) = 1$.

2. *Ikeda's conjecture on the Period of the Duke-Imamoḡlu-Ikeda lift*

Put $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$, where 1_n and O_n denotes the unit matrix and the zero matrix of degree n, respectively. Furthermore, put

$$\Gamma^{(n)} = Sp_n(\mathbf{Z}) = \{M \in GL_{2n}(\mathbf{Z}) \mid J_n[M] = J_n\}.$$

Let $\mathbf{H}_n$ be Siegel's upper half-space of degree n. Let l be an integer or half integer. For a congruence subgroup Γ of $\Gamma^{(n)}$, we denote by $\mathfrak{M}_l(\Gamma)$ the space of holomorphic modular forms of weight l for Γ. We denote by $\mathfrak{S}_l(\Gamma)$ the subspace of $\mathfrak{M}_l(\Gamma)$ consisting of cusp forms. For $F,\, G \in \mathfrak{S}_l(\Gamma)$ we define the Petersson product $\langle F, G\rangle$ by

$$\langle F, G\rangle = [\Gamma^{(n)} : \Gamma\{\pm 1_{2n}\}]^{-1} \int_{\Gamma\backslash\mathbf{H}_n} F(Z)\overline{G(Z)} \det(\mathrm{Im}(Z))^l d^*Z,$$

where d^*Z denotes the invariant volume element on $\mathbf{H}_n$ defined as usual. We call $\langle F, F\rangle$ the period of F. For a positive integer N, let

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \;\middle|\; C \equiv O_n \bmod N \right\},$$

and in particular put $\Gamma_0(N) = \Gamma_0^{(1)}(N)$. Let p be a prime number. For a non-zero element $a \in \mathbf{Q}_p$ we put $\chi_p(a) = 1, -1,$ or 0 according as $\mathbf{Q}_p(a^{1/2}) = \mathbf{Q}_p$, $\mathbf{Q}_p(a^{1/2})$ is an unramified quadratic extension of $\mathbf{Q}_p$, or $\mathbf{Q}_p(a^{1/2})$ is a ramified quadratic extension of $\mathbf{Q}_p$. We note that $\chi_p(D) = \left(\frac{D}{p}\right)$ if D is a fundamental discriminant. For an element T of $\mathcal{L}_{n,p}^\times$ with n even, put $\xi_p(T) = \chi_p((-1)^{n/2}\det T)$. Let T be an element of $\mathcal{L}_n^\times$. Then $(-1)^{n/2}\det(2T) \equiv 0$ or $\equiv 1 \bmod 4$, and we define $\mathfrak{d}_T$ and $\mathfrak{f}_T$ as $\mathfrak{d}_T = \mathfrak{d}_{(-1)^{n/2}\det(2T)}$ and $\mathfrak{f}_T = \mathfrak{f}_{(-1)^{n/2}\det(2T)}$, respectively. For an element T of $\mathcal{L}_{n,p}^\times$, there exists an element $\widetilde{T}$ of $\mathcal{L}_n^\times$ such that $\widetilde{T} \sim_{\mathbf{Z}_p} T$. We then put $\mathfrak{e}_p(T) = \nu_p(\mathfrak{f}_{\widetilde{T}})$, and $[\mathfrak{d}_T] = \mathfrak{d}_{\widetilde{T}} \bmod \mathbf{Z}_p^{*\square}$. They do not depend on the choice of $\widetilde{T}$. We note that $(-1)^{n/2}\det(2T)$ can be expressed as $(-1)^{n/2}\det(2T) = dp^{2\mathfrak{e}_p(T)} \bmod \mathbf{Z}_p^{*\square}$ for any $d \in [\mathfrak{d}_T]$.

For each $T \in \mathcal{L}_{n,p}^\times$ we define the local Siegel series $b_p(T, s)$ by

$$b_p(T, s) = \sum_{R \in S_n(\mathbf{Q}_p)/S_n(\mathbf{Z}_p)} \mathbf{e}_p(\mathrm{tr}(TR))p^{-\nu_p(\mu_p(R))s},$$

where $\mu_p(R) = [R\mathbf{Z}_p^n + \mathbf{Z}_p^n : \mathbf{Z}_p^n]$. We remark that there exists a unique polynomial $F_p(T, X)$ in X such that

$$b_p(T, s) = F_p(T, p^{-s})\frac{(1 - p^{-s})\prod_{i=1}^{n/2}(1 - p^{2i-2s})}{1 - \xi_p(T)p^{n/2-s}}$$

(cf. Kitaoka [**26**]). We then define a polynomial $\widetilde{F}_p(T, X)$ in X and X^{-1} as

$$\widetilde{F}_p(T, X) = X^{-\mathfrak{e}_p(T)}F_p(T, p^{-(n+1)/2}X).$$

We remark that $\widetilde{F}_p(B, X^{-1}) = \widetilde{F}_p(B, X)$ (cf. [**18**]).

Now let k be a positive even integer. Let

$$h(z) = \sum_{\substack{m \in \mathbf{Z}_{>0} \\ (-1)^{n/2} m \equiv 0,1 \bmod 4}} c_h(m)\mathbf{e}(mz)$$

be a Hecke eigenform in the Kohnen plus space $\mathfrak{S}^+_{k-n/2+1/2}(\Gamma_0(4))$ and

$$f(z) = \sum_{m=1}^{\infty} c_f(m)\mathbf{e}(mz)$$

the primitive form in $\mathfrak{S}_{2k-n}(\Gamma^{(1)})$ corresponding to h under the Shimura correspondence (cf. Kohnen [**28**]). For the precise definition of the Kohnen plus space, we give it in Section 3 in more general settings. Let $\alpha_p \in \mathbf{C}$ such that $\alpha_p + \alpha_p^{-1} = p^{-k+n/2+1/2} c_f(p)$, which we call the Satake p-parameter of f. Then for a Dirichlet character χ we define Hecke's L-function $L(s, f, \chi)$ twisted by χ as

$$L(s, f, \chi) = \prod_p \{(1 - \alpha_p p^{-s+k-n/2-1/2}\chi(p))(1 - \alpha_p^{-1} p^{-s+k-n/2-1/2}\chi(p))\}^{-1}.$$

In particular, if χ is the principal character we write $L(s, f, \chi)$ as $L(s, f)$ as usual. We define a Fourier series $I_n(h)(Z)$ in $Z \in \mathbf{H}_n$ by

$$I_n(h)(Z) = \sum_{T \in (\mathcal{L}_n)_{>0}} c_{I_n(h)}(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where

$$c_{I_n(h)}(T) = c_h(|\mathfrak{d}_T|)\mathfrak{f}_T^{k-n/2-1/2} \prod_p \widetilde{F}_p(T, \alpha_p).$$

Then Ikeda [**14**] showed the following:

Theorem 2.1. *$I_n(h)(Z)$ is a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$ whose standard L-function coincides with*

$$\zeta(s) \prod_{i=1}^{n} L(s + k - i, f).$$

We note that $I_2(h)$ coincides with the Saito-Kurokawa lift of h. Originally, starting from a primitive form g in $\mathfrak{S}_{2k-n}(\Gamma^{(1)})$, Ikeda constructed the $I_n(\widetilde{g})$, where $\widetilde{g}$ is a Hecke eigenform in $\mathfrak{S}^+_{k-n/2+1/2}(\Gamma_0(4))$ corresponding to g under the Shimura correspondence. We note that $\widetilde{g}$ is uniquely determined, only up to constant multiple, by g, and therefore so is $I_n(\widetilde{g})$.

To formulate Ikeda's conjecture, put

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2) \quad \text{and} \quad \Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1).$$

We note that $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Furthermore put

$$\xi(s) = \Gamma_{\mathbf{R}}(s)\zeta(s) \quad \text{and} \quad \widetilde{\xi}(s) = \Gamma_{\mathbf{C}}(s)\zeta(s).$$

For a Dirichlet character χ put

$$\Lambda(s, f, \chi) = \frac{\Gamma_{\mathbf{C}}(s)L(s, f, \chi)}{\tau(\chi)},$$

where $\tau(\chi)$ is the Gauss sum of χ. In particular, we simply write $\Lambda(s,\, f, \chi)$ as $\Lambda(s,\, f)$ if χ is the principal character. Furthermore, we define the adjoint L-function $L(s,\, f,\, \mathrm{Ad})$ as

$$L(s, f, \mathrm{Ad}) = \prod_p \{(1-\alpha_p^2 p^{-s})(1-p^{-s})(1-\alpha_p^{-2}p^{-s})\}^{-1},$$

and put

$$\Lambda(s,\, f,\, \mathrm{Ad}) = \Gamma_{\mathbf{R}}(s+1)\Gamma_{\mathbf{C}}(s+2k-n-1)L(s,\, f,\, \mathrm{Ad}),$$

and

$$\widetilde{\Lambda}(s,\, f,\, \mathrm{Ad}) = \Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s+2k-n-1)L(s,\, f,\, \mathrm{Ad}).$$

We note that

$$\Lambda(1-s,\, f,\, \mathrm{Ad}) = \Lambda(s,\, f,\, \mathrm{Ad}).$$

Then Ikeda [**15**] among others proposed the following conjecture:

Conjecture A. *We have*

$$\frac{\langle I_n(h),\, I_n(h)\rangle}{\langle h,\, h\rangle} = 2^{\alpha(n,\,k)}\Lambda(k,\, f)\widetilde{\xi}(n)\prod_{i=1}^{n/2-1}\widetilde{\Lambda}(2i+1,\, f,\, \mathrm{Ad})\,\widetilde{\xi}(2i),$$

where $\alpha(n,k) = -(n-3)(k-n/2)-n+1$.

Remark. *The primitive form f as well as $I_n(h)$ is uniquely determined by h. Therefore there is no ambiguity in the above formulation. Conjecture A is compatible with the period formula for the Saito-Kurokawa lift proved by Kohnen and Skoruppa [**30**] (see also Oda [**33**]). In [**15**], Ikeda proposed a more general conjecture for the period of the Miyawaki-Ikeda lift. We also remark that he constructed a lifting from an elliptic modular form to the space of Hermitian modular forms, and proposed a conjecture similar to the above (cf. [**16**]).*

Now our main result in this paper is the following:

Theorem 2.2. *Conjecture A holds true for any positive even integer n.*

By the above theorem, we obtain the following result:

Theorem 2.3. *Let the notation be as above. Let D be a fundamental discriminant such that $(-1)^{n/2}D > 0$. For $i = 1, ..., n/2-1$, put $\mathbf{L}(2i+1,\, f,\, \mathrm{Ad}) = \widetilde{\Lambda}(2i+1,\, f,\, \mathrm{Ad})/\langle f,\, f\rangle$. Then*

$$\frac{|c_h(|D|)|^2\langle f,\, f\rangle^{n/2}}{\langle I_n(h),\, I_n(h)\rangle} = \frac{\sqrt{-1}^{a_n}\,|D|^{k-n/2}\Lambda(k-n/2, f, (\frac{D}{*}))}{2^{b_{n,k}}\Lambda(k,\, f)\widetilde{\xi}(n)\prod_{i=1}^{n/2-1}\mathbf{L}(2i+1,\, f,\, \mathrm{Ad})\widetilde{\xi}(2i)},$$

where $a_n = 0$ or 1 according as $n \equiv 0 \bmod 4$ or $n \equiv\ 2 \bmod 4$, and $b_{n,k}$ is some integer depending only on n and k.

Proof. By [**31**, Theorem 1], for any such D we have

$$\frac{|c_h(|D|)|^2}{\langle h, h\rangle} = \frac{\sqrt{-1}^{a_n}\,2^{k-n/2-1}|D|^{k-n/2}\Lambda(k-n/2, f, (\frac{D}{*}))}{\langle f, f\rangle}.$$

Thus, by Theorem 2.2, the assertion holds. □

It is well-known that the value $\dfrac{\Lambda(k-n/2, f, (\frac{D}{*}))}{\sqrt{-1}^{n/2}\Lambda(k,\, f)}$ and the values $\mathbf{L}(2i+1,\, f,\, \mathrm{Ad})$ for $i = 1, ..., n/2-1$ are algebraic numbers and belong to the Hecke field $\mathbf{Q}(f)$ if $k > n$ (cf. Shimura [**36**, **37**]). Thus we obtain

Corollary. *Assume that $k > n$ and that all the Fourier coefficients of h belong to $\mathbf{Q}(f)$. Then the ratio $\dfrac{\langle f,\, f\rangle^{n/2}}{\langle I_n(h),\, I_n(h)\rangle}$ belongs to $\mathbf{Q}(f)$.*

We note that we can multiply some non-zero complex number c with h so that all the Fourier coefficients of ch belong to $\mathbf{Q}(f)$. We also note that the above result has been proved by Furusawa [**8**] in case $n = 2$, and by Choie and Kohnen [**6**] under the assumption $k > 2n$ in general case. Thus Theorem 2.3 and its corollary can be regarded as a refined version of their results.

3. *Rankin-Selberg series of the image of the first Fourier-Jacobi coefficient of the Duke-Imamoḡlu-Ikeda lift under the Ibukiyama isomorphism*

To prove Conjecture A, we rewrite it in terms of the residue of the Rankin-Selberg series of a certain half-integral weight Siegel modular form. Let l an m be positive integers. Let $F(Z)$ be an element of $\mathfrak{S}_{l-1/2}(\varGamma_0^{(m)}(4))$. Then $F(Z)$ has the following Fourier expansion:

$$F(Z) = \sum_{A \in (\mathcal{L}_m)_{>0}} c_F(A)\mathbf{e}(\mathrm{tr}(AZ))$$

We define the Rankin-Selberg series $R(s, F)$ of F as

$$R(s, F) = \sum_{A \in (\mathcal{L}_m)_{>0}/SL_m(\mathbf{Z})} \frac{|c_F(A)|^2}{e(A)(\det A)^s},$$

where $e(A) = \#\{X \in SL_m(\mathbf{Z}) \mid A[X] = A\}$.

Put

$$\mathcal{L}'_m = \{A \in \mathcal{L}_m \mid A \equiv -\, {}^t rr \bmod 4\mathcal{L}_m \text{ for some } r \in \mathbf{Z}^m\}.$$

For $A \in \mathcal{L}'_m$, the integral vector $r \in \mathbf{Z}^m$ in the above definition is uniquely determined modulo $2\mathbf{Z}^m$ by A, and is denoted by r_A. Moreover it is easily shown that the matrix

$$\begin{pmatrix} 1 & r_A/2 \\ {}^t r_A/2 & ({}^t r_A r_A + A)/4 \end{pmatrix}$$

which will be denoted by $A^{(1)}$, belongs to $\mathcal{L}_{m+1}$, and that its $SL_{m+1}(\mathbf{Z})$-equivalence class is uniquely determined by A. In particular, if m is odd and $A \in (\mathcal{L}'_m)^{\times}$, put $\mathfrak{d}_A^{(1)} = \mathfrak{d}_{A^{(1)}}$, and $\mathfrak{f}_A^{(1)} = \mathfrak{f}_{A^{(1)}}$. Now we define the generalized Kohnen plus space of weight $l-1/2$ for $\varGamma_0^{(m)}(4)$ as

$$\mathfrak{S}^+_{l-1/2}(\varGamma_0^{(m)}(4)) = \{F \in \mathfrak{S}_{l-1/2}(\varGamma_0^{(m)}(4)) \,|\, c_F(A) = 0 \text{ unless } A \in (-1)^l \mathcal{L}'_m\}.$$

Now, for the rest of this section, suppose that l is even. Then there exists an isomorphism from the space of Jacobi forms of index 1 to the generalized Kohnen plus space due to Ibukiyama. To explain this, let $\varGamma_J^{(m)} = \varGamma^{(m)} \ltimes \mathrm{H}_m(\mathbf{Z})$, where $\mathrm{H}_m(\mathbf{Z})$ is the subgroup of the Heisenberg group $\mathrm{H}_m(\mathbf{R})$ consisting of all elements with integral entries.

Let $J_{l,N}^{\mathrm{cusp}}(\Gamma_J^{(m)})$ denote the space of Jacobi cusp forms of weight l and index N for the Jacobi group $\Gamma_J^{(m)}$. Let $\phi(Z,z) \in J_{l,1}^{\mathrm{cusp}}(\Gamma_J^{(m)})$. Then we have the following Fourier expansion:

$$\phi(Z,\, z) = \sum_{\substack{T \in \mathcal{L}_m,\, r \in \mathbf{Z}^m, \\ 4T - {}^t rr > 0}} c_\phi(T,r)\mathbf{e}(\mathrm{tr}(TZ) + r\,{}^t z).$$

We say that two elements (T,r) and (T',r') of $\mathcal{L}_m \times \mathbf{Z}^m$ are $SL_m(\mathbf{Z})$-equivalent and write $(T,r) \sim (T',r')$ if there exists an element $g \in SL_m(\mathbf{Z})$ such that $T' - {}^t r'r'/4 = (T - {}^t rr/4)[g]$. We then define a Dirichlet series $R(s,\phi)$ as

$$R(s,\phi) = \sum_{(T,r)} \frac{|c_\phi(T,r)|^2}{e(T - {}^t rr/4)(\det(T - {}^t rr/4))^s},$$

where (T,r) runs over a complete set of representatives of $SL_m(\mathbf{Z})$-equivalence classes of $\mathcal{L}_m \times \mathbf{Z}^m$ such that $T - {}^t rr/4 \in (\mathcal{L}_m)_{>0}$. Now $\phi(Z,z)$ can also be expressed as follows:

$$\phi(Z,z) = \sum_{r \in \mathbf{Z}^m/2\mathbf{Z}^m} h_r(Z)\theta_r(Z,z),$$

where $h_r(Z)$ is a holomorphic function on $\mathbf{H}_m$, and

$$\theta_r(Z,z) = \sum_{\lambda \in M_{1,m}(\mathbf{Z})} \mathbf{e}(\mathrm{tr}(Z[{}^t(\lambda + 2^{-1}r)]) + 2(\lambda + 2^{-1}r)\,{}^t z).$$

We note that $h_r(Z)$ have the following Fourier expansion:

$$h_r(Z) = \sum_T c_\phi(T,r)\mathbf{e}(\mathrm{tr}((T - {}^t rr/4)Z)),$$

where T runs over all elements of $\mathcal{L}_m$ such that $T - {}^t rr/4$ is positive definite. Put $\mathbf{h}(Z) = (h_r(Z))_{r \in \mathbf{Z}^m/2\mathbf{Z}^m}$. Then $\mathbf{h}$ is a vector valued modular form of weight $l - 1/2$ for $\Gamma^{(m)}$, that is, for each $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{(m)}$ we have

$$\mathbf{h}(\gamma(Z)) = J(\gamma,Z)\mathbf{h}(Z).$$

Here $J(\gamma,Z)$ is an $m \times m$ matrix whose entries are holomorphic functions on $\mathbf{H}_m$ such that ${}^t\overline{J(\gamma,Z)}J(\gamma,Z) = |j(\gamma,Z)|^{2l-1}1_m$, where $j(\gamma,Z) = \det(CZ + D)$. In particular, we have

$$\sum_{r \in \mathbf{Z}^m/2\mathbf{Z}^m} h_r(\gamma(Z))\overline{h_r(\gamma(Z))} = |j(\gamma,Z)|^{2l-1} \sum_{r \in \mathbf{Z}^m/2\mathbf{Z}^m} h_r(Z)\overline{h_r(Z)}.$$

We then put

$$\sigma_m(\phi)(Z) = \sum_{r \in \mathbf{Z}^m/2\mathbf{Z}^m} h_r(4Z).$$

Then Ibukiyama [**9**] showed the following:

The mapping σ_m gives a $\mathbf{C}$-linear isomorphism

$$\sigma_m : J_{l,1}^{\mathrm{cusp}}(\Gamma_J^{(m)}) \simeq \mathfrak{S}_{l-1/2}^{+}(\Gamma_0^{(m)}(4)),$$

which is compatible with the actions of Hecke operators.

We call σ_m the Ibukiyama isomorphism. We note that

$$\sigma_m(\phi) = \sum_{A \in (\mathcal{L}'_m)_{>0}} c_\phi((A + {}^t r_A r_A)/4, r_A)\mathbf{e}(\mathrm{tr}(AZ)),$$

where $r = r_A$ denotes an element of $\mathbf{Z}^m$ such that $A + {}^t r_A r_A \in 4\mathcal{L}_m$. This r_A is uniquely determined up to modulo $2\mathbf{Z}^m$, and $c_\phi((A + {}^t r_A r_A)/4, r_A)$ does not depend on the choice of the representative of r_A mod $2\mathbf{Z}^m$. Furthermore, we have

$$R(s, \sigma_m(\phi)) = \sum_{A \in (\mathcal{L}'_m)_{>0}/SL_m(\mathbf{Z})} \frac{|c_\phi((A + {}^t rr)/4, r)|^2}{e(A) \det A^s},$$

and hence

$$R(s, \phi) = 2^{2sm} R(s, \sigma_m(\phi)).$$

Now for ϕ, $\psi \in J_{l,1}^{\mathrm{cusp}}(\Gamma_J^{(m)})$ we define the Petersson product of ϕ and ψ by

$$\langle \phi, \psi \rangle = \int_{\Gamma_J^{(m)} \backslash (\mathbf{H}_m \times \mathbf{C}^m)} \phi(Z, z)\overline{\psi(Z, z)} \det(v)^{l-m-2} \exp(-4\pi v^{-1}[{}^t y])\, dudvdxdy,$$

where $Z = u + \sqrt{-1}v \in \mathbf{H}_m$, $z = x + \sqrt{-1}y \in \mathbf{C}^m$. Now we consider the analytic properties of $R(s, \phi)$.

Proposition 3.1. *Let $\phi(Z, z) \in J_{l,1}^{\mathrm{cusp}}(\Gamma_J^{(m)})$. Put*

$$\mathcal{R}(s, \phi) = \gamma_m(s)\xi(2s + m + 2 - 2l) \prod_{i=1}^{[m/2]} \xi(4s + 2m + 4 - 4l - 2i) R(s, \phi),$$

where

$$\gamma_m(s) = 2^{1-2sm} \prod_{i=1}^{m} \Gamma_{\mathbf{R}}(2s - i + 1).$$

Then the following assertions hold:

(1) *$\mathcal{R}(s, \phi)$ has a meromorphic continuation to the whole s-plane, and has the following functional equation:*

$$\mathcal{R}(2l - 3/2 - m/2 - s, \phi) = \mathcal{R}(s, \phi).$$

(2) *$\mathcal{R}(s, \phi)$ is holomorphic for $\mathrm{Re}(s) > l - 1/2$, and has a simple pole at $s = l - 1/2$ with the residue $2^{m+1} \prod_{i=1}^{[m/2]} \xi(2i + 1)\langle \phi, \phi \rangle$.*

Proof. The assertions can be proved in the same manner as in Kalinin [**17**], but for the convenience of readers we here give an outline of the proof. We define the non-holomorphic Siegel Eisenstein series $E^{(m)}(Z, s)$ by

$$E^{(m)}(Z, s) = (\det \mathrm{Im}(Z))^s \sum_{M \in \Gamma_\infty^{(m)} \backslash \Gamma^{(m)}} |j(M, Z)|^{-2s},$$

where $\Gamma_\infty^{(m)} = \left\{ \begin{pmatrix} A & B \\ O_m & D \end{pmatrix} \in \Gamma^{(m)} \right\}$. For the $\phi(Z, z)$ let $\mathbf{h}(Z) = (h_r(Z))_{r \in \mathbf{Z}^m/2\mathbf{Z}^m}$ be as above. Since $\mathbf{h}$ is a vector valued modular form for $\Gamma^{(m)}$, we can apply the Rankin-Selberg method and we obtain

$$\mathcal{R}(s, \phi) = \int_{\Gamma^{(m)} \backslash \mathbf{H}_m} \sum_{r \in \mathbf{Z}^m/2\mathbf{Z}^m} h_r(Z)\overline{h_r(Z)} \mathrm{Im}(Z)^{l-1/2} \mathcal{E}^{(m)}(Z, s) d^*Z,$$

where

$$\mathcal{E}^{(m)}(Z, s) = \xi(2s + m + 2 - 2l) \prod_{i=1}^{[m/2]} \xi(4s + 2m + 4 - 4l - 2i) E^{(m)}(Z, s + m/2 + 1 - l).$$

It is well-known that $\mathcal{E}^{(m)}(Z, s)$ has a meromorphic continuation to the whole s-plane, and has the following functional equation:

$$\mathcal{E}^{(m)}(Z, 2l-3/2-m/2-s) = \mathcal{E}^{(m)}(Z, s).$$

Thus the first assertion (1) holds. Furthermore it is holomorphic for $\mathrm{Re}(s) > l-1/2$, and has a simple pole at $s = l-1/2$ with the residue $\prod_{i=1}^{[m/2]} \xi(2j+1)$. We note that

$$\langle \phi,\, \phi \rangle = 2^{-m-1} \int_{\Gamma^{(m)} \backslash \mathbf{H}_m} \sum_{r \in \mathbf{Z}^m/2\mathbf{Z}^m} h_r(Z)\overline{h_r(Z)} \mathrm{Im}(Z)^{l-1/2} d^*Z.$$

Thus the second assertion (2) holds. □

For $F \in \mathfrak{S}^+_{l-1/2}(\Gamma_0^{(m)}(4))$ put

$$\mathcal{R}(s, F) = \prod_{i=1}^{m} \Gamma_{\mathbf{R}}(2s-i+1)\xi(2s+m+2-2l) \prod_{i=1}^{[m/2]} \xi(4s+2m+4-4l-2i) R(s, F).$$

We note that

$$\mathcal{R}(s, \sigma_m(\phi)) = 2^{-1}\mathcal{R}(s, \phi)$$

for $\phi \in J_{l,\,1}^{\mathrm{cusp}}(\Gamma_J^{(m)})$. Thus we obtain

Corollary. *Let the notation and the assumption be as in Proposition 3.1. Then $\mathcal{R}(s, \sigma_m(\phi))$ has a meromorphic continuation to the whole s-plane, and has the following functional equation:*

$$\mathcal{R}(2l-3/2-m/2-s, \sigma_m(\phi)) = \mathcal{R}(s, \sigma_m(\phi)).$$

Furthermore it is holomorphic for $\mathrm{Re}(s) > l-1/2$, and has a simple pole at $s = l-1/2$ with the residue $2^m \prod_{i=1}^{[m/2]} \xi(2i+1)\langle \phi, \phi \rangle$.

Let n and k be positive even integers. Let h be a Hecke eigenform in $\mathfrak{S}^+_{k-n/2+1/2}(\Gamma_0(4))$, and f and $I_n(h)$ be as in Section 2. Write $Z \in \mathbf{H}_n$ as $Z = \begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}$ with $\tau \in \mathbf{H}_{n-1}$, $z \in \mathbf{C}^{n-1}$ and $\tau' \in \mathbf{H}_1$. Then we have the following Fourier-Jacobi expansion of $I_n(h)$:

$$I_n(h)\left(\begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}\right) = \sum_{N=0}^{\infty} \phi_{I_n(h),N}(\tau,\, z)\mathbf{e}(N\tau'),$$

where $\phi_{I_n(h),N}(\tau,\, z)$ is called the N-th Fourier-Jacobi coefficient of $I_n(h)$ and defined by

$$\phi_{I_n(h),N}(\tau,\, z) = \sum_{\substack{T \in \mathcal{L}_{n-1},\, r \in \mathbf{Z}^{n-1}, \\ 4NT - {}^t rr > 0}} c_{I_n(h)}\left(\begin{pmatrix} N & r/2 \\ {}^t r/2 & T \end{pmatrix}\right) \mathbf{e}(\mathrm{tr}(T\tau) + r\,{}^t z).$$

We easily see that $\phi_{I_n(h),N}$ belongs to $J_{k,\,N}^{\mathrm{cusp}}(\Gamma_J^{(n-1)})$ for each $N \in \mathbf{Z}_{>0}$. Under the above notation, we will prove the following theorem in Section 6:

Theorem 3.2.

$$\begin{aligned} &\mathrm{Res}_{s=k-1/2}\mathcal{R}(s, \sigma_{n-1}(\phi_{I_n(h),1})) \\ &= 2^{\beta(n,\,k)}\langle h,\, h \rangle \prod_{i=1}^{n/2-1} \widetilde{\xi}(2i)\xi(2i+1)\widetilde{\Lambda}(2i+1,\, f,\, \mathrm{Ad}), \end{aligned}$$

where $\beta(n,k) = -(n-4)k + (n^2-5n+2)/2$.

Then we can show the following:

THEOREM 3.3. *Under the above notation and the assumption, Theorem 3.2 implies Conjecture A.*

Proof. By [**23**, Corollary to Main Theorem] with a minor correction (see the remark below), we have

$$\frac{\langle I_n(h),\, I_n(h)\rangle}{\langle \phi_{I_n(h),1},\, \phi_{I_n(h),1}\rangle} = 2^{-k+n-1}\Lambda(k,\, f)\widetilde{\xi}(n).$$

Thus Conjecture A holds true if and only if

$$\langle \phi_{I_n(h),1},\, \phi_{I_n(h),1}\rangle = 2^{-k(n-4)+n(n-7)/2+2}\langle h,\, h\rangle \prod_{i=1}^{n/2-1} \widetilde{\xi}(2i)\widetilde{\Lambda}(2i+1,\, f,\, \mathrm{Ad}).$$

On the other hand, by Corollary to Proposition 3.1 we have

$$\mathrm{Res}_{s=k-1/2}\mathcal{R}(s, \sigma_{n-1}(\phi_{I_n(h),1})) = 2^{n-1}\langle \phi_{I_n(h),1},\, \phi_{I_n(h),1}\rangle \prod_{i=1}^{n/2-1} \xi(2i+1).$$

Thus the assertion holds. □

REMARK. *In [**23**], we incorrectly quoted Yamazaki's result in [**38**]. Indeed "$\langle F, G\rangle$" on [**23**, p.2026, l.14] should read "$2^{-1}\langle F, G\rangle$" (cf. Krieg [**29**]) and therefore "2^{2k-n+1}" on [**23**, p.2027, l.7] should read "2^{2k-n}".*

4. *Reduction to local computations*

To prove Theorem 3.2, we give an explicit formula for $R(s, \sigma_{n-1}(\phi_{I_n(h),1}))$ for the first Fourier-Jacobi coefficient $\phi_{I_n(h),1}$ of $I_n(h)$. To do this, we reduce the problem to local computations.

For $a, b \in \mathbf{Q}_p^\times$ let $(a, b)_p$ the Hilbert symbol on $\mathbf{Q}_p$. Following Kitaoka [**27**], we define the Hasse invariant $\varepsilon(A)$ of $A \in S_m(\mathbf{Q}_p)^\times$ by

$$\varepsilon(A) = \prod_{1\le i\le j\le m} (a_i, a_j)_p$$

if A is equivalent to $a_1\perp\cdots\perp a_m$ over $\mathbf{Q}_p$ with some $a_1, \cdots, a_m \in \mathbf{Q}_p^\times$. We note that this definition does not depend on the choice of $a_1, \cdots, a_m$.

Now put

$$\mathcal{L}'_{m,p} = \{A \in \mathcal{L}_{m,p} \mid A \equiv -\,{}^t rr \bmod 4\mathcal{L}_{m,p} \text{ for some } r \in \mathbf{Z}_p^m\}.$$

Furthermore we put $S_m(\mathbf{Z}_p)_e = 2\mathcal{L}_{m,p}$ and $S_m(\mathbf{Z}_p)_o = S_m(\mathbf{Z}_p) \setminus S_m(\mathbf{Z}_p)_e$. We note that $\mathcal{L}'_{m,p} = \mathcal{L}_{m,p} = S_m(\mathbf{Z}_p)$ if $p \neq 2$. Let $T \in \mathcal{L}'_{m-1,p}$. Then there exists an element $r \in \mathbf{Z}_p^{m-1}$ such that $\begin{pmatrix} 1 & r/2 \\ {}^t r/2 & (T+{}^t rr)/4 \end{pmatrix}$ belongs to $\mathcal{L}_{m,p}$. As is easily shown, r is uniquely determined by T, up to modulo $2\mathbf{Z}_p^{m-1}$, and is denoted by r_T. Moreover as will be shown in the next lemma, $\begin{pmatrix} 1 & r_T/2 \\ {}^t r_T/2 & (T+{}^t r_T r_T)/4 \end{pmatrix}$ is uniquely determined by T, up to $GL_m(\mathbf{Z}_p)$-equivalence, and is denoted by $T^{(1)}$.

LEMMA 4.1. (cf. [**25**, Lemma 3.1]) *Let m be a positive integer.*

(1) *Let A and B be elements of $\mathcal{L}'_{m-1,p}$. Then* $\begin{pmatrix} 1 & r_A/2 \\ {}^t r_A/2 & (A+{}^t r_A r_A)/4 \end{pmatrix} \sim \begin{pmatrix} 1 & r_B/2 \\ {}^t r_B/2 & (B+{}^t r_B r_B)/4 \end{pmatrix}$ *if $A \sim B$.*

(2) *Let $A \in \mathcal{L}'_{m-1,p}$.*

(2.1) *Let $p \neq 2$. Then* $A^{(1)} \sim \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$.

(2.2) *Let $p = 2$. If $r_A \equiv 0 \bmod 2$, then $A \sim 4B$ with $B \in \mathcal{L}_{m-1,2}$, and* $A^{(1)} \sim \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$. *In particular,* $\mathrm{ord}(\det B) \geq m$ *or* $\geq m+1$ *according as m is even or odd.* *If $r_A \not\equiv 0 \bmod 2$, then $A \sim a \perp 4B$ with $a \equiv -1 \bmod 4$ and $B \in \mathcal{L}_{m-2,2}$ and we have* $A^{(1)} \sim \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & (a+1)/4 & 0 \\ 0 & 0 & B \end{pmatrix}$. *In particular,* $\mathrm{ord}(\det B) \geq m$ *or* $\geq m-1$ *according as m is even or odd.*

Now let m be a positive even integer. For $T \in (\mathcal{L}'_{m-1,p})^\times$, we define $[\mathfrak{d}_T^{(1)}]$ and $\mathfrak{e}_T^{(1)}$ as $[\mathfrak{d}_{T^{(1)}}]$ and $\mathfrak{e}_{T^{(1)}}$, respectively. These do not depend on the choice of r_T. We note that $(-1)^{m/2}\det T = 2^{m-2}dp^{2\mathfrak{e}_T^{(1)}} \bmod \mathbf{Z}_p^{*\square}$ for any $d \in [\mathfrak{d}_T^{(1)}]$. We define a polynomial $F_p^{(1)}(T,X)$ in X, and a polynomial $\widetilde{F}_p^{(1)}(T,X)$ in X and X^{-1} by

$$F_p^{(1)}(T,X) = F_p(T^{(1)},X),$$

and

$$\widetilde{F}_p^{(1)}(T,X) = X^{-\mathfrak{e}_p^{(1)}(T)}F_p^{(1)}(T,p^{-(n+1)/2}X).$$

Let B be a half-integral matrix B over $\mathbf{Z}_p$ of degree m. Let $p \neq 2$. Then

$$\widetilde{F}_p^{(1)}(B,X) = \widetilde{F}_p(1 \perp B, X).$$

Let $p = 2$. Then

$$\widetilde{F}_2^{(1)}(B,X) = \begin{cases} \widetilde{F}_2(\begin{pmatrix} 1 & 1/2 \\ 1/2 & (a+1)/4 \end{pmatrix} \perp B', X) & \text{if } B = a \perp 4B' \\ & \quad \text{with } a \equiv -1 \bmod 4, \\ \widetilde{F}_2(1 \perp B', X) & \text{if } B = 4B'. \end{cases}$$

Now for each $T \in S_m(\mathbf{Z}_p)_e^\times$ put

$$F_p^{(0)}(T,X) = F_p(2^{-\delta_{2,p}}T,X)$$

and

$$\widetilde{F}_p^{(0)}(T,X) = \widetilde{F}_p(2^{-\delta_{2,p}}T,X).$$

We define $[\mathfrak{d}_T^{(0)}]$ and $\mathfrak{e}_T^{(0)}$ as $[\mathfrak{d}_{2^{-\delta_{2,p}}T}]$ and $\mathfrak{e}_{2^{-1}T}$, respectively. We note that $(-1)^{m/2}\det T = dp^{2\mathfrak{e}_T^{(0)}} \bmod \mathbf{Z}_p^{*\square}$ for any $d \in [\mathfrak{d}_T^{(0)}]$.

Now let m and l be positive integers such that $m \geq l$. Then for non-degenerate symmetric matrices A and B of degree m and l respectively with entries in $\mathbf{Z}_p$ we define the local density $\alpha_p(A,B)$ and the primitive local density $\beta_p(A,B)$ representing B by A as

$$\alpha_p(A,B) = 2^{-\delta_{m,l}} \lim_{a\to\infty} p^{a(-ml+l(l+1)/2)} \#\mathcal{A}_a(A,B),$$

and

$$\beta_p(A,B) = 2^{-\delta_{m,l}} \lim_{a\to\infty} p^{a(-ml+l(l+1)/2)} \#\mathcal{B}_a(A,B),$$

where

$$\mathcal{A}_a(A,B) = \{X \in M_{ml}(\mathbf{Z}_p)/p^a M_{ml}(\mathbf{Z}_p) \mid A[X] - B \in p^a S_l(\mathbf{Z}_p)_e\},$$

and

$$\mathcal{B}_a(A,B) = \{X \in \mathcal{A}_a(A,B) \mid \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p}(X \bmod p) = l\}.$$

In particular we write $\alpha_p(A) = \alpha_p(A,A)$. Furthermore put

$$M(A) = \sum_{A' \in \mathcal{G}(A)} \frac{1}{e(A')}$$

for a positive definite symmetric matrix A of degree $n-1$ with entries in $\mathbf{Z}$, where $\mathcal{G}(A)$ denotes the set of $SL_{n-1}(\mathbf{Z})$-equivalence classes belonging to the genus of A. Then by Siegel's main theorem on the quadratic forms, we obtain

$$M(A) = 2^{2-n} e_{n-1} \kappa_{n-1} \det A^{n/2} \prod_p \alpha_p(A)^{-1}$$

where $e_{n-1} = 1$ or 2 according as $n = 2$ or not, and

$$\kappa_{n-1} = \prod_{i=1}^{(n-2)/2} \Gamma_{\mathbf{C}}(2i)$$

(cf. [**27**, Theorem 6.8.1]). Put

$$\mathcal{F}_p = \{d_0 \in \mathbf{Z}_p \mid \nu_p(d_0) \le 1\}$$

if p is an odd prime, and

$$\mathcal{F}_2 = \{d_0 \in \mathbf{Z}_2 \mid d_0 \equiv 1 \bmod 4, \text{ or } d_0/4 \equiv -1 \bmod 4, \text{ or } \nu_2(d_0) = 3\}.$$

For $d \in \mathbf{Z}_p^\times$ put

$$S_m(\mathbf{Z}_p, d) = \{T \in S_m(\mathbf{Z}_p) \mid (-1)^{[(m+1)/2]} \det T = p^{2i} d \bmod \mathbf{Z}_p^{*\square} \text{ with some } i \in \mathbf{Z}\},$$

and $S_m(\mathbf{Z}_p, d)_x = S_m(\mathbf{Z}_p, d) \cap S_m(\mathbf{Z}_p)_x$ for $x = e$ or o. We note that $S_m(\mathbf{Z}_p, d) = S_m(\mathbf{Z}_p, p^j d)$ for any even integer j. If m is even, put $\mathcal{L}_{m,p}^{(0)} = S_m(\mathbf{Z}_p)_e^\times$ and $\mathcal{L}_{m-1,p}^{(1)} = (\mathcal{L}'_{m-1,p})^\times$. We also define $\mathcal{L}_{m-j,p}^{(j)}(d) = S_{m-j}(\mathbf{Z}_p, d) \cap \mathcal{L}_{m-j,p}^{(j)}$ for $j = 0, 1$. Let m be an even integer. For $d_0 \in \mathcal{F}_p, l = 0, 1$ and $j = 0, 1$, define a rational number $\kappa(d_0, m-j, l) = \kappa_p(d_0, m-j, l)$ as

$$\kappa(d_0, m-j, l) = \begin{cases} \{(-1)^{lm(m-2)/8} 2^{-(m-2)(m-1)/2}\}^{\delta_{2,p}} & \\ \quad \times ((-1)^{m/2}, (-1)^{m/2} d_0)_p^l \, p^{-(m/2-1)l\nu(d_0)} & \text{if } j = 1 \\ & \\ \{(-1)^{m(m+2)/8} ((-1)^{m/2} 2, \, d_0)_2\}^{l\delta_{2,p}} & \text{if } j = 0. \end{cases}$$

Let $\iota_{m,p}$ be the constant function on $\mathcal{L}_{m,p}^\times$ taking the value 1, and $\varepsilon_{m,p}$ the function on $\mathcal{L}_{m,p}^\times$ assigning the Hasse invariant of A for $A \in \mathcal{L}_{m,p}^\times$. We sometimes drop the suffix and write $\iota_{m,p}$ as ι_p or ι and so on if there is no fear of confusion. From now on we sometimes write $\omega = \varepsilon^l$ with $l = 0$ or 1 according as $\omega = \iota$ or ε. Let n be an even integer. For $d_0 \in \mathcal{F}_p$ and $\omega = \varepsilon^l$ with $l = 0, 1$ we define a formal power series $H_{n-1,p}(d_0, \omega, X, Y, t) \in \mathbf{C}[X, X^{-1}, Y, Y^{-1}][[t]]$ by

$$H_{n-1,p}(d_0, \omega, X, Y, t) = \kappa(d_0, n-1, l)^{-1} t^{\delta_{2,p}(2-n)}$$
$$\times \sum_{A \in \mathcal{L}_{n-1,p}^{(1)}(d_0)/GL_{n-1}(\mathbf{Z}_p)} \frac{\widetilde{F}_p^{(1)}(A, X) \widetilde{F}_p^{(1)}(A, Y)}{\alpha_p(A)} \varepsilon(A)^l t^{\nu_p(\det A)}.$$

We call $H_{n-1,p}(d_0, \omega, X, Y, t)$ a formal power series of Rankin-Selberg type. An explicit formula for $H_{n-1,p}(d_0, \omega_p, X, Y, t)$ will be given in the next section. Let $\mathcal{F}$ denote the set of fundamental discriminants, and for $l = \pm 1$, put

$$\mathcal{F}^{(l)} = \{d_0 \in \mathcal{F} \mid l d_0 > 0\}.$$

Now let h be a Hecke eigenform in $\mathfrak{S}^{+}_{k-n/2+1/2}(\Gamma_0(4))$, and $f, I_n(h), \phi_{I_n(h),1}$ and $\sigma_{n-1}(\phi_{I_n(h),1})$ be as in Section 3. Then we have

THEOREM 4.2. *Let the notation and the assumption be as above. Then for* $\mathrm{Re}(s) \gg 0$, *we have*

$$R(s, \sigma_{n-1}(\phi_{I_n(h),1})) = \frac{e_{n-1}}{2}\kappa_{n-1}2^{(-s-1/2)(n-2)}$$
$$\times \left\{ \sum_{d_0 \in \mathcal{F}^{((-1)^{n/2})}} |c_h(|d_0|)|^2 |d_0|^{n/2-k+1/2} \prod_p H_{n-1,p}(d_0, \iota_p, \alpha_p, \alpha_p, p^{-s+k-1/2}) \right.$$
$$\left. + (-1)^{n(n-2)/8} \sum_{d_0 \in \mathcal{F}^{((-1)^{n/2})}} |c_h(|d_0|)|^2 |d_0|^{-k+3/2} \prod_p H_{n-1,p}(d_0, \varepsilon_p, \alpha_p, \alpha_p, p^{-s+k-1/2}) \right\},$$

where α_p *is the Satake* p*-parameter of* f. *Moreover we have*

$$R(s, h) = \kappa_1 \sum_{d_0 \in \mathcal{F}^{((-1)^{n/2})}} |c_h(|d_0|)|^2 |d_0|^{n/2-k+1/2} \prod_p H_{1,p}(d_0, \iota_p, \alpha_p, \alpha_p, p^{-s+k-n/2+1/2}).$$

Proof. Let $T \in (\mathcal{L}'_{n-1})_{>0}$. Then it follows from Lemma 4.1 that the T-th Fourier coefficient $c_{\sigma_{n-1}(\phi_{I_n(h),1})}(T)$ of $\sigma_{n-1}(\phi_{I_n(h),1})$ is uniquely determined by the genus to which T belongs, and, by definition, it can be expressed as

$$c_{\sigma_{n-1}(\phi_{I_n(h),1})}(T) = c_{I_n(h)}(T^{(1)}) = c_h(|\mathfrak{d}_T^{(1)}|)(\mathfrak{f}_T^{(1)})^{k-n/2-1/2} \prod_p \widetilde{F}^{(1)}(T, \alpha_p).$$

We also note that $\prod_p ((-1)^{n/2}, (-1)^{n/2}d_0)_p = 1$ for any $d_0 \in \mathcal{F}^{((-1)^{n/2})}$, and hence

$$\prod_p \kappa_p(d_0, n-1, 0) = 2^{-(n-2)(n-1)/2}$$

and

$$\prod_p \kappa_p(d_0, n-1, 1) = 2^{-(n-2)(n-1)/2}(-1)^{n(n-2)/8}|d_0|^{-n/2+1}.$$

Thus, by using the same method as in [**13**, Proposition 2.2], similarly to [**10**, Theorem 3.3 (1)], and [**11**, Theorem 3.2], we obtain the first assertion. Similarly we can prove the second assertion. □

5. *Formal power series associated with local Siegel series*

Throughout this section we fix a positive even integer n. In this section we give an explicit formula for $H_{n-1}(d_0, \omega, X, Y, t) = H_{n-1,p}(d_0, \omega, X, Y, t)$ for $\omega = \iota, \varepsilon$ (cf. Theorem 5.3.1). The idea is to rewrite $H_{n-1}(d_0, \omega, X, Y, t)$ in terms of various power series. Henceforth, for a $GL_m(\mathbf{Z}_p)$-stable subset $\mathcal{B}$ of $S_m(\mathbf{Q}_p)$, we simply write $\sum_{T \in \mathcal{B}}$ instead of $\sum_{T \in \mathcal{B}/\sim}$ if there is no fear of confusion. We also simply write ν_p as ν and the others if the prime number p is clear from the context.

5.1. *Formal power series of Andrianov type*

For an $m \times m$ half-integral matrix B over $\mathbf{Z}_p$, let $(\overline{W}, \overline{q})$ denote the quadratic space over $\mathbf{Z}_p/p\mathbf{Z}_p$ defined by the quadratic form $\overline{q}(\mathbf{x}) = B[\mathbf{x}] \bmod p$, and define the radical $R(\overline{W})$ of $\overline{W}$ by

$$R(\overline{W}) = \{\mathbf{x} \in \overline{W} \mid \overline{B}(\mathbf{x}, \mathbf{y}) = 0 \text{ for any } \mathbf{y} \in \overline{W}\},$$

where $\overline{B}$ denotes the associated symmetric bilinear form of $\overline{q}$. We then put $l_p(B) = \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} R(\overline{W})^{\perp}$, where $R(\overline{W})^{\perp}$ is the orthogonal complement of $R(\overline{W})^{\perp}$ in $\overline{W}$. Furthermore, in case $l_p(B)$ is even, put $\overline{\xi}_p(B) = 1$ or -1 according as $R(\overline{W})^{\perp}$ is hyperbolic or not. In case $l_p(B)$ is odd, we put $\overline{\xi}_p(B) = 0$. Here we make the convention that $\xi_p(B) = 1$ if $l_p(B) = 0$. We note that $\overline{\xi}_p(B)$ is different from the $\xi_p(B)$ in general, but they coincide if $B \in \mathcal{L}_{m,p} \cap \frac{1}{2}GL_m(\mathbf{Z}_p)$.

Let m be a positive even integer. For $B \in \mathcal{L}_{m-1,p}^{(1)}$ put $B^{(1)} = \begin{pmatrix} 1 & r/2 \\ {}^t r/2 & (B + {}^t rr)/4 \end{pmatrix}$, where r is an element of $\mathbf{Z}_p^{m-1}$ such that $B + {}^t rr \in 4\mathcal{L}_{m-1,p}$. Then we put $\xi_p^{(1)}(B) = \xi_p(B^{(1)})$ and $\overline{\xi}_p^{(1)}(B) = \overline{\xi}_p(B^{(1)})$. These do not depend on the choice of r, and we have $\xi_p^{(1)}(B) = \chi_p((-1)^{m/2} \det B)$. Let $p \neq 2$. Then an element B of $\mathcal{L}_{m-1,p}^{(1)}$ is equivalent, over $\mathbf{Z}_p$, to $\Theta \perp pB_1$ with $\Theta \in GL_{m-n_1-1}(\mathbf{Z}_p) \cap S_{m-n_1-1}(\mathbf{Z}_p)$ and $B_1 \in S_{n_1}(\mathbf{Z}_p)^{\times}$. Then $\overline{\xi}_p(B) = 0$ if n_1 is odd, and $\overline{\xi}_p^{(1)}(B) = \chi_p((-1)^{(m-n_1)/2} \det \Theta)$ if n_1 is even. Let $p = 2$. Then an element $B \in \mathcal{L}_{m-1,2}^{(1)}$ is equivalent, over $\mathbf{Z}_2$, to a matrix of the form $2\Theta \perp B_1$, where $\Theta \in GL_{m-n_1-2}(\mathbf{Z}_2) \cap S_{m-n_1-2}(\mathbf{Z}_2)_e$ and B_1 is one of the following three types:

(I) $B_1 = a \perp 4B_2$ with $a \equiv -1 \bmod 4$, and $B_2 \in S_{n_1}(\mathbf{Z}_2)_e^{\times}$;

(II) $B_1 \in 4S_{n_1+1}(\mathbf{Z}_2)^{\times}$;

(III) $B_1 = a \perp 4B_2$ with $a \equiv -1 \bmod 4$, and $B_2 \in S_{n_1}(\mathbf{Z}_2)_o$.

Then $\overline{\xi}_2^{(1)}(B) = 0$ if B_1 is of type (II) or type (III). Let B_1 be of type (I). Then $(-1)^{(m-n_1)/2} a \det \Theta \bmod (\mathbf{Z}_2^*)^{\square}$ is uniquely determined by B and we have

$$\overline{\xi}_2^{(1)}(B) = \chi_2((-1)^{(m-n_1)/2} a \det \Theta).$$

Suppose that $p \neq 2$, and let $\mathcal{U} = \mathcal{U}_p$ be a complete set of representatives of $\mathbf{Z}_p^*/(\mathbf{Z}_p^*)^{\square}$. Then, for each positive integer l and $d \in \mathcal{U}_p$, there exists a unique, up to $\mathbf{Z}_p$-equivalence, element of $S_l(\mathbf{Z}_p) \cap GL_l(\mathbf{Z}_p)$ whose determinant is $(-1)^{[(l+1)/2]} d$, which will be denoted by $\Theta_{l,d}$. Suppose that $p = 2$, and put $\mathcal{U} = \mathcal{U}_2 = \{1, 5\}$. Then for each positive even integer l and $d \in \mathcal{U}_2$ there exists a unique, up to $\mathbf{Z}_2$-equivalence, element of $S_l(\mathbf{Z}_2)_e \cap GL_l(\mathbf{Z}_2)$ whose determinant is $(-1)^{l/2} d$, which will be also denoted by $\Theta_{l,d}$. In particular, if p is any prime number and l is even, we put $\Theta_l = \Theta_{l,1}$ We make the convention that $\Theta_{l,d}$ is the empty matrix if $l = 0$. For an element $d \in \mathcal{U}$ we use the same symbol d to denote the coset $d \bmod (\mathbf{Z}_p^*)^{\square}$.

We put $\mathcal{D}_{l,i} = GL_n(\mathbf{Z}_p) \begin{pmatrix} 1_{l-i} & 0 \\ 0 & p1_i \end{pmatrix} GL_l(\mathbf{Z}_p)$ for $0 \leq i \leq l$. Suppose that r is a positive even integer. For $j = 0, 1, \xi = \pm 1$ and $T \in \mathcal{L}_{r-j,p}^{(j)}$, we define a polynomial $\widetilde{F}_p^{(j)}(T, \xi, X)$ in X and X^{-1} by

$$\widetilde{F}_p^{(j)}(T, \xi, X) = X^{-\mathfrak{e}^{(j)}(T)} F_p^{(j)}(T, \xi X).$$

We note that $\widetilde{F}_p^{(j)}(T, \xi, X) = \xi^{e^{(j)}(T)} \widetilde{F}_p^{(j)}(T, \xi X)$, and in particular $\widetilde{F}_p^{(j)}(T, 1, X)$ coincides with $\widetilde{F}_p^{(j)}(T, X)$. We also define a polynomial $\widetilde{G}_p^{(j)}(T, \xi, X, t)$ in X, X^{-1} and t by

$$\text{(5-1)} \quad \widetilde{G}_p^{(j)}(T, \xi, X, t) = \sum_{i=0}^{r-j} (-1)^i p^{i(i-1)/2} t^i \sum_{D \in GL_{r-j}(\mathbf{Z}_p) \backslash \mathcal{D}_{r-j,i}} \widetilde{F}_p^{(j)}(T[D^{-1}], \xi, X),$$

and put $\widetilde{G}_p^{(j)}(T, X, t) = \widetilde{G}_p^{(j)}(T, 1, X, t)$. We also define a polynomial $G_p^{(j)}(T, X)$ in X by

$$(5\text{-}2)\quad G_p^{(j)}(T,X)=\sum_{i=0}^{r-j}(-1)^i p^{i(i-1)/2}(X^2p^{r+1-j})^i \sum_{D\in GL_{r-j}(\mathbf{Z}_p)\backslash \mathcal{D}_{r-j,i}} F_p^{(j)}(T[D^{-1}],X).$$

We note that

$$\widetilde{G}_p^{(j)}(T,X,1)=X^{-\mathfrak{e}^{(j)}(T)}G_p^{(j)}(T,Xp^{-(m+1)/2}).$$

Now for an element $T\in\mathcal{L}_{r-1,p}^{(1)}$ we define a polynomial $B_p^{(1)}(T,t)$ in t by

$$(5\text{-}3)\quad B_p^{(1)}(T,t)=\frac{(1-\xi_p(T^{(1)})p^{-r/2+1/2}t)\prod_{i=1}^{(r-2)/2}(1-p^{-2i+1}t^2)}{G_p^{(1)}(T,p^{-r+1/2}t)}.$$

Then by [**25**, Lemma 4.2.1], we have the following:

Lemma 5.1.1. *Let n be the fixed positive even integer. Let $B\in\mathcal{L}_{n-1,p}^{(1)}$.*
(1) Let $p\neq 2$, and suppose that $B=\Theta_{n-n_1-1,d}\perp pB_1$ with $d\in\mathcal{U}$ and $B_1\in S_{n_1}(\mathbf{Z}_p)^{\times}$. Then

$$B_p^{(1)}(B,t)=\begin{cases}(1-\overline{\xi}_p^{(1)}(B)p^{(n_1-n+1)/2}t)\displaystyle\prod_{i=1}^{(n-n_1-2)/2}(1-p^{-2i+1}t^2) & \text{if } n_1 \text{ even},\\ \displaystyle\prod_{i=1}^{(n-n_1-1)/2}(1-p^{-2i+1}t^2) & \text{if } n_1 \text{ odd}.\end{cases}$$

(2) Let $p=2$, and suppose that $B=2\Theta\perp B_1\in\mathcal{L}'_{n-1,2}$ with $\Theta\in S_{n-n_1-2}(\mathbf{Z}_2)_e\cap GL_{n-n_1-2}(\mathbf{Z}_2)$ and $B_1\in S_{n_1+1}(\mathbf{Z}_2)^{\times}$. Then

$$B_p^{(1)}(B,t)=\begin{cases}(1-\overline{\xi}_2^{(1)}(B)p^{(n_1-n+1)/2}t)\displaystyle\prod_{i=1}^{(n-n_1-2)/2}(1-p^{-2i+1}t^2) & \text{if } B_1 \text{ is of type (I)},\\ \displaystyle\prod_{i=1}^{(n-n_1-2)/2}(1-p^{-2i+1}t^2) & \text{if } B_1 \text{ is of type (II) or (III)}.\end{cases}$$

Let m be a positive even integer and $j=0,1$. For a non-degenerate half-integral matrix T over $\mathbf{Z}_p$ of degree $m-j$, put

$$R^{(j)}(T,X,t)=\sum_{W\in M_{m-j}(\mathbf{Z}_p)^{\times}/GL_{m-j}(\mathbf{Z}_p)}\widetilde{F}_p^{(j)}(T[W],X)t^{\nu(\det W)}.$$

This type of formal power series was first introduced by Andrianov [**1**] to study the standard L-function of Siegel modular form of integral weight. Therefore we call it the formal power series of Andrianov type. (See also Böcherer [**2**].) The following proposition follows from [**25**, Lemma 4.1.1 (1)].

Proposition 5.1.2. *Let m be a positive even integer and $j=0$ or 1. Let $T\in\mathcal{L}_{m-j,p}^{(j)}$. Then*

$$\sum_{B\in\mathcal{L}_{m-j,p}^{(j)}}\frac{\widetilde{F}_p^{(j)}(B,X)\alpha_p(T,B)}{\alpha_p(B)}t^{\nu(\det B)}=t^{\nu(\det T)}R^{(j)}(T,X,p^{-m+j}t^2).$$

The following theorem is due to [**24**].

THEOREM 5.1.3. *Let T be an element of $\mathcal{L}_{n-1,p}^{(1)}$. Then*

$$R^{(1)}(T,X,t) = \frac{B_p^{(1)}(T,p^{n/2-1}t)\widetilde{G}_p^{(1)}(T,X,t)}{\prod_{j=1}^{n-1}(1-p^{j-1}X^{-1}t)(1-p^{j-1}Xt)}.$$

In [**4**], Böcherer and Sato got a similar formula for $T \in \mathcal{L}_{n,p}$. We note that the above formula for $p \neq 2$ can be derived directly from [**37**, Theorem 20.7] (see also Zhuravlëv [**40**]). However, we note that we cannot use their results to prove the above formula for $p=2$.

Then for $d_0 \in \mathcal{F}_p$ and $\omega = \varepsilon^l$ with $l=0,1$, we define a formal power series $\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)$ in t by

$$\widetilde{R}_{n-1}(d_0,\omega,X,Y,t) = \kappa(d_0,n-1,l)^{-1}t^{\delta_{2,p}(2-n)} \sum_{B' \in \mathcal{L}_{n-1,p}^{(1)}(d_0)} \frac{\widetilde{G}_p^{(1)}(B',X,p^{-n}Yt^2)}{\alpha_p(B')}$$
$$\times Y^{-\mathfrak{e}^{(1)}(B')}t^{\nu(\det B')}B_p^{(1)}(B',p^{-n/2-1}Yt^2)G_p^{(1)}(B',p^{-(n+1)/2}Y)\omega(B').$$

More precisely this is an element of $\mathbf{C}[X,X^{-1},Y^{1/2},Y^{-1/2}][[t]]$. Now by Theorem 5,2,5, we can rewrite $H_{n-1}(\omega,d_0,X,Y,t)$ in terms of $\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)$ in the following way:

THEOREM 5.1.4. *For $\omega = \varepsilon^l$, we have*

$$H_{n-1}(d_0,\omega,X,Y,t) = \frac{\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)}{\prod_{j=1}^{n}(1-p^{j-1-n}XYt^2)(1-p^{j-1-n}X^{-1}Yt^2)}.$$

Proof. By [**25**, Lemma 4.2.2], we have

$$\kappa(d_0,n-1,l)t^{\delta_{2,p}(n-2)}H_{n-1}(d_0,\omega,X,Y,t) = \sum_{B \in \mathcal{L}_{n-1,p}^{(1)}(d_0)} \frac{\widetilde{F}_p^{(1)}(B,X)}{\alpha_p(B)}\omega(B)t^{\nu(\det B)}$$
$$\times \sum_{B' \in \mathcal{L}_{n-1,p}^{(1)}} \frac{Y^{-\mathfrak{e}^{(1)}(B')}G_p^{(1)}(B',p^{-(n+1)/2}Y)\alpha_p(B',B)}{\alpha_p(B')}(p^{-1}Y)^{(\nu(\det B)-\nu(\det B'))/2}.$$

Let B and B' be elements of $\mathcal{L}_{n-1,p}^{(1)}$, and suppose that $\alpha_p(B',B) \neq 0$. Then we note that $B \in \mathcal{L}_{n-1,p}^{(1)}(d_0)$ if and only if $B' \in \mathcal{L}_{n-1,p}^{(1)}(d_0)$. Hence by Proposition 5.1.2 and Theorem 5.1.3

we have

$$\begin{aligned}
&\kappa(d_0, n-1, l)t^{\delta_{2,p}(n-2)}H_{n-1}(d_0, \omega, X, Y, t)\\
&= \sum_{B' \in \mathcal{L}^{(1)}_{n-1,p}(d_0)} \frac{G_p^{(1)}(B', p^{-(n+1)/2}Y)Y^{-\mathfrak{e}^{(1)}(B')}}{\alpha_p(B')}(pY^{-1})^{\nu(\det B')/2}\omega(B')\\
&\quad\times \sum_{B \in \mathcal{L}^{(1)}_{n-1,p}} \frac{\widetilde{F}_p^{(1)}(B, X)\alpha_p(B', B)}{\alpha_p(B)}(t^2p^{-1}Y)^{\nu(\det B)/2}\\
&= \sum_{B' \in \mathcal{L}^{(1)}_{n-1,p}(d_0)} \frac{G_p^{(1)}(B', p^{-(n+1)/2}Y)Y^{-\mathfrak{e}^{(1)}(B')}}{\alpha_p(B')}t^{\nu(\det B')}\omega(B')R^{(1)}(B', X, t^2Yp^{-n})\\
&= \sum_{B' \in \mathcal{L}^{(1)}_{n-1,p}(d_0)} \frac{\widetilde{G}_p^{(1)}(B', X, p^{-n}Yt^2)}{\alpha_p(B')}\omega(B')Y^{-\mathfrak{e}^{(1)}(B')}t^{\nu(\det B')}\\
&\quad\times \frac{B_p^{(1)}(B', p^{-n/2-1}Yt^2)G_p^{(1)}(B', p^{-(n+1)/2}Y)}{\prod_{j=1}^n(1-p^{j-1-n}XYt^2)(1-p^{j-1-n}X^{-1}Yt^2)}.
\end{aligned}$$

This proves the assertion. □

The polynomials $G_p^{(1)}(T, X)$ and $B_p^{(1)}(T, t)$ are expressed explicitly, and in particular they are determined by $[\mathfrak{d}_T]$ and the p-rank of T (cf. [**25**, Lemma 4.2.1] and Lemma 5.1.1). Thus we can rewrite the power series $\widetilde{R}_{n-1}(d_0, \omega, X, Y, t)$ in a more concise form (cf. Corollary to Theorem 5.2.8.)

5.2. *Formal power series of Koecher-Maaß type and of modified Koecher-Maaß type*

Let r be a positive even integer. For $d_0 \in \mathcal{F}_p, j = 0, 1$ and $l = 0, 1$, we define a formal power series $P^{(j)}_{r-j}(d_0, \omega, \xi, X, t)$ in t by

$$P^{(j)}_{r-j}(d_0, \omega, \xi, X, t) = \kappa(d_0, r-j, l)^{-1}t^{-a_{r-j,p}} \sum_{B \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} \frac{\widetilde{F}_p^{(j)}(B, \xi, X)}{\alpha_p(B)}\omega(B)t^{\nu(\det B)} \tag{5-4}$$

for $\omega = \varepsilon^l$ with $l = 0, 1$, where $a_{i,p} = \delta_{2,p}(i-1)$ or 1 according as i is odd or even. In particular we put $P^{(j)}_{r-j}(d_0, \omega, X, t) = P^{(j)}_{r-j}(d_0, \omega, 1, X, t)$. This type of formal power series appears in an explicit formula for the Koecher-Maaß series associated with the Siegel Eisenstein series and the Duke-Imamoḡlu-Ikeda lift (cf. [**11**, **12**] and [**25**]). Therefore we say that this formal power series is of Koecher-Maaß type.

For a variable Y we introduce the symbol $Y^{1/2}$ so that $(Y^{1/2})^2 = Y$, and for an integer a write $Y^{a/2} = (Y^{1/2})^a$. Under this convention, we can write $Y^{-\mathfrak{e}^{(j)}(T)}t^{\nu(\det T)}$ as $Y^{a_{r-j,p}/2}Y^{\nu(d_0)/2}(Y^{-1/2}t)^{\nu(\det T)}$ if $T \in \mathcal{L}^{(j)}_{r-j,p}(d_0)$, and we sometimes write a power series

$$P(Y, t) = \sum_{T \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} a(T, Y)Y^{-\mathfrak{e}^{(j)}(T)}t^{\nu(\det T)} \in \mathbf{C}[Y, Y^{-1}][[t]]$$

as

$$P(Y, t) = Y^{a_{r-j,p}/2}Y^{\nu(d_0)/2} \sum_{T \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} a(T, Y)(Y^{-1/2}t)^{\nu(\det T)}.$$

For $T \in \mathcal{L}^{(j)}_{r-j,p}$ let $\widetilde{G}^{(j)}_p(T,\xi,X,t)$ be the polynomial defined in the previous subsection. Moreover for $\xi = \pm 1$, and $j = 0,1$, we define a formal power series $\widetilde{P}^{(j)}_{r-j}(n; d_0, \omega, \xi, X, Y, t)$ in t by

$$\text{(5-5)}\quad \widetilde{P}^{(j)}_{r-j}(n; d_0, \omega, \xi, X, Y, t) = \kappa(d_0, r-j, l)^{-1}(tY^{-1/2})^{-a_{r-j,p}} Y^{\nu(d_0)/2}$$
$$\times \sum_{B' \in \mathcal{L}^{(j)}_{r,p}(d_0)} \frac{\widetilde{G}^{(j)}_p(B', \xi, X, p^{-n}t^2Y)}{\alpha_p(B')} \omega(B')(tY^{-1/2})^{\nu(\det B')}$$

for $\omega = \varepsilon^l$. Here we make the convention that $\widetilde{P}^{(0)}_0(n; d_0, \omega, \xi, X, Y, t) = 1$ or 0 according as $\nu(d_0) = 0$ or not. We say that the series $\widetilde{P}^{(j)}_{r-j}(n; d_0, \omega, \xi, X, Y, t)$ is of modified Koecher-Maaß type. The relation between $\widetilde{P}^{(j)}_{r-j}(n; d_0, \omega, \xi, X, Y, t)$ and $P^{(j)}_{r-j}(d_0, \omega, \xi, X, t)$ will be given in the following proposition:

Proposition 5.2.1. *Let r be a positive even integer. Let $\omega = \varepsilon^l$ with $l = 0,1$, and $j = 0,1$. Then*

$$\widetilde{P}^{(j)}_{r-j}(n; d_0, \omega, \xi, X, Y, t) = Y^{\nu(d_0)/2} P^{(j)}_{r-j}(d_0, \omega, \xi, X, tY^{-1/2}) \prod_{i=1}^{r-j}(1 - t^4 p^{-n-r+j-2+i}).$$

Proof. For $i = 0, ..., r-j$ put

$$\widetilde{P}^{(j)}_{r-j,i}(d_0, \omega, \xi, X, t) = \kappa(d_0, r-j, l)^{-1} t^{-a_{r-j,p}}$$
$$\times \sum_{B \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} \sum_{D \in GL_{r-j}(\mathbf{Z}_p)\backslash \mathcal{D}_{r-j,i}} \frac{\widetilde{F}^{(j)}_p(B[D^{-1}], \xi, X)}{\alpha_p(B)} \omega(B) t^{\nu(\det B)}.$$

Then we have

$$\widetilde{P}^{(j)}_{r-j}(n; d_0, \omega, \xi, X, Y, t) = \sum_{i=0}^{r-j} (-1)^i p^{i(i-1)/2} (p^{-n}t^2Y)^i Y^{\nu(d_0)/2} \widetilde{P}^{(j)}_{r-j,i}(d_0, \omega, \xi, X, tY^{-1/2}).$$

We have

$$\widetilde{P}^{(j)}_{r-j,i}(d_0, \omega, \xi, X, t) = \sum_{B \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} \frac{\omega(B)}{\alpha_p(B)} t^{\nu(\det B)}$$
$$\times \sum_{B' \in \mathcal{L}^{(j)}_{r-j,p}} \widetilde{F}^{(j)}_p(B', \xi, X) \#(\widetilde{\Omega}(B', B, i)/GL_{r-j}(\mathbf{Z}_p)),$$

where $\widetilde{\Omega}(B', B, i) = \{D \in \mathcal{D}_{r-j,i} \mid B'[D^{-1}] \sim B\}$. Hence by [**25**, Lemma 4.1.1 (2)], we have

$$\widetilde{P}^{(j)}_{r-j,i}(d_0, \omega, \xi, X, t)$$
$$= \sum_{B \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} \frac{1}{\alpha_p(B)} \sum_{B' \in \mathcal{L}^{(j)}_{r-j,p}} \frac{\widetilde{F}^{(j)}_p(B', \xi, X)\alpha_p(B', B, i)}{\alpha_p(B')} \omega(B) p^{-(\nu(\det B) - \nu(\det B'))/2} t^{\nu(\det B)},$$

where

$$\alpha_p(B', B, i) = 2^{-1} \lim_{e \to \infty} p^{-(r-j)(r-j-1)e/2} \#\{\overline{X} \in \mathcal{A}_e(B', B) \mid X \in \mathcal{D}_{r-j,i}\}.$$

Let B and B' be elements of $\mathcal{L}^{(j)}_{r-j,p}$, and suppose that $\alpha_p(B',B,i) \neq 0$. Then we note that $B \in \mathcal{L}^{(j)}_{r-j,p}(d_0)$ if and only if $B' \in \mathcal{L}^{(j)}_{r-j,p}(d_0)$. Hence by [**25**, Lemma 4.1.1 (1)], we have

$$\begin{aligned}
&\widetilde{P}^{(j)}_{r-j,i}(d_0,\omega,\xi,X,t)\\
&= \sum_{B' \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} \frac{\widetilde{F}^{(j)}_p(B',\xi,X)}{\alpha_p(B')} p^{\nu(\det B')/2}\omega(B') \sum_{B \in \mathcal{L}^{(j)}_{r-j,p}} (tp^{-1/2})^{\nu(\det B)} \frac{\alpha_p(B',B,i)}{\alpha_p(B)}\\
&= \sum_{B' \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} \frac{\widetilde{F}^{(j)}_p(B',\xi,X)}{\alpha_p(B')} p^{\nu(\det B')/2}(tp^{-1/2})^{\nu(\det B')}(t^2p^{-r+j-1})^i \#(GL_{r-j}(\mathbf{Z}_p)\backslash \mathcal{D}_{r-j,i}).
\end{aligned}$$

By [**1**, Lemma 3.2.18], we have

$$\#(GL_{r-j}(\mathbf{Z}_p)\backslash \mathcal{D}_{r-j,i}) = \frac{\phi_{r-j}(p)}{\phi_i(p)\phi_{r-j-i}(p)}.$$

Hence

$$\begin{aligned}
&\widetilde{P}^{(j)}_{r-j,i}(d_0,\omega,\xi,X,t)\\
&= \sum_{B' \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} \frac{\widetilde{F}^{(j)}_p(B',\xi,X)}{\alpha_p(B')} \omega(B') t^{\nu(\det B')} \frac{\phi_{r-j}(p)}{\phi_i(p)\phi_{r-j-i}(p)}(t^2p^{-r+j-1})^i\\
&= \frac{\phi_{r-j}(p)}{\phi_i(p)\phi_{r-j-i}(p)} P^{(j)}_{r-j}(d_0,\omega,\xi,X,t)(t^2p^{-r+j-1})^i.
\end{aligned}$$

Thus, by [**1**, (3.2.34)], we have

$$\begin{aligned}
&\widetilde{P}^{(j)}_{r-j}(n;d_0,\omega,\xi,X,t)\\
&= Y^{\nu(d_0)/2}\sum_{i=0}^{r-j}(-1)^i p^{i(i+1)/2}(p^{-n-r+j-2}t^4)^i \frac{\phi_{r-j}(p)}{\phi_i(p)\phi_{r-j-i}(p)} P^{(j)}_{r-j}(d_0,\omega,\xi,X,tY^{-1/2})\\
&= Y^{\nu(d_0)/2}P^{(j)}_{r-j}(d_0,\omega,\xi,X,tY^{-1/2}) \prod_{i=1}^{r-j}(1-t^4p^{-n-r+j-2+i}).
\end{aligned}$$

□

We give explicit formulas for $P^{(j)}_{r-j}(d_0,\varepsilon^l,\xi,X,t)$ for $j=0,1, l=0,1$ and $\xi=\pm 1$. From now on we for $a \in \mathbf{Q}_p^\times$, we simply write $\chi_p(a)$ as $\chi(a)$ if the prime number p is clear from the context.

THEOREM 5.2.2. *Let $d_0 \in \mathcal{F}_p$ and $\xi_0 = \chi(d_0)$.*
(1) *Let r be even. Then*

$$\begin{aligned}
P^{(0)}_r(d_0,\iota,X,t) &= \frac{(p^{-1}t)^{\nu(d_0)}}{\phi_{r/2-1}(p^{-2})(1-p^{-r/2}\xi_0)}\\
&\times \frac{(1+t^2p^{-r/2-3/2})(1+t^2p^{-r/2-5/2}\xi_0^2) - \xi_0 t^2 p^{-r/2-2}(X+X^{-1}+p^{1/2-r/2}+p^{-1/2+r/2})}{(1-p^{-2}Xt^2)(1-p^{-2}X^{-1}t^2)\prod_{i=1}^{r/2}(1-t^2p^{-2i-1}X)(1-t^2p^{-2i-1}X^{-1})},
\end{aligned}$$

and

$$P^{(0)}_r(d_0,\varepsilon,X,t) = \frac{\xi_0^2}{\phi_{r/2-1}(p^{-2})(1-p^{-r/2}\xi_0)\prod_{i=1}^{r/2}(1-t^2p^{-2i}X)(1-t^2p^{-2i}X^{-1})}.$$

(2) *Let r be even. Then*

$$P_{r-1}^{(1)}(d_0,\iota,X,t) = \frac{(p^{-1}t)^{\nu(d_0)}(1-\xi_0 t^2 p^{-5/2})}{(1-t^2p^{-2}X)(1-t^2p^{-2}X^{-1})\prod_{i=1}^{(r-2)/2}(1-t^2p^{-2i-1}X)(1-t^2p^{-2i-1}X^{-1})\phi_{(r-2)/2}(p^{-2})},$$

and

$$P_{r-1}^{(1)}(d_0,\varepsilon,X,t) = \frac{(p^{-1}t)^{\nu(d_0)}(1-\xi_0 t^2 p^{(-1/2-r)})}{\prod_{i=1}^{r/2}(1-t^2p^{-2i}X)(1-t^2p^{-2i}X^{-1})\phi_{(r-2)/2}(p^{-2})}.$$

Proof. The assertions (1) and (2) are due to [**22**, Proposition 4.3], and to [**25**, Theorem 4.4.1], respectively. □

Corollary. *Let $\xi = \pm 1$.*
(1) *Let r be even. Then*

$$P_r^{(0)}(d_0,\iota,\xi,X,t) = \frac{(p^{-1}t)^{\nu(d_0)}}{\phi_{r/2-1}(p^{-2})(1-p^{-r/2}\xi_0)}$$
$$\times \{(1+t^2p^{-r/2-3/2}\xi)(1+t^2p^{-r/2-5/2}\xi\xi_0^2) - \xi_0 t^2 p^{-r/2-2}(X+X^{-1}+p^{1/2-r/2}\xi+p^{-1/2+r/2}\xi)\}$$
$$\times \frac{1}{(1-p^{-2}Xt^2)(1-p^{-2}X^{-1}t^2)\prod_{i=1}^{r/2}(1-t^2p^{-2i-1}X)(1-t^2p^{-2i-1}X^{-1})},$$

and

$$P_r^{(0)}(d_0,\varepsilon,\xi,X,t) = \frac{\xi_0^2}{\phi_{r/2-1}(p^{-2})(1-p^{-r/2}\xi_0)\prod_{i=1}^{r/2}(1-t^2p^{-2i}X)(1-t^2p^{-2i}X^{-1})}.$$

(2) *Let r be even. Then*

$$P_{r-1}^{(1)}(d_0,\iota,\xi,X,t) = \frac{(p^{-1}t)^{\nu(d_0)}(1-\xi_0 t^2 p^{-5/2}\xi)}{(1-t^2p^{-2}X)(1-t^2p^{-2}X^{-1})\prod_{i=1}^{(r-2)/2}(1-t^2p^{-2i-1}X)(1-t^2p^{-2i-1}X^{-1})\phi_{(r-2)/2}(p^{-2})},$$

and

$$P_{r-1}^{(1)}(d_0,\varepsilon,\xi,X,t) = \frac{(p^{-1}t)^{\nu(d_0)}(1-\xi_0 t^2 p^{(-1/2-r)}\xi)}{\prod_{i=1}^{r/2}(1-t^2p^{-2i}X)(1-t^2p^{-2i}X^{-1})\phi_{(r-2)/2}(p^{-2})}.$$

Proof. Put

$$S_{r-j}^{(j)}(d_0,\omega,\xi,X,t) = \sum_{T\in\mathcal{L}_{r-j,p}^{(j)}} \frac{\widetilde{F}(T,\xi,X)}{\alpha_p(T)} t^{\mathfrak{e}^{(j)}(T)},$$

and

$$S_{r-j}^{(j)}(d_0,\omega,X,t) = \sum_{T\in\mathcal{L}_{r-j,p}^{(j)}} \frac{\widetilde{F}(T,X)}{\alpha_p(T)} t^{\mathfrak{e}^{(j)}(T)}.$$

Then we have
$P_{r-j}^{(j)}(d_0,\omega,\xi,X,t) = t^{\nu(d_0)}S_{r-j}^{(j)}(d_0,\omega,\xi,X,t^2)$ and $P_{r-j}^{(j)}(d_0,\omega,X,t) = t^{\nu(d_0)}S_{r-j}^{(j)}(d_0,\omega,X,t^2)$.
By definition we have

$$S_{r-j}^{(j)}(d_0,\omega,\xi,X,t^2) = S_{r-j}^{(j)}(d_0,\omega,\xi X,\xi t^2).$$

Thus the assertion follows from the above theorem. □

Now let r be an even integer, and for $j=0,1$, we consider partial series of $\widetilde{P}^{(j)}_{r-j}(n;d_0,\omega,\xi,X,Y,t)$: Let $\xi=\pm 1$. and $l=0,1$. First let $p\neq 2$. Then put

$$\text{(5-6-1)}\quad Q_r^{(0)}(n;d_0,\varepsilon^l,\xi,X,Y,t)=Y^{\nu(d_0)/2} \times \sum_{B'\in S_r(\mathbf{Z}_p,d_0)\cap S_r(\mathbf{Z}_p)} \frac{\widetilde{G}_p^{(0)}(pB',\xi,X,p^{-n}t^2Y)}{\alpha_p(pB')}\varepsilon(pB')^l(tY^{-1/2})^{\nu(\det pB')},$$

and

$$\text{(5-6-2)}\quad Q_{r-1}^{(1)}(n;d_0,\varepsilon^l,\xi,X,Y,t)=\kappa(d_0,r-1,l)^{-1}Y^{\nu(d_0)/2} \times \sum_{B'\in p^{-1}S_{r-1}(\mathbf{Z}_p,d_0)\cap S_{r-1}(\mathbf{Z}_p)} \frac{\widetilde{G}_p^{(1)}(pB',\xi,X,p^{-n}t^2Y)}{\alpha_p(pB')}\varepsilon(pB')^l(tY^{-1/2})^{\nu(\det pB')},$$

where $\widetilde{G}_p^{(j)}(pB',\xi,X,p^{-n}t^2Y)$ is the polynomial defined in (5-1) for $j=0,1$. Next let $p=2$. Then put

$$\text{(5-6-3)}\quad Q_{r-1}^{(1)}(n;d_0,\varepsilon^l,\xi,X,Y,t)=\kappa(d_0,r-1,l)^{-1}(tY^{-1/2})^{\delta_{2,p}(2-n)}Y^{\nu(d_0)/2} \times \sum_{B'\in S_{r-1}(\mathbf{Z}_2,d_0)\cap S_{r-1}(\mathbf{Z}_2)} \frac{\widetilde{G}_2^{(1)}(4B',\xi,X,2^{-n}t^2Y)}{\alpha_2(4B')}\varepsilon(4B')^l(tY^{-1/2})^{\nu_2(\det(4B'))},$$

and

$$\text{(5-6-4)}\quad Q_r^{(0)}(n;d_0,\varepsilon^l,\xi,X,Y,t)=\kappa(d_0,r,l)^{-1}Y^{\nu(d_0)/2} \times \sum_{B'\in S_r(\mathbf{Z}_2,d_0)\cap S_r(\mathbf{Z}_2)_e} \frac{\widetilde{G}_2^{(0)}(2B',\xi,X,2^{-n}t^2Y)}{\alpha_2(2B')}\varepsilon(B')^l(tY^{-1/2})^{\nu(\det(2B'))},$$

where $\widetilde{G}_2^{(1)}(4B',\xi,X,2^{-n}t^2Y)$ and $\widetilde{G}_2^{(0)}(2B',\xi,X,2^{-n}t^2Y)$ are the polynomials defined in (5-1). Here we make the convention that $Q_0^{(0)}(n;d_0,\varepsilon^l,\xi,X,Y,t)=1$ or 0 according as $\nu(d_0)=0$ or not.

A non-degenerate square matrix $D=(d_{ij})_{m\times m}$ with entries in $\mathbf{Z}_p$ is said to be reduced if D satisfies the following two conditions:

(a) For $i=j$, $d_{ii}=p^{e_i}$ with a non-negative integer e_i;

(b) For $i\neq j$, d_{ij} is a non-negative integer satisfying $d_{ij}\leq p^{e_j}-1$ if $i<j$ and $d_{ij}=0$ if $i>j$.

It is well known that we can take the set of all reduced matrices as a complete set of representatives of $GL_m(\mathbf{Z}_p)\backslash M_m(\mathbf{Z}_p)^{\times}$.

To consider the relation between

$$\widetilde{P}^{(j)}_{r-j}(n;d_0,\varepsilon^l,\xi,X,Y,t)\ \text{ and }\ Q^{(j)}_{r-j}(n;d_0,\varepsilon^l,\xi,X,Y,t),$$

and to express $\widetilde{R}_{n-1}(d_0,\varepsilon^l,X,Y,t)$ in terms of $\widetilde{P}^{(j)}_{r-j}(n;d_0,\varepsilon^l,\xi,X,Y,t)$, we give some preliminary results.

LEMMA 5.2.3. *Let $p\neq 2$. Let m be an even integer, and r an integer such that $0\leq r\leq m$. Let $d\in\mathcal{U}$ and $\xi_0=\pm 1$.*

(1) *Suppose that r is even.*

(1.1) *Let $B'\in S_r(\mathbf{Z}_p)^{\times}$. Then*

$$\widetilde{G}_p^{(0)}(\Theta_{m-r,d}\perp pB',\xi_0,X,t)=\widetilde{G}_p^{(0)}(pB',\xi_0\chi(d),X,t).$$

(1.2) *Let* $B' \in S_{r-1}(\mathbf{Z}_p)^{\times}$. *Then*

$$\widetilde{G}_p^{(1)}(\Theta_{m-r,d} \bot pB', \xi_0, X, t) = \widetilde{G}_p^{(1)}(pdB', \xi_0, X, t).$$

(2) *Suppose that* r *is odd.*

(2.1) *Let* $B' \in S_r(\mathbf{Z}_p)^{\times}$. *Then*

$$\widetilde{G}_p^{(0)}(\Theta_{m-r,d} \bot pB', \xi_0, X, t) = \widetilde{G}_p^{(1)}(-pdB', \xi_0, X, t).$$

(2.2) *Let* $B' \in S_{r-1}(\mathbf{Z}_p)^{\times}$. *Then*

$$\widetilde{G}_p^{(1)}(\Theta_{m-r,d} \bot pB', \xi_0, X, t) = \widetilde{G}_p^{(0)}(pB', \xi_0\chi(d), X, t).$$

(3) *Suppose that* r *is even. Then we have*

$$\widetilde{G}_p^{(0)}(d'B, \xi_0, X, t) = \widetilde{G}_p^{(0)}(B, \xi_0, X, t)$$

for $d' \in \mathbf{Z}_p^*$, *and* $B \in S_r(\mathbf{Z}_p)^{\times}$.

Proof. Let $m-r$ be even. By [**20**, Proposition 3.2], we have

$$\widetilde{F}_p^{(0)}(\Theta_{m-r,d} \bot pB', \xi_0, X) = \widetilde{F}_p^{(0)}(pB', \xi_0\chi(d), X)$$

for $B' \in S_r(\mathbf{Z}_p)^{\times}$. We note that

$$\widetilde{G}_p^{(0)}(\Theta_{m-r,d} \bot pB', \xi_0, X, t) = \sum_{i=0}^{m} (-1)^i p^{i(i-1)/2} t^i \times \sum_{D \in GL_m(\mathbf{Z}_p) \backslash \widetilde{\Omega}^{(0)}(\Theta_{m-r,d} \bot pB', i)} \widetilde{F}_p^{(0)}((\Theta_{m-r,d} \bot pB')[D^{-1}], \xi_0, X),$$

where for $j = 0, 1$ and $B \in \mathcal{L}_{m-j,p}^{(j)}$ put

$$\widetilde{\Omega}^{(j)}(B, i) = \{W \in \mathcal{D}_{m-j,i} \mid B[W^{-1}] \in \mathcal{L}_{m-j,p}^{(j)}\}.$$

Thus the assertion (1.1) follows from [**25**, Lemma 4.1.2 (3)]. Furthermore we have

$$\begin{aligned} \widetilde{F}_p^{(1)}(\Theta_{m-r,d} \bot pB', \xi_0, X) &= \widetilde{F}_p(1 \bot \Theta_{m-r,d} \bot pB', \xi_0, X) \\ &= \widetilde{F}_p(d \bot \Theta_{m-r} \bot pB', \xi_0, X) = \widetilde{F}_p(1 \bot \Theta_{m-r} \bot pdB', \xi_0, X) \\ &= \widetilde{F}_p(1 \bot pdB', \xi_0, X) = \widetilde{F}_p^{(1)}(pdB', \xi_0, X) \end{aligned}$$

for $B' \in S_{r-1}(\mathbf{Z}_p)^{\times}$. Thus the assertion (1.2) follows from [**25**, Lemma 4.1.2 (3)]. The other assertions can be proved in a similar way. □

LEMMA 5.2.4. *Let* $p = 2$. *Let* m *and* r *be even integers such that* $0 \le r \le m$, *and* $\xi_0 = \pm 1$.

(1) *Let* $d \in \mathcal{U}$.

(1.1) *Let* $B' \in S_r(\mathbf{Z}_2)^{\times}$. *Then*

$$\widetilde{G}_2^{(0)}(\Theta_{m-r,d} \bot 2B', \xi_0, X, t) = \widetilde{G}_2^{(0)}(2B', \xi_0\chi(d), X, t),$$

(1.2) *Let* $B' \in S_{r-1}(\mathbf{Z}_2)^{\times}$. *Then*

$$\widetilde{G}_2^{(1)}(2\Theta_{m-r,d} \bot 4B', \xi_0, X, t) = \widetilde{G}_2^{(1)}(4dB', \xi_0, X, t).$$

(2)

(2.1) *Let* $a \in \mathcal{U}$ *and* $B' \in S_r(\mathbf{Z}_2)^{\times}$. *Then*

$$\widetilde{G}_2^{(1)}(-a \bot 2\Theta_{m-r-2} \bot 4B', \xi_0, X, t) = \widetilde{G}_2^{(0)}(2B', \xi_0\chi(a), X, t).$$

(2.2) *Let $B' \in S_{r-1}(\mathbf{Z}_2)^{\times}$ and $a \in \mathbf{Z}_2^*$. Then*

$$\widetilde{G}_2^{(0)}(\Theta_{m-r}\perp 2a\perp 2B', \xi_0, X, t) = \widetilde{G}_2^{(1)}(4aB', \xi_0, X, t).$$

(3) *We have*

$$\widetilde{G}_2^{(0)}(d'B, \xi_0, X, t) = \widetilde{G}_2^{(0)}(B, \xi_0, X, t)$$

for $d' \in \mathbf{Z}_2^*$, and $B \in S_r(\mathbf{Z}_2)^{\times}$.
(4) *Let $u_0 \in \mathbf{Z}_2^*$ and $B_1 \in S_{r-2}(\mathbf{Z}_2)^{\times}$. Then*

$$\widetilde{G}_2^{(1)}(u_0\perp 5B_1, \xi_0, X, t) = \widetilde{G}_2^{(1)}(u_0\perp B_1, \xi_0, X, t).$$

Proof. All the assertions except (4) can be proved in a way similar to Lemma 5.2.3. To prove (4), we first note that $GL_{m-1}(\mathbf{Z}_2)\backslash\widetilde{\Omega}(u_0\perp 5B_1, i) = GL_{m-1}(\mathbf{Z}_2)\backslash\widetilde{\Omega}(u_0\perp B_1, i)$ for $i = 0, \cdots, m-1$. Hence it suffices to prove

$$\widetilde{F}_2^{(1)}((u_0\perp 5B_1)[D^{-1}], \xi_0, X) = \widetilde{F}_2^{(1)}((u_0\perp B_1)[D^{-1}], \xi_0, X)$$

for $D \in \widetilde{\Omega}(u_0\perp B_1, i)$. We may assume that D is reduced. Since we have $u_0 \in \mathbf{Z}_2^*$ we have $D = \begin{pmatrix} 1 & d \\ O & D_1 \end{pmatrix}$ with $d \in M_{1,m-2}(\mathbf{Z}_2)$ and $M_{m-2}(\mathbf{Z}_2)$. We also note that $2D_1^{-1} \in M_{m-2}(\mathbf{Z}_2)$. We have

$$\begin{aligned}\widetilde{F}_2^{(1)}((u_0\perp 5B_1)[D^{-1}], \xi_0, X) &= \widetilde{F}_2((1\perp u_0\perp 5B_1)[1\perp D^{-1}], \xi_0, X) \\ &= \widetilde{F}_2((1\perp u_0\perp 5B_1)\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -dD_1^{-1} \\ 0 & 0 & D_1^{-1} \end{pmatrix}\right], \xi_0, X) \\ &= \widetilde{F}_2((5\perp 5u_0\perp B_1)\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -dD_1^{-1} \\ 0 & 0 & D_1^{-1} \end{pmatrix}\right], \xi_0, X).\end{aligned}$$

We can easily see that there exits an element $U = (u_{ij}) \in GL_2(\mathbf{Z}_2)$ such that $(1\perp u_0)[U] = 5\perp 5u_0$ and $u_{12} \equiv 0, u_{22} \equiv 1 \bmod 2$. Then we have

$$\widetilde{F}_2^{(1)}((u_0\perp 5B_1)[D^{-1}], \xi_0, X) = \widetilde{F}_2((1\perp u_0\perp B_1)[(1\perp D^{-1})V], \xi_0, X),$$

where $V = \begin{pmatrix} u_{11} & u_{12} & -u_{12}dD_1^{-1} \\ u_{21} & u_{22} & -u_{22}dD_1^{-1}+dD_1^{-1} \\ 0 & 0 & 1_{m-2} \end{pmatrix}$. By construction, we have $V \in GL_m(\mathbf{Z}_2)$, and hence we have

$$\widetilde{F}_2((1\perp u_0\perp B_1)[(1\perp D^{-1})])V], \xi_0, X) = \widetilde{F}_2^{(1)}((u_0\perp B_1)[D^{-1}], \xi_0, X).$$

□

Let $\widetilde{R}_{n-1}(d_0, \omega, X, Y, t)$ be the formal power series defined at the beginning of Section 5. We express $\widetilde{R}_{n-1}(d_0, \omega, X, Y, t)$ in terms of $Q_{2r}^{(0)}(n; d_0d, \omega, \chi(d), X, Y, t)$ and $Q_{2r+1}^{(1)}(n; d_0, \omega, 1, X, Y, t)$. Henceforth, for $d_0 \in \mathcal{F}_p$ and non-negative integers m, r such that $r \le m$, put $\mathcal{U}(m, r, d_0) = \{1\}, \mathcal{U} \cap \{d_0\}$, or $\mathcal{U}$ according as $r = 0$, $r = m \ge 1$, or $1 \le r \le m-1$. Moreover, we sometimes abbreviate $S_r(\mathbf{Z}_p)$ and $S_r(\mathbf{Z}_p, d)$ as $S_{r,p}$ and $S_{r,p}(d)$, respectively. Furthermore we abbreviate $S_r(\mathbf{Z}_2)_x$ and $S_r(\mathbf{Z}_2, d)_x$ as $S_{r,2;x}$ and $S_{r,2}(d)_x$, respectively, for $x = e, o$.

Theorem 5.2.5. *Let $d_0 \in \mathcal{F}_p$, and $\xi_0 = \chi(d_0)$. For $d \in \mathcal{U}(n-1, n-2r-1, d_0)$ put*

$$D_{2r}(d_0, d, Y, t) = \frac{1 - \xi_0 p^{-1/2}Y}{1 - p^{r-1/2}\chi(d)Y}(1 - p^{-n-1/2+r}\chi(d)Yt^2).$$

(1) *Let $\omega = \iota$, or $\nu(d_0) = 0$. Then*

$$\widetilde{R}_{n-1}(d_0, \omega, X, Y, t) = \sum_{r=0}^{(n-2)/2} \frac{\prod_{i=1}^{r}(1 - p^{2i-1}Y^2)\prod_{i=1}^{(n-2r-2)/2}(1 - p^{-2i-n-1}Y^2t^4)}{2^{1-\delta_{0,r}}\phi_{(n-2r-2)/2}(p^{-2})}$$
$$\times \sum_{d \in \mathcal{U}(n-1,n-2r-1,d_0)} D_{2r}(d_0, d, Y, t) Q_{2r}^{(0)}(n; d_0 d, \omega, \chi(d), X, Y, t)$$
$$+ \sum_{r=0}^{(n-2)/2} \frac{\prod_{i=1}^{r}(1 - p^{2i-1}Y^2)\prod_{i=1}^{(n-2r-2)/2}(1 - p^{-2i-n-1}Y^2t^4)}{\phi_{(n-2r-2)/2}(p^{-2})}$$
$$\times (1 - \xi_0 p^{-1/2}Y) Q_{2r+1}^{(1)}(n; d_0, \omega, 1, X, Y, t).$$

(2) *Let $\nu(d_0) > 0$. Then*

$$\widetilde{R}_{n-1}(d_0, \varepsilon, X, Y, t) = \sum_{r=0}^{(n-2)/2} \frac{\prod_{i=1}^{r}(1 - p^{2i-1}Y^2)\prod_{i=1}^{(n-2r-2)/2}(1 - p^{-2i-n-1}Y^2t^4)}{\phi_{(n-2r-2)/2}(p^{-2})}$$
$$\times (1 - \xi_0 p^{-1/2}Y) Q_{2r+1}^{(1)}(n; d_0, \varepsilon, 1, X, Y, t).$$

Proof. Let $p \neq 2$. Let B be a symmetric matrix of degree $2r$ or $2r+1$ with entries in $\mathbf{Z}_p$. Then we note that $\Theta_{n-2r-2,d} \perp pB$ belongs to $\mathcal{L}_{n-1,p}(d_0)$ if and only if $B \in S_{2r+1,p}(p^{-1}d_0 d) \cap S_{2r+1,p}$, and that $\Theta_{n-2r-1,d} \perp pB$ belongs to $\mathcal{L}_{n-1,p}(d_0)$ if and only if $B \in S_{2r,p}(d_0 d) \cap S_{2r,p}$. Thus by the theory of Jordan decompositions (cf. [**27**, Theorem 5.3.1]), for $\omega = \varepsilon^l$ we have

$$\widetilde{R}_{n-1}(d_0, \omega, X, Y, t) = \kappa(d_0, n-1, l)^{-1}(tY^{-1/2})^{\delta_{2,p}(2-n)}$$
$$\times \Bigg\{ \sum_{r=0}^{(n-2)/2} \sum_{d \in \mathcal{U}(n-1,n-2r-2,d_0)} \sum_{B' \in p^{-1}S_{2r+1,p}(d_0 d) \cap S_{2r+1,p}} \frac{G_p^{(1)}(\Theta_{n-2r-2,d} \perp pB', p^{-(n+1)/2}Y)}{\alpha_p(\Theta_{n-2r-2,d} \perp pB')}$$
$$\times B_p^{(1)}(\Theta_{n-2r-2,d} \perp pB', p^{-n/2-1}Yt^2) \widetilde{G}_p^{(1)}(\Theta_{n-2r-2,d} \perp pB', 1, X, p^{-n}t^2 Y)$$
$$\times \omega(\Theta_{n-2r-2,d} \perp pB')(tY^{-1/2})^{\nu(\det(pB'))}$$
$$+ \sum_{r=0}^{(n-2)/2} \sum_{d \in \mathcal{U}(n-1,n-2r-1,d_0)} \sum_{B' \in S_{2r,p}(d_0 d) \cap S_{2r,p}} \frac{G_p^{(1)}(\Theta_{n-2r-1,d} \perp pB', p^{-(n+1)/2}Y)}{\alpha_p(\Theta_{n-2r-1,d} \perp pB')}$$
$$\times B_p^{(1)}(\Theta_{n-2r-1,d} \perp pB', p^{-n/2-1}Yt^2) \widetilde{G}_p^{(1)}(\Theta_{n-2r-1,d} \perp pB', 1, X, p^{-n}t^2 Y)$$
$$\times \omega(\Theta_{n-2r-1,d} \perp pB')(tY^{-1/2})^{\nu(\det(pB'))} \Bigg\},$$

where $\widetilde{G}_p^{(1)}(*, *, X, p^{-n/2}t^2Y)$, $G_p^{(1)}(*, p^{-(n+1)/2})$, and $B_p(*, p^{-n/2-1}Yt^2)$ are those defined in (5-1), (5-2), and (5-3), respectively. By [**25**, Lemma 4.2.1] and Lemma 5.1.1 we have

$$G_p^{(1)}(\Theta_{n-2r-2,d} \perp pB', p^{-(n+1)/2}Y) B_p^{(1)}(\Theta_{n-2r-2,d} \perp pB', p^{-n/2-1}Yt^2)$$
$$= \prod_{i=1}^{r}(1 - p^{2i-1}Y^2) \prod_{i=1}^{(n-2r-2)/2} (1 - p^{-2i-n-1}Y^2t^4)(1 - \xi_0 p^{-1/2}Y),$$

and

$$G_p^{(1)}(\Theta_{n-2r-1,d} \perp pB', p^{-(n+1)/2}Y) B_p^{(1)}(\Theta_{n-2r-1,d} \perp pB', p^{-n/2-1}Yt^2)$$

$$= \prod_{i=1}^{r-1}(1-p^{2i-1}Y^2) \prod_{i=1}^{(n-2r-2)/2} (1-p^{-2i-n-1}Y^2t^4)D_{2r}(d_0,d,Y,t).$$

Put $H^{(1)}_{2i-1,\xi}(B) = \widetilde{G}^{(1)}_p(B,\xi,X,p^{-n}t^2Y)$ for $B \in S_{2i-1}(\mathbf{Z}_p)^\times$, and $H^{(0)}_{2i,\xi}(B) = \widetilde{G}^{(0)}_p(B,\xi,X,p^{-n}t^2Y)$ for $B \in S_{2i}(\mathbf{Z}_p)^\times$ and $\xi = \pm 1$. Then $H^{(1)}_{2i-1,\xi}$ and $H^{(0)}_{2i,\xi}$ are $GL_{2i-1}(\mathbf{Z}_p)$ -invariant functions on $S_{2i-1}(\mathbf{Z}_p)^\times$ with values in $\mathbf{C}[X,X^{-1},Y,Y^{-1},t]$ and satisfy the conditions (H-p-1) $\sim$ (H-p-5) in [**25**, Section 4] by virtue of Lemma 5.2.3. Thus the assertion (1) in case $p \neq 2$ follows from [**25**, Propositions 4.3.3 and 4.3.4].

Next let $p = 2$. Let B be a symmetric matrix of degree $2r$ or $2r+1$ with entries in $\mathbf{Z}_2$, and $d \in \mathcal{U}$. We note that $2\Theta_{n-2r-2,d} \perp 4B$ belongs to $\mathcal{L}_{n-1,2}(d_0)$ if and only if $B \in S_{2r+1,2}(d_0d) \cap S_{2r+1,2}$, and that $-d\perp 2\Theta_{n-2r-2}\perp 4B$ belongs to $\mathcal{L}_{n-1,2}(d_0)$ if and only if $B \in S_{2r+2,2}(d_0d) \cap S_{2r+2,2}$. Then, similarly to the above, we have

$$\begin{aligned}
&\widetilde{R}_{n-1}(d_0,\omega,X,Y,t) = \kappa(d_0,n-1,l,tY^{-1/2})^{-1}\\
&\times \Bigg\{ \sum_{r=0}^{(n-2)/2} \Bigg(\sum_{d\in\mathcal{U}(n-1,n-2r-2,d_0)} \sum_{B'\in S_{2r+1,2}(d_0d)\cap S_{2r+1,2;e}} G^{(1)}_p(2\Theta_{n-2r-2,d}\perp 4B', 2^{-(n+1)/2}Y)\\
&\qquad \times B^{(1)}_p(2\Theta_{n-2r-2,d}\perp 4B', p^{-n/2-1}Yt^2)\frac{\widetilde{G}^{(1)}_2(2\Theta_{n-2r-2,d}\perp 4B',1,X,2^{-n}t^2Y)}{\alpha_2(2\Theta_{n-2r-2,d}\perp 4B')}\\
&\qquad\quad \times\omega(2\Theta_{n-2r-2,d}\perp 4B')(tY^{-1/2})^{\nu(\det(4B'))+n-2r-2}\\
&\quad + \sum_{B'\in S_{2r+1,2}(d_0)\cap S_{2r+1,2;o}} G^{(1)}_p(2\Theta_{n-2r-2}\perp 4B', 2^{-(n+1)/2}Y)\\
&\qquad \times B^{(1)}_p(2\Theta_{n-2r-2}\perp 4B', p^{-n/2-1}Yt^2)\frac{\widetilde{G}^{(1)}_2(2\Theta_{n-2r-2}\perp 4B',1,X,2^{-n}t^2Y)}{\alpha_2(2\Theta_{n-2r-2}\perp 4B')}\\
&\qquad\quad \times\omega(2\Theta_{n-2r-2}\perp 4B')(tY^{-1/2})^{\nu(\det(4B'))+n-2r-2}\\
&\quad + \sum_{B'\in S_{2r+2,2}(d_0)\cap S_{2r+2,2;o}} G^{(1)}_p(-1\perp 2\Theta_{n-2r-4}\perp 4B', 2^{-(n+1)/2}Y)\\
&\qquad \times B^{(1)}_p(-1\perp 2\Theta_{n-2r-4}\perp 4B', p^{-n/2-1}Yt^2)\frac{\widetilde{G}^{(1)}_2(-1\perp 2\Theta_{n-2r-4}\perp 4B',1,X,2^{-n}t^2Y)}{\alpha_2(-1\perp 2\Theta_{n-2r-4}\perp 4B')}\\
&\qquad\quad \times\omega(-1\perp 2\Theta_{n-2r-4}\perp 4B')(tY^{-1/2})^{\nu(\det(4B'))+n-2r-4}\Bigg)\\
&\quad + \sum_{r=0}^{(n-2)/2} \sum_{d\in\mathcal{U}(n-1,n-2r-1,d_0)} \sum_{B'\in S_{2r,2}(d_0d)\cap S_{2r,2;e}} G^{(1)}_p(-d\perp 2\Theta_{n-2r-2}\perp 4B', 2^{-(n+1)/2}Y)\\
&\qquad \times B^{(1)}_p(-d\perp 2\Theta_{n-2r-2}\perp 4B', p^{-n/2-1}Yt^2)\frac{\widetilde{G}^{(1)}_2(-d\perp 2\Theta_{n-2r-2}\perp 4B',1,X,2^{-n}t^2Y)}{\alpha_2(-d\perp 2\Theta_{n-2r-2}\perp 4B')}\\
&\qquad\quad \times\omega(-d\perp 2\Theta_{n-2r-2}\perp 4B')(tY^{-1/2})^{\nu(\det(4B'))+n-2r-2}\Bigg\}.
\end{aligned}$$

Thus the assertion (1) in case $p = 2$ can be proved by using [**25**, Lemma 4.2.1], Lemmas 5.1.1 and 5.2.4, and [**25**, Propositions 4.3.3 and 4.3.4] in the same way as above. Similarly the assertion (2) can be proved. □

Now let $\widetilde{P}^{(1)}_{m-1}(n;d_0,\omega,\eta,X,Y,t)$ and $\widetilde{P}^{(0)}_m(n;d_0,\omega,\eta,X,Y,t)$ be those defined in (5-5), and $Q^{(1)}_{2r+1}(n;d_0,\omega,\eta,X,Y,t)$ and $Q^{(0)}_{2r}(n;d_0d,\omega,\eta,X,Y,t)$ be those defined in (5-6-1) $\sim$

(5-6-4). Then to rewrite the above theorem, first we express $\widetilde{P}^{(1)}_{m-1}(n;d_0,\omega,\eta,X,Y,t)$ and $\widetilde{P}^{(0)}_{m}(n;d_0,\omega,\eta,X,Y,t)$ in terms of $Q^{(1)}_{2r+1}(n;d_0,\omega,\eta,X,Y,t)$ and $Q^{(0)}_{2r}(n;d_0d,\omega,\eta,X,Y,t)$.

PROPOSITION 5.2.6. *Let m be an even integer. Let $d_0 \in \mathcal{F}_p$, and $\eta=\pm 1$.*
(1) (1.1) Let $l=0$ or $\nu(d_0)=0$. Then

$$\widetilde{P}^{(1)}_{m-1}(n;d_0,\varepsilon^l,\eta,X,Y,t) = \sum_{r=0}^{(m-2)/2} \frac{Q^{(1)}_{2r+1}(n;d_0,\varepsilon^l,\eta,X,Y,t)}{\phi_{(m-2-2r)/2}(p^{-2})}$$
$$+\sum_{r=0}^{(m-2)/2} \sum_{d\in\mathcal{U}(m-1,m-1-2r,d_0)} \frac{Q^{(0)}_{2r}(n;d_0d,\varepsilon^l,\eta\chi(d),X,Y,t)}{2^{1-\delta_{0,r}}\phi_{(m-2-2r)/2}(p^{-2})}.$$

(1.2) Let $\nu(d_0)\geq 1$. Then

$$Q^{(0)}_{2r}(n;d_0d,\varepsilon,\eta\chi(d),X,Y,t)=0$$

for any d and

$$\widetilde{P}^{(1)}_{m-1}(n;d_0,\varepsilon,\eta,X,Y,t) = \sum_{r=0}^{(m-2)/2} \frac{Q^{(1)}_{2r+1}(n;d_0,\varepsilon,\eta,X,Y,t)}{\phi_{(m-2-2r)/2}(p^{-2})}.$$

(2) (2.1) Let $l=0$ or $\nu(d_0)=0$. Then

$$\widetilde{P}^{(0)}_{m}(n;d_0,\varepsilon^l,\eta,X,Y,t)$$
$$=\sum_{r=0}^{m/2} \sum_{d\in\mathcal{U}(m,m-2r,d_0)} \frac{1+p^{(-m+2r)/2}\chi(d)}{2^{1-\delta_{0,r}+\delta_{0,m}}\phi_{(m-2r)/2}(p^{-2})} Q^{(0)}_{2r}(n;d_0d,\varepsilon^l,\eta\chi(d),X,Y,t)$$
$$+\sum_{r=0}^{(m-2)/2} \frac{1}{\phi_{(m-2r)/2}(p^{-2})} Q^{(1)}_{2r+1}(n;d_0,\varepsilon^l,\eta,X,Y,t).$$

(2.2) Let $\nu(d_0)>0$. Then

$$\widetilde{P}^{(0)}_{m}(n;d_0,\varepsilon,\eta,X,Y,t)=0.$$

Proof. The assertions can be proved in a way similar to Theorem 5.2.5. □

COROLLARY. *Let r be a non-negative integer. Let d_0 be an element of $\mathcal{F}_p$ and $\xi=\pm 1$.*
(1) Let $l=0$ or $\nu(d_0)=0$. Then

$$Q^{(0)}_{2r}(n;d_0,\varepsilon^l,\xi,X,Y,t)$$
$$=\sum_{m=0}^{r} \sum_{d\in\mathcal{U}(2r,2m,d_0)} \frac{(-1)^m(\chi(d)+p^{-m})p^{-m^2}}{2^{1-\delta_{0,r-m}+\delta_{0,r}}\phi_m(p^{-2})} \widetilde{P}^{(0)}_{2r-2m}(n;d_0d,\varepsilon^l,\xi\chi(d),X,Y,t)$$
$$+\sum_{m=0}^{r-1} \frac{(-1)^{m+1}p^{-m-m^2}}{\phi_m(p^{-2})} \widetilde{P}^{(1)}_{2r-2m-1}(n;d_0,\varepsilon^l,\xi,X,Y,t)),$$

and

$$Q^{(1)}_{2r+1}(n;d_0,\varepsilon^l,\xi,X,Y,t) = \sum_{m=0}^{r} \frac{(-1)^m p^{-m-m^2}}{\phi_m(p^{-2})} \widetilde{P}^{(1)}_{2r-2m+1}(n;d_0,\varepsilon^l,\xi,X,Y,t)$$
$$+\sum_{m=0}^{r} \sum_{d\in\mathcal{U}(2r+1,2m+1,d_0)} \frac{(-1)^{m+1}p^{-m-m^2}}{2^{1-\delta_{0,r-m}}\phi_m(p^{-2})} \widetilde{P}^{(0)}_{2r-2m}(n;d_0d,\varepsilon^l,\xi\chi(d),X,Y,t)).$$

(2) *Let $\nu(d_0) > 0$. We have*

$$Q_{2r+1}^{(1)}(n; d_0, \varepsilon, \xi, X, Y, t) = \sum_{m=0}^{r} \frac{(-1)^m p^{m-m^2}}{\phi_m(p^{-2})} \widetilde{P}_{2r+1-2m}^{(1)}(n; d_0, \varepsilon, \xi, X, Y, t),$$

and

$$Q_{2r}^{(0)}(n; d_0, \varepsilon, \xi, X, Y, t) = 0.$$

Proof. We prove the assertion (1) by induction on r. Clearly the assertion holds for $r = 0$. Let $r \geq 1$ and suppose that the assertion holds for any $r' < r$. Fix l and we simply write $Q_{2i-j}^{(j)}(n; d, \varepsilon^l, \xi, X, Y, t)$ and $\widetilde{P}_{2i-j}^{(j)}(n; d, \varepsilon^l, \xi, X, Y, t)$ as $Q_{2i-j}^{(j)}(d; \xi)$ and $\widetilde{P}_{2i-j}^{(j)}(d; \xi)$, respectively. Then by Proposition 5.2.6 and the induction hypothesis we have

$$\begin{aligned}
Q_{2r+1}^{(1)}(d_0; \xi) = \widetilde{P}_{2r+1}^{(1)}(d_0; \xi) - \sum_{i=1}^{r} \frac{1}{\phi_i(p^{-2})} &\left\{ \sum_{j=0}^{r-i} \frac{(-1)^j p^{-j-j^2}}{\phi_j(p^{-2})} \widetilde{P}_{2r-2i-2j+1}^{(1)}(d_0; \xi) \right. \\
&\left. + \sum_{j=0}^{r-i} \sum_{d' \in \mathcal{U}(2r-2i+1, 2j+1, d_0)} \frac{(-1)^{j+1} p^{-j-j^2}}{2^{1-\delta_{0,r-i-j}} \phi_j(p^{-2})} \widetilde{P}_{2r-2i-2j}^{(0)}(d_0 d'; \xi\chi(d')) \right\} \\
- \sum_{i=0}^{r-1} \sum_{d \in \mathcal{U}(2r+1, 2i+1, d_0)} &\frac{1}{2^{1-\delta_{0,r-i}} \phi_i(p^{-2})} \\
\times &\left\{ \sum_{j=0}^{r-i} \sum_{d' \in \mathcal{U}(2r-2i, 2j, d_0 d)} \frac{(-1)^j (\chi(d') + p^{-j}) p^{-j^2}}{2^{1-\delta_{0,r-i-j}+\delta_{0,r-i}} \phi_j(p^{-2})} \widetilde{P}_{2r-2i-2j}^{(0)}(d_0 d d'; \xi\chi(d)\chi(d')) \right. \\
&\left. + \sum_{j=0}^{r-i-1} \frac{(-1)^{j+1} p^{-j-j^2}}{\phi_j(p^{-2})} \widetilde{P}_{2r-2i-2j-1}^{(1)}(d_0 d; \xi\chi(d)) \right\}.
\end{aligned}$$

By Proposition 5.2.1 and Corollary to Theorem 5.2.2 we have

$$\widetilde{P}_{2r-2i-2j-1}^{(1)}(d_0 d; \xi\chi(d)) = \widetilde{P}_{2r-2i-2j-1}^{(1)}(d_0; \xi)$$

for $d \in \mathcal{U}(2r+1, 2i+1, d_0)$ and hence

$$\sum_{d \in \mathcal{U}(2r+1, 2i+1, d_0)} \widetilde{P}_{2r-2i-2j-1}^{(1)}(d_0 d; \xi\chi(d)) = 0.$$

Moreover we have

$$\begin{aligned}
&\sum_{d \in \mathcal{U}(2r+1, 2i+1, d_0)} \sum_{d' \in \mathcal{U}(2r-2i, 2j, d_0 d)} \frac{(-1)^j (\chi(d') + p^{-j}) p^{-j^2}}{2^{1-\delta_{0,r-i}} 2^{1-\delta_{0,r-i-j}+\delta_{0,r-i}}} \widetilde{P}_{2r-2i-2j}^{(0)}(d_0 d d'; \xi\chi(d)\chi(d')) \\
&= \sum_{d'' \in \mathcal{U}(2r, 2i+2j, d_0)} \frac{(-1)^j p^{-j-j^2}}{2^{1-\delta_{0,r-i-j}}} \widetilde{P}_{2r-2i-2j}^{(0)}(d_0 d''; \xi\chi(d'')).
\end{aligned}$$

Hence we have

$$\begin{aligned}
Q_{2r+1}^{(1)}(d_0; \xi) = \widetilde{P}_{2r+1}^{(1)}(d_0; \xi) &+ \sum_{m=1}^{r} \widetilde{P}_{2r-2m+1}^{(1)}(d_0; \xi) A_m \\
&- \sum_{m=1}^{r} \sum_{d \in \mathcal{U}(2r+1, 2m+1, d_0)} \frac{1}{2^{1-\delta_{0,r-m}}} \widetilde{P}_{2r-2m}^{(0)}(d_0 d; \xi\chi(d)) A_m \\
&- \sum_{m=0}^{r} \sum_{d \in \mathcal{U}(2r, 2m, d_0)} \frac{1}{2^{1-\delta_{0,r-m}}} \widetilde{P}_{2r-2m}^{(0)}(d_0 d; \xi\chi(d)) B_m,
\end{aligned}$$

where $A_m = -\sum_{j=0}^{m-1} \dfrac{(-1)^j p^{-j-j^2}}{\phi_{m-j}(p^{-2})\phi_j(p^{-2})}$ and $B_m = -\sum_{j=0}^{m} \dfrac{(-1)^j p^{-j-j^2}}{\phi_{m-j}(p^{-2})\phi_j(p^{-2})}$. We have $A_m = \dfrac{(-1)^m p^{-m-m^2}}{\phi_m(p^{-2})}$ for $m \geq 1$, and $B_m = 1$ or 0 according as $m = 0$ or $m \geq 1$. Thus we get the desired result for $Q^{(1)}_{2r+1}(n; d_0, \varepsilon^l, \xi, X, Y, t)$. We also get the result for $Q^{(0)}_{2r}(n; d_0, \varepsilon^l, \xi, X, Y, t)$, and this completes the induction. Similarly the assertion (2) can be proved.

$\square$

The following lemma follows from [**12**, Lemma 3.4]:

LEMMA 5.2.7. *Let l be a positive integer, and q, U and Q variables. Then*

$$\prod_{i=1}^{l}(1 - U^{-1}Qq^{-i+1})U^l$$
$$= \sum_{m=0}^{l} \frac{\phi_l(q^{-1})}{\phi_{l-m}(q^{-1})\phi_m(q^{-1})} \prod_{i=1}^{l-m}(1 - Qq^{-i+1}) \prod_{i=1}^{m}(1 - Uq^{i-1})(-1)^m q^{(m-m^2)/2}.$$

The following corollary follows directly from the above lemma, and will be used in the proof of Theorem 5.2.8.

COROLLARY. *Let t and Y be variables, and p a prime number.*
(1) *For non-negative integers l and i_0 such that $i_0 + 1 \leq l \leq (n-2)/2$ we have*

$$\sum_{m=0}^{(n-2-2l)/2} (-1)^m p^{m-m^2} \frac{\prod_{i=i_0}^{l+m-1}(1 - p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-2)/2}(1 - p^{-2i-n-1}Y^2t^4)}{\phi_m(p^{-2})\phi_{(n-2-2l)/2-m}(p^{-2})}$$

$$= \frac{\prod_{i=i_0}^{l-1}(1 - p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2)/2}(1 - p^{-2l-n-2i}t^4)(p^{2l-1}Y^2)^{(n-2l-2)/2}}{\phi_{(n-2-2l)/2}(p^{-2})}.$$

(2) *For a non-negative integer $l \leq (n-2)/2$ we have*

$$\sum_{m=0}^{(n-2-2l)/2} (-1)^m p^{m-m^2} \frac{\prod_{i=1}^{l+m}(1 - p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-2)/2}(1 - p^{-2i-n-1}Y^2t^4)}{\phi_m(p^{-2})\phi_{(n-2-2l)/2-m}(p^{-2})}$$

$$= \frac{\prod_{i=1}^{l}(1 - p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2)/2}(1 - p^{-2l-n-2i-2}t^4)(p^{2l+1}Y^2)^{(n-2l-2)/2}}{\phi_{(n-2-2l)/2}(p^{-2})}.$$

(3) *For a non-negative integer $l \leq (n-4)/2$ we have*

$$\sum_{m=0}^{(n-4-2l)/2} (-1)^m p^{m-m^2} \frac{\prod_{i=1}^{l+m}(1 - p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-4)/2}(1 - p^{-2i-n-1}Y^2t^4)}{\phi_m(p^{-2})\phi_{(n-4-2l)/2-m}(p^{-2})}$$

$$= \frac{\prod_{i=1}^{l}(1 - p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-4)/2}(1 - p^{-2l-n-2i-2}t^4)(p^{2l+1}Y^2)^{(n-2l-4)/2}}{\phi_{(n-4-2l)/2}(p^{-2})}.$$

In the equations listed above, we understand that the product $\prod_{i=a}^{b}(*) = 1$ if $a > b$.

THEOREM 5.2.8. *Let the notation be as in Theorem 5.2.5.*
(1) *Suppose that* $\nu(d_0)=0$. *Put* $\xi_0=\chi(d_0)$. *Then*

$$\begin{aligned}
&\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)=(1-p^{-n}t^2)\\
&\times\Biggl\{\sum_{l=0}^{(n-2)/2}\sum_{d\in\mathcal{U}(n-1,n-1-2l,d_0)}\widetilde{P}_{2l}^{(0)}(n;d_0d,\omega,\chi(d),X,Y,t)\prod_{i=1}^{(n-2-2l)/2}(1-p^{-2l-n-2i}t^4)\\
&\quad\times\frac{(p^{2l-1}Y^2)^{(n-2l-2)/2}\prod_{i=0}^{l-1}(1-p^{2i-1}Y^2)p^{l-1/2}\chi(d)Y(1+\chi(d)Yp^{l-1/2})}{2^{1-\delta_{0,l}}(1+\xi_0p^{-1/2}Y)\phi_{(n-2l-2)/2}(p^{-2})}\\
&\quad+\sum_{l=0}^{(n-4)/2}\widetilde{P}_{2l+1}^{(1)}(n;d_0,\omega,1,X,Y,t)\prod_{i=2}^{(n-4-2l)/2}(1-p^{-2l-n-2i}t^4)\\
&\quad\times\frac{(p^{2l-1}Y^2)^{(n-2l-2)/2}\prod_{i=1}^{l}(1-p^{2i-1}Y^2)(1-\xi_0p^{-1/2}Y)(1+p^{-2l-2}t^2)}{\phi_{(n-2l-2)/2}(p^{-2})}\Biggr\}.
\end{aligned}$$

(2) *Suppose that* $\nu(d_0)>0$ *and* $\omega=\iota$. *Put* $\xi_0=\chi(d_0)$. *Then*

$$\begin{aligned}
&\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)=(1-p^{-n}t^2)\\
&\times\Biggl\{\sum_{l=1}^{(n-2)/2}\sum_{d\in\mathcal{U}(n-1,n-1-2l,d_0)}\widetilde{P}_{2l}^{(0)}(n;d_0d,\omega,\chi(d),X,Y,t)\prod_{i=1}^{(n-2-2l)/2}(1-p^{-2l-n-2i}t^4)\\
&\quad\times\frac{(p^{2l-1}Y^2)^{(n-2l-2)/2}\prod_{i=1}^{l-1}(1-p^{2i-1}Y^2)p^{l-1/2}\chi(d)Y(1+\chi(d)Yp^{l-1/2})}{2\phi_{(n-2l-2)/2}(p^{-2})}\\
&\quad+\sum_{l=0}^{(n-4)/2}\widetilde{P}_{2l+1}^{(1)}(n;d_0,\omega,1,X,Y,t)\prod_{i=2}^{(n-4-2l)/2}(1-p^{-2l-n-2i}t^4)\\
&\quad\times\frac{(p^{2l-1}Y^2)^{(n-2l-2)/2}\prod_{i=1}^{l}(1-p^{2i-1}Y^2)(1+p^{-2l-2}t^2)}{\phi_{(n-2l-2)/2}(p^{-2})}\Biggr\}.
\end{aligned}$$

(3) *Suppose that* $\nu(d_0)>0$ *and* $\omega=\varepsilon$. *Then*

$$\begin{aligned}
&\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)=\sum_{l=0}^{(n-2)/2}\widetilde{P}_{2l+1}^{(1)}(n;d_0,\omega,1,X,Y,t)\\
&\times\frac{(p^{2l+1}Y^2)^{(n-2l-2)/2}\prod_{i=1}^{l}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2-2l)/2}(1-p^{-2l-n-2i-2}t^4)}{\phi_{(n-2-2l)/2}(p^{-2})}.
\end{aligned}$$

Proof. Suppose that $\nu(d_0)=0$ or $\omega=\iota$. Then by (1) of Theorem 5.2.5 and (1) of Corollary to Proposition 5.2.6, we have

$$\begin{aligned}
&\widetilde{R}_{n-1}(d_0,\omega;X,Y,t)\\
&=\sum_{r=0}^{(n-2)/2}\frac{\prod_{i=1}^{r}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2r-2)/2}(1-p^{-2i-n-1}Y^2t^4)}{2^{1-\delta_{0,r}}\phi_{(n-2r-2)/2}(p^{-2})}\sum_{d_1\in\mathcal{U}(n-1,n-2r-1,d_0)}D_{2r}(d_0,d_1,Y,t)\\
&\quad\times\Bigg\{\sum_{m=0}^{r}\sum_{d_2\in\mathcal{U}(2r,2m,d_0d_1)}\frac{(-1)^m(\chi(d_2)+p^{-m})p^{-m^2}}{2^{1-\delta_{0,r-m}+\delta_{0,r}}\phi_m(p^{-2})}\widetilde{P}^{(0)}_{2r-2m}(n;d_0d_1d_2,\omega,\chi(d_1)\chi(d_2),X,Y;t)\\
&\qquad+\sum_{m=0}^{r-1}\frac{(-1)^{m+1}p^{-m-m^2}}{\phi_m(p^{-2})}\widetilde{P}^{(1)}_{2r-2m-1}(n;d_0d_1,\omega,\chi(d_1),X,Y;t)\Bigg\}\\
&\quad+\sum_{r=0}^{(n-2)/2}\frac{\prod_{i=1}^{r}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2r-2)/2}(1-p^{-2i-n-1}Y^2t^4)}{\phi_{(n-2r-2)/2}(p^{-2})}\\
&\quad\times(1-\xi_0p^{-1/2}Y)\Bigg\{\sum_{m=0}^{r}\frac{(-1)^m p^{-m}p^{-m^2}}{\phi_m(p^{-2})}\widetilde{P}^{(1)}_{2r+1-2m}(n;d_0,\omega,1,X,Y,t)\\
&\qquad+\sum_{m=0}^{r}\sum_{d_2\in\mathcal{U}(2r+1,2m+1,d_0)}\frac{(-1)^{m+1}p^{-m-m^2}}{2^{1-\delta_{0,r-m}}\phi_m(p^{-2})}\widetilde{P}^{(0)}_{2r-2m}(n;d_0d_2,\omega,\chi(d_2),X,Y,t)\Bigg\}.
\end{aligned}$$

By Proposition 5.2.1 and Corollary to Theorem 5.2.2, for any $d_1\in\mathcal{U}$ we have

$$\widetilde{P}^{(1)}_{2r+1-2m}(n;d_0d_1,\omega,\chi(d_1),X,Y,t)=\widetilde{P}^{(1)}_{2r+1-2m}(n;d_0,\omega,1,X,Y,t).$$

Moreover, if $r>m\geq 0$, then $\mathcal{U}(n-1,n-2r-1,d_0)=\mathcal{U}(2r+1,2m+1,d_0d_1)=\mathcal{U}$. Hence

$$\begin{aligned}
\text{(A)}\quad&\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)\\
&=\sum_{m=0}^{(n-2)/2}S(n;m,d_0,Y)\frac{\prod_{i=1}^{m}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2-2m)/2}(1-p^{-2i-n-1}Y^2t^4)p^{-m^2}(-1)^m}{\phi_m(p^{-2})\phi_{(n-2)/2-m}(p^{-2})}\\
&\quad+\sum_{l=1}^{(n-2)/2}\sum_{d\in\mathcal{U}}\widetilde{P}^{(0)}_{2l}(n;d_0d,\omega,\chi(d),X,Y,t)\\
&\quad\times\sum_{m=0}^{(n-2-2l)/2}\Bigg\{\frac{1}{2}\sum_{d_1\in\mathcal{U}(2l+2m,2m,d_0)}D_{2l+2m}(d_0,d_1,Y,t)\\
&\qquad\qquad\times(\chi(d_1)\chi(d)+p^{-m})(-1)^m p^{-m^2}-(1-\xi_0p^{-1/2}Y)(-1)^m p^{-m-m^2}\Bigg\}\\
&\qquad\times\frac{\prod_{i=1}^{l+m}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-2)/2}(1-p^{-2i-n-1}Y^2t^4)}{2\phi_m(p^{-2})\phi_{(n-2-2l)/2-m}(p^{-2})}\\
&\quad+\sum_{l=0}^{(n-2)/2}\widetilde{P}^{(1)}_{2l+1}(n;d_0,\omega,1,X,Y,t)\\
&\quad\times\Bigg\{\sum_{m=0}^{(n-2-2l)/2}((1-\xi_0p^{-1/2}Y)(-1)^m p^{-m-m^2})
\end{aligned}$$

$$\times \frac{\prod_{i=1}^{l+m}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-2)/2}(1-p^{-2i-n-1}Y^2t^4)}{\phi_m(p^{-2})\phi_{(n-2-2l)/2-m}(p^{-2})}$$
$$-\sum_{m=0}^{(n-4-2l)/2}\frac{1}{2}\sum_{d\in\mathcal{U}}D_{2l+2m+2}(d_0,d,Y,t)(-1)^m p^{-m-m^2}$$
$$\left.\times \frac{\prod_{i=1}^{l+m}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-4)/2}(1-p^{-2i-n-1}Y^2t^4)}{\phi_m(p^{-2})\phi_{(n-4-2l)/2-m}(p^{-2})}\right\},$$

where

$$S(n;m,d_0,Y)$$
$$=\sum_{d_1\in\mathcal{U}(n-1,n-2m-1,d_0)}\frac{(\chi(d_0d_1)+p^{-m})D_{2m}(d_0,d_1,Y,t)}{2}-(1-\xi_0p^{-1/2}Y)p^{-m} \text{ or } 0$$

according as $\nu(d_0)=0$ or not. We have

$$\begin{aligned}&(1-\xi_0p^{-1/2}Y)p^{-m}(1-p^{-2n+2l+2m+1}Y^2t^4)\\&\quad-\frac{1}{2}\sum_{d_1\in\mathcal{U}}D_{2l+2m+2}(d_0,d_1,Y,t)p^{-m}(1-p^{-2n+2m+2l+2})(1-p^{2l+2m+1}Y^2)\\&=(1-p^{-1/2}\xi_0Y)p^{-n+m+2l+2}(1-p^{-2n+2m+2l+1}Y^2t^4)\\&\quad+(1-p^{-1/2}\xi_0Y)(1-p^{-n+2m+2l+2})p^{2l+m-n+1}Y^2t^2(1-p^{-n}t^2)\end{aligned}$$

for any $0\le l\le (n-2)/2$ and $0\le m\le (n-2l-2)/2$. Furthermore we have

$$\begin{aligned}&\frac{1}{2}\sum_{d_1\in\mathcal{U}}D_{2l+2m}(d_0,d_1,Y,t)(\chi(d_1)\chi(d)+p^{-m})-(1-\xi_0p^{-1/2}Y)p^{-m}\\&=\frac{\chi(d)(1-p^{-1/2}\xi_0Y)(1-p^{-n}t^2)p^{l+m-1/2}Y(1+\chi(d)Yp^{l-1/2})}{1-p^{2l+2m-1}Y^2}\end{aligned}$$

for any $1\le l\le (n-2)/2, 0\le m\le (n-2l-2)/2$ and $d\in\mathcal{U}$. Suppose that $\nu(d_0)=0$. Then for any m we have

$$\begin{aligned}&\frac{1}{2}\sum_{d_1\in\mathcal{U}(n-1,n-2m-1,d_0)}D_{2m}(d_0,d_1,Y,t)(\chi(d_1)\chi(d_0)+p^{-m})\\&\qquad-(1-\xi_0p^{-1/2}Y)p^{-m}=\frac{\xi_0p^{m-1/2}Y(1-p^{-1}Y^2)(1-p^{-n}t^2)}{1-p^{2m-1}Y^2}.\end{aligned}$$

We remark that $\mathcal{U}(n-1, n-2l-1, d_0) = \mathcal{U}$ for $l > 0$ and $\mathcal{U}(n-1, n-1, d_0) = \{d_0\}$. Hence

$$\widetilde{R}_{n-1}(d_0, \omega, X, Y, t) = \xi_0 p^{-1/2} Y (1 - p^{-n}t^2)$$
$$\times \sum_{l=0}^{(n-2)/2} \sum_{d \in \mathcal{U}(n-1,n-2l-1,d_0)} \widetilde{P}_{2l}^{(0)}(n; d_0 d, \omega, \chi(d), X, Y, t)(1 - p^{-n}t^2) p^{l-1/2} \chi(d) Y \frac{1 + \chi(d) Y p^{l-1/2}}{1 + \xi_0 p^{-1/2} Y}$$
$$\times \sum_{m=0}^{(n-2-2l)/2} (-1)^m p^{m-m^2} \frac{\prod_{i=0}^{l+m-1}(1 - p^{2i-1}Y^2) \prod_{i=1}^{(n-2l-2m-2)/2}(1 - p^{-2i-n-1}Y^2 t^4)}{2^{1-\delta_{0,l}} \phi_m(p^{-2}) \phi_{(n-2-2l)/2-m}(p^{-2})}$$
$$+ \sum_{l=0}^{(n-2)/2} \widetilde{P}_{2l+1}^{(1)}(n; d_0, \omega, 1, X, Y, t) \Bigg\{ (1 - \xi_0 p^{-1/2} Y) p^{-n+2+2l}$$
$$\times \sum_{m=0}^{(n-2-2l)/2} (-1)^m p^{m-m^2} \frac{\prod_{i=1}^{l+m}(1 - p^{2i-1}Y^2) \prod_{i=1}^{(n-2l-2m-2)/2}(1 - p^{-2i-n-1}Y^2 t^4)}{\phi_m(p^{-2}) \phi_{(n-2-2l)/2-m}(p^{-2})}$$
$$+ Y^2 t^2 p^{2l-n+1}(1 - \xi_0 p^{-1/2} Y)(1 - p^{-n}t^2)$$
$$\times \sum_{m=0}^{(n-4-2l)/2} (-1)^m p^{m-m^2} \frac{\prod_{i=1}^{l+m}(1 - p^{2i-1}Y^2) \prod_{i=1}^{(n-2l-2m-4)/2}(1 - p^{-2i-n-1}Y^2 t^4)}{\phi_m(p^{-2}) \phi_{(n-4-2l)/2-m}(p^{-2})} \Bigg\}.$$

Thus the assertion (1) follows from Corollary to Lemma 5.2.7. Similarly the assertion (2) can be proved.

Suppose that $\nu(d_0) > 0$ and $\omega = \varepsilon$. Then by (2) of Theorem 5.2.5 and (2) of Corollary to Proposition 5.2.6, we have

$$\widetilde{R}_{n-1}(d_0, \omega, X, Y, t) = \sum_{l=0}^{(n-2)/2} \widetilde{P}_{2l+1}^{(1)}(n; d_0, \omega, 1, X, Y, t)$$
$$\times \sum_{m=0}^{(n-2-2l)/2} ((-1)^m p^{m-m^2}) \frac{\prod_{i=1}^{l+m}(1 - p^{2i-1}Y^2) \prod_{i=1}^{(n-2l-2m-2)/2}(1 - p^{-2i-n-1}Y^2 t^4)}{\phi_m(p^{-2}) \phi_{(n-2-2l)/2-m}(p^{-2})}.$$

Thus the assertion (3) follows from Corollary to Lemma 5.2.7. □

By Proposition 5.2.1 we immediately obtain:

Corollary. *Let the notation be as in Theorem 5.2.8.*
(1) *Suppose that* $\nu(d_0) = 0$. *Then*

$$\widetilde{R}_{n-1}(d_0, \omega, X, Y, t) = (1 - p^{-n}t^2) \prod_{i=1}^{n/2-1} (1 - p^{-2n+2i}t^4)$$
$$\times \Bigg\{ \sum_{l=0}^{(n-2)/2} \prod_{i=1}^{l} (1 - p^{-n-2l-3+2i}t^4) \sum_{d \in \mathcal{U}(n-1,n-1-2l,d_0)} P_{2l}^{(0)}(d_0 d, \omega, \chi(d), X, tY^{-1/2})$$
$$\times \frac{(p^{2l-1}Y^2)^{(n-2l-2)/2} \prod_{i=0}^{l-1}(1 - p^{2i-1}Y^2) p^{l-1/2} \chi(d) Y (1 + \chi(d) Y p^{l-1/2})}{2^{1-\delta_{0,l}}(1 + \xi_0 p^{-1/2} Y) \phi_{(n-2l-2)/2}(p^{-2})}$$
$$+ \sum_{l=0}^{(n-2)/2} \prod_{i=1}^{l} (1 - p^{-n-2l-3+2i}t^4) P_{2l+1}^{(1)}(d_0, \omega, 1, X, tY^{-1/2})$$
$$\times \frac{(p^{2l-1}Y^2)^{(n-2l-2)/2} \prod_{i=1}^{l}(1 - p^{2i-1}Y^2)(1 - \xi_0 p^{-1/2} Y)(1 + p^{-2l-2}t^2)}{\phi_{(n-2l-2)/2}(p^{-2})} \Bigg\}.$$

(2) *Suppose that* $\nu(d_0) > 0$ *and* $\omega = \iota$. *Put* $\xi_0 = \chi(d_0)$. *Then*

$$\widetilde{R}_{n-1}(d_0,\omega,X,Y,t) = (1-p^{-n}t^2)\prod_{i=1}^{n/2-1}(1-p^{-2n+2i}t^4)$$
$$\times\Bigg\{\sum_{l=1}^{(n-2)/2}\prod_{i=1}^{l}(1-p^{-n-2l-3+2i}t^4)\sum_{d\in\mathcal{U}(n-1,n-1-2l,d_0)}P_{2l}^{(0)}(d_0d,\omega,\chi(d),X,tY^{-1/2})$$
$$\times\frac{(p^{2l-1}Y^2)^{(n-2l-2)/2}\prod_{i=1}^{l-1}(1-p^{2i-1}Y^2)p^{l-1/2}\chi(d)Y(1+\chi(d)Yp^{l-1/2})}{2\phi_{(n-2l-2)/2}(p^{-2})}$$
$$+\sum_{l=0}^{(n-2)/2}\prod_{i=1}^{l}(1-p^{-n-2l-3+2i}t^4)P_{2l+1}^{(1)}(d_0,\omega,1,X,tY^{-1/2})$$
$$\times\frac{(p^{2l-1}Y^2)^{(n-2l-2)/2}\prod_{i=1}^{l}(1-p^{2i-1}Y^2)(1+p^{-2l-2}t^2)}{\phi_{(n-2l-2)/2}(p^{-2})}\Bigg\}.$$

(3) *Suppose that* $\nu(d_0) > 0$ *and* $\omega = \varepsilon$. *Then*

$$\widetilde{R}_{n-1}(d_0,\omega,X,Y,t) = (1-\xi_0p^{-1/2}Y)\prod_{i=1}^{n/2}(1-p^{-2n+2i-2}t^4)\sum_{l=0}^{(n-2)/2}P_{2l+1}^{(1)}(d_0,\omega,1,X,tY^{-1/2})$$
$$\times\frac{(p^{2l+1}Y^2)^{(n-2l-2)/2}\prod_{i=1}^{l}(1-p^{2i-1}Y^2)\prod_{i=1}^{l}(1-p^{-2l-n+2i-3}t^4)}{\phi_{(n-2-2l)/2}(p^{-2})}.$$

5.3. *Explicit formulas for formal power series of Rankin-Selberg type*

We prove our main result in this section.

THEOREM 5.3.1. *Let* $d_0 \in \mathcal{F}_p$ *and put* $\xi_0 = \chi(d_0)$.
(1) *We have*

$$H_{n-1}(d_0,\iota,X,Y,t)$$
$$= \phi_{(n-2)/2}(p^{-2})^{-1}(p^{-1}t)^{\nu(d_0)}(1-p^{-n}t^2)\prod_{i=1}^{n/2-1}(1-p^{-2n+2i}t^4)$$
$$\times\frac{(1+p^{-2}t^2)(1+p^{-3}\xi_0^2t^2)-p^{-5/2}t^2\xi_0(X+X^{-1}+Y+Y^{-1})}{(1-p^{-2}XYt^2)(1-p^{-2}XY^{-1}t^2)(1-p^{-2}X^{-1}Yt^2)(1-p^{-2}X^{-1}Y^{-1}t^2)}$$
$$\times\frac{1}{\prod_{i=1}^{n/2-1}(1-p^{-2i-1}XYt^2)(1-p^{-2i-1}XY^{-1}t^2)(1-p^{-2i-1}X^{-1}Yt^2)(1-p^{-2i-1}X^{-1}Y^{-1}t^2)}.$$

(2) *We have*

$$H_{n-1}(d_0,\varepsilon,X,Y,t)$$
$$= \phi_{(n-2)/2}(p^{-2})^{-1}(1-p^{-n}t^2)\prod_{i=1}^{n/2-1}(1-p^{-2n+2i}t^4)(tp^{-1})^{\nu(d_0)}$$
$$\times\frac{(1+p^{-n}t^2)(1+p^{-n-1}\xi_0^2t^2)-p^{-1/2-n}t^2\xi_0(X+X^{-1}+Y+Y^{-1})}{(1-p^{-n}XYt^2)(1-p^{-n}XY^{-1}t^2)(1-p^{-n}X^{-1}Yt^2)(1-p^{-n}X^{-1}Y^{-1}t^2)}$$
$$\times\frac{1}{\prod_{i=1}^{n/2-1}(1-p^{-2i}XYt^2)(1-p^{-2i}XY^{-1}t^2)(1-p^{-2i}X^{-1}Yt^2)(1-p^{-2i}X^{-1}Y^{-1}t^2)}.$$

Proof. First suppose that $\omega = \iota$. For an integer l put

$$V(l,X,Y,t) = (1-t^2p^{-2}XY^{-1})(1-t^2p^{-2}X^{-1}Y^{-1})\prod_{i=1}^{l}(1-t^2p^{-2i-1}XY^{-1})(1-t^2p^{-2i-1}X^{-1}Y^{-1}).$$

For $d \in \mathcal{U}$, put $\eta_d = \chi(d)$. Then by Theorem 5.2.2, and (1) of Corollary to Theorem 5.2.8, we have

$$\widetilde{R}_{n-1}(d_0,\omega,X,Y,t) = (1-p^{-n}t^2)\prod_{i=1}^{(n-2)/2}(1-p^{-2n+2i}t^4)\Bigg\{\frac{(p^{-1}Y^2)^{(n-2)/2}\xi_0}{\phi_{(n-2)/2}(p^{-2})}$$
$$+\frac{(p^{-1}Y^2)^{(n-2)/2}(1-p^{-1/2}\xi_0Y)(1+p^{-2}t^2)(p^{-1}tY^{-1/2})^{\nu(d_0)}(1-p^{-5/2}\xi_0t^2Y^{-1})}{(1-t^2p^{-2}XY^{-1})(1-t^2p^{-2}X^{-1}Y^{-1})}$$
$$+\sum_{l=1}^{(n-2)/2}\sum_{d\in\mathcal{U}(n-1,n-1-2l,d_0)}\frac{\prod_{i=1}^{l}(1-p^{-n-2l-3+2i}t^4)(p^{-1}tY^{-1/2})^{\nu(d_0)}S_{2l}^{(0)}(d_0d,\iota,\eta_d,X,Y,t^2)}{V(l,X,Y,t)}$$
$$+\sum_{l=1}^{(n-2)/2}\frac{\prod_{i=2}^{l}(1-p^{-n-2l-3+2i}t^4)(p^{-1}tY^{-1/2})^{\nu(d_0)}S_{2l+1}^{(1)}(d_0,\iota,X,Y,t^2)}{V(l-1,X,Y,t)}\Bigg\},$$

where $S_{2r}^{(0)}(d_0d,\iota,\eta_d,X,Y,t)$ and $S_{2r+1}^{(1)}(d_0,\iota,X,Y,t)$ are polynomials in t of degree at most 2. We note that $\xi_0 = 0$ if $\nu(d_0) > 0$. Hence $\widetilde{R}_{n-1}(d_0,\iota,X,Y,t)$ can be expressed as

$$\widetilde{R}_{n-1}(d_0,\iota,X,Y,t) = \frac{(1-p^{-n}t^2)\prod_{i=1}^{(n-2)/2}(1-p^{-n-2i}t^4)(p^{-1}tY^{-1/2})^{\nu(d_0)}S(d_0,\iota,X,Y,t^2)}{\phi_{(n-2)/2}(p^{-2})V((n-2)/2,X,Y,t)},$$

where $S(d_0,\iota,X,Y,t)$ is a polynomial in t of degree at most n. Moreover it can be expressed as

(D) $$S(d_0,\iota,X,Y,t^2) = \frac{(p^{-1}Y^2)^{(n-2)/2}\prod_{i=1}^{(n-2)/2}(1-t^2p^{-2i-1}XY^{-1})(1-t^2p^{-2i-1}X^{-1}Y^{-1})}{\phi_{(n-2)/2}(p^{-2})}$$
$$\times\xi_0p^{-1/2}Y(1-t^2p^{-2}XY^{-1})(1-t^2p^{-2}X^{-1}Y^{-1})$$
$$+\frac{(p^{-1}Y^2)^{(n-2)/2}\prod_{i=1}^{(n-2)/2}(1-t^2p^{-2i-1}XY^{-1})(1-t^2p^{-2i-1}X^{-1}Y^{-1})}{\phi_{(n-2)/2}(p^{-2})}$$
$$\times(1-p^{-1/2}\xi_0Y)(1+p^{-2}t^2)(1-p^{-5/2}\xi_0t^2Y^{-1})$$
$$+(1-p^{-n-3}t^4)U(d_0,X,Y,\iota,t^2)$$

with $U(d_0,\iota,X,Y,t)$ a polynomial in t. Hence by Theorem 5.1.4 we have

$$H_{n-1}(d_0,\iota,X,Y,t) = (p^{-1}t)^{\nu(d_0)}(1-p^{-n}t^2)\prod_{i=1}^{n/2-1}(1-p^{-2n+2i}t^4)$$
$$\times\frac{S(d_0,\iota,X,Y,t^2)}{(1-p^{-2}XYt^2)(1-p^{-2}XY^{-1}t^2)(1-p^{-2}X^{-1}Yt^2)(1-p^{-2}X^{-1}Y^{-1}t^2)}$$
$$\times\frac{1}{\prod_{i=1}^{n/2-1}(1-p^{-2i-1}XYt^2)(1-p^{-2i-1}XY^{-1}t^2)(1-p^{-2i-1}X^{-1}Yt^2)(1-p^{-2i-1}X^{-1}Y^{-1}t^2)}$$

$$\times \frac{1}{\prod_{i=1}^{(n-2)/2}(1-p^{-2i}XYt^2)(1-p^{-2i}X^{-1}Yt^2)}.$$

Hence the power series $H_{n-1}(d_0, \iota, X, Y, t)$ is a rational function in X, Y and t, and is invariant under the transformation $Y \mapsto Y^{-1}$. This implies that the reduced denominator of the rational function $H_{n-1}(d_0, \iota, X, Y, t)$ in t is at most

$$(1-p^{-2}XYt^2)(1-p^{-2}XY^{-1}t^2)(1-p^{-2}X^{-1}Yt^2)(1-p^{-2}X^{-1}Y^{-1}t^2)$$
$$\times \prod_{i=1}^{n/2-1}(1-p^{-2i-1}XYt^2)(1-p^{-2i-1}XY^{-1}t^2)(1-p^{-2i-1}X^{-1}Yt^2)(1-p^{-2i-1}X^{-1}Y^{-1}t^2)$$

and therefore we have

$$\text{(E)} \qquad S(d_0, \iota, X, Y, t^2) = \sum_{i=0}^{2} a_i(d_0, X, Y)t^{2i} \frac{\prod_{i=1}^{(n-2)/2}(1-p^{-2i}XYt^2)(1-p^{-2i}X^{-1}Yt^2)}{\phi_{(n-2)/2}(p^{-2})},$$

where $a_i(d_0, X, Y)$ $(i = 0, 1, 2)$ is a polynomial in $X + X^{-1}$ and $Y + Y^{-1}$. First assume $\nu(d_0) = 0$. Then we can easily see $a_0(d_0, X, Y) = 1$. Then by substituting $\pm p^{(n+3)/2}$ for t^2 in (D) and (E), and comparing them, we obtain

$$\begin{aligned} &1 \pm a_1(d_0, X, Y)p^{(n+3)/2} + a_2(d_0, X, Y)p^{n+3} \\ &\quad = 1 \pm (p^{(n-3)/2} + p^{(n-1)/2} - \xi_0 p^{n/2-1}(X + X^{-1} + Y + Y^{-1})) + p^{n-2}. \end{aligned}$$

Hence $a_1(d_0, X, Y) = p^{-2} + p^{-3} - p^{-5/2}(X + X^{-1} + Y + Y^{-1})\xi_0$ and $a_2(d_0, X, Y) = p^{-5}$. This proves the assertion in case $\nu(d_0) = 0$. Next assume $\nu(d_0) > 0$. Then in the same manner as above we have $a_0(d_0, X, Y) = 1$, and

$$1 \pm a_1(d_0, X, Y)p^{(n+3)/2} + a_2(d_0, X, Y)p^{n+3} = 1 \pm p^{(n-1)/2}$$

Hence $a_2(d_0, X, Y) = 0$ and $a_1(d_0, X, Y) = p^{-2}$. This proves the assertion in case $\nu(d_0) > 0$.

Similarly the assertion for $\nu(d_0) = 0$ and $\omega = \varepsilon$ can be proved. Next suppose that $\nu(d_0) > 0$ and $\omega = \varepsilon$. Then the assertion can be proved similarly by using Theorems 5.1.4 and 5.2.2, and (2) of Corollary to Theorem 5.2.8. □

6. *Proof of Theorem 3.2*

Now we give an explicit form of $R(s, \sigma_{n-1}(\phi_{I_n(h),1}))$ for the first Fourier-Jacobi coefficient $\phi_{I_n(h),1}$ of the Duke-Imamoḡlu-Ikeda lift.

PROPOSITION 6.1. *Let k and n be positive even integers. Given a Hecke eigenform $h \in \mathfrak{S}^+_{k-n/2+1/2}(\Gamma_0(4))$, let $f \in \mathfrak{S}_{2k-n}(\Gamma^{(1)})$ be the primitive form as in Section 2. Then*

$$\begin{aligned} R(s, h) = L(2s - 2k + n + 1, f, \mathrm{Ad}) &\sum_{d_0 \in \mathcal{F}^{(-1)^{n/2}}} |c_h(|d_0|)|^2 |d_0|^{-s} \\ \times \prod_p \{(1 + p^{-2s+2k-n-1})&(1 + p^{-2s+2k-n-2}\chi_p(d_0)^2) - 2p^{-2s+k-n/2-1}\chi_p(d_0)c_f(p)\}. \end{aligned}$$

Proof. The assertion can be proved by Theorems 4.2 and 5.3.1. □

THEOREM 6.2. *Let k and n be positive even integers. Given a Hecke eigenform $h \in \mathfrak{S}^+_{k-n/2+1/2}(\Gamma_0(4))$, let $f \in \mathfrak{S}_{2k-n}(\Gamma^{(1)})$ and $\phi_{I_n(h),1} \in J^{\mathrm{cusp}}_{k,1}(\Gamma^{(n-1),J})$ be as in Section 2 and*

Section 3, respectively. Put $\lambda_n = \frac{e_{n-1}}{2}\prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)$*. Then, we have*

$$R(s,\sigma_{n-1}(\phi_{I_n(h),1})) = \lambda_n 2^{(-s-1/2)(n-2)}\zeta(2s+n-2k+1)^{-1}\prod_{i=1}^{\frac{n-2}{2}}\zeta(4s+2n-4k+2-2i)^{-1}$$

$$\times\{R(s-n/2+1,h)\zeta(2s-2k+3)\prod_{i=1}^{\frac{n-2}{2}}L(2s-2k+2i+2,f,\mathrm{Ad})\zeta(2s-2k+2i+2)$$

$$+(-1)^{n(n-2)/8}R(s,h)\zeta(2s-2k+n+1)\prod_{i=1}^{\frac{n-2}{2}}L(2s-2k+2i+1,f,\mathrm{Ad})\zeta(2s-2k+2i+1)\}.$$

Proof. The assertion follows directly from Theorems 4.2 and 5.3.1, and Proposition 6.1. □

Proof of Theorem 3.2. The assertion trivially holds if $n=2$. Suppose that $n \geq 4$. By Theorem 6.2 we have

$$\text{(F)}\qquad \mathcal{R}(s,\sigma_{n-1}(\phi_{I_n(h),1})) = \prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)2^{(-s-1/2)(n-2)}\mathcal{T}(s)$$

$$\times\left\{\mathcal{U}(s)^{-1}\mathcal{R}(s-n/2+1,h)\prod_{i=1}^{\frac{n-2}{2}}\widetilde{\Lambda}(2s-2k+2i+2,f,\mathrm{Ad})\xi(2s-2k+2i+2)\right.$$

$$\left.+(-1)^{n(n-2)/8}R(s,h)\prod_{i=1}^{\frac{n-2}{2}}L(2s-2k+2i+1,f,\mathrm{Ad})\zeta(2s-2k+2i+1)\right\},$$

where

$$\mathcal{T}(s) = \Gamma_{\mathbf{R}}(2s+n-2k+1)\prod_{i=1}^{(n-2)/2}\Gamma_{\mathbf{R}}(4s+2n-4k+2-2i)\prod_{i=1}^{n-1}\Gamma_{\mathbf{R}}(2s-i+1),$$

and

$$\mathcal{U}(s) = \Gamma_{\mathbf{R}}(2s-2k+3)\Gamma_{\mathbf{R}}(2s-n+2)$$
$$\times\prod_{i=1}^{(n-2)/2}(\Gamma_{\mathbf{C}}(2s-2k+2i+2)\Gamma_{\mathbf{C}}(2s-n+2i+1)\Gamma_{\mathbf{R}}(2s-2k+2i+2)).$$

We note that $\mathcal{R}(s,h)$ is holomorphic at $s=k-1/2$. Thus by taking the residue of the both-sides of (F) at $s=k-1/2$, we get

$$\mathrm{Res}_{s=k-1/2}\mathcal{R}(s,\sigma_{n-1}(\phi_{I_n(h),1})) = 2^{-k(n-2)}\prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)\frac{\mathcal{T}(k-1/2)}{\mathcal{U}(k-1/2)}$$

$$\times\mathrm{Res}_{s=k-n/2+1/2}\mathcal{R}(s,h)\prod_{i=1}^{\frac{n-2}{2}}\widetilde{\Lambda}(2i+1,f,\mathrm{Ad})\xi(2i+1).$$

We easily see that

$$\frac{\mathcal{T}(k-1/2)}{\mathcal{U}(k-1/2)} = 2^{(n-1)(n-2)/2}.$$

By [**31**, Theorem 1], we have

$$\mathrm{Res}_{s=k-n/2+1/2}\mathcal{R}(s,h) = 2^{2k-n}\langle h, h\rangle.$$

Thus we complete the proof.

References

1. A. N. Andrianov, *Quadratic forms and Hecke operators*, Grundl. Math. Wiss., 286, Springer-Verlag, Berlin, 1987.

2. S. Böcherer, *Eine Rationalitätsatz für formale Heckereihen zur Siegelschen Modulgruppe*, Abh. Math. Sem. Univ. Hamburg **56** (1986), 35–47.

3. S. Böcherer, N. Dummigan, and R. Schulze-Pillot, *Yoshida lifts and Selmer groups*, J. Math. Soc. Japan **64**(2012), 1353-1405.

4. S. Böcherer and F. Sato, *Rationality of certain formal power series related to local densities*, Comment. Math. Univ. St.Paul. **36** (1987), 53–86.

5. J. Brown, *Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture*, Compos. Math. **143** (2007), no. 2, 290–322.

6. Y. Choie and W. Kohnen, *On the Petersson norm of certain Siegel modular forms*, Ramanujan J. **7** (2003), 45–48.

7. K. Doi, H. Hida and H. Ishii, *Discriminant of Hecke fields and twisted adjoint L-values for $GL(2)$*, Invent. Math. **134** (1998), 547–577.

8. M. Furusawa, *On Petersson norms for some liftings*, Math. Ann. **248** (1984), 543–548.

9. T. Ibukiyama, *On Jacobi forms and Siegel modular forms of half integral weights*, Comment. Math. Univ. St.Paul. **41** (1992), no. 2, 109–124.

10. T. Ibukiyama and H. Katsurada, *An explicit formula for Koecher-Maaß Dirichlet series for Eisenstein series of Klingen type*, J. Number Theory, **102** (2003), 223–256.

11. T. Ibukiyama and H. Katsurada, *An explicit formula for Koecher-Maaß Dirichlet series for the Ikeda lifting*, Abh. Math. Sem. Hamburg **74** (2004), 101–121.

12. T. Ibukiyama and H. Katsurada, *Koecher-Maaß series for real analytic Siegel Eisenstein series*, Automorphic forms and zeta functions, 170–197, World Sci. Publ., Hackensack, NJ, 2006.

13. T. Ibukiyama and H. Saito, *On zeta functions associated to symmetric matrices. I. An explicit form of zeta functions*, Amer. J. Math. **117** (1995), 1097–1155.

14. T. Ikeda, *On the lifting of elliptic modular forms to Siegel cusp forms of degree $2n$*, Ann. of Math. **154** (2001), no. 3, 641–681.

15. T. Ikeda, *Pullback of the lifting of elliptic cusp forms and Miyawaki's conjecture*, Duke Math. J. **131** (2006), no. 3, 469–497.

16. T. Ikeda, *On the lifting of hermitian modular forms*, Compositio Math. **144** (2008) 1107-1154.

17. V. L. Kalinin, *Analytic properties of the convolution products of genus g*, Math. USSR Sbornik **48** (1984), 193–200.

18. H. Katsurada, *An explicit formula for Siegel series*, Amer. J. Math. **121** (1999), 415–452.

19. H. Katsurada, *Congruence of Siegel modular forms and special values of their standard zeta functions*, Math. Z. **259** (2008), 97–111.

20. H. Katsurada, *Exact standard zeta values of Siegel modular forms* Experiment. Math. **19**(2010),65-76.

21. H. Katsurada, *Congruence between Duke-Imamoḡlu-Ikeda lifts and non-Duke-Imamoḡlu-Ikeda lifts*, Preprint, 2013, arXiv:1101.3377v1[math. NT].

22. H. Katsurada, *Explicit formulas for the twisted Koecher-Maass series of the Duke-Imamoglu-Ikeda lift and their applications*, in Math. Z. 276(2014) 1049-1075.

23. H. Katsurada and H. Kawamura, *A certain Dirichlet series of Rankin-Selberg type associated with the Ikeda lifting*, J. Number Theory **128** (2008), 2025–2052.

24. H. Katsurada and H. Kawamura, *On the Andrianov-type identity for power series attached to Jacobi forms and its applications*, Acta Arith. **145** (2010), 233–265.

25. H. Katsurada and H. Kawamura, *Koecher-Maaß series of a certain half-integral weight modular form related to the Duke-Imamoḡlu-Ikeda lift*, Acta. Arith. 162(2014) 1-42.

26. Y. Kitaoka, *Dirichlet series in the theory of Siegel modular forms*, Nagoya Math. J. **95** (1984), 73–84.

27. Y. Kitaoka, *Arithmetic of quadratic forms*, Cambridge Tracts in Mathematics, 106. Cambridge University Press, Cambridge, 1993

28. W. Kohnen, *Modular forms of half-integral weight on $\Gamma_0(4)$*, Math. Ann. **248** (1980), 249–266.

29. A. Krieg, *A Dirichlet series for modular forms of degree n*, Acta Arith. **59** (1991), 243–259.

30. W. Kohnen and N.-P. Skoruppa, *A certain Dirichlet series attached to Siegel modular forms of degree 2*, Invent. Math. **95** (1989), 541–558.

31. W. Kohnen and D. Zagier, *Values of L-series of modular forms at the center of the critical strip*, Invent. Math. **64** (1981), 175–198.

32. A. Murase and T. Sugano, *Inner product formula for Kudla lift*, Automorphic forms and zeta functions, 280–313, World Sci. Publ., Hackensack, NJ, 2006.

33. T. Oda, *On the poles of Andrianov L-functions*, Math. Ann. **256** (1981), 323–340.

34. S. Rallis, *L-functions and Oscillator representation*, Lecture Notes in Math., Vol. 1245, Springer-Verlag, Berlin, 1987.

35. R. Schulze-Pillot, *Local theta correspondence and the theta lifting of Duke-Imamoglu and Ikeda*, Osaka J. Math. **45**(2008), 965-971.

36. G. Shimura, *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. **29** (1976), no. 6, 783–804

37. G. Shimura *Arithmeticity in the theory of automorphic forms*, Mathematical Surveys and Monographs, 82, Amer. Math. Soc., 2000.

38. T. Yamazaki, *Rankin-Selberg method for Siegel cusp forms*, Nagoya Math. J. **120** (1990), 35–49.

39. D. Zagier, *Modular forms whose Fourier coefficients involve zeta functions of quadratic fields*, Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), pp. 105–169. Lecture Notes in Math., Vol. 627, Springer, Berlin, 1977.

40. V. G. Zhuravlëv, *Euler expansions of theta-transformations of Siegel modular forms of half integer weight and their analytic properties*, Math. USSR Sbornik **51** (1985), 169–190.

Hidenori Katsurada
Muroran Institute of Technology
27-1 Mizumoto, Muroran 050-8585, Japan

hidenori@mmm.muroran-it.ac.jp

Hisa-aki Kawamura
Department of Mathematics, Graduate School of Science, Hiroshima University
1-7-1 Kagamiyama, Higashi-Hiroshima 739-8521, Japan

hisa@hiroshima-u.ac.jp