A Constructive Study of the Functions at the Points of Discontinuity in the Theory of Stieltjes Integration

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Abstract

When we test some limiting deformation to fit for the expression

$$f(\xi)\,dG(\xi)=f(\xi)\{G(\xi_+)-G(\xi_-)\}$$

there emerges a constructive problem between the functions $f_1(x)$ and $G_1(x)$ which are taken in this process of deformation. Two important constructions are defined and discussed in this paper.

1. Introduction

When there is given a function of a variable of bounded variation $G(x)$, we may define a function of a set $\tilde{G}(e)$ as follows:

(i) $\tilde{G}(e)=G(b_+)-G(a_-)$ for $e=[a, b]$;

(ii) $\tilde{G}(e)=G(b_-)-G(a_+)$ for $e=(a, b)$;

(iii) $\tilde{G}(e)=G(b_+)-G(a_+)$ for $e=(a, b]$;

(iv) $\tilde{G}(e)=G(b_-)-G(a_-)$ for $e=[a, b)$;

where] and [mean closed;) and (mean open; $a_+\equiv a+0$ and $a_-\equiv a-0$; and basing on these four definitions make a system of additive values $\tilde{G}(e)$ corresponding to any Borel sets. This system is no other than the system which bases on the following two:

(i) $\tilde{G}(e)=G(b_-)-G(a_+)$ for $e=(a, b)$;

(ii) $\tilde{G}(P_t)=G(t_+)-G(t_-)$, P_t denoting the point $x=t$;

because we have then, for $e=[a, b]$

$$\begin{aligned}\tilde{G}(e)&=G(b_-)-G(a_+)+G(P_a)+G(P_b)\\&=G(b_-)-G(a_+)+G(a_+)-G(a_-)+G(b_+)-G(b_-)\\&=G(b_+)-G(a_-)\end{aligned}$$

and similarly (iii) and (iv) are implied. It is evident that no other system can be defined than that of $\tilde{G}(e)$ above shown when we demand:

$$\tilde{G}(e)=\overline{W}(G)+\underline{W}(G),$$

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$\overline{W}(G)$ and $\underline{W}(G)$ being the upper and lower variations of $G(x)$ over an open set e respectively.

At the points of discontinuity the Stieltjes integration thus gets the convenient representation by the above modification:

$$f(x)\,dG = f(x)\{G(x_+)-G(x_-)\}. \tag{1, 1}$$

If we conform to the rule (1, 1) and compute any integral of Stieltjes type, there may be left no question by itself, but when we try to test some constructive interpretation on the formula (1, 1) we come across a special problem.

Let us suppose any pair of curves γ_δ and Γ_δ of which γ_δ passes through the three points $\{\xi-\delta, f(\xi_-)\}$, $\{\xi, f(\xi)\}$, $\{\xi+\delta, f(\xi_+)\}$ and Γ_δ passes through the three points $\{\xi-\delta, G(\xi_-)\}$, $\{\xi, G(\xi)\}$, $\{\xi+\delta, G(\xi_+)\}$ $(\delta>0)$, and define tow functions $f_1(x, \delta)$ and $G_1(x, \delta)$ by the curves γ_δ and Γ_δ respectivlye, then we may have a Stieltjes integral:

$$J_\delta = \int_{\xi-\delta}^{\xi+\delta} f_1(x, \delta)\,dG_1(x, \delta).$$

After positing so, let us define the relations between $f_1(x, 1)$ and $f_1(x, \delta)$ and between $G_1(x, 1)$ and $G_1(x, \delta)$ by

$$f_1(x, \delta) = f_1(x_1, 1),$$

and $$G_1(x, \delta) = G_1(x_1, 1);$$

where $$x_1 = \frac{1}{\delta}(x-\xi)+\xi \quad \text{i. e.} \quad x-\xi=(x_1-\xi)\,\delta.$$

Then it can be easily seen that

$$J_1 = J_\delta = \lim_{\delta=0} J_\delta \tag{1, 2}$$

whenever the limit exists.

If we demand the construction

$$\lim_{\delta=0} J_\delta = f(\xi)\{G(\xi_+)-G(\xi_-)\}, \tag{1, 3}$$

there will arise the problem to determine the relevant functions f_1 and G_1 satisfying (1, 3), or to determine the relevant curves γ_1 and Γ_1 to give the functions f_1 and G_1 satisfying (1, 3). Being informed by (1, 2), we can use the formula:

$$J_1 = f(\xi)\{G(\xi_+)-G(\xi_-)\}$$

in place of (1, 3). In this paper we consider the continuous curves γ_1 and Γ_1 as differentiable except the finite number of points. Let us write $f_1(x)$ and $G_1(x)$ in lieu of $f_1(x, 1)$ and $G_1(x, 1)$.

2. Linear Construction.

This construction is defined by

$$f_1(x)=t\{f(\xi)-f(\xi_-)\}+f(\xi) \quad \text{and}$$
$$G_1(x)=t\{G(\xi)-G(\xi_-)\}+G(\xi) \quad \text{in} \quad \xi-1\leqslant x=\xi+t\leqslant\xi \ ;$$
$$f_1(x)=t\{f(\xi_+)-f(\xi)\}+f(\xi) \quad \text{and}$$
$$G_1(x)=t\{G(\xi_+)-G(\xi)\}+G(\xi) \quad \text{in} \quad \xi\leqslant x=\xi+t\leqslant\xi+1 \ ;$$

and
$$f(\xi)\{G(\xi_+)-G(\xi_-)\}=J_1=\int_{\xi-1}^{\xi+1} f_1(x)\,dG_1$$
$$=\int_{\xi-1}^{\xi+1} f_1(x)\,G_1'(x)\,dx$$

where $G_1'(x)\equiv dG_1(x)/dx$. In this case, since

$$J_1=\int_{-1}^{0}[t\{f(\xi)-f(\xi_-)\}+f(\xi)]\{G(\xi)-G(\xi_-)\}\,dt$$
$$+\int_0^1[t\{f(\xi_+)-f(\xi)\}+f(\xi)]\{G(\xi_+)-G(\xi)\}\,dt$$
$$=\frac{1}{2}\{f(\xi_-)+f(\xi)\}\{G(\xi)-G(\xi_-)\}$$
$$+\frac{1}{2}\{f(\xi)+f(\xi_+)\}\{G(\xi_+)-G(\xi)\}$$
$$=\frac{1}{2}[f(\xi_-)\{G(\xi)-G(\xi_-)\}+f(\xi)\{G(\xi_+)-G(\xi_-)\}$$
$$+f(\xi_+)\{G(\xi_+)-G(\xi)\}],$$

it must be

$$f(\xi)\{G(\xi_+)-G(\xi_-)\}=f(\xi_-)\{G(\xi)-G(\xi_-)\}+f(\xi_+)\{G(\xi_+)-G(\xi)\}. \tag{2, 1}$$

Therefore, when $G(\xi_+)\neq G(\xi_-)$ we have

$$f(\xi)=\lambda f(\xi_-)+\mu f(\xi_+) \quad ; \quad \lambda+\mu=1.$$

Especially, it will be notable that it is sufficient for this case:

$$f(\xi)=\frac{1}{2}\{f(\xi_-)+f(\xi_+)\}, \qquad G(\xi)=\frac{1}{2}\{G(\xi_-)+G(\xi_+)\}.$$

This means both f and G are *regular* at the point $x=\xi$.

(2, 1) shows the special possibility of λ, μ, so there arises the question *for the value given by an arbitrary pair of* λ, μ $(\lambda+\mu=1)$: *Does there exist any construction* (γ_1, Γ_1) *or not?*

3. Quasi-linear Construction.

This construction is defined by

$$
\begin{aligned}
f_1(x) &= \frac{1+t}{1-\alpha}(k-a)+a && \text{in} \quad -1 \leqslant t \leqslant -\alpha \leqslant 0, \ (t=x-\xi) \ ; \\
&= \frac{t-1}{\beta-1}(k-b)+b && \text{in} \quad 0 < \beta \leqslant t \leqslant 1 \ ; \\
&= k && \text{in} \quad -\alpha \leqslant t \leqslant \beta \ ; \\
G_1(x) &= t(K-A)+K && \text{in} \quad -1 \leqslant t \leqslant 0 \ ; \\
&= t(B-K)+K && \text{in} \quad 0 \leqslant t \leqslant 1 \ ;
\end{aligned}
$$

$$k=f(\xi), \quad a=f(\xi_-), \quad b=f(\xi_+) \quad ; \quad K=G(\xi), \quad A=G(\xi_-), \quad B=G(\xi_+).$$

In this case, since

$$
\begin{aligned}
J_1 = \int_{\xi-1}^{\xi+1} f_1(x)\,dG_1 &= \int_{-1}^{-\alpha} \left\{ \frac{k-a}{1-\alpha}(1+t)+a \right\} (K-A)\,dt \\
&+ \int_{\beta}^{1} \left\{ \frac{k-b}{\beta-1}(t-1)+b \right\} (B-K)\,dt + \int_{-\alpha}^{0} k(K-A)\,dt + \int_{0}^{\beta} k(B-K)\,dt \\
&= \left\{ \frac{k-a}{1-\alpha}\frac{1}{2}(1-\alpha)^2 + a(1-\alpha) \right\}(K-A) + \left\{ \frac{k-b}{\beta-1}\left(\frac{-1}{2}\right)(\beta-1)^2 + b(1-\beta) \right\}(B-K) \\
&\qquad + k\{(K-A)\alpha + (B-K)\beta\} \\
&= \left\{ \frac{1}{2}(K-A)(1-\alpha) + \frac{1}{2}(B-K)(1-\beta) + (K-A)\alpha + (B-K)\beta \right\} k \\
&\qquad + \frac{1}{2}(K-A)(1-\alpha)\,a + \frac{1}{2}(B-K)(1-\beta)\,b,
\end{aligned}
$$

to be $J_1=f(\xi)\{G(\xi_+)-G(\xi_-)\}=(B-A)\,k$ it must be

$$\frac{(K-A)\ (1-\alpha)}{(B-K)\ (1-\beta)} = \frac{\lambda}{\mu} \ . \tag{3, 1}$$

Since $\dfrac{1-\alpha}{1-\beta} > 0$ we must have $\dfrac{K-A}{B-K} \cdot \dfrac{\lambda}{\mu} > 0$

i. e.

$$\frac{G(\xi)-G(\xi_-)}{G(\xi_+)-G(\xi)} \cdot \frac{\lambda}{\mu} > 0 \tag{3, 2}$$

when $\lambda\mu \neq 0$ and $G(\xi_+) \neq G(\xi)$. On this condition, evidently (3, 1) is solved for α and β.

If $B=K$, we have

$$(B-A)(1-\alpha)k=(K-A)(1-\alpha)a \ , \ [1-\alpha \neq 0]$$

and
$$B-A=K-A,$$

so that it must be $\lambda=1$ and $\mu=0$. Similarly, when $K=A$ it must be $\lambda=0$ and $\mu=1$.

Consequently, we see that *there exists a quasi-linear construction when* (3, 2) *is satisfied and* $\lambda\mu\neq 0$ *for the given* $f(\xi)=\lambda f(\xi_-)+\mu f(\xi_+)$ $(\lambda+\mu=1)$. When the condition (3, 2) does not hold, if we change the value of $G(\xi)$ to fit (3, 2) for the given pair of λ, μ we may find the value $0<\alpha$, $\beta<1$ to suffice a quasi-linear construction on condition that $\lambda\mu\neq 0$ and $G(\xi_-)\neq G(\xi_+)$.

When $a-b\neq 0$ we may find the expression

$$k=\lambda a+\mu b \quad (\lambda+\mu=1)$$

is possible for any given value of k, because it is solved by

$$\lambda=\frac{k-b}{a-b} \ , \quad \mu=1-\lambda.$$

This being so, we will find *all the cases of* $f(\xi)$ *are involved in the form* $f(\xi)=\lambda f(\xi_-)+\mu f(\xi_+)$ *except when* $f(\xi_-)=f(\xi_+)$. For this exceptive case we have, in the quasi-linear construction,

$$\begin{aligned}&[\{G(\xi)-G(\xi_-)\}(1-\alpha)+\{G(\xi_+)-G(\xi)\}(1-\beta)]f(\xi)\\&\qquad=[\{G(\xi)-G(\xi_-)\}(1-\alpha)+\{G(\xi_+)-G(\xi)\}(1-\beta)]f(\xi_-),\end{aligned}$$

so that we may have a quasi-linear construction for any value of $f(\xi)$

when
$$\{G(\xi)-G(\xi_-)\}\{G(\xi)-G(\xi_+)\}>0 \qquad (3,\ 3)$$

But it must be
$$f(\xi)=f(\xi_-)=f(\xi_+)$$

if (3, 3) does not hold.

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