A Study of the Categories of Summation

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A Study of the Categories of Summation

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Abstract

The categories are specified by the primitive constitution by which the aggregated total or the limiting state of a set are mathematically conceived as existent. It is emphasized that the conceptions of the total and of the limiting state are not always coincident. Enumerable set is elucidated on its specially important construction.

1. Introduction. In studying the sets, it may be said that the hardest floorer for students is the conceptive uncertainty of a sum. So, to have a categorical study of the summation is considerable specially in point that, then a primitive light may be caught in some way rather apart from the traditional course. In this paper, the effectiveness of the categories of summation is studied in relation of the theory of measure.

The notion of a total is exposed specially on its proper operation, as it is, to be devided into its parts; while the notion of a limit of aggregating the elements is not always found to coincide with the total.

The probabilistic method gives often the most exact scale to distinguish any two ways of conception of a set. N be a subset of the set M, the elements of which are presumed to be equi-probable at the selection of an element. Then, let's denote as

$$\text{Prob.}\,(x \in N) = \mathfrak{p}\,(N/M)$$

when x is restricted to be selected from within the set M. If the value of $\mathfrak{p}\,(N/M)$ be uniquely determined, then we say *the concomitance N/M is determined in the a priori probabilism.*

As ZENON suggested, we cannot count up the total length 1 from the measure zero of each point contained in it. Nevertheless, with B. CAVALIERI, we may affirm the length 1 consists of the points contained in it. In effect, the formulation

$$\mathfrak{S}\mu_P = 1$$

μ_P indicating the dimension of the point $P^{1)}$, gives us the category of sum-

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1) About the symbols $\mathfrak{S}$ and μ_P, cf. e. g. Y. Kinokuniya: *A Course of Radonian Calculus* (1953) (this booklet will be obtained at Maruzen, Sapporo Japan).

mation that is exactly the same fact as in case of CAVALIERI. Such is the construction of the length 1 that is equivalent to the conception of the total of the points which are defined to make up the length 1

In the theory of a priori measure and in the a priori probabilism, discussions are made specially on the consistence of the following two destinations:

$\mathfrak{D}_1$: $\overline{\overline{N}} < \overline{\overline{M}}$ *and* $\overline{\overline{M}} \geqslant \aleph_0$, *then[2] it is destined that*

$$\widetilde{m}(N)/\widetilde{m}(M) = 0 \tag{1, 1}_1$$

and

$$\mathfrak{p}(N/M) = 0. \tag{1, 1}_2$$

$\mathfrak{D}_2$:   *M be a bounded infinite set and F be any measurable set for which the concomitance F/M is determined, then we have*

$$\varlimsup_{F \subset M} \widetilde{m}(F) = \widetilde{m}(M) \tag{1, 2}_1$$

and

$$\varlimsup_{F \subset M} \mathfrak{p}(F/M) = 1. \tag{1, 2}_2$$

In the above, $\widetilde{m}$ indicates the a priori measure and it is appointed that

$$F \subset M \;\rhd\; M - F \neq \text{void}.$$

$\mathfrak{D}_1$ is called the *Null-Measure Assertion of the Second Species* and $\mathfrak{D}_2$ is called the *Pan-Measurability*. In the theory of a priori measure, the pan-measurability is induced from several axioms[3] posited as fundamental ones to make up the theory. The relations $(1, 1)_1$ and $(1, 2)_1$ are studied in the theory of a priori measure, and the relations $(1, 1)_2$ and $(1, 2)_2$ are studied in the a priori probabilism, respectively.

The category of a total of elements M is thought to be certain when, for any element x, it is decided either

$$x \in M \text{ or } x \bar{\in} M.$$

The conception of a summation is thought to be caused when a set of suffices X is given and a set M_x uniquely corresponds to each suffix $x \in X$. In this case, denoting as

$$\widetilde{M}_Y = \bigcup_{y \in Y} M_y$$

for any subset $Y \subset X$, the category of the summation

$$\widetilde{M} = \mathfrak{S}_{x \in X} M_x = \bigcup_{x \in X} M_x$$

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2) The symbol $\overline{\overline{M}}$ indicates the cardinal number of the set M.

3) Cf. Y. Kinokuniya: *Logical Construction of the Theory of A Priori Measure*, Mem. Muroran Univ. Vol. 2, No. 3, p. 786 (268) *(Theorem 2)*.

is said *effective with respect to the measure* $\tilde{\gamma}$ when the following two conditions are satisfied :

    (i) $\widetilde{M}$ *is the total of all the elements contained in either* M_x $(x \in X)$;

    (ii) $\mathfrak{D}_2$ *is consistent with respect to* $\tilde{\gamma}$, i.e.

$$\overline{\lim_{Y \subset x}} \; \tilde{\gamma}(M_Y) = \tilde{\gamma}(\widetilde{M}) \; ; \tag{1, 3}$$

$\tilde{\gamma}$ *is restricted to a non-negative measure. The condition* (1, 3) *is equivalent to the condition*

$$\overline{\lim_{Y \subset x}} \; \tilde{\gamma}(\widetilde{M} - \widetilde{M}_Y) = 0.$$

In Paragraph 5, there is introduced the logical term *"Accommodation"*, by which some important relations are designated and dealt with. The criticism on the theory of ordinal numbers is not referred to, since we may then digress from what is schemed in this paper.

2. **Enumerable Set.**   When the elements a_k $(k = 1, 2, \cdots)$ are given, the set

$$M = (a_1, \; a_2, \; \cdots)$$

is thought to be the total of the elements on condition that any element x is decided either

$$x \in M \text{ or } x \bar{\in} M.$$

Let us denote as

$$M_n = (a_1, \; a_2, \; \cdots, a_n) \; (n = 1, \; 2, \; \cdots),$$

then it may not be stated as

$$\lim_{n \to \infty} M_n = M$$

in the probabilism, because it is evidently seen that

$$\mathfrak{p}(M_n/M) = 0$$

hence

$$\lim \; \mathfrak{p}(M_n/M) = 0 \neq 1 = \mathfrak{p}(M/M).$$

By the way, if we denote by N_λ the total of the elements $a_\lambda, a_{2\lambda}, \cdots$, it may be naturally accepted that

$$\mathfrak{p}(N_\lambda/M) = 1/\lambda.$$

In effect, as far as we merely write in line

$$a_1, \; a_2, \; \cdots,$$

there may not be denied the arrangement to continue such as

$$a_1, a_2, \cdots, a_\omega, a_{\omega+1}, \cdots, a_{n\omega}, \cdots, a_{\omega^2}, a_{\omega^2+1}, \cdots.$$

This being so, our criticism shall find its principal object in the tail part

$$M - M_n (n = 1, 2, \cdots).$$

In order to establish a relative construction between the values $\mathfrak{p}(M/M) = 1$ and $\mathfrak{p}(a_n/M) = 0$, we may do it well by introducing the application system

$$\pi_n = \mathfrak{p}(a_n/M)$$

instead of the formulas $\mathfrak{p}(a_n/M) = 0$, to make up the new category of summation by the formula

$$\underset{a_n \in M}{\mathfrak{S}} \pi_n = 1. \tag{2, 1}$$

It is very important that the summation (2, 1) might not be understood as of a mere enumerable construction in the classic sense.

In the classic analysis, it was decided that any enumerable set of real numbers must be of zero measure. The reason was:

$$0 \leqslant m(a_n) < \frac{1}{2^n} \varepsilon \quad \text{for any } \varepsilon > 0$$

so that

$$0 \leqslant \sum m(a_n) < \varepsilon.$$

However, this method is discovered now difficult because of the above-stated reasoning about the summation (2, 1). While, in the theory of a priori measure, we assert the zero measure of an enumerable set by means of the destination $\mathfrak{D}_1$, the null-measure assertion of the 2nd species.

3. **Arranged Summation.** $\mathfrak{A}$ be a set of ordinal numbers and M_α be the set corresponds to the number $a \in \mathfrak{A}$, then the summation

$$\widetilde{M}_\beta = \underset{\alpha < \beta}{\cup} M_\alpha$$

comes into question; in this case, let us say the sum-set $\widetilde{M}_\beta$ is posited in the *category of arranged summation*. The effectiveness of the conception $\widetilde{M}_\beta$ shall be observed in the respect whether the relation

$$\overline{\underset{\beta < \gamma}{\lim}} \, \widetilde{m}(\widetilde{M}_\beta) = \widetilde{m}(\widetilde{M}_\gamma) \tag{3, 1}$$

effects or not.

When $\widetilde{m}(\widetilde{M}_\gamma) > 0$, the problem is transferred to the probabilism, because we then directly have

$$\widetilde{m}(\widetilde{M}_\beta)/\widetilde{m}(\widetilde{M}_\gamma) = \mathfrak{p}(\widetilde{M}_\beta/\widetilde{M}_\gamma)$$

In this case, if it is observed that

$$m(\widetilde{M}_\beta) = 0 \quad \text{for each } \beta < \gamma,$$

we may never attain the effectiveness of the relation (3, 1). Besides, if

$$\widetilde{m}(M_\alpha) = 0 \quad \text{for each } \alpha < \gamma$$

and the destination

$$\varlimsup_{\alpha < \beta} \widetilde{m}(\widetilde{\boldsymbol{M}}_\alpha) = \widetilde{m}(\widetilde{\boldsymbol{M}}_\beta)$$

is conformed by each $\beta < \alpha$, then it may be concluded by means of the transfinite induction that

$$\widetilde{m}(\widetilde{\boldsymbol{M}}_\beta) = 0 \text{ for each } \beta < \gamma.$$

Thus we have :

   Proposition 1. *As far as the relation of pan-measurability is conformed, the arranged summation does not produce a set of positive measure from the sets of zero measure.*

   If we, from the first, restrict our observation within the a priori probabilism, the conditions are left in a more general scale, because we may then get along, paying no attention on the measure $\widetilde{m}(\widetilde{\boldsymbol{M}}_\gamma)$. In this case, the pan-measurability is defined by the relation

$$\varlimsup_{\alpha < \beta} \mathfrak{p}(\widetilde{\boldsymbol{M}}_\alpha / \widetilde{\boldsymbol{M}}_\gamma) = \mathfrak{p}(\widetilde{\boldsymbol{M}}_\beta / \widetilde{\boldsymbol{M}}_\gamma).$$

According to $\mathfrak{p}(N/M) = 0$ or > 0, let us say N/M is a *zero-concomitance* or a *positive concomitance* respectively. Then, by the evident analogous process, we may have :

   Proposition 2. *As far as the relation of pan-measurability is conformed, the arranged summation does not produce a positive concomitance from the zero-concomitances.*

   4. Co-ordinate Metamorphism. When between two sets of points $\boldsymbol{M}$ and $\boldsymbol{N}$ the relation

$$\overline{\overline{\boldsymbol{M}}} = \overline{\overline{\boldsymbol{N}}}$$

is observed, we say the two sets are *co-ordinate* with each other. As G. CANTOR defined, this means that there exists a bi-univoque mapping

$$y = f(x) \ (x \in \boldsymbol{M}, \ y \in \boldsymbol{N}). \tag{4, 1}$$

In the theory of a priori measure, we interpret that the inversion number of $\boldsymbol{M}$ and $\boldsymbol{N}$ are then equal, so that denoting by

$$\mathfrak{n}_f(\boldsymbol{N})$$

the inversion number of $\boldsymbol{N}$ given by the mapping (4, 1), we may have

$$\mathfrak{n}_f(\boldsymbol{N}) = \mathfrak{n}(\boldsymbol{M}) \tag{4, 2$_1$}$$

and, denoting by $f^{-1}(y) = x$ the inversive mapping of (4, 1), we may have

$$\mathfrak{n}_{f^{-1}}(\boldsymbol{M}) = \mathfrak{n}(\boldsymbol{N}). \tag{4, 2$_2$}$$

   It is remarkable that the case

$$\mathfrak{n}(M) \neq \mathfrak{n}(N)$$

is possible despite of the simultaneous relations $(4, 2)_1$ and $(4, 2)_2$. In effect, the a priori measure of M and N are expressed in the forms

$$\widetilde{m}(M) = \mu \cdot \mathfrak{n}(M)$$

and

$$\widetilde{m}(N) = \mu \cdot \mathfrak{n}(N)$$

respectively, μ indicating the point-dimension of each point in the space.

In case

$$\mathfrak{n}_f(N) \neq \mathfrak{n}(N),$$

the values of $\mathfrak{n}_f(N)$ and $\mathfrak{n}(N)$ may not be applied under the same system of point-dimension. So, we introduce a system of non-negative application $\gamma_{f(P)}$ to be applied in the formulation

$$\widetilde{m}(N) = \underset{P \in M}{\mathfrak{S}} \gamma_{f(P)}. \tag{4, 3}$$

We then presume that the system $\gamma_{f(P)}$ is uniquely determined simultaneously with the mapping f.

In case $M = (0, 1)$ and $N = (0, 2)$, a famous mapping of G. CANTOR is given by the function

$$y = f(x) = 2x \quad (x \in M, \; y \in N)$$

by which he determined as $\overline{\overline{M}} = \overline{\overline{N}}$. Then, the above-stated application $\gamma_{f(x)}$ is determined as

$$\gamma_{f(x)} = 2\mu.$$

In effect, when the point x runs through the interval $(0, 1)$, the corresponding point y runs through the interval $(0, 2)$ with redoubled velocity, so that the point $(y)^{4)}$ may be measured as of the doubled scale.

In the a priori probabilism, an enumerable set $M = (a_1, a_2, \cdots)$ shows properties analogous to the above-stated facts. In this case, by the bi-univoque mapping

$$\varphi(a_k) = a_{k\lambda} \quad (k = 1, 2, \cdots)$$

M is transferred into its subsequence $N_\lambda = (a_\lambda, a_{2\lambda}, \cdots)$, so that we may here introduce the pair of application systems (π_k) and $(\omega_{\varphi(a_k)})$ to be applied in the formulation

$$\underset{a_k \in M}{\mathfrak{S}} \pi_k = \mathfrak{p}(M/M) = 1$$

and

$$\underset{a_k \in M}{\mathfrak{S}} \omega_{\varphi(a_k)} = \mathfrak{p}(N_\lambda/M) = \frac{1}{\lambda}.$$

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4) In the theory of a priori measure, the one-dimensional point of the abscissa x is denoted as (x) or P_x.

Moreover, it may be posited as

$$\mathfrak{p}(M/M) = \pi \cdot \mathfrak{n}(M) \quad (\pi = \pi_k; \; k = 1, 2, \cdots)$$

and

$$\mathfrak{p}(N_\lambda/M) = \omega \cdot \mathfrak{n}_\varphi(M) \quad (\omega = \omega_{\varphi(a_k)}; \; k = 1, 2, \cdots),$$

so that it may be analysed as

$$\omega = \frac{1}{\lambda}\pi$$

and

$$\mathfrak{n}_\varphi(M) = \lambda \mathfrak{n}(N_\lambda).$$

About the existence of the system $\gamma_{f(x)}$, we may refer to the following conditions:

(i) $f(x_1) \neq f(x_2)$ *when* $x_1 \neq x_2$;

(ii) *the value of* $\widetilde{m}(N) = \widetilde{m}(f(M))$ *is readily existent, provided especially with its total occupation*

$$((N)) = ((f(M))).$$

But, if perfectly, we may not pass through without any axiomatization.

Puting back to the case the formulas (4, 1)–(4, 3) are effective, if $\widetilde{m}(N) = 0$ whereas $\widetilde{m}(M) > 0$, we have

$$\mathfrak{n}(N)/\mathfrak{n}(M) = 0$$

under the same normal[5] system of point-dimension μ; because the measures are then expressed as

$$\widetilde{m}(M) = \mu \cdot \mathfrak{n}(M) \text{ and } \widetilde{m}(N) = \mu \cdot \mathfrak{n}(N).$$

Therefore, denoting as

$$E\left(P: \; P \in M \text{ and } \frac{1}{k} < \frac{\gamma_{f(P)}}{\mu}\right) = \overset{(k)}{N} \quad (k = 1, 2, \cdots)$$

and

$$E\left(P: \; P \in M \text{ and } 0 = \frac{\gamma_{f(P)}}{\mu}\right) = N_0,$$

we may have

$$\frac{\mathfrak{n}_f(\overset{(k)}{N})}{\mathfrak{n}_f(N)} = \frac{\mathfrak{n}_f(\overset{(k)}{N})}{\mathfrak{n}(M)} = 0 \tag{4, 4}$$

because

$$\frac{\mathfrak{n}_f(\overset{(k)}{N})}{\mathfrak{n}(M)} = \frac{\mu \cdot \mathfrak{n}_f(\overset{(k)}{N})}{\mu \cdot \mathfrak{n}(M)} < \frac{\underset{P \in \overset{(k)}{N}}{k \lessgtr \gamma_{f(P)}}}{\mu \cdot \mathfrak{n}(M)} = k \frac{\widetilde{m}(\overset{(k)}{N})}{\widetilde{m}(M)} \leqslant k \frac{\widetilde{m}(N)}{\widetilde{m}(M)} = 0.$$

Then, on account of (4, 4)

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5) When $\mu_P = \mu_Q$ for each P and Q of the space, the system of point-dimension μ_P is said normal, and is briefly represented by μ on condition that $\mu = \mu_P$ for each P.

$$\lim_{k \to \infty} \frac{\mathfrak{n}_f(\overset{(k)}{N})}{\mathfrak{n}_f(N)} = 0. \tag{4, 5}$$

On the other hand

$$N = N_0 + \lim_{k \to \infty} \overset{(k)}{N},$$

so that

$$\mathfrak{n}_f(N) = \mathfrak{n}_f(N_0) + \lim_{k \to \infty} \mathfrak{n}_f(\overset{(k)}{N}). \tag{4, 6}$$

Hence, by (4, 5) and (4, 6), we may have

$$\mathfrak{n}_f(N_0)/\mathfrak{n}_f(N) = 1$$

i. e.

$$\mathfrak{n}_f(N) = \mathfrak{n}_f(N_0). \tag{4, 7}$$

In such a case as (4, 7), we say *the set N_0 consists of almost all elements of the set N in respect to the inversion $\mathfrak{n}_f$*; then we may state the above result as follows:

Proposition 3. *If there exists the system of non-negative application $r_f(P)$ in respect to the two sets M and N for which it is observed that*

$$\widetilde{m}(N) = 0 \quad and \quad \widetilde{m}(M) > 0,$$

then almost all points of the set N suffer measure cohesion[6] in respect to the inversion $\mathfrak{n}_f$.

If the system of non-negative application $r_{f(P)}$ exists and is regular (i.e. $0 < r_{f(P)}/\mu < \infty$), then $f(P)$ is called a *measure-regular mapping* or briefly a *regular mapping*. In case of Proposition 3, the mapping may not be expected necessarily to be a regular one, since $r_{f(P)}$ can be infinitesimal even when $r_{f(P)}/\mu = \infty$.

5. Theory of Accommodation. A fact $\mathfrak{b}$, being surmised by the fact $\mathfrak{a}$ in the theory $\mathfrak{A}$, be presumed effective in the theory $\mathfrak{B}$, then we say, $\mathfrak{b}$ *is an accommodation in $\mathfrak{B}$ caused by $\mathfrak{a}$ in $\mathfrak{A}$*. If the formula

$$p(\mathfrak{a}, \mathfrak{b}) = 0 \tag{5, 1}$$

is needed in order that the facts $\mathfrak{a} \in \mathfrak{A}$ and $\mathfrak{b} \in \mathfrak{B}$ may accord with each other, then the formula (5, 1) is called a *conformity contributed to $\mathfrak{b}$ by $\mathfrak{a}$*.

The inversion number $\mathfrak{n}(M)$ in the theory of a apriori measure may be regarded as an accommodation caused by the notion of the number of the elements in the theory of finite sets. The normal system of point-dimension μ is an accommodation caused by the homogenity of the Euclidian space; and the formula

$$\widetilde{m}(M) = \mu \cdot \mathfrak{n}(M)$$

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6) $r_{f(P)}/\mu = 0$, then P is said suffers *measure cohesion* by the mapping $f(P)$. Cf. e.g. ibid. with 1.

is then the conformity contributed to the a priori measure $\widetilde{m}(M)$ of a set M.

With respect to the mapping

$$y = f(x) = 2x$$

we should not overlook the difference of the respective velocities of the points (y) and (x), as stated in the previous paragraph. This is the difference of spacing characters of the two points. The corresponding accommodation caused by this phasic property is the notion of the occupation of a point (x), denoted as $((x))$, and the point-dimension μ_x of (x) is expressed in the form

$$\widetilde{m}\left[((x))\right] = \mu_x,$$

which is the conformity contributed to $((x))$ in the theory of a priori measure.

6. **Transmutation of Concomitance.** When the normal system μ in the x-space is transferred by a bi-univoque mapping $y=f(x)$ to an application system $\gamma_{f(x)}$ in the y-space, the concomitance

$$N/M$$

is transferred to the concomitance

$$f(N)/f(M).$$

It is then to be computed by the definition as

$$\mathfrak{p}(N/M) = \frac{\widetilde{m}(N)}{\widetilde{m}(M)} = \frac{\mathfrak{n}(N)}{\mathfrak{n}(M)}$$

and

$$\mathfrak{p}(f(N)/f(M)) = \widetilde{m}(f(N))/\widetilde{m}(f(M)).$$

In this process, it is remarkable that the equation

$$\mathfrak{p}(N/M) = \mathfrak{p}(f(N)/f(M))$$

may not be generally satisfied, since the equation

$$\frac{\widetilde{m}(N)}{\widetilde{m}(f(N))} = \frac{\widetilde{m}(M)}{\widetilde{m}(f(M))}$$

may not be generally effected. Hence there is presumed a transmutation of concomitance, which may be defined, as it is, by the expression

$$\mathfrak{p}(N/M) \longrightarrow \mathfrak{p}(f(N)/f(M))$$

or briefly by the expression

$$N/M \longrightarrow f(N)/f(M).$$

Under the normal system μ we have

$$\mathfrak{p}(N/M) = \mathfrak{n}(N)/\mathfrak{n}(M)$$

whereas under the system $\gamma_{f(x)}$ we may not generally have the analogous relation

$$\mathfrak{p}(f(N)/f(M)) = \mathfrak{n}_f(N)/\mathfrak{n}_f(M)$$

to be effective; because, when $x_1 \neq x_2$, the value of the ratio

$$\gamma_{f(x_1)}/\gamma_{f(x_2)}$$

may not be generally equal to 1. In this case, the normal system

$$\pi = \pi_x = \mu/\widetilde{m}(M)$$

is transferred to the system

$$\omega_{f(x)} = \gamma_{f(x)}/\widetilde{m}(f(M)).$$

As it is computed as

$$\pi = \pi_{x_1} = \pi_{x_2} \quad \text{for each } x_1,\ x_2 \in M$$

the system π is applied just in accord with the equi-probabilism, but the transmuted system $\omega_{f(x)}$ no more holds the homogenity.

In effect, the occupation $((f(x)))$ is found in diverse spacing, so that the concomitance $f(x)/f(M)$ should be determined in proportion to the size of the spacing, i.e. the value of $\gamma_{f(x)}$. Such means that the occupation of a point is turned to be observed in the same property as a set, though it is not originally defined as a set.

On examination, it may be said that the transmuted application $\omega_{f(x)}$ has two ways to be interpreted: as an accommodation caused by the notion of a point to fill up the space, and as an accommodation caused by the set-theoretical determination in the classic probabilism. This is homologous to the fact that a point is understood at the same time as a mere element, which has no size to be measured, and as the limit of a sequence of intervals (or neighbourhoods). It will be very difficult to establish the notion of probability of a point without any use of the set-theoretical probabilism. Viewed from this angle, a point cannot be independently posited from the notion of devision of the total. Besides, such may also be an unavoidable destiny for the conception of a continuum. Thus, the conceptions of a total and of a limiting object are exposed to be originally distinguished.