Measure-theoretical Decomposition of a Set

Measure-theoretical Decomposition of a Set

Yoshio Kinokuniya*

Abstract

If a set M is posited accompanied with the (non-negative) value defined by

$$\tilde{\gamma}(M) = \underset{P \in M}{\mathfrak{S}} \gamma_P ,$$

some important singularities are found when a decomposition applied to devide M. The notion of *overflow* and that of *a quantitative metamorphosis* are defined to analyse the set M in a clearer aspect.

1. Simple and Singular Decomposition

In this paper the applicaton γ_P (the measure quantity given to the point P) is supposed non-negative and $\neq \infty$, and the normal system μ** is given as the standard scale to be compared with any application system ; the points and the sets are restricted to be observed in a finite-dimensional Euclidian space E. γ_P is called *non-negative* when

$$\gamma_P \geqslant \circledcirc = \text{empty null}^{1)} ,$$

and *positive* when

$$\gamma_P > \circledcirc ,$$

therefore γ_P may be infinitesimal even when it is called as positive ; but, when a quantity f is expressed as

$$f > 0 ,$$

then f is destined as a positive real number, which is of the same notion as in the classical analysis.

---

* 紀国谷芳雄

** In case of a normal system μ, if μ_P is the measure of the point P, then

$$\mu_P = \mu_Q = \mu \quad \text{for all the points } P, Q,$$

and the measure

$$\tilde{m}(M) = \underset{P \in M}{\mathfrak{S}} \mu_P$$

gives the same value as the Euclidian volume-value, whenever M is measurable in the Euclidian volume-theory.

When there exists at least one sequence of sets $\{M_k\}$ such that

$$\text{(i)}\quad M_k \cap M_j = \text{void whenever } k \neq j\,;$$
$$\text{(ii)}\quad M = \sum M_k\,;$$
$$\text{(iii)}\quad \tilde{\gamma}(M_k) = \underset{P\in M}{\mathfrak{S}}\, \gamma_P < \infty \text{ for each } k = 1,\ 2, \cdots,$$

then M is said to be *of a simple decomposition in respect to the system* $\{\gamma_P\}$ or briefly to be *simple in respect to* $\tilde{\gamma}$. When there exists no such sequence of sets as the above-stated, then M is said to be *of a singular decomposition* or briefly to be *singular in respect to* $\tilde{\gamma}$. Let E_∞ be the aggregation of the points for which

$$\gamma_P/\mu = \infty$$

then it is easily verified that

$$E_\infty \cap M \neq \text{void}$$

when M is singular, though in general this cannot make a sufficient condition for the singularity. When it is observed for each point P that

$$(0 \leqslant)\, \gamma_P/\mu < \infty$$

then $\{\gamma_P\}$ is said to make *a regular system.*

2. Overflow

To study about a singular set in respect to $\tilde{\gamma}$, we find it appropriate to introduce the following notion to build up a clearer aspect on a decomposition of a set. If a set V contains the point P as its inner point, then V is called a neighborhood of P. If it is observed for any neighborhood V of a fixed point P that

$$\tilde{\gamma}(V) = \infty\,,$$

then P is called *a point of overflow*, and the aggregation of all the points of overflow in E is called the *overflow* in respect to $\tilde{\gamma}$.

Proposition 1.—*If a set M be singular in respect to $\tilde{\gamma}$, the aggregation of the points of overflow contained in M has a power strictly larger than enumerability.*

Demonstration. Suppose the points of overfiow contained in M are exhausted by

$$Q_1,\quad Q_2, \cdots$$

and

$$M^* = M - \sum \{Q_k\}$$

then, if $P \in M^*$, there exists a neighborhood $V(P)$ of P such that

$$\tilde{\gamma}(V(P)) < \infty$$

(because the point P is not a point of overflow). Therefore, by the covering theorem of Lindeloef, there exists a sequence of points $\{P_k\} \subseteq M^*$, such that

$$M^* \subseteq \sum V(P_k)$$

and
$$\tilde{\gamma}(V(P_k)) < \infty .$$

Then
$$M \subseteq \sum V(P_k) + \sum \{Q_k\} ,$$

so that the set M is discovered as of a simple decomposition. But this is contradictory to the assumption that M is singular.⌋

Proposition 2.—*Any inner point of E_∞ is a point of overflow.*

Demonstration. Let P be an inner point of E_∞ and V be a neighborhood of P, then there exists another neighborhood U of P such that

$$V \supset U \quad \text{and} \quad E_\infty \supset U .$$

It is evident that

$$0 < \tilde{m}U = \mathop{\mathfrak{S}}_{P \in U} \mu_P \qquad (\mu_P = \mu) ,$$

so that, by the definition of E_∞, we see

$$\tilde{\gamma}(V) \geqslant \tilde{\gamma}(U) \geqslant \tilde{m}U \cdot \infty = \infty .$$

As V may be an arbitrary neighborhood of P, this induces that the point P is a point of overflow.⌋

Moreover, the following fact is directly gained from the definition:

Proposition 3.—*The overflow is a closed set.*

3. Integral and the Singularity

Let $f(P)$ be a non-negative function of a point P and define the integral

$$\tilde{\gamma}(f, M) = \mathop{\mathfrak{S}}_{P \in M} \gamma_P f(P) ,$$

then, if we denote it as

$$M_k = M_k(f) = \left\{ P \middle/ P \in M,\ \frac{1}{k} \leqslant f(P) < \frac{1}{k-1} \right\}^* ,$$

we have
$$\tilde{\gamma}(f, M) \geqslant \sum \frac{1}{k} \tilde{\gamma}(M_k) .$$

* The symbol $\{P/\mathfrak{p}\}$ indicates the aggregation of the points P which satisfy the property $\mathfrak{p}$.

Therefore, if
$$\tilde{\gamma}(f, M) < \infty$$
and
$$M_P(f) = \{P/ P\in M,\ f(P) > 0\}\,,$$
the sequence $\{M_k\}$ gives a simple decomposition of $M_P(f)$; so, we have:

Proposition 4.—*Let M be a singular set in respect to $\tilde{\gamma}$ and $f(P)$ be a non-negative (real-valued) function for which it is observed that*
$$\tilde{\gamma}(f, M) < \infty\,,$$
then $M_P(f)$ is simple, so that the set
$$M - M_P(f) = \{P/ P\in M,\ f(P) = 0\}$$
must be singular in respect to $\tilde{\gamma}$.

A point of overflow is not to be generally excepted from the supporting domain of the above-stated integration. For instance: for a fixed point Q, let's write as
$$M_{Q,k} = \left\{P\Big/ \frac{1}{k+1} \leqslant |P-Q| < \frac{1}{k}\right\} \qquad (k = 1, 2, \cdots)$$
and define the $\tilde{\gamma}$-values such as
$$\tilde{\gamma}(M_{Q,k}) = \frac{1}{k}$$
or rather
$$M_{Q,k} \ni P \rhd \gamma_P = \frac{\mu}{k}\Big/ \tilde{m}(M_{Q,k})$$
and
$$\gamma_Q = ◎\,,$$
then Q is found as a point of overflow in respect to this $\tilde{\gamma}$. In this case may be found many pairs of a non-negative function ϕ and a neighborhood $V(Q)$ of Q such as
$$\phi(Q) \neq 0 \quad \text{and} \quad \tilde{\gamma}(\phi, V(Q)) < \infty\,.$$
It is, however, to be noticed that the system $\{\gamma_P\}$ defined above, makes a regular system (say, $\gamma_P/\mu \neq \infty$).

4. Singular System and Singularity

Let the sets E_k and S_k $(k=1, 2, \cdots)$ be defined by
$$E_k = \{P/ k-1 \leqslant \gamma_P/\mu < k\}$$
and
$$S_k = \{P/ |P| \leqslant k)\,,$$
then, if γ_P are regular everywhere in E, any set M in E has a simple decomposition

by the enumerable ensemble of the sets

$$M_{k,j} = M \cap E_k \cap (S_j - S_{j-1}) \qquad (k = 1, 2, \cdots ; j = 2, 3, \cdots)$$

and

$$M_{k,1} = M \cap E_k \cap S_1 .$$

Therefore:

Proposition 5.—*If there exists a singular set in respect to* $\tilde{\gamma}$, $\{\gamma_P\}$ *cannot be a regular system.*

Now let's denote it as

$$E_R = E - E_\infty ,$$

then, by a slightly modified application of Prop. 5, it may be verified that E_R cannot be a singular set. Moreover, since the aggregation E_I of the isolated points of E_∞ is at most enumerable, the set

$$E_R + E_I = \dot{E}_R \qquad (4, 1)$$

must be simple in respect to $\tilde{\gamma}$. So, we consequently see:

Proposition 6.—*If* E *is singular in respect to* $\tilde{\gamma}$, *the set* E'_∞ *(the derived set from* E_∞*) must be of a power larger than enumerability. The set* $\dot{E}_R$ *defined by* (4, 1) *is in any case a simple set in respect to* $\tilde{\gamma}$.

By this theorem we have a necessary condition for E to be singular in respect to $\tilde{\gamma}$; but, to tell the truth, this is not a sufficient one. To reach a clearer aspect on the singular sets, the notion of overflow may serve as an indispensable medium; in effect, in the previous sections we have to some extent realized several interesting relations between this notion and E_∞. However, it seems what is waiting for us is a discovery of an essential chaos. It is similar to what has been destined to the theory of sets. For instance, where

$$\gamma / \mu = \infty$$

an inversion number $\mathfrak{n}$[2] which satisfies the relation

$$0 < \mathfrak{n} \cdot \gamma < \infty$$

may be an object set beyond the scale of our decision. This is the first gape for our logic.

5. Quantitative Metamorphosis

In case of an a priori measure $\tilde{m}$[2] applied in an Euclidian space E of finite dimensions, E is treated as a simple set in respect to $\tilde{m}$, because E may be then devided into an enumerable ensemble of rectangles of finite diagonal. From the physical viewpoint, we may posit the space E as being filled up with a sort of homogeneous medium and interpret the $\widetilde{m}$-value of a set M $\widetilde{m}(M)$ as the weight-measure of this medium filled in M. This interpretation is not a new one, but, for instance, in a part of the ergodicity theory several authors used this type of expression.

Now let's suppose a quantitative transition of the medium, which may not necessarily be continuous. After such a transition is to be found a new repartition of the medium; in other words, we shall then find each point in a different distribution of the medium-mass from the first homogeneous one. So, if we denote the new share-value for a point P as π_P, these make together a non-negative application system. In addition, such a transition causes a quantitative transmutation of the space E; so we call this transmutation a *quantitative metamorphosis* of the space E.

Ultimately, we may understand the mass-value π_P as the mass of the medium filled in the point-occupation[3] $((P))$ of the size μ. The case $\pi_P=\infty$ is excluded according to the promise at the beginning of this paper. When $\pi_P=\circledcirc$, the occupation $((P))$ is naturally understood as being left vacant. Let the space E transmuted by a metamorphosis be denoted by E_π and let it be accomodated with the share-value

$$\widetilde{\pi}(M)=\underset{P\in M}{\mathfrak{S}}\pi_P$$

to measure the medium-mass filled in a set M.

Proposition 7.—*If there is a family of sets $\{M_\tau\}_{\tau\in I}$ in the space E such that*

$$\widetilde{\pi}(M_\tau)>0 \quad \textit{for all}\ \tau\in I,$$

the indices' set I must be enumerable.

Demonstration. Devide E into an enumerable ensemble of rectangles $\{R_k\}$ of the same form and of the same (finite) size, and search for the former position of the medium filled in M_τ in which it was to be found before the metamorphosis. Then, as easily verified, there must be found at least one R_k in which the former mass of the medium of M_τ was measured as strictly positive before the metamorphosis. In addition, the number of M_τ thus corresponding to the same R_k is at most enumerable, because positive numbers can make by summation a finite positive number only in case of at most enumerable density. So, putting the two enumerabilities together, the density of I is concluded as to be also of enumerability.⌋

6. Repartition Postulate

E being transmuted to E_π by a metamorphosis, there exists at least one repartition $\{E_\tau\}_{\tau\in I}$ such that

$$E_\tau\cap E_\kappa=\text{void} \quad \text{for}\ \tau\neq\kappa,$$

$$E=\cup E_\tau$$

and $$0 < \tilde{\pi}(E_\tau) < \infty \text{ for all } \tau . \tag{6, 1}$$

If we merely demand such conditions, there may be no other way than to posit some postulate which gives them as satisfied. But, when

$$\pi_P < \infty \text{ for all } P, \tag{6, 2}$$

these conditions are satisfied if we define it as

$$E_P = \{P\} \text{ and } I = E,$$

except the alteration of (6, 1) by

$$0 \leqslant \tilde{\pi}(E_\tau) < \infty .$$

Since, if there can be no such set that

$$0 < \tilde{\pi}(M) < \infty , \tag{6, 3}$$

the condition (6, 2) induces that

$$(\tilde{\pi}(E_\pi) =)^* \ \tilde{\pi}(E) = 0 ,$$

there must exist such a set M as described in (6, 3). This being so, it may not be a big venture to assert a disjoint repartition $\{E_\tau\}$ as existent in such a manner as described in (6, 1); in effect there may be found no positive standpoint to deny this assertion. Ultimately, we may regard the above-stated postulate about the repartition of E as collateral to the condition (6, 2), which is assumed throughout this paper.

It is remarkable that, when E_π is provided with a disjoint repartition $\{E_\tau\}$ satisfying the condition (6, 1), E_π is simple in respect to $\tilde{\pi}$; the demonstration can be directly gained from Prop. 7. So, with regard to the above-stated collateral relation between (6, 2) and (6, 1), we may believe that, if $\{\pi_P\}$ makes an application system (with the restriction given in Section 1 : say, finite and non-negative), then E_π is simple in respect to $\tilde{\pi}$.

7. Constructive Aspect

In case of a regular system $\{\gamma_P\}$, it is specially distinguishable that we may then have an enumerable decomposition of E by the sets

$$E_k = \left\{P / \, k-1 \leqslant \frac{\gamma_P}{\mu} < k \right) \qquad (k = 1, 2, \cdots) ;$$

* E and E_π may be identified in point that E is the occupation of E_π.

in effect, when it is combined with the spherical decomposition given by

$$S_k = \{P/ k-1 \leqslant |P-Q| < k) \qquad (k=1, 2, \cdots)$$

Q being a fixed point, the space E may be found in a very concise construction.

In case of E_π, the repartition $\{E_k\}$ given in the previous section as to satisfy (6, 1), gives us an effectual method to analyse the frontier circumstance of ∞, which is denoted as

$$(\rbrack \infty \lbrack)$$

in the theory of a priori measure[5]. Let an increasing sequence of sets be defined by

$$\tilde{E}_k = \bigcup_{j \leqslant k} E_j \qquad (k=1, 2, \cdots),$$

then the medium filled in E may be exhausted by the process

$$\lim (E-\tilde{E}_k).$$

This being so, it must be confessed that we are assuming the part $(\rbrack \infty \lbrack)$ is either vacant or holding an invariant state of the medium. However, if we renounce the standard restriction

$$⊚ \leqslant \pi_P < \infty$$

and allow the case

$$\pi_P = \infty$$

as possible, the part $(\rbrack \infty \lbrack)$ must be the source of this pile of the medium; so the above-mentioned invariant state must be broken. This is the second gape for our logic.

8. Stochastic Metamorphosis

In the theory of a priori measure the occupation of a point is posited as provided with a geometric shape; in case of the rectangular coordinates it is elucidated as an infinitesimal parallelogram and so on. Therefore, when a quantitative transmutation is assumed to be realized by a metamorphosis, it may need some distortion if the direction of transition of the medium does not accord with any direction of the axes. In such a case, it is appropriate to apply the notion of a distribution of the theory of probability; in other words, the transmutation caused by a metamorphosis can be understood as a change of the distribution of the medium, the formula of which may be given as follows:

$$\pi_P/\mu = \lambda(P)$$

$\lambda(P)$ indicating the density of the distribution of the medium.

Conversely, the notion of a metamorphosis may be utilized in the probabilistic theory of distributions. For this purpose, the total mass of the medium should naturally be considered equal to 1. It is critic that the starting stage of the distribution is then to be given as homogeneous, whereas it is generally believed in the classical analysis that the homogeneous (probabilistic) distribution is impossible. Hence, at the starting stage, the measure of probability at the point P

$$\pi_P$$

is to make a non-negative homogeneous application system on condition that

$$1 = \tilde{\pi}(E) = \mathfrak{S}\, \pi_P \,. \tag{8, 1}$$

E being a finite-dimensional Euclidian space, if there exists a metamorphosis

$$\pi_P \longrightarrow \omega_P \tag{8, 2}$$

true fo our expectance, it shall be a singular one, because then, if

$$\tilde{\omega}(M) = \mathop{\mathfrak{S}}_{P \in M} \omega_P > 0 \tag{8, 3}$$

for a set M of a finite measure (say, $0 \leqslant \tilde{m}(M) < \infty$), it may be directly induced that at least at one point P we have

$$\omega_P / \pi_P = \infty \,.$$

Despite of such a condition, we cannot deny the metamorphosis as impossible, because, if we admit both of (8, 1) and (8, 3) as simultaneously existent, there may be no other physical way than to admit the metamorphosis (8, 2) to make an intermediation between the two states.

To permit the metamorphosis (8, 2), we have only to remark that the principal part on this transportation is $(]\infty[)$. It may be better understood by way of the calculation on the state (8, 1) itself. In effect, if

$$\tilde{\pi}(F) > 0$$

it must be that

$$\tilde{\pi}(F) = \tilde{\pi}(F \cap (]\infty[))\,;$$

because, on denoting it as

$$E_n = \{P / \, |P| \leqslant n\} \qquad (n = 1, 2, \cdots)\,,$$

we have

$$0 \leqslant \tilde{\pi}(F \cap E_n) \leqslant \tilde{\pi}(E_n) = 0$$

for all n, so that

$$\tilde{\pi}(F) = \lim \tilde{\pi}(F \cap (E - E_n)).$$

Besides, the inlaying of the medium of F into the set M may be interpreted as a transmutation of the point-measures

$$\pi_P \longrightarrow \omega_P .$$

$\tilde{\gamma}$-measurability is realized when the upper- and lower- destinations for the value of $\tilde{\gamma}(M)$ are found to be equal. When the two destinations give different values, the distribution share of the points of M is taken to be indeterminate in total. This will be the most characteristic difference of our view from the classical one. In our analysis, the indeterminate $\tilde{\gamma}$-value for a set M is regarded as oscillating between its upper- and lower-destinations.

(Received Apr. 16, 1960)

References

1) Kinokuniya, Y.: Mem. Muroran Univ. Eng. **2**, 511 (1956)
2) Kinokuniya, Y.: Mem. Muroran Univ. Eng. **2**, 221 (1955)
3) Kinokuniya, Y.: Men. Muroran Inst. Tech. **3**, 207 (1958)
4) Kinokuniya, Y.: A Course of Radonian Calculus, p. 5 (Sapporo, 1953)
5) Kinokuniya, Y.: Mem. Muroran Univ. Eng. **2** (3), 263 (1957)

Mathematical Seminar in the Muroran Inst. Eng., Hokkaido