Mean-Position

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Abstract

Lagrange's θ in the remainder's expression in Taylor's expansion $\frac{1}{\lfloor n}f^{(n)}(x+\theta h)h^n$ has been discussed especially accurately by Rothe, but when we emphasize the sense of "position" for $x, x+\theta h, x+h$, then are found some important new aspects for these calculi.

The famous classic theorem on the remainder's form in TAYLOR's expansion which LAGRANGE left behind, has been applied in various ways and in various fields. I, too, owe very much to the theorem and ROTHE's what I am going to state in this paper. But I mean to show here the importance of "position" and that is the reason why the title "Mean-Position" should have been chosen for this paper.

1. Mean-Position. When the difference of two values of a function $f(x)$ at two different points $x, x+h$ is written as

$$f(x+h)-f(x) = h^n f^{(n)}(x+\theta h)\cdot\frac{1}{\lfloor n} \tag{1, 1}$$

where $0<\theta<1$ and $f^{(n)}(x) \equiv \dfrac{d^n f(x)}{dx^n}$, it makes a special case of LAGRANGE's formula for

$$f'(x) = f''(x) = \cdots = f^{(n-1)}(x) = 0. \tag{1, 2}$$

In this case the position $\qquad x+\theta h \qquad$ (1, 3)

may be called "mean-position of n-th order".

Firstly let us take up the case $n=1$, when we can write down

$$f(x+h)-f(x) = hf'(x+\theta h) \qquad (0<\theta<1)$$

$$= hf'(x)+h^{n+1}\frac{f^{(n+1)}(x+\theta_1 h)}{\lfloor n+1} \qquad (0<\theta_1<1)$$

$$f'(x+\theta h) = f'(x)+(\theta h)^n\frac{f^{(n+1)}(x+\theta_2 h)}{\lfloor n} \qquad (0<\theta_2<1)$$

if $f''(x) = f'''(x) = \cdots = f^{(n-1)}(x) = 0$, $f^{(n)}(x) \neq 0$, and $f^{(n+1)}(x)$ is continuous at x. Hence

$$\frac{1}{n+1}\;\frac{f^{(n+1)}(x+\theta_1 h)}{f^{(n+1)}(x+\theta_2 h)} = \theta^n$$

, therefore $\lim_{h=0} \theta(x, h) = \sqrt[n]{1/(n+1)}$ (1, 4)

This has been firstly shown by ROTHE[1], besides which the following is thinkable. i. e.: For $\phi(n) \equiv \sqrt[n]{1/(n+1)}$

$$\phi(1)(=1/2)<\phi(2)<\cdots \quad <\phi(n)<\phi(n+1)< \quad \cdots<1 \qquad (1, 5)$$

and $\lim_{n=\infty} \phi(n) = 1,$

This result can be made applicable in the following experiment such as: Supposing a potential by $f(x)$, we understand by $f'(x)$ the force at the position x and by

$$\frac{f(x+h)-f(x)}{h} \ (=f'(x+\theta h))$$

the mean-quotient of the potential slope between x and $x+h$, when *the mean-position $x+\theta h$ is inclined not to get near to x beyond the middle-point of x and $x+h$, if the force itself is not invariant in the neighborhood of x.*

2. θ of n-th order. The θ shown in (1, 1) has been also discussed accurately by ROTHE[2], who has proved that

$$\lim \theta(x,h) = \frac{1}{n+1}, \text{ if } \theta \text{ is of } n\text{-th order} . \qquad (2, 1)$$

This result is effected under the condition

$$f^{(n+1)}(x) \neq 0 , \qquad (2, 2)$$

when it runs

$$f(x+h)-f(x) = f'(x)h+f''(x)\frac{h^2}{\lfloor 2} + \cdots + f^{(n-1)}(x) \cdot \frac{h^{n-1}}{\lfloor n-1}$$
$$+f^{(n)}(x+\theta h)\cdot \frac{h^n}{\lfloor n} . \qquad (2, 3)$$

But, if we include the restriction (1, 2) (say: $f'(x) = \cdots = f^{(n-1)}(x) = 0$) we can use the relation

$$f(x+h)-f(x) = \int_0^h dh_1 \int_0^{h_1} dh_2 \cdots \int_0^{h_{n-1}} f^{(n)}(x+h_n)dh_n \equiv \int_0^y f^{(n)}(x+\xi)d^n\xi .$$

therefore if an arbitrary function is taken up which satisfies the condition that $\phi'(0) \neq 0, \quad \phi(0) = 0$ (2, 4)
and $\phi'(x)$ is continuous in $|x| < \delta$ (δ: a real positive number) it is effected that

1) ROTHE: Tohoku Math. Journ., 29 (1928), p. 145; also TAKASU: Biseki. Shingi, I (1930), p. 176.
2) ROTHE: Math. Zeitschr., Bd. 9 (1921), S. 309; also TAKASU: Biseki. Shingi, I, p. 173.

$$\phi(\theta h) = \frac{\int_0^h \phi(\xi) d^n \xi}{\int_0^h d^n \xi} \qquad (0 < \theta < 1) \tag{2, 5}$$

and $\lim_{h=0} \theta(h) = \dfrac{1}{n+1}$

3. **Supplementary Remarks.** It is well-known that if $\phi(x)$ is bounded and continuous in $0<x<h$

$$\int_0^h \phi(x) d^n x = \frac{1}{\lfloor n-1} \int_0^h (h-t)^{n-1} \phi(t) dt \text{ [1]} . \tag{3, 1}$$

Still more it may be worth mention that if we put the summations

$$S_1 = \lim_{N=\infty} e \sum_{\nu=0}^{N} \phi(\nu e)$$

$$S_2 = \lim_{N=\infty} e^2 \sum_{\nu=0}^{N} \sum_{\lambda=0}^{\nu} \phi(\lambda e)$$

$$S_3 = \lim_{N=\infty} e^3 \sum_{\nu=0}^{N} \sum_{\lambda=0}^{\nu} \sum_{\kappa=0}^{\lambda} \phi(\kappa e)$$

$$\cdots$$

$$\text{etc.} \; ; \; Ne = h \; ,$$

then
$$S_n = \frac{1}{\lfloor n-1} \int_0^h (h-t)^{n-1} \phi(t) dt \; , \tag{3, 2}$$

For the case $f(0)=0$, $f''(0) \neq 0$ we have known that

$$f(x) = f'(\theta x) x \qquad (0<\theta<1)$$

$$\lim_{x=0} \theta(x) = 1/2 \; ,$$

but we should not suppose here

$$\frac{f(x)}{x} = \frac{f(x/2)}{x/2} + \langle x^2 \rangle$$

because: $\dfrac{f(x)}{x} - \dfrac{f(\theta x)}{\theta x} = \dfrac{1}{\lfloor 2} f''(0)(1-\theta)x + \dfrac{1}{\lfloor 3} f'''(0)(1-\theta^2)x^2 + \cdots$

therefore if we set $\quad 0 = \dfrac{1}{2} f''(0)(1-\theta)x + \langle x^2 \rangle$

then $\quad \lim_{x=0} \theta(x) = 1 \neq 1/2$

inevitably.

On $\theta(x,h)$ for $h \to \infty$, we can reach some interesting facts by giving a few restrictions for $f(x)$.

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1) KOWALEWSKI: Integralgleichungen (1930), S. 34.