Fundamental Viewpoints in the Theory of A Priori Measure

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	作成者: 紀國谷, 芳雄
	メールアドレス:
	所属:
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Fundamental Viewpoints in the Theory of A Priori Measure

Yoshio Kinokuniya*

Abstract

The present author has decided to establish his theory of *a priori measure* basing on four principal hypotheses. The important characteristics will be observed in the assertion of null measure for any set, the power of which is really less than that of continuum and the complete exclusion of non-measurable sets. Some remarks on a study of the occupation of a point are made in supplement.

1. Introduction. In several previous memoirs, in introducing a measure of point called *point-dimension*, so as to define a measure of a set of points called *a priori measure*, I have intended to study the relative structure between the theory of sets and the theory of integral. Recently I had the good fortune to find some important conditions to make the set-theorogical aspect very simple, so that in this paper remarks may be made about a new system of hypotheses, establishing the foundation of our theory of a priori measure and giving a new light on the theory of sets.

We restrict our investigations within the Euclidian space of finite dimension. As for the set of real numbers, the points $\mathbf{P}_x$ of which the abscissa is x, is supposed to possess an infinitesimal space called the *occupation* of $\mathbf{P}_x$

$$(x-0,\ x+0)=((x)) \tag{1,1}$$

and the point-dimension of $\mathbf{P}_x$

$$\mu_x$$

is considered as the measure of $((x))$; i.e. we posit them in the relation

$$\mu_x=\widetilde{\mathrm{m}}((x)). \tag{1,2}$$

With $\widetilde{\mathrm{m}}$ we will indicate an a priori measure; as has been stated several times in the previous memoirs, measure of a set is given by the formula

$$\widetilde{\mathrm{m}}(\mathbf{M})=\underset{\mathbf{P}\in\mathbf{M}}{\mathfrak{S}}\ \mu_P$$

μ_P being the point-dimension of the point P.

* 紀国谷芳雄

When the point-dimensions are uniformly equal for each point of the space, it is said they make a *normal system* of point-dimension. When a space $\mathbf{E}_1$ is put in biunivoquely continuous correspondence with the space $\mathbf{E}$ for which a normal system of point-dimension μ is given, and if the relation

$$\mu_1(\mathbf{P}_1)=\lambda(\mathbf{P})\mu(\mathbf{P})$$
$$0<\lambda(\mathbf{P})<\infty \text{ [1]}$$

(μ_1 designates the transformed point-dimension in the space $\mathbf{E}_1$ by this correspondence) is satisfied for each point $\mathbf{P}$ in $\mathbf{E}$, it is said that μ_1 makes a *regular system* in $\mathbf{E}_1$; in other words, the occupation of $\mathbf{P}_1(\mathbf{P})$(the corresponding point in $\mathbf{E}_1$ to the point $\mathbf{P}$ in $\mathbf{E}$) is changed in its size by the measure proportion $\lambda(\mathbf{P})$ to be compared with the original occupation of $\mathbf{P}$. These are the facts I stated already in the previous memoirs of mine; in this paper some structural proprieties of an occupation of point shall be investigated, too.

2. **Fundamental System of Hypotheses.** The following system of hypotheses gives many conveniences, if we adopt it to provide for the sets considered; so, I have decided to take it as the fundamental base to establish the theory of a priori measure. It consists of four hypotheses devided into two groups The sets are taken in a finite-dimensional Euclidian space.

I. UNDER A NORMAL SYSTEM OF POINT-DIMENSION

I, 1) *A set* $\mathbf{M}$ *is measurable a priori with respect to a normal system* μ *when and only when the measure is given by the formula:*

$$\widetilde{\mathrm{m}}(\mathbf{M})=\mathfrak{n}(\mathbf{M})\mu.$$

$\mathfrak{n}(\mathbf{M})$ *is the inversion number of* $\mathbf{M}$, which has been defined to indicate the number of the points contained in $\mathbf{M}$. *When* $\mathbf{M}$ *is* $\mathfrak{L}$*-measurable* $\mathbf{M}$ *is measurable a priori too and* $\widetilde{\mathrm{m}}(\mathbf{M})$ *is equal to the* $\mathfrak{L}$*-measure of* $\mathbf{M}$.

I, 2) *On denoting with* ψ_i *the cardinal of the infinite set* $\mathbf{M}_i$ $(i=1,2)$, *if*

$$\psi_1<\psi_2$$

it is destined that

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1 $\lambda(P)$ may not be necessarily continuous of P in spite of continuity of the correspondence between the points in $\mathbf{E}$ and $\mathbf{E}_1$

$$\tilde{m}(\mathbf{M}_1)/\tilde{m}(\mathbf{M}_2)=0$$

with respect to a normal system, whether $\mathbf{M}_i$ *is measurable a priori or not.*

II. FOR THE GENERAL CASE OF DIMENSION SYSTEM

II, 1) *If both of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be measurable a priori, the sets*

$$\mathbf{M}_1 \pm \mathbf{M}_2$$

are measurable a priori, too.

II, 2) *If for any set measurable a priori* $\mathbf{F}$ *contained in the given set* $\mathbf{M}$, *the relation*

$$\tilde{m}(\mathbf{F})=0$$

be observed, then it must be that the set $\mathbf{M}$ *is a priori measurable and*

$$\tilde{m}(\mathbf{M})=0.$$

As a measure we mean a non-negative value for any case, so that it may be direct from I, 1) that $\tilde{m}(\mathbf{M})$ is an additive function of a set; i. e. when $\mathbf{M}_1$ and $\mathbf{M}_2$ are a priori measurable and $\mathbf{M}_1 \cap \mathbf{M}_2=0$, we have $\tilde{m}(\mathbf{M}_1+\mathbf{M}_2)=\tilde{m}(\mathbf{M}_1)+\tilde{m}(\mathbf{M}_2)$. I, 2) may be induced from I, 1), but I put it up here in regard to its importance. When $\lim\limits_{k} \tilde{m}(\mathbf{M}_k)=u$ is found on a certain structure, by which the elements of $\lim \mathbf{M}_k=\mathbf{M}$ are distinguished within the limit of enumerability, we will say that *the inversion number* $n(\mathbf{M})$ *is determined* and $\mathbf{M}$ *is measurable a priori.* Then the following results are directly obtained.

When the sets $\mathbf{M}_1 \supset \mathbf{M}_2 \supset \cdots \supset \mathbf{M}_k \supset \mathbf{M}_{k+1} \supset \cdots$ are all measurable a priori, it is easily seen that the product of them

$$\mathbf{M}=\prod_{k=1}^{\infty} \mathbf{M}_k$$

is measurable a priori too and

$$\tilde{m}(\mathbf{M})=\lim_{k\to\infty} \tilde{m}(\mathbf{M}_k).$$

As for the sets $\mathbf{M}_1 \subset \mathbf{M}_2 \subset \cdots \subset \mathbf{M}_k \subset \mathbf{M}_{k+1} \subset \cdots$ measurable apriori, if $\tilde{m}(\mathbf{M}_k)$ are uniformly bounded above, it is proved that the reunion

$$\mathbf{M}=\sum_{k=1}^{\infty}\mathbf{M}_k$$

is measurable and

$$\widetilde{\mathrm{m}}(\mathbf{M})=\lim_{k\to\infty}\widetilde{\mathrm{m}}(\mathbf{M}_k).$$

To induce II, 1) from the standpoint of I, 1) will be impossible with no auxiliary assumptions.

II, 2) is important specially in point that it leads us to the exclusion of non-measurable set. Under a normal system of point-dimension, it is remarkable that if the condition of II, 2) be satisfied, for any sequence of inversion numbers

$$\mathfrak{n}_1\leqq\mathfrak{n}_2\leqq\cdots\leqq\mathfrak{n}$$

($\mathfrak{n}$ being the supposed inversion number for $\mathbf{M}$) we shall have

$$\mathfrak{n}_1\mu=\mathfrak{n}_2\mu=\cdots=0.$$

3. On Null Measure Assertion. *When the cardinal of the set* $\mathbf{N}$ *is really less than that of continuum, N is measvrable a priori and*

$$\widetilde{\mathrm{m}}(\mathbf{N})=0.$$

This proposition is *Null Measure Assertion*, but to tell the truth, it needs some conditions to be effectively consistent. When the space considered is provided with a normal system of point-dimension, we see the assertion is valid, on account of the hypothesis I, 2), since as is well known there exists a set of continuum power of which the measure is observed as zero in the sense of $\mathfrak{L}$-measurability. In this section I will show that the assertion is consistent with respect to a regular system of point-dimension.

If $\mathbf{N}$ be a set of points in the space $\mathbf{E}$, of which the cardinal is really less than that of continuum and $\mathbf{E}$ be provided with a regular system of point-dimension; then it is direct that for any pair of points x, $x'\in\mathbf{E}$ we have

$$0<\frac{\mu_x}{\mu_{x'}}<\infty. \tag{3.1}$$

Besides, we may suppose with no loss of generality that $\mathbf{E}$ is the linear space of real numbers $(-\infty,\infty)$, and $\mathbf{N}$ is bounded; i. e.

$$\mathbf{N}\subset(a,b)=\mathbf{I},\ (-\infty<a<b<\infty).$$

If ξ is a point in $\mathbf{N}$, on account of (3,1) a positive integer n exists for any point $x \in \mathbf{I}-\mathbf{N}$, such that

$$\frac{1}{n-1} > \frac{\mu_x}{\mu_\xi} \geqslant \frac{1}{n} .$$

Let this number n be denoted as $n(x, \xi)$, and let the set of the points x ($\in \mathbf{I}-\mathrm{N}$) for which $n(x,\xi)=k$ be denoted as $\mathbf{X}(\xi, k)$ ($k=1,2,3,\cdots$).
Then we have

$$\sum_{k=1}^{\infty} \mathbf{X}(\xi, k) = \mathbf{I}-\mathbf{N} \tag{3,2}$$

because if not so, there exists a point $x \in \mathbf{I}-\mathbf{N}-\sum_{k=1}^{\infty}\mathbf{X}(\xi, k)$ such as

$$\frac{\mu_x}{\mu_\xi}=0 \text{ or } \infty ;$$

this is contradictory to (3,1).

As the power of the set $\mathbf{I}-\mathbf{N}$ is apparently equal to that of continuum, there exists a set

$$\mathbf{X}(\xi, \kappa)$$

the power of which is equal to that of continuum. Then, on account of the definition of $\mathbf{X}(\xi, \kappa)$, we have

$$\widetilde{\mathfrak{m}}\left\{\mathbf{X}(\xi, \kappa)\right\} \geqslant \frac{1}{\kappa}\, \mathfrak{n}\left\{\mathbf{X}(\xi, \kappa)\right\} \mu_\xi.$$

On the other hand, there exists at least one point $\xi \in \mathbf{N}$, for which

$$\mu_\xi \geqslant \frac{\widetilde{\mathfrak{m}}(\mathbf{N})}{\mathfrak{n}(\mathbf{N})} \tag{3,3}$$

because $\widetilde{\mathfrak{m}}(\mathbf{N}) = \underset{\xi \in \mathbf{N}}{\mathfrak{S}} \mu_\xi$. Therefore, supposing the point ξ satisfies the inequality (3,3) in advance, we have

$$\widetilde{\mathfrak{m}}\left\{\mathbf{X}(\xi, \kappa)\right\} \geqslant \frac{1}{\kappa}\, \mathfrak{n}\left\{\mathbf{X}(\xi, \kappa)\right\} \frac{\widetilde{\mathfrak{m}}(\mathbf{N})}{\mathfrak{n}(\mathbf{N})} .$$

By I,2) we see directly

$$\frac{\mathfrak{n}\{\mathbf{X}(\xi, \kappa)\}}{\mathfrak{n}(\mathbf{N})} = \infty$$

so that we may have:

$$b - a \geqslant \widetilde{\mathfrak{m}}\ \{\mathbf{X}(\xi, \kappa)\} \geqslant \infty\, \widetilde{\mathfrak{m}}(\mathbf{N}) \geqslant 0,$$

that means

$$\widetilde{\mathrm{m}}(\mathbf{N})=0. \quad \text{Q.E.D.}$$

Since what is mentioned above is verified by using the symbols $\widetilde{\mathrm{m}}(\mathbf{N})$ or $\widetilde{\mathrm{m}}\{\mathbf{X}(\xi,\kappa)\}$, it seems we may not assert the result for the general case where the sets $\mathbf{N}$ or $\mathbf{X}(\xi,\kappa)$ may not be always posited as measurable from the first. But, as a matter of fact, the verifying composition mentioned above can be held unchanged on symbolical formalism, so that we may admit the result gained above to be valid generally.

4. **Exclusion of Non-measurability.** When the notion of inversion number was introduced in a previous memoir of mine, I thought in private I could set measurability to be equivalent to conceivability of a set by means of this notion. But I have changed my mind recently when I found I could help the absurdity of measurability by excluding non-measurability by means of the hypothesis II,2).

About a sequence of measurable sets;

$$\mathrm{B}_1 \subset \mathrm{B}_2 \subset \cdots \subset \mathrm{B}_k \subset \mathrm{B}_{k+1} \subset \cdots \subset \mathbf{M},$$

if the set $\mathbf{M}$ is bounded[2] and the space in which $\mathbf{M}$ is given is provided with a normal system of point-dimension, $\widetilde{\mathrm{m}}(\mathbf{M})$ may not be larger than a certain finite number, so that we may not find the disjoint measurable sets $\mathbf{L}_k \subset \mathbf{M}-\mathbf{B}_k$ such as

$$\widetilde{\mathrm{m}}(\mathbf{L}_k) > \varepsilon > 0$$

for an infinite number of k, for any positive number ε fixed. Using this fact, we do not find it difficult to prove:

Proposition: *For any bounded set in a Euclidian space of finite dimension provided with a normal system of point-dimension, there can be found a sequence of a priori measurable sets* $\mathbf{B}_1 \subseteq \mathbf{B}_2 \subseteq \cdots \subseteq \mathbf{B}_k \subseteq \mathbf{B}_{k+1} \subseteq \cdots \subseteq \mathbf{M}$ *so that the set*

$$\mathbf{M}-\sum_{k=1}^{\infty}\mathbf{B}_k \tag{4,1}$$

2 Under the general system of point-dimension a *bounded set* may be defined as a set which is contained in a certain a priori measurable set.

may contain no subset which is measurable a priori with a positive measure; i.e. *for any a priori measurable set* **F** *contained in* $\mathbf{M}-\sum_{k=1}^{\infty}\mathrm{B}_k$ *we have* $\tilde{\ }(\mathbf{F})=0$.

Then, on acount of II,2) we see directly that the set (4,1) is of null measure; and consequently we conclude that the set $\mathbf{M}=\Sigma\mathbf{B}_k+(\mathbf{M}-\Sigma\mathbf{B}_k)$ is measurable a priori, since $\Sigma\,\mathbf{B}_k$ is measurable a priori as verified in the section 2. Thus it is observed that *non-measurability is excluded from our conception of a set,* on our course of study based on the four hypotheses I,1)---II,2).

It is interesting that the denial of non-measurability by means of the hypothesis II,2) is very similar to that of any other parallel lines than the equidistant one by the hypothesis of Euclidian parallelism.

Moreover, we may find any univoque real function $f(\mathbf{P})$ to be measurable in our sense, when we take an application

$$\gamma_P \ (\mathbf{P}\epsilon\mathbf{M},\ \mathbf{M} \text{ being a bounded set})$$

to indicate the general system of point-dimension; because, then the function of a set

$$\tilde{\gamma}(\mathbf{M})=\mathfrak{S}\ \gamma_P$$

is promised to satisfy the axioms (II,1) and (II,2) as the representation of the a priori measure of **M** with respect to the system γ_P, and consequently the set of the points for which

$$y-\varepsilon < f(\mathbf{P}) \leqslant y+\varepsilon$$

should be measurable a priori, on condition that the support of $f(\mathbf{P})$ is a bounded set. Those being so, we have:

PROPOSITION : *If the function of a set*

$$\tilde{\gamma}(\mathbf{M})=\mathfrak{S}\ \gamma_P\ (\geqslant 0)$$

be generally regarded as a priori measure, and if the real function $f(\mathrm{P})$ *is bounded in module and has its support to be a bounded set with respect to the system* γ_P, *then the integral*

$$\tilde{\gamma}(f(\mathbf{P}),\mathbf{M})=\mathfrak{S}\ f(\mathbf{P})\ \gamma_P$$

exists in the sense of the generalized Lebesgue composition with respect to γ.

The demonstration is direct. As the sets

$$\mathbf{M}_{n,\varepsilon}(f)=(\mathbf{P}\,;\,y_n<f(\mathbf{P})\leqslant y_{n+1})\cap\mathbf{M}$$

$(y_{n+1}=y_n+\varepsilon)$ are all a priori measurable as stated above, both of fhe sums

$$\Sigma\, y_n\gamma(\mathbf{M}_{n,\varepsilon}(f))=\underline{J}$$

$$\Sigma\, y_{n+1}\,\gamma(\mathbf{M}_{n,\varepsilon}(f))=\overline{J}$$

exist and tend monotonely to the same limit J, which must be the value to be represented in the form

$$J=\mathfrak{S}\,f(\mathbf{P})\,\gamma_P=\tilde{\gamma}(f(\mathbf{P}),\mathbf{M})\qquad \text{Q. E. D.}$$

5. Law of Absorption. Our theory of a priori measure is not only the development reduced from the fundamental system of hypotheses I, 1) --- II, 2), but it contains many delicate ideas which seem very natural to our intuition. Among them the notion of occupation of a point is a specially difficult one.

By the occupation

$$((x))\;=\;(x-0,\,x+0)\tag{5.1}$$

the author means that $((x))$ contains all the possible spacing regarded as lying between the limiting points $x-0$ and $x+0$, and he has asserted that

$$\left.\begin{aligned}\mu_x&=\tilde{\mathrm{m}}(x-0,\;x+0)=2\tilde{\mathrm{m}}(x-\tfrac{0}{2},\;x+\tfrac{0}{2})\\&=2\tilde{\mathrm{m}}(x+0^2,\;x+0)\,.\end{aligned}\right\}\tag{5,2}$$

Such are of the new categories that have never appeared in any classical books, but are considerred very efficient to establish the conception of continuum.

The point

$$x+\lambda\varepsilon\qquad(\varepsilon>0,\;1>\lambda>0)$$

is distinct from the point x, because there is observed the distance $\lambda\varepsilon$ between them; but, when we take the limiting process $\varepsilon\to0$ the limiting point

$$x+\lambda 0$$

should be regarded as belonging to the occupation (5.1), whereas the position $x+\lambda 0$ itself may not be regarded as overlapping exactly with the position x. Thus the notion of occupation $((x))$ is seen to be different essentially from that of position. The law mentioned above [say, $x+\lambda 0\in((x))$] is called *Law of Absorption.*

In fact it seems very natural that on the process

$$x' \rightarrow x$$

generlly we should have

$$\lim x' \in ((x)), \tag{5,3}$$

whereas, to conform to the calculating process (5,2) it must be qxactly

$$x + \frac{0}{2} \in ((x)) \quad \text{but} \quad x + 2 \cdot 0 \ \bar{\in} ((x))$$

on the definition (5,1). To remove such a contradiction it may be reasonable if we consider that the designation $x + \frac{0}{2}$ or $x + 2 \cdot 0$ may not indicate the simple limiting process of the types

$$x + \frac{\varepsilon}{2} \text{ or } x + 2\varepsilon \qquad (\varepsilon \rightarrow 0),$$

but they may suggest some structural relation of the occupations $((x)) = (x-2\ 0,\ x+2.0),\ (x-\frac{0}{2},\ x+0)$ or $(x-\frac{0}{2},\ x+2{\cdot}0)$ etc. to the formula

$$\mathbf{I} = (0,1) = \underset{x \in \mathbf{I}}{\mathfrak{S}} ((x)).$$

If we accept this distinction, it will be to indicate the fact (5,3) by the term "law of absorption" generally.

6. Resilience. On the study of continuum, it has been an important remark that any point $x \in (-\infty, \infty)$ has no contiguous point. G. Cantor posited to take the three points $x-0$, x and $x+0$ as the same to indicate the position of the point $\mathbf{P}_x$, but it is well known in the theories of integral and real functions, to distinguish these three is necessary in some cases. The source of the discussions on well-ordered sets too, may be understood to have lain in the absurdity of contiguity of the real numbers.

The first observation of the contiguous state of the real numbers has been made with respect to the law of absorption from our point of view, and then an inversion of this law is posited to make the notion of *resilience*; in other words, we elucidate the absurd contiguity of the real numbers to be caused by the resilience of each point. An occupation may be considered as dwelling in its expansive state only when it is considered to have some mechanical propriety --- say, *resilience*.

On the elementary plane geometry, we learned a famous casuistic process, to verify any length larger than the proper length of a line segment l, to be possibly adoptable as the measure of l, by using auxiliary lines parallel to each side of a triangle of which l is the base. From our standpoint of view, this is not a mere paradox, but it may be valid when we bestow each point of l with two directions of resilience parallel to each sides of the triangle.

In the general Euclidian space of finite dimension, each point is considered to have its resilience expansive in the directions of the coordinate-axes so that the aggregative structure of the space may be observed to make a continuum very naturally.

Besides the notion of resilience we shall have another inversion of the law of absorption, which is found to be needed when the absorbed limiting point is considered to be separated from inward the occupation, on moving along the inverse prosess of the limiting given in the first. Such is a phenomenon to be observed in mechanical historicity; we study it for instance on the observation of the histories of distribution and call it *Law of Dissolution*.

You will perfectly understand the ideas described in this paper if you will refer to the following works by the same author.

i) *On Continuum*, Mem. Muroran Univ. Eng. Vol. I, No. 3 (1952);

ii) *A Course of Radonian Calculus* (1953) (this booklet will be obtained at Maruzen, Sapporo Japan);

iii) *A Synthetic Light on the Distributions and their Stochasticity*, Mem. Muroran Univ. Eng. Vol. I, No. 5 (1954).

Mathematical Seminar in the Muroran Univ. Eng., Hokkaido

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