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2-adic properties for the numbers of involutions in the alternating groups.

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Abstract

We study the 2-adic properties for the numbers of involutions in the alternative groups, and give an affirmative answer to a conjecture of D. Kim and J. S. Kim [14]. Some analogous and general results are also presented.

1 Introduction

Let S_n be the symmetric group of degree n , and let A_n be the alternating group of degree n . Let ϵ be the identity of a group. Given a positive integer m , we denote by $a_n(m)$ the number of permutations $\sigma \in S_n$ such that $\sigma^m = \epsilon$. Let p be a prime. By definition and Wilson's theorem, $a_p(p) = 1 + (p-1)! \equiv 0 \pmod{p}$. Moreover, $a_n(m) \equiv 0 \pmod{\gcd(m, n!)}$ by a theorem of Frobenius (see, *e.g.*, [10]).

Let u be a positive integer. There exist remarkable p -adic properties of $a_n(p^u)$ (cf. Theorems 4.2–4.4). The beginning of them is due to H. Ochiai [16] and K. Conrad [4]. For each integer a , $\text{ord}_p(a)$ denotes the exponent of p in the decomposition of a into prime factors. As a pioneer work, the formula

$$\text{ord}_p(a_n(p)) \geq \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{n}{p^2} \right\rfloor$$

(cf. Corollary 4.5) was given in [6, 7, 9], which was also shown by various methods (cf. [4, 11, 13, 14]); moreover, the equality holds for all n such that $n - \lfloor n/p^2 \rfloor p^2 \leq p-1$ (see, *e.g.*, [6, 11, 13]). When $p = 2$, this formula was found by S. Chowla, I. N.

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Herstein, and W. K. Moore [2]. The precise formula for $\text{ord}_2(a_n(2))$ is known as

$$\text{ord}_2(a_n(2)) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor + 1 & \text{if } n \equiv 3 \pmod{4}, \\ \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor & \text{otherwise} \end{cases}$$

(cf. Example 4.6). The value of $\text{ord}_2(a_n(4))$ is also determined (cf. Proposition 4.7).

We denote by $t_n(m)$ the number of even permutations $\sigma \in A_n$ such that $\sigma^m = \epsilon$. Recently, D. Kim and J. S. Kim [14] proved that for any nonnegative integer y ,

$$\text{ord}_2(t_{4y}(2)) = y + \chi_o(y), \text{ord}_2(t_{4y+2}(2)) = \text{ord}_2(t_{4y+3}(2)) = y,$$

where $\chi_o(y) = 1$ if y is odd, and $\chi_o(y) = 0$ if y is even. They also conjectured that for any nonnegative integer y , there exists a 2-adic integer α satisfying

$$\text{ord}_2(t_{4y+1}(2)) = y + \chi_o(y) \cdot (\text{ord}_2(y + \alpha) + 1)$$

(see [14, Conjecture 5.6]). According to [14], $\alpha = 1 + 2 + 2^3 + 2^8 + 2^{10} + \dots$ satisfies the condition for all $y \leq 1000$. In this paper, we solve affirmatively their conjecture (cf. Theorem 5.1), and present some analogous and general results, including the result for $\text{ord}_2(t_n(4))$ (cf. Theorems 5.4). We adapt K. Conrad's methods presented in [4] to the case of $t_n(2^u)$.

Sections 2–5 are devoted to the study of $\text{ord}_p(a_n(p^u))$ and $\text{ord}_2(t_n(2^u))$. In addition to the above results, we also show that, if $r = 0$ or $r = 1$, then there exists a 2-adic integer α_r such that

$$\text{ord}_2(t_{2^{u+1}y+r}(2^u)) = (2^{u+1} - u - 2)y + \chi_o(y) \cdot (\text{ord}_2(y + \alpha_r) + u)$$

for any nonnegative integer y (cf. Theorem 5.6).

Let $C_p \wr S_n$ be the wreath product of C_p by S_n , where C_p is a cyclic group of order p , and let $C_2 \wr A_n$ be the wreath product of C_2 by A_n . We are also interested in the number of elements x of these wreath products such that $x^m = \epsilon$. Let $b_n(p^u)$ be the number of elements x of $C_p \wr S_n$ such that $x^{p^u} = \epsilon$, and let $q_n(2^u)$ be the number of elements x of $C_2 \wr A_n$ such that $x^{2^u} = \epsilon$. In Sections 6–8, we focus on the p -adic properties of $b_n(p^u)$ and the 2-adic properties of $q_n(2^u)$. When $u = 1$, we are successful in finding the fact that

$$\text{ord}_p(b_n(p)) = n - \left\lfloor \frac{n}{p} \right\rfloor \quad \text{and} \quad \text{ord}_2(q_n(2)) = \left\lfloor \frac{n+1}{2} \right\rfloor + \chi_o\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$$

(cf. Examples 7.4 and 8.2). The former fact with $p = 2$ is due to T. Yoshida [20]. The results for $\text{ord}_p(b_n(p^u))$ and $\text{ord}_2(q_n(2^u))$ with $u \geq 2$ are similar to those for $\text{ord}_p(a_n(p^{u-1}))$ and $\text{ord}_2(t_n(2^{u-1}))$, while there are slight differences between the proofs (cf. Example 7.5, Proposition 7.6, Theorems 8.3, 8.5, and 8.7).

2 Generating functions

For each $\sigma \in S_n$, $\sigma^{p^u} = \epsilon$ if and only if the cycle type of σ is of the form

$$(1^{j_0}, p^{j_1}, \dots, (p^u)^{j_u}),$$

where $j_0, j_1, \dots, j_u$ are nonnegative integers satisfying $\sum_k j_k p^k = n$. Since the number of such a permutations is $n! / \prod_{k=0}^u p^{kj_k} j_k!$ (see, *e.g.*, [12, Lemma 1.2.15] or [18, Chap. 4 §2]), it follows that

$$a_n(p^u) = \sum_{j_0+j_1p+\dots+j_u p^u=n} \frac{n!}{\prod_{k=0}^u p^{kj_k} j_k!}. \quad (1)$$

Set $a_n^0(p^u) = a_n(p^u)$, and define

$$a_n^1(p^u) = \sum_{j_0+j_1p+\dots+j_u p^u=n} \frac{(-1)^{j_0+j_1+\dots+j_u} n!}{\prod_{k=0}^u p^{kj_k} j_k!}. \quad (2)$$

Then we have

$$t_n(p^u) = \frac{a_n^0(p^u) + (-1)^n a_n^1(p^u)}{2}. \quad (3)$$

(Obviously, $a_n(p^u) = t_n(p^u)$ if $p \neq 2$.) Let $\natural$ denotes both 0 and 1. We always assume that $a_0^\natural(p^u) = 1$. By Eqs. (1)–(3), we have

$$\sum_{n=0}^{\infty} \frac{a_n^\natural(p^u)}{n!} X^n = \exp \left((-1)^\natural \sum_{k=0}^u \frac{1}{p^k} X^{p^k} \right) \quad (4)$$

and

$$\sum_{n=0}^{\infty} \frac{t_n(2^u)}{n!} X^n = \frac{1}{2} \exp \left(\sum_{k=0}^u \frac{1}{2^k} X^{2^k} \right) + \frac{1}{2} \exp \left(X - \sum_{k=1}^u \frac{1}{2^k} X^{2^k} \right)$$

(see also [3] and [18, Chap. 4, Problem 22]). Let $\{c_n^\natural\}_{n=0}^\infty$ be a sequence given by

$$\sum_{n=0}^{\infty} c_n^\natural X^n = \exp \left((-1)^\natural \sum_{k=0}^{\infty} \frac{1}{p^k} X^{p^k} \right). \quad (5)$$

Then by [5, Proposition 1] (see also [15, p. 97, Exercise 18]), $c_n^\natural \in \mathbb{Z}_p \cap \mathbb{Q}$, where $\mathbb{Z}_p$ is the ring of p -adic integers. When $\natural = 0$, this formal power series is called the Artin-Hasse exponential (cf. [5], [15, Chap. IV §2], [19, §48]). We write $c_n = c_n^0$ for the sake of simplicity. By definition, $c_r = a_r(p^u)/r!$ for any nonnegative integer r less than p^{u+1} . According to Mathematica, we have the following lemma.

Lemma 2.1 *If $p = 2$, then the values of $c_r^{\natural}$ for integers r with $0 \leq r \leq 17$ are as follows :*

r	0	1	2	3	4	5	6	7	8	9	10	11
c_r^0	1	1	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{7}{15}$	$\frac{16}{45}$	$\frac{67}{315}$	$\frac{88}{315}$	$\frac{617}{2835}$	$\frac{2626}{14175}$	$\frac{18176}{155925}$
c_r^1	1	-1	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{45}$	$-\frac{5}{63}$	$-\frac{8}{105}$	$\frac{43}{405}$	$-\frac{74}{14175}$	$-\frac{559}{17325}$

r	12	13	14	15	16	17
c_r^0	$\frac{6949}{66825}$	$\frac{423271}{6081075}$	$\frac{2172172}{42567525}$	$\frac{19151162}{638512875}$	$\frac{58438907}{638512875}$	$\frac{899510224}{10854718875}$
c_r^1	$\frac{697}{18711}$	$-\frac{13232}{552825}$	$-\frac{30727}{14189175}$	$\frac{450991}{49116375}$	$-\frac{5519014}{91216125}$	$\frac{8250311}{144729585}$

For any nonnegative integer r less than p^{u+1} , we set

$$H_{u,r}^{\natural}(X) = \sum_{y=0}^{\infty} \frac{a_{p^{u+1}y+r}^{\natural}(p^u)}{(p^{u+1}y+r)!} (-(-1)^{\natural} p^{u+1})^y X^y,$$

and define a sequence $\{d_{n,r}^{\natural}\}_{n=0}^{\infty}$ by

$$\sum_{n=0}^{\infty} d_{n,r}^{\natural} X^n = \left(\sum_{j=0}^{\infty} c_{p^{u+1}j+r}^{\natural} (-(-1)^{\natural} p^{u+1})^j X^j \right) \exp \left(\sum_{i=1}^{\infty} \frac{\varepsilon^{\natural} p^{p^i(u+1)}}{p^{u+i+1}} X^{p^i} \right),$$

where $\varepsilon^{\natural} = -1$ if $p = 2$ and $\natural = 0$, and $\varepsilon^{\natural} = +1$ otherwise.

Lemma 2.2 *Let r be a nonnegative integer less than p^{u+1} . Then*

$$H_{u,r}^{\natural}(X) = \exp(X) \sum_{n=0}^{\infty} d_{n,r}^{\natural} X^n.$$

Proof. Using Eqs. (4) and (5), we have

$$\sum_{n=0}^{\infty} \frac{a_n^{\natural}(p^u)}{n!} X^n = \left(\sum_{n=0}^{\infty} c_n^{\natural} X^n \right) \exp \left(-(-1)^{\natural} \sum_{k=u+1}^{\infty} \frac{1}{p^k} X^{p^k} \right).$$

This formula yields

$$\begin{aligned} \sum_{y=0}^{\infty} \frac{a_{p^{u+1}y+r}^{\natural}(p^u)}{(p^{u+1}y+r)!} X^{p^{u+1}y+r} &= \left(\sum_{j=0}^{\infty} c_{p^{u+1}j+r}^{\natural} X^{p^{u+1}j+r} \right) \\ &\quad \times \exp \left(-(-1)^{\natural} \sum_{i=0}^{\infty} \frac{1}{p^{u+i+1}} X^{p^{u+i+1}} \right). \end{aligned}$$

Omit X^r and substitute $(-(-1)^{\natural}p^{u+1})X$ for $X^{p^{u+1}}$. Then we have

$$\begin{aligned} \sum_{y=0}^{\infty} \frac{a_{p^{u+1}y+r}^{\natural}(p^u)}{(p^{u+1}y+r)!} (-(-1)^{\natural}p^{u+1})^y X^y &= \left(\sum_{j=0}^{\infty} c_{p^{u+1}j+r}^{\natural} (-(-1)^{\natural}p^{u+1})^j X^j \right) \\ &\times \exp \left(-(-1)^{\natural} \sum_{i=0}^{\infty} \frac{(-(-1)^{\natural}p^{u+1})^{p^i}}{p^{u+i+1}} X^{p^i} \right). \end{aligned}$$

This completes the proof. $\square$

Remark 2.3 In [4], Conrad has given the equation in Lemma 2.2 with $\natural = 0$.

3 Fundamental facts

In this section, we provide four fundamental facts for the study of $\text{ord}_p(a_n^{\natural}(p^u))$ and $\text{ord}_p(t_n(p^u))$. The next lemma is well-known (cf. [8, Problems 164 and 165], [15, p. 7, Exercise 14], [19, Lemma 25.5]).

Lemma 3.1 *Suppose that $n = n_0 + n_1p + n_2p^2 + \cdots \neq 0$, where n_i , $i = 0, 1, \dots$, are nonnegative integers less than p . Then*

$$\text{ord}_p(n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor = \frac{n - n_0 - n_1 - n_2 - \cdots}{p - 1} \leq \frac{n - 1}{p - 1}.$$

For each non-zero p -adic integer $x = \sum_{i=0}^{\infty} x_i p^i$ with $0 \leq x_i \leq p - 1$, we denote by $\text{ord}_p(x)$ the first index i such that $x_i \neq 0$. The p -adic absolute value of a p -adic integer x is given by

$$|x|_p = \begin{cases} p^{-\text{ord}_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We define a subring $\mathbb{Z}_p\langle X \rangle$ of $\mathbb{Z}_p[[X]]$ by

$$\mathbb{Z}_p\langle X \rangle = \left\{ \sum_{n=0}^{\infty} m_n X^n \in \mathbb{Z}_p[[X]] \mid \lim_{n \rightarrow \infty} |m_n|_p = 0 \right\}.$$

For each $g(X) = \sum_{n=0}^{\infty} g_n X^n \in \mathbb{Z}_p[[X]]$, $g(X) + p^{k_1} X^{k_2} \mathbb{Z}_p\langle X \rangle$ denotes the set of all formal power series $f(X) = \sum_{n=0}^{\infty} f_n X^n$ such that $f(X) - g(X) \in p^{k_1} X^{k_2} \mathbb{Z}_p\langle X \rangle$, where k_1 and k_2 are nonnegative integers.

Lemma 3.2 *Let k be a positive integer, and let a be a p -adic integer such that $\text{ord}_p(a) = k$. Excepting the case where $p = 2$ and $k = 1$,*

$$\exp(aX) \in 1 + aX + \frac{a^2}{2} X^2 + \frac{a^3}{6} X^3 + p^{2k+1} X^4 \mathbb{Z}_p\langle X \rangle.$$

Proof. Observe that

$$\exp(aX) - 1 - aX - \frac{a^2}{2}X^2 - \frac{a^3}{6}X^3 = p^{2k}X^3 \sum_{n=1}^{\infty} p^{-2k} \frac{a^{n+3}}{(n+3)!} X^n.$$

Then it follows from Lemma 3.1 that

$$\text{ord}_p \left(p^{-2k} \frac{a^{n+3}}{(n+3)!} \right) \geq k(n+1) - \frac{n+2}{p-1} = \left(k - \frac{1}{p-1} \right) n + \left(k - \frac{2}{p-1} \right).$$

This completes the proof. $\square$

The next lemma is essentially due to K. Conrad [4] (see also [19, Theorem 54.4]).

Lemma 3.3 *Let $\sum_{n=0}^{\ell} m_n X^n$ be a polynomial of degree ℓ with coefficients in $\mathbb{Z}_p$, and let $\sum_{n=1}^{\infty} w_n X^n \in p^k X \mathbb{Z}_p \langle X \rangle$, k a nonnegative integer. Define a sequence $\{d_n\}_{n=0}^{\infty}$ by $d_0 = m_0$ and $d_n = m_n + w_n$ for $n = 1, 2, \dots$. Then there exists a p -adic analytic function $g(X) \in \mathbb{Z}_p \langle X \rangle$ such that*

$$\sum_{n=0}^{\infty} \frac{g(n)}{n!} X^n = \exp(X) \sum_{n=0}^{\infty} d_n X^n \quad \text{and} \quad g(X) \in \sum_{i=0}^{\ell} m_i i! \binom{X}{i} + p^k X \mathbb{Z}_p \langle X \rangle,$$

where

$$\binom{X}{i} = \frac{X(X-1) \cdots (X-i+1)}{i!}, \quad i = 1, 2, \dots, \quad \text{and} \quad \binom{X}{0} = 1.$$

Proof. Define a formal series

$$f(X) = \sum_{i=0}^{\infty} d_i i! \binom{X}{i}.$$

For any nonnegative integer i , we have

$$\sum_{n=0}^{\infty} \frac{i! \binom{n}{i}}{n!} X^n = \exp(X) \cdot X^i,$$

which is extended to the formula

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!} X^n = \exp(X) \sum_{n=0}^{\infty} d_n X^n$$

by $\mathbb{Z}_p$ -linearity. For each positive integer i , let $\{k_{in}\}_{n=1}^{\infty}$ be a sequence given by

$$\sum_{n=1}^{\infty} k_{in} X^n = i! \binom{X}{i}.$$

Then $k_{in} \in \mathbb{Z}$, and $k_{in} = 0$ if $n \geq i + 1$. Since $\lim_{n \rightarrow \infty} |w_n|_p = 0$, it follows that

$$f(x) - \sum_{i=0}^{\ell} m_i i! \binom{x}{i} = \sum_{i=1}^{\infty} w_i i! \binom{x}{i} = \sum_{i=1}^{\infty} \sum_{n=1}^i w_i k_{in} x^n = \sum_{n=1}^{\infty} \left(\sum_{i=n}^{\infty} w_i k_{in} \right) x^n$$

for any $x \in \mathbb{Z}_p$. In particular, $\sum_{i=n}^{\infty} w_i k_{in} \in p^k \mathbb{Z}_p$ for any positive integer n . Moreover, $\lim_{n \rightarrow \infty} |\sum_{i=n}^{\infty} w_i k_{in}|_p = 0$. Now define a formal power series

$$g(X) = \sum_{i=0}^{\ell} m_i i! \binom{X}{i} + \sum_{n=1}^{\infty} \left(\sum_{i=n}^{\infty} w_i k_{in} \right) X^n.$$

Then $f(n) = g(n)$ for $n = 0, 1, 2, \dots$. This completes the proof. $\square$

The following theorem is part of [8, Theorem 6.2.6] (see also [15, Chap. IV Theorem 14]).

Theorem 3.4 (p -adic Weierstrass Preparation Theorem) *Let*

$$f(X) = \sum f_n X^n$$

be a power series with coefficients in the field $\mathbb{Q}_p$ of p -adic numbers such that $\lim_{n \rightarrow \infty} |f_n|_p = 0$. Let N be the number defined by

$$|f_N|_p = \max |f_n|_p \quad \text{and} \quad |f_n|_p < |f_N|_p \text{ for all } n > N.$$

Then there exists a polynomial

$$k_0 + k_1 X + k_2 X^2 + \dots + k_N X^N$$

of degree N with coefficients in $\mathbb{Q}_p$, and a formal power series

$$1 + m_1 X + m_2 X^2 + \dots$$

with coefficients in $\mathbb{Q}_p$, satisfying

- (i) $f(X) = (k_0 + k_1 X + k_2 X^2 + \dots + k_N X^N)(1 + m_1 X + m_2 X^2 + \dots)$,
- (ii) $|k_N|_p = \max |k_n|_p$,
- (iii) $\lim_{n \rightarrow \infty} |m_n|_p = 0$,
- (iv) $|m_n|_p < 1$ for all $n \geq 1$.

4 p -adic properties of $a_n(p^u)$

We define a sequence $\{e_n^\natural\}_{n=0}^\infty$ by

$$\sum_{n=0}^\infty e_n^\natural X^n = \exp \left(\sum_{i=2}^\infty \frac{\varepsilon^\natural p^{p^i(u+1)}}{p^{u+i+1}} X^{p^i} \right),$$

so that for any nonnegative integer r less than p^{u+1} ,

$$\sum_{n=0}^\infty d_{n,r}^\natural X^n = \left(\sum_{j=0}^\infty c_{p^{u+1}j+r}^\natural (-(-1)^\natural p^{u+1})^j X^j \right) \exp \left(\frac{\varepsilon^\natural p^{p(u+1)}}{p^{u+2}} X^p \right) \sum_{n=0}^\infty e_n^\natural X^n.$$

To give p -adic properties of $a_n(p^u)$, we need the following.

Lemma 4.1 $\sum_{n=0}^\infty e_n^\natural X^n \in 1 + p^{3u+1} X \mathbb{Z}_p \langle X \rangle$.

Proof. If $i \geq 2$, then $p^i = (1 + p - 1)^i \geq i(p - 1) + p \geq i + 2 \geq 4$, and thereby,

$$\begin{aligned} \text{ord}_p \left(\frac{p^{p^i(u+1)}}{p^{u+i+1}} \right) &= p^i(u+1) - (u+i+1) \\ &= p^i u + p^i - (u+i+1) \\ &\geq 4u + (i+2) - (u+i+1) \\ &= 3u + 1. \end{aligned}$$

Hence the assertion follows from Lemma 3.2. This completes the proof. $\square$

The results are divided into three theorems, which generalize part of the results proved by K. Conrad [4] (see also [11, 16]).

Theorem 4.2 Suppose that $p \geq 3$. Let r be a nonnegative integer less than p^{u+1} . Then there exists a p -adic analytic function $g_r(X) \in \mathbb{Z}_p \langle X \rangle$ such that

$$g_r(y) = \frac{a_{p^{u+1}y+r}(p^u)}{(p^{u+1}y+r)!} (-p^{u+1})^y y!$$

for any nonnegative integer y and

$$g_r(X) \in c_r - c_{p^{u+1}+r} p^{u+1} X + p^{2u+1} X \mathbb{Z}_p \langle X \rangle.$$

Proof. Using Lemmas 3.2 and 4.1, we have

$$\begin{aligned} \sum_{n=0}^\infty d_{n,r}^0 X^n &= \left(\sum_{j=0}^\infty c_{p^{u+1}j+r} (-p^{u+1})^j X^j \right) \exp \left(\frac{p^{p(u+1)}}{p^{u+2}} X^p \right) \sum_{n=0}^\infty e_n^0 X^n \\ &\in c_r - c_{p^{u+1}+r} p^{u+1} X + p^{2u+1} X \mathbb{Z}_p \langle X \rangle. \end{aligned}$$

Hence the assertion follows from Lemmas 2.2 and 3.3. This completes the proof. $\square$

Theorem 4.3 *Suppose that $p = 2$ and $u \geq 2$. Let r be a nonnegative integer less than 2^{u+1} . Then there exists a 2-adic analytic function $g_r^{\natural}(X) \in \mathbb{Z}_2\langle X \rangle$ such that*

$$g_r^{\natural}(y) = \frac{a_{2^{u+1}y+r}^{\natural}(2^u)}{(2^{u+1}y+r)!} (-(-1)^{\natural} 2^{u+1})^y y!$$

for any nonnegative integer y and

$$g_r^{\natural}(X) \in c_r^{\natural} (1 - (-1)^{\natural} 2^u X(X-1) + 2^{2u-1} X(X-1)(X-2)(X-3)) \\ - (-1)^{\natural} c_{2^{u+1}+r}^{\natural} 2^{u+1} X + 2^{2u+1} X \mathbb{Z}_2\langle X \rangle.$$

Proof. By definition,

$$\sum_{n=0}^{\infty} d_{n,r}^{\natural} X^n = \left(\sum_{j=0}^{\infty} c_{2^{u+1}j+r}^{\natural} (-(-1)^{\natural} 2^{u+1})^j X^j \right) \exp(-(-1)^{\natural} 2^u X^2) \sum_{n=0}^{\infty} e_n^{\natural} X^n.$$

(Note that $\varepsilon^{\natural} = -(-1)^{\natural}$ if $p = 2$.) Using Lemma 3.2, we have

$$\exp(-(-1)^{\natural} 2^u X^2) \in 1 - (-1)^{\natural} 2^u X^2 + 2^{2u-1} X^4 + 2^{2u+1} X^6 \mathbb{Z}_2\langle X \rangle.$$

Moreover, it follows from Lemma 4.1 that

$$\sum_{i=0}^{\infty} d_{n,r}^{\natural} X^n \in c_r^{\natural} (1 - (-1)^{\natural} 2^u X^2 + 2^{2u-1} X^4) \\ - (-1)^{\natural} c_{2^{u+1}+r}^{\natural} 2^{u+1} X + 2^{2u+1} X \mathbb{Z}_2\langle X \rangle.$$

Hence the assertion follows from Lemmas 2.2 and 3.3. This completes the proof. $\square$

Theorem 4.4 *Suppose that $p = 2$ and $u = 1$. Let r be a nonnegative integer less than 4. Then there exists a 2-adic analytic function $g_r^{\natural}(X) \in \mathbb{Z}_2\langle X \rangle$ such that*

$$g_r^{\natural}(y) = \frac{a_{4y+r}^{\natural}(2)}{(4y+r)!} ((-1)^{\natural} 4)^y y!$$

for any nonnegative integer y and

$$g_r^{\natural}(X) \in c_r^{\natural} (1 - 2X + 4\delta_{\natural 1} X(X-1) - 4X(X-1)(X-2)(X-3)) \\ + (-1)^{\natural} 4c_{4+r}^{\natural} X + 8X \mathbb{Z}_2\langle X \rangle,$$

where δ is the Kronecker delta.

Proof. Substituting $-X$ for X in Lemma 2.2, we have

$$\begin{aligned} \sum_{y=0}^{\infty} \frac{a_{4y+r}^{\natural}(2)}{(4y+r)!} ((-1)^{\natural}4)^y X^y &= \exp(X) \exp(-2X - (-1)^{\natural}2X^2) \\ &\times \left(\sum_{j=0}^{\infty} c_{4j+r}^{\natural} ((-1)^{\natural}4)^j X^j \right) \sum_{n=0}^{\infty} e_n^{\natural} (-1)^n X^n. \end{aligned} \quad (6)$$

Moreover, it follows from Eq. (4) with $p = 2$ and $u = 2$ that

$$\begin{aligned} \exp(-2X - (-1)^{\natural}2X^2) &= \exp(-2X + 2X^2 + 4X^4) \exp(-4\delta_{\natural 0}X^2 - 4X^4) \\ &= \left(\sum_{n=0}^{\infty} \frac{a_n(4)}{n!} (-2X)^n \right) \exp(-4\delta_{\natural 0}X^2 - 4X^4). \end{aligned}$$

By Lemma 3.1 and Theorem 4.3,

$$\text{ord}_2 \left(\frac{a_n(4)}{n!} (-2)^n \right) = \text{ord}_2(a_n(4)) + \text{ord}_2 \left(\frac{(-2)^n}{n!} \right) \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor - 2 \left\lfloor \frac{n}{8} \right\rfloor + 1$$

if $n \geq 1$ (see also Proposition 4.7). Observe that

$$\text{ord}_2 \left(\frac{a_n(4)}{n!} (-2)^n \right) \geq 4$$

if $n \geq 4$. Then, since $a_0(4) = a_1(4) = 1$, $a_2(4) = 2$, and $a_3(4) = 4$, we have

$$\sum_{n=0}^{\infty} \frac{a_n(4)}{n!} (-2X)^n \in 1 - 2X + 4X^2 + 16X\mathbb{Z}_2\langle X \rangle.$$

This, combined with Lemma 3.2, yields

$$\exp(-2X - (-1)^{\natural}2X^2) \in (1 - 2X + 4X^2)(1 - 4\delta_{\natural 0}X^2 - 4X^4) + 8X\mathbb{Z}_2\langle X \rangle.$$

Hence it follows from Lemma 4.1 that

$$\begin{aligned} \exp(-2X - (-1)^{\natural}2X^2) &\left(\sum_{j=0}^{\infty} c_{4j+r}^{\natural} ((-1)^{\natural}4)^j X^j \right) \sum_{n=0}^{\infty} e_n^{\natural} (-1)^n X^n \\ &\in c_r^{\natural} (1 - 2X + 4\delta_{\natural 1}X^2 - 4X^4) + (-1)^{\natural}4c_{4+r}^{\natural}X + 8X\mathbb{Z}_2\langle X \rangle. \end{aligned}$$

The assertion now follows from Lemma 3.3 and Eq. (6). $\square$

Let r be a nonnegative integer less than p^{u+1} . By Lemma 3.1,

$$\text{ord}_p \left(\frac{(p^{u+1}y + r)!}{p^{(u+1)y}y!} \right) = \sum_{j=1}^u \left[\frac{p^{u+1}y + r}{p^j} \right] - uy = \left\{ \frac{p^{u+1} - 1}{p - 1} - (u + 1) \right\} y + \text{ord}_p(r!)$$

for any nonnegative integer y . Combining this fact with Theorems 4.2, 4.3, and 4.4, we obtain the following.

Corollary 4.5 ([13]) *Let r be a nonnegative integer less than p^{u+1} . Then*

$$\begin{aligned} \text{ord}_p(a_{p^{u+1}y+r}(p^u)) &\geq \sum_{j=1}^u \left\lfloor \frac{p^{u+1}y+r}{p^j} \right\rfloor - uy \\ &= \left\{ \frac{p^{u+1}-1}{p-1} - (u+1) \right\} y + \text{ord}_p(r!) \end{aligned}$$

for any nonnegative integer y . Moreover, if $\text{ord}_p(c_r) \leq u$, then

$$\begin{aligned} \text{ord}_p(a_{p^{u+1}y+r}(p^u)) &= \sum_{j=1}^u \left\lfloor \frac{p^{u+1}y+r}{p^j} \right\rfloor - uy + \text{ord}_p(c_r) \\ &= \left\{ \frac{p^{u+1}-1}{p-1} - (u+1) \right\} y + \text{ord}_p(r!) + \text{ord}_p(c_r) \end{aligned}$$

for any nonnegative integer y .

Example 4.6 ([6, 13, 14, 16]) Suppose that $p = 2$ and $u = 1$. By Lemma 2.1 and Corollary 4.5,

$$\text{ord}_2(a_n(2)) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor + 1 & \text{if } n \equiv 3 \pmod{4}, \\ \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor & \text{otherwise.} \end{cases}$$

Proposition 4.7 Suppose that $p = 2$ and $u = 2$, and let r be a nonnegative integer less than 8. For any nonnegative integer y ,

$$\begin{aligned} \text{ord}_2(a_{8y+r}(4)) &= \left\lfloor \frac{8y+r}{2} \right\rfloor + \left\lfloor \frac{8y+r}{4} \right\rfloor - 2y + \text{ord}_2(c_r) \\ &= 4y + \text{ord}_2(r!) + \text{ord}_2(c_r), \end{aligned}$$

that is, the values of $\text{ord}_2(a_{8y+r}(4)) - 4y$, $0 \leq r \leq 7$, are as follows :

r	0	1	2	3	4	5	6	7
$\text{ord}_2(a_{8y+r}(4)) - 4y$	0	0	1	2	4	3	8	4

Proof. If $r \neq 6$, then the proposition follows from Lemma 2.1 and Corollary 4.5. By Theorem 4.3, there exists a 2-adic analytic function $g_6^0(X) \in \mathbb{Z}_2\langle X \rangle$ such that

$$g_6^0(y) = \frac{a_{8y+6}(4)}{(8y+6)!} (-8)^y y!$$

for any nonnegative integer y and

$$g_6^0(X) \in c_6(1 - 4X(X-1) + 8X(X-1)(X-2)(X-3)) - 8c_{14}X + 2^5X\mathbb{Z}_2\langle X \rangle.$$

Let y be a nonnegative integer. We have $\text{ord}_2(a_{8y+6}(4)) = 4y + 4 + \text{ord}_2(g_6^0(y))$. Since $c_6 = 16/45$ and $c_{14} = 2172172/42567525$, it follows that $\text{ord}_2(g_6^0(y)) = 4$. Hence $\text{ord}_2(a_{8y+6}(4)) = 4y + 8$. This completes the proof. $\square$

5 2-adic properties of $t_n(2^u)$

The first statement of the following theorem is due to D. Kim and J. S. Kim [14], and the second one is an affirmative answer to a conjecture of them.

Theorem 5.1 *Suppose that $p = 2$ and $u = 1$. Then the following statements hold for any nonnegative integer y .*

- (a) $\text{ord}_2(t_{4y}(2)) = y + \chi_o(y)$, $\text{ord}_2(t_{4y+2}(2)) = \text{ord}_2(t_{4y+3}(2)) = y$.
- (b) *There exists a 2-adic integer α such that*

$$\text{ord}_2(t_{4y+1}(2)) = y + \chi_o(y) \cdot (\text{ord}_2(y + \alpha) + 1).$$

Proof. Keep the notation of Theorem 4.4, and let y be a nonnegative integer. Then by Eq. (3), we have

$$t_{4y+r}(2) = \frac{(4y+r)!}{4^y \cdot y!} \cdot \frac{g_r^0(y) + (-1)^{r+y} g_r^1(y)}{2}.$$

Now set $L_{r,y}(X) = (g_r^0(X) + (-1)^{r+y} g_r^1(X))/2$. Then there exists a 2-adic analytic function $M_{r,y}(X) \in \mathbb{Z}_2\langle X \rangle$ such that

$$\begin{aligned} L_{r,y}(X) &= c_r^0 \frac{1 - 2X - 4X(X-1)(X-2)(X-3)}{2} \\ &\quad + (-1)^{r+y} c_r^1 \frac{1 - 2X + 4X(X-1) - 4X(X-1)(X-2)(X-3)}{2} \\ &\quad + 2(c_{4+r}^0 - (-1)^{r+y} c_{4+r}^1)X + 4XM_{r,y}(X). \end{aligned}$$

Moreover, it follows from Lemma 2.1 that

$$\begin{aligned} L_{0,y}(y) &\equiv L_{1,y}(y) \equiv 1 \pmod{4}, \\ L_{2,y}(y) &\equiv \frac{1}{2} \pmod{2}, \quad L_{3,y}(y) \equiv \frac{1}{6} \pmod{2} \end{aligned}$$

if y is even, and

$$\begin{aligned} L_{0,y}(y) &\equiv -2y^2 \pmod{4}, \quad L_{1,y}(y) \equiv \frac{38}{15}y - 2y^2 \pmod{4}, \\ L_{2,y}(y) &\equiv \frac{1}{2} - y \pmod{2}, \quad L_{3,y}(y) \equiv \frac{1}{2} - y \pmod{4} \end{aligned}$$

if y is odd. Since $\text{ord}_2((4y+r)!/4^y \cdot y!) = y + \text{ord}_2(r!)$, it follows that

$$\text{ord}_2(t_{4y+r}(2)) = \begin{cases} y + \chi_o(y) & \text{if } r = 0, \\ y & \text{if } r = 1 \text{ and } y \text{ is even,} \\ y & \text{if } r = 2 \text{ or } r = 3. \end{cases}$$

Assume that y is odd. Then by Lemma 2.1,

$$L_{1,y}(X) = -2X(X-1) + \frac{8}{15}X + 4XM_{1,y}(X) = \frac{38}{15}X - 2X^2 + 4XM_{1,y}(X).$$

Hence it follows from Theorem 3.4 that there exists a polynomial

$$k_0 + k_1X + k_2X^2$$

of degree 2 with coefficients in $\mathbb{Q}_2$, and a power series

$$1 + m_1X + m_2X^2 + \dots$$

with coefficients in $\mathbb{Q}_2$, satisfying the conditions (i)–(iv) with $f(X) = L_{1,y}(X)$, $N = 2$, and $p = 2$. We have $k_0 = 0$, $k_1 \equiv 38/15 \pmod{4}$, and $k_2 \equiv -2 - k_1m_1 \pmod{4}$. Now set $\lambda = 2^{-1}k_2$. Then $\text{ord}_2(\lambda) = 0$, because $\text{ord}_2(m_1) > 0$. Observe that $\alpha := 2^{-1}k_1\lambda^{-1} \in \mathbb{Z}_2$ and

$$L_{1,y}(X) = 2\lambda X(X + \alpha)(1 + m_1X + m_2X^2 + \dots).$$

Then we have

$$\text{ord}_2(t_{4y+1}) = y + 1 + \text{ord}_2(y + \alpha).$$

This completes the proof. $\square$

Remark 5.2 According to Mathematica,

$$\alpha \equiv 1 + 2 + 2^3 + 2^8 + 2^{10} + 2^{12} \pmod{2^{14}}.$$

The following lemma is an immediate consequence of Eq. (3) and Theorem 4.3.

Lemma 5.3 *Suppose that $p = 2$ and $u \geq 2$. Let r be a nonnegative integer less than 2^{u+1} , and let y be a nonnegative integer. Then there exists a 2-adic analytic function $M_{r,y}(X) \in \mathbb{Z}_2\langle X \rangle$ such that*

$$t_{2^{u+1}y+r}(2^u) = \frac{(2^{u+1}y+r)!}{2^{(u+1)y} \cdot y!} \cdot L_{r,y}(y)$$

with

$$\begin{aligned} L_{r,y}(X) = & (-1)^y c_r^0 \frac{1 - 2^u X(X-1) + 2^{2u-1} X(X-1)(X-2)(X-3)}{2} \\ & + (-1)^r c_r^1 \frac{1 + 2^u X(X-1) + 2^{2u-1} X(X-1)(X-2)(X-3)}{2} \\ & + 2^u (-(-1)^y c_{2^{u+1}+r}^0 + (-1)^r c_{2^{u+1}+r}^1) X + 2^{2u} X M_{r,y}(X). \end{aligned}$$

Moreover, $\text{ord}_2(t_{2^{u+1}y+r}(2^u)) = (2^{u+1} - u - 2)y + \text{ord}_2(r!) + \text{ord}_2(L_{r,y}(y))$.

We set $\chi_e(y) = 1 - \chi_o(y)$ for any nonnegative integer y .

Theorem 5.4 *Suppose that $p = 2$ and $u = 2$. Then the following statements hold for any nonnegative integer y .*

$$\begin{aligned} \text{(a)} \quad \text{ord}_2(t_{8y+2}(4)) &= \text{ord}_2(t_{8y+3}(4)) = 4y, \quad \text{ord}_2(t_{8y+4}(4)) = 4y + 2, \\ \text{ord}_2(t_{8y+5}(4)) &= 4y + 3 + \chi_e(y), \quad \text{ord}_2(t_{8y+6}(4)) = 4y + 3, \\ \text{ord}_2(t_{8y+7}(4)) &= 4y + 4 + \chi_e(y). \end{aligned}$$

(b) If $r = 0$ or $r = 1$, then there exists a 2-adic integer α_r such that

$$\text{ord}_2(t_{8y+r}(4)) = 4y + \chi_o(y) \cdot (\text{ord}_2(y + \alpha_r) + 2).$$

Proof. Keep the notation of Lemma 5.3 with $u = 2$. Then by Lemma 2.1,

$$\begin{aligned} L_{0,y}(y) &\equiv L_{1,y}(y) \equiv 1 \pmod{8}, & L_{2,y}(y) &\equiv \frac{1}{2} \pmod{4}, \\ L_{3,y}(y) &\equiv L_{4,y}(y) \equiv \frac{1}{6} \pmod{4}, & L_{5,y}(y) &\equiv \frac{2}{15} \pmod{8}, \\ L_{6,y}(y) &\equiv \frac{17}{90} \pmod{4}, & L_{7,y}(y) &\equiv \frac{46}{315} \pmod{8} \end{aligned}$$

if y is even, and

$$\begin{aligned} L_{0,y}(y) &\equiv 4y \left(y - \frac{251}{315} \right) \pmod{16}, & L_{1,y} &\equiv 4y \left(y - \frac{2519}{2835} \right) \pmod{16}, \\ L_{2,y}(y) &\equiv L_{3,y}(y) \equiv L_{4,y}(y) \equiv -\frac{1}{2} \pmod{4}, & L_{5,y}(y) &\equiv -\frac{1}{3} \pmod{4}, \\ L_{6,y}(y) &\equiv -\frac{1}{6} \pmod{4}, & L_{7,y}(y) &\equiv -\frac{1}{15} \pmod{4} \end{aligned}$$

if y is odd. This, combined with Lemma 5.3, yields the statement (a). The proof of the statement (b) is analogous to that of Theorem 5.1, while the assertion is a special case of Theorem 5.6. This completes the proof. $\square$

Remark 5.5 According to Mathematica,

$$\alpha_0 \equiv 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^7 + 2^9 + 2^{10} + 2^{12} + 2^{13} + 2^{14} + 2^{15} \pmod{2^{17}}$$

and

$$\alpha_1 \equiv 1 + 2 + 2^4 + 2^7 + 2^8 \pmod{2^{12}}.$$

The statement (b) of Theorem 5.4 is extended to a result for $\text{ord}_2(t_{2^{u+1}y+r}(2^u))$ with $u \geq 3$ and $r = 0$ or $r = 1$.

Theorem 5.6 *Suppose that $p = 2$ and $u \geq 2$. Let y be a nonnegative integer. If $r = 0$ or $r = 1$, then there exists a 2-adic integer α_r such that*

$$\text{ord}_2(t_{2^{u+1}y+r}(2^u)) = (2^{u+1} - u - 2)y + \chi_o(y) \cdot (\text{ord}_2(y + \alpha_r) + u).$$

Moreover, if $\text{ord}_2(c_{2^{u+1}+r}^0 + (-1)^r c_{2^{u+1}+r}^1) = 0$ with $r = 0$ or $r = 1$, then

$$\text{ord}_2(t_{2^{u+1}y+r}(2^u)) = (2^{u+1} - u - 2)y + \chi_o(y) \cdot u.$$

Proof. Keep the notation of Lemma 5.3. Since $c_0^0 = c_0^1 = c_1^0 = 1$ and $c_1^1 = -1$ by Lemma 2.1, it follows from Lemma 5.3 that the assertion holds if y is even. Assume that y is odd. Then

$$L_{r,y}(X) = 2^u(-1 + \hat{c}_{2^{u+1}+r})X + 2^u X^2 + 2^{2u} X M_{r,y}(X),$$

where $\hat{c}_{2^{u+1}+r} = c_{2^{u+1}+r}^0 + (-1)^r c_{2^{u+1}+r}^1$. In each of the cases where $r = 0$ and $r = 1$, it follows from Theorem 3.4 that there exists a polynomial

$$k_0 + k_1 X + k_2 X^2$$

of degree 2 with coefficients in $\mathbb{Q}_2$, and a power series

$$1 + m_1 X + m_2 X^2 + \dots$$

with coefficients in $\mathbb{Q}_2$, satisfying the conditions (i)–(iv) with $f(X) = L_{r,y}(X)$, $N = 2$, and $p = 2$. We have $k_0 = 0$, $k_1 \equiv 2^u(-1 + \hat{c}_{2^{u+1}+r}) \pmod{2^{2u}}$, and $k_2 \equiv 2^u - k_1 m_1 \pmod{2^{2u}}$. Now set $\lambda_r = 2^{-u} k_2$. Then $\text{ord}_2(\lambda_r) = 0$, because $\text{ord}_2(m_1) > 0$. Observe that $\alpha_r := 2^{-u} k_1 \lambda_r^{-1} \in \mathbb{Z}_2$ and

$$L_{r,y}(X) = 2^u \lambda_r X(X + \alpha_r)(1 + m_1 X + m_2 X^2 + \dots).$$

Combining this fact with Lemma 5.3, we conclude that

$$\text{ord}_2(t_{2^{u+1}y+r}(2^u)) = (2^{u+1} - u - 2)y + \text{ord}_2(y + \alpha_r) + u$$

Moreover, if $\text{ord}_2(\hat{c}_{2^{u+1}+r}) = 0$, then $\text{ord}_2(\alpha_r) > 0$, and thereby, $\text{ord}_2(y + \alpha_r) = 0$. This completes the proof. $\square$

6 Wreath products

Let G be a finite group, and let K be a subgroup of S_n . The wreath product $G \wr K$ of G by K is defined to be the set

$$G \wr K = \{(g_1, \dots, g_n)\sigma \mid (g_1, \dots, g_n) \in G^{(n)} \text{ and } \sigma \in K\},$$

where $G^{(n)}$ is the direct product of n copies of G , with multiplication given by

$$(g_1, \dots, g_n)\sigma(h_1, \dots, h_n)\tau = (g_1 h_{\sigma^{-1}(1)}, \dots, g_n h_{\sigma^{-1}(n)})\sigma\tau.$$

Let m be a positive integer. We set

$$a(G \wr K, m) = \#\{(g_1, \dots, g_n)\sigma \in G \wr K \mid ((g_1, \dots, g_n)\sigma)^m = \epsilon\}.$$

Lemma 6.1 *Let $\tau \in S_n$ be a cycle of length ℓ . Then $((g_1, \dots, g_n)\tau)^m = \epsilon$ if and only if ℓ divides m and $(g_i g_{\tau^{-1}(i)} \cdots g_{\tau^{-\ell+1}(i)})^{m/\ell} = \epsilon$ for all $i = 1, 2, \dots, n$.*

Proof. The proof is straightforward. $\square$

Let $\{\ell_0, \ell_1, \dots, \ell_s\}$ be the set of divisors of a positive integer m . We quote the following (cf. [12, Lemma 4.2.10]).

Lemma 6.2 *The number of elements $(g_1, \dots, g_n)\sigma$ of $G \wr S_n$ such that the cycle type of σ is $(\ell_0^{j_0}, \ell_1^{j_1}, \dots, \ell_s^{j_s})$ and $((g_1, \dots, g_n)\sigma)^m = \epsilon$ is*

$$n! \prod_{k=0}^s \frac{|G|^{(\ell_k-1)j_k} a(G, m/\ell_k)^{j_k}}{\ell_k^{j_k} j_k!},$$

where $a(G, m/\ell_k) = \#\{g \in G \mid g^{m/\ell_k} = \epsilon\}$.

Proof. Let k be a nonnegative integer less than or equal to s , and let $\tau = (i_1 \cdots i_{\ell_k})$ be a cycle of length ℓ_k . Then it follows from Lemma 6.1 that the number of elements $(g_1, \dots, g_n)$ of $G^{(n)}$ such that $((g_1, \dots, g_n)\tau)^m = \epsilon$ and $g_i = \epsilon$ for all $i \neq i_1, \dots, i_{\ell_k}$ is $|G|^{\ell_k-1} a(G, m/\ell_k)$. Thus the lemma holds. $\square$

By Lemma 6.2, we have

$$b_n(p^u) = a(C_p \wr S_n, p^u) = \sum_{j_0+j_1p+\dots+j_up^u=n} n! \left(\prod_{k=0}^u \frac{p^{p^k j_k}}{p^{k j_k} j_k!} \right) \frac{1}{p^{j_u}}. \quad (7)$$

Set $b_n^0(p^u) = b_n(p^u)$, and define

$$b_n^1(p^u) = \sum_{j_0+j_1p+\dots+j_up^u=n} (-1)^{j_0+j_1+\dots+j_u} n! \left(\prod_{k=0}^u \frac{p^{p^k j_k}}{p^{k j_k} j_k!} \right) \frac{1}{p^{j_u}}. \quad (8)$$

Then by Lemma 6.2, we have

$$q_n(p^u) = a(C_p \wr A_n, p^u) = \frac{b_n^0(p^u) + (-1)^n b_n^1(p^u)}{2}. \quad (9)$$

(Obviously, $b_n(p^u) = q_n(p^u)$ if $p \neq 2$.) Let $\natural$ denotes both 0 and 1. We always assume that $b_0^\natural(p^u) = 1$. By Eqs. (7)–(9), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{b_n^\natural(p^u)}{n!} X^n &= \exp \left((-1)^\natural \sum_{k=0}^{u-1} \frac{p^{p^k}}{p^k} X^{p^k} + (-1)^\natural \frac{p^{p^u}}{p^{u+1}} X^{p^u} \right), \quad (10) \\ \sum_{n=0}^{\infty} \frac{q_n(2^u)}{n!} X^n &= \frac{1}{2} \exp \left(\sum_{k=0}^{u-1} \frac{2^{2^k}}{2^k} X^{2^k} + \frac{2^{2^u}}{2^{u+1}} X^{2^u} \right) \\ &\quad + \frac{1}{2} \exp \left(2X - \sum_{k=1}^{u-1} \frac{2^{2^k}}{2^k} X^{2^k} - \frac{2^{2^u}}{2^{u+1}} X^{2^u} \right) \end{aligned}$$

(cf. [1], [17, Proposition 3.4]). Moreover, by Eq. (5), we have

$$\sum_{n=0}^{\infty} c_n^{\natural}(pX)^n = \exp \left((-1)^{\natural} \sum_{k=0}^{\infty} \frac{p^{p^k}}{p^k} X^{p^k} \right). \quad (11)$$

Recall that $\varepsilon^{\natural} = -1$ if $p = 2$ and $\natural = 0$, and $\varepsilon^{\natural} = +1$ otherwise. For any nonnegative integer r less than p^u , we set

$$\tilde{H}_{u,r}^{\natural}(X) = \sum_{y=0}^{\infty} \frac{b_{p^u y+r}^{\natural}(p^u)}{(p^u y+r)!} \left(-(-1)^{\natural} \frac{p^{u+1}}{p^{p^u}(p-1)} X \right)^y,$$

and define a sequence $\{\tilde{d}_{n,r}^{\natural}\}_{n=0}^{\infty}$ by

$$\sum_{n=0}^{\infty} \tilde{d}_{n,r}^{\natural} X^n = \left(\sum_{j=0}^{\infty} c_{p^u j+r}^{\natural} p^r \left(-(-1)^{\natural} \frac{p^{u+1}}{p-1} X \right)^j \right) \exp \left(\sum_{i=1}^{\infty} \frac{\varepsilon^{\natural} p^{p^i(u+1)}}{p^{u+i}(p-1)^{p^i}} X^{p^i} \right).$$

Lemma 6.3 *Let r be a nonnegative integer less than p^u . Then*

$$\tilde{H}_{u,r}^{\natural}(X) = \exp(X) \sum_{n=0}^{\infty} \tilde{d}_{n,r}^{\natural} X^n.$$

Proof. Using Eqs. (10) and (11), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{b_n^{\natural}(p^u)}{n!} X^n &= \left(\sum_{n=0}^{\infty} c_n^{\natural}(pX)^n \right) \exp \left(-(-1)^{\natural} \frac{p^{p^u}}{p^u} X^{p^u} \right) \\ &\quad \times \exp \left((-1)^{\natural} \frac{p^{p^u}}{p^{u+1}} X^{p^u} \right) \exp \left(-(-1)^{\natural} \sum_{k=u+1}^{\infty} \frac{p^{p^k}}{p^k} X^{p^k} \right). \end{aligned}$$

This formula yields

$$\begin{aligned} \sum_{y=0}^{\infty} \frac{b_{p^u y+r}^{\natural}(p^u)}{(p^u y+r)!} X^{p^u y+r} &= \left(\sum_{j=0}^{\infty} c_{p^u j+r}^{\natural} p^{p^u j+r} X^{p^u j+r} \right) \exp \left(-(-1)^{\natural} \frac{p^{p^u}}{p^u} X^{p^u} \right) \\ &\quad \times \exp \left((-1)^{\natural} \frac{p^{p^u}}{p^{u+1}} X^{p^u} \right) \exp \left(-(-1)^{\natural} \sum_{i=1}^{\infty} \frac{p^{p^{u+i}}}{p^{u+i}} X^{p^{u+i}} \right). \end{aligned}$$

Omit X^r and substitute $(-(-1)^{\natural} p^{u+1} X / p^{p^u}(p-1))^{1/p^u}$ for X . Then we have

$$\begin{aligned} \sum_{y=0}^{\infty} \frac{b_{p^u y+r}^{\natural}(p^u)}{(p^u y+r)!} \left(-(-1)^{\natural} \frac{p^{u+1}}{p^{p^u}(p-1)} X \right)^y &= \left(\sum_{j=0}^{\infty} c_{p^u j+r}^{\natural} p^r \left(-(-1)^{\natural} \frac{p^{u+1}}{p-1} X \right)^j \right) \\ &\quad \times \exp(X) \exp \left(\sum_{i=1}^{\infty} \frac{-(-1)^{\natural} \cdot (-(-1)^{\natural})^{p^i} p^{p^i(u+1)}}{p^{u+i}(p-1)^{p^i}} X^{p^i} \right). \end{aligned}$$

This completes the proof. $\square$

7 p -adic properties of $b_n(p^u)$

In order to analyze $\sum_{n=0}^{\infty} \tilde{d}_{n,r}^{\natural} X^n$, we define a sequence $\{\tilde{e}_n^{\natural}\}_{n=0}^{\infty}$ by

$$\sum_{n=0}^{\infty} \tilde{e}_n^{\natural} X^n = \exp \left(\sum_{i=2}^{\infty} \frac{\varepsilon^{\natural} p^{p^i(u+1)}}{p^{u+i}(p-1)^{p^i}} X^{p^i} \right).$$

The proof of the following lemma is analogous to that of Lemma 4.1.

Lemma 7.1 $\sum_{n=0}^{\infty} \tilde{e}_n^{\natural} X^n \in 1 + p^{3u+2} X \mathbb{Z}_p \langle X \rangle$.

We are now in position to state a p -adic property of $b_n(p^u)$.

Theorem 7.2 *Let r be a nonnegative integer less than p^u . Then there exists a p -adic analytic function $g_r^{\natural}(X) \in \mathbb{Z}_p \langle X \rangle$ such that*

$$g_r^{\natural}(y) = \frac{b_{p^u y + r}^{\natural}(p^u)}{(p^u y + r)!} \left(-(-1)^{\natural} \frac{p^{u+1}}{p^{p^u}(p-1)} \right)^y y!$$

for any nonnegative integer y and

$$g_r^{\natural}(X) \in c_r^{\natural} p^r \left\{ 1 + \varepsilon^{\natural} \frac{p^{(u+1)(p-1)}}{(p-1)^p} X(X-1)(X-2) \cdots (X-p+1) \right\} \\ -(-1)^{\natural} c_{p^u+r}^{\natural} \frac{p^{u+1+r}}{p-1} X + p^{2u+1+r} X \mathbb{Z}_p \langle X \rangle.$$

Proof. Using Lemmas 3.2 and 7.1, we have

$$\sum_{n=0}^{\infty} \tilde{d}_{n,r}^{\natural} X^n = \left(\sum_{j=0}^{\infty} c_{p^u j + r}^{\natural} p^r \left(-(-1)^{\natural} \frac{p^{u+1}}{p-1} X \right)^j \right) \\ \times \exp \left(\frac{\varepsilon^{\natural} p^{p(u+1)}}{p^{u+1}(p-1)^p} X^p \right) \sum_{n=0}^{\infty} \tilde{e}_n^{\natural} X^n \\ \in c_r^{\natural} p^r \left\{ 1 + \varepsilon^{\natural} \frac{p^{(u+1)(p-1)}}{(p-1)^p} X^p \right\} \\ -(-1)^{\natural} c_{p^u+r}^{\natural} \frac{p^{u+1+r}}{p-1} X + p^{2u+1+r} X \mathbb{Z}_p \langle X \rangle.$$

Hence the assertion follows from Lemmas 3.3 and 6.3. This completes the proof. $\square$

This theorem, together with Lemma 3.1, yields the following.

Corollary 7.3 *Let r be a nonnegative integer less than p^u . Then*

$$\begin{aligned} \text{ord}_p(b_{p^u y + r}(p^u)) &\geq \sum_{j=0}^{u-1} \left[\frac{p^u y + r}{p^j} \right] - uy \\ &= \left\{ \frac{p^u - 1}{p - 1} + p^u - (u + 1) \right\} y + r + \text{ord}_p(r!) \end{aligned}$$

for any nonnegative integer y . If $\text{ord}_p(c_r) \leq u$, then

$$\begin{aligned} \text{ord}_p(b_{p^u y + r}(p^u)) &= \sum_{j=0}^{u-1} \left[\frac{p^u y + r}{p^j} \right] - uy + \text{ord}_p(c_r) \\ &= \left\{ \frac{p^u - 1}{p - 1} + p^u - (u + 1) \right\} y + r + \text{ord}_p(r!) + \text{ord}_p(c_r) \end{aligned}$$

for any nonnegative integer y .

Example 7.4 Suppose that $u = 1$. Then for any nonnegative integer r less than p , we have $\text{ord}_p(c_r) = 0$. Hence

$$\text{ord}_p(b_n(p)) = n - \left\lfloor \frac{n}{p} \right\rfloor \quad \text{and} \quad \text{ord}_2(b_n(2)) = \left\lfloor \frac{n + 1}{2} \right\rfloor.$$

Example 7.5 Suppose that $p = 2$ and $u = 2$. By Lemma 2.1 and Corollary 7.3,

$$\text{ord}_2(b_n(4)) = \begin{cases} n + \left\lfloor \frac{n}{2} \right\rfloor - 2 \left\lfloor \frac{n}{4} \right\rfloor + 1 & \text{if } n \equiv 3 \pmod{4}, \\ n + \left\lfloor \frac{n}{2} \right\rfloor - 2 \left\lfloor \frac{n}{4} \right\rfloor & \text{otherwise.} \end{cases}$$

Proposition 7.6 *Suppose that $p = 2$ and $u = 3$, and let r be a nonnegative integer less than 8. For any nonnegative integer y ,*

$$\begin{aligned} \text{ord}_2(b_{8y+r}(8)) &= 8y + r + \left\lfloor \frac{8y+r}{2} \right\rfloor + \left\lfloor \frac{8y+r}{4} \right\rfloor - 3y + \text{ord}_2(c_r) \\ &= 11y + r + \text{ord}_2(r!) + \text{ord}_2(c_r), \end{aligned}$$

that is, the values of $\text{ord}_2(b_{8y+r}(8)) - 11y - r$, $0 \leq r \leq 7$, are the following :

r	0	1	2	3	4	5	6	7
$\text{ord}_2(b_{8y+r}(8)) - 11y - r$	0	0	1	2	4	3	8	4

Proof. If $r \neq 6$, then the theorem follows from Lemma 2.1 and Corollary 7.3. By Lemma 2.1 and Theorem 7.2, there exists a 2-adic analytic function $g_6^0(X) \in \mathbb{Z}_2\langle X \rangle$ such that

$$g_6^0(y) = \frac{b_{8y+6}(8)}{(8y+6)!} \left(-\frac{1}{16} \right)^y y!$$

for any nonnegative integer y and

$$g_6^0(X) \in 2^6 \cdot \frac{16}{45}(1 - 16X(X - 1)) - 2^{10} \cdot \frac{2172172}{42567525}X + 2^{13}X\mathbb{Z}_2\langle X \rangle.$$

Hence Lemma 3.1 implies that $\text{ord}_2(b_{8y+6}(8)) = 11y + 4 + \text{ord}_2(g_6^0(y)) = 11y + 14$ for any nonnegative integer y . This completes the proof. $\square$

8 2-adic properties of $q_n(2^u)$

The following lemma is an immediate consequence of Eq. (9) and Theorem 7.2.

Lemma 8.1 *Suppose that $p = 2$. Let r be a nonnegative integer less than 2^u , and let y be a nonnegative integer. Then there exists a 2-adic analytic function $M_{r,y}(X) \in \mathbb{Z}_2\langle X \rangle$ such that*

$$q_{2^u y + r}(2^u) = \frac{(2^u y + r)!}{2^{uy} \cdot y!} \cdot 2^{(2^u - 1)y} \cdot L_{r,y}(y)$$

with

$$L_{r,y}(X) = (-1)^y 2^r c_r^0 \frac{1 - 2^{u+1}X(X - 1)}{2} + (-2)^r c_r^1 \frac{1 + 2^{u+1}X(X - 1)}{2} \\ + 2^{u+r}(-(-1)^y c_{2^u+r}^0 + (-1)^r c_{2^u+r}^1)X + 2^{2u+r}XM_{r,y}(X).$$

Moreover, $\text{ord}_2(q_{2^u y + r}(2^u)) = (2^{u+1} - u - 2)y + \text{ord}_2(r!) + \text{ord}_2(L_{r,y}(y))$.

Example 8.2 Suppose that $p = 2$ and $u = 1$. Let r be a nonnegative integer less than 2, and let y be a nonnegative integer. By Lemma 2.1 and Lemma 8.1, we have

$$\text{ord}_2(q_{2y+r}(2)) = y + \left\lfloor \frac{r+1}{2} \right\rfloor + \chi_o(y) = \begin{cases} y & \text{if } y \text{ is even and if } r = 0, \\ y + 1 & \text{if } y \text{ is even and if } r = 1, \\ y + 1 & \text{if } y \text{ is odd and if } r = 0, \\ y + 2 & \text{if } y \text{ is odd and if } r = 1. \end{cases}$$

We conclude this paper with the following three results for $\text{ord}_2(q_n(2^u))$.

Theorem 8.3 *Suppose that $p = 2$ and $u = 2$. Then the following statements hold for any nonnegative integer y .*

- (a) $\text{ord}_2(q_{4y}(4)) = 4y + 2\chi_o(y)$, $\text{ord}_2(q_{4y+2}(4)) = 4y + 2$, $\text{ord}_2(q_{4y+3}(4)) = 4y + 3$.
- (b) *There exists a 2-adic integer β such that*

$$\text{ord}_2(q_{4y+1}(4)) = 4y + 1 + \chi_o(y) \cdot (\text{ord}_2(y + \beta) + 3).$$

Proof. Keep the notation of Lemma 8.1 with $u = 2$. Set $h_{r,y} = \text{ord}_2(L_{r,y}(y))$. Then by Lemma 2.1,

$$h_{0,y}(y) = 0, \quad h_{1,y} = h_{2,y} = 1, \quad h_{3,y} = 2$$

if y is even, and

$$h_{0,y} = 2, \quad L_{1,y}(y) \equiv 16y \left(y - \frac{13}{15} \right) \pmod{32}, \quad h_{2,y} = 1, \quad h_{3,y} = 2$$

if y is odd. Thus the statement (a) follows from Lemma 8.1. The proof of the statement (b) is analogous to that of Theorem 5.1(b), while the assertion is a special case of Theorem 8.7. This completes the proof. $\square$

Remark 8.4 According to Mathematica,

$$\beta \equiv 1 + 2^2 + 2^3 + 2^4 + 2^6 + 2^7 + 2^8 \pmod{2^{13}}.$$

Theorem 8.5 Suppose that $p = 2$ and $u = 3$. Then the following statements hold for any nonnegative integer y .

$$\begin{aligned} \text{(a)} \quad & \text{ord}_2(q_{8y+2}(8)) = 11y + 2, \text{ord}_2(q_{8y+3}(8)) = 11y + 3, \text{ord}_2(q_{8y+4}(8)) = 11y + 6, \\ & \text{ord}_2(q_{8y+5}(8)) = 11y + 8 + \chi_e(y), \text{ord}_2(q_{8y+6}(8)) = 11y + 9, \\ & \text{ord}_2(q_{8y+7}(8)) = 11y + 11 + \chi_e(y). \end{aligned}$$

(b) If $r = 0$ or $r = 1$, then there exists a 2-adic integer β_r such that

$$\text{ord}_2(q_{8y+r}(8)) = 11y + r + \chi_o(y) \cdot (\text{ord}_2(y + \beta_r) + 4).$$

Proof. Keep the notation of Lemma 8.1 with $u = 3$. Set $h_{r,y} = \text{ord}_2(L_{r,y}(y))$. Then by Lemma 2.1,

$$h_{0,y} = 0, \quad h_{1,y} = h_{2,y} = 1, \quad h_{3,y} = 2, \quad h_{4,y} = 3, \quad h_{5,y} = 6, \quad h_{6,y} = 5, \quad h_{7,y} = 8$$

if y is even, and

$$\begin{aligned} L_{0,y}(y) &\equiv 16y \left(y - \frac{283}{315} \right) \pmod{64}, \quad L_{1,y}(y) \equiv 32y \left(y - \frac{2677}{2835} \right) \pmod{128}, \\ h_{2,y} &= 1, \quad h_{3,y} = 2, \quad h_{4,y} = 3, \quad h_{5,y} = h_{6,y} = 5, \quad h_{7,y} = 7 \end{aligned}$$

if y is odd. Thus the statement (a) follows from Lemma 8.1. The proof of the statement (b) is analogous to that of Theorem 5.1(b), while the assertion is a special case of Theorem 8.7. This completes the proof. $\square$

Remark 8.6 According to Mathematica,

$$\beta_0 \equiv 1 + 2 + 2^2 + 2^3 + 2^4 + 2^6 + 2^8 + 2^9 \pmod{2^{12}}$$

and

$$\beta_1 \equiv 1 + 2^3 + 2^4 + 2^5 + 2^6 + 2^8 + 2^{10} + 2^{11} + 2^{12} \pmod{2^{14}}.$$

The statement (b) both of Theorems 8.3 and 8.5 is extended to a result for $\text{ord}_2(q_{2^u y+r}(2^u))$ with $u \geq 4$ and $r = 0$ or $r = 1$.

Theorem 8.7 *Suppose that $p = 2$ and $u \geq 2$. Let y be a nonnegative integer. If $r = 0$ or $r = 1$, then there exists a 2-adic integer β_r such that*

$$\text{ord}_2(q_{2^u y+r}(2^u)) = (2^{u+1} - u - 2)y + r + \chi_o(y) \cdot (\text{ord}_2(y + \beta_r) + u + 1).$$

Moreover, if $\text{ord}_2(c_{2^u+r}^0 + (-1)^r c_{2^u+r}^1) = 0$ with $r = 0$ or $r = 1$, then

$$\text{ord}_2(q_{2^u y+r}(2^u)) = (2^{u+1} - u - 2)y + r + \chi_o(y) \cdot u.$$

Proof. Keep the notation of Lemma 8.1. Since $c_0^0 = c_0^1 = c_1^0 = 1$ and $c_1^1 = -1$ by Lemma 2.1, it follows from Lemma 8.1 that the assertion holds if y is even. Assume that y is odd. Then

$$\begin{aligned} L_{0,y}(X) &= 2^{u+1}X(X-1) + 2^u(c_{2^u}^0 + c_{2^u}^1)X + 2^{2u}XM_{0,y}(X), \\ L_{1,y}(X) &= 2^{u+2}X(X-1) + 2^{u+1}(c_{2^u+1}^0 - c_{2^u+1}^1)X + 2^{2u+1}XM_{1,y}(X). \end{aligned}$$

Hence, if $\text{ord}_2(c_{2^u+r}^0 + (-1)^r c_{2^u+r}^1) = 0$, then the assertion follows from Lemma 8.1. Suppose that $\text{ord}_2(c_{2^u+r}^0 + (-1)^r c_{2^u+r}^1) > 0$. Then by an argument analogous to that in the proof of Theorem 5.6, we have

$$\text{ord}_2(L_{r,y}(y)) = r + \text{ord}_2(y + \beta_r) + u + 1$$

for some $\beta_r \in \mathbb{Z}_2$. Hence the assertion follows from Lemma 8.1. This completes the proof. $\square$

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REFERENCES

1. N. Chigira, The solutions of $x^d = 1$ in finite groups, J. Algebra **180** (1996), 653–661.
2. S. Chowla, I. N. Herstein, and W. K. Moore, On recursions connected with symmetric groups I, Canad. J. Math. **3** (1951), 328–334.
3. S. Chowla, I. N. Herstein, and W. R. Scott, The solutions of $x^d = 1$ in symmetric groups, Norske Vid. Selsk. Forh. (Trondheim) **25** (1952), 29–31.
4. K. Conrad, p -adic properties of truncated Artin-Hasse coefficients, 1998, preprint.

5. J. Dieudonné, On the Artin-Hasse exponential series, *Proc. Amer. Math. Soc.* **8** (1957), 210–214.
6. A. Dress and T. Yoshida, On p -divisibility of the Frobenius numbers of symmetric groups, 1991, unpublished note.
7. B. Dwork, A note on the p -adic gamma function, *Groupe d'étude d'Analyse ultramétrique*, 9e année, 1981/82, fasc. 3, n° J5, 10 pp., Inst. Henri Poincaré, Paris, 1983.
8. F. Q. Gouvêa, p -adic Numbers, 2nd ed., Universitext, Springer-Verlag, New York, 1997.
9. M. Grady and M. Newman, Residue periodicity in subgroup counting functions; in :“The Rademacher Legacy to Mathematics,” *Contemp. Math.* **166** (1994), 265–273.
10. M. Hall Jr., *The Theory of Groups*, 2nd edition, Chelsea, New York, 1976.
11. H. Ishihara, H. Ochiai, Y. Takegahara, and T. Yoshida, p -divisibility of the number of solutions of $x^p = 1$ in a symmetric group, *Ann. Comb.* **5** (2001), 197–210.
12. G. D. James and A. Kerber, *The Representation Theory of the Symmetric Group*, *Encyclopedia of mathematics and its applications*, Vol. 16, Addison-Wesley, Reading, MA, 1981.
13. H. Katsurada, Y. Takegahara, and T. Yoshida, The number of homomorphisms from a finite abelian group to a symmetric group, *Comm. Algebra* **28** (2000), 2271–2290.
14. D. Kim and J. S. Kim, A combinatorial approach to the power of 2 in the number of involutions, *J. Combin. Theory Ser. A* **117** (2010), 1082–1094.
15. N. Koblitz, p -adic Numbers, p -adic Analysis, and Zeta-Functions, 2nd ed., Springer-Verlag, New York, 1984.
16. H. Ochiai, A p -adic property of the Taylor series of $\exp(x + x^p/p)$, *Hokkaido Math. J.* **28** (1999), 71–85.
17. S. Okada, Wreath products by the symmetric groups and product posets of Young's lattices, *J. Combin. Theory Ser. A* **55** (1990), 14–32.
18. J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.
19. W. H. Schikhof, *Ultrametric Calculus*, Cambridge University Press, Cambridge, 1984.
20. T. Yoshida, Private discussions.