On One Method of Solving Stress Problems in Cylindrical Co-ordinates by Means of Finite Fourier Hankel Transforms (Part 1)

On One Method of Solving Stress Problems in Cylindrical Co-ordinates by Means of Finite Fourier Hankel Transforms (Part I)

Sumio G. Nomachi

Abstract

The purpose of this paper is first to present the general solutions according to the finite Fourier Hankel transforms of the three-dimensional stress problems concerning a cylindrical body submitted to forces on its boundaries. Then making use of the solutions, stress problems of a hollow cylinder having uniform pressure on its inner and outer surfaces are analyzed, and cylinders with axialsymmetrical deformation are dealt with.

1. Introduction

Many problems in stress analysis which are of practical importance are concerned with a solid of revolution. For problems of this kind it is often convenient to use cylindrical co-ordinates. By means of Galerkin's vector[1] or Love's strain function[2] described in cylindrical co-ordinates, interesting problems have been solved. The procedure herein proposed is a kind of operator calculs. First of all we make the integral transforms with respect to the three components of forces around an element of solid by the kernel L. Then integrating by parts we get the integral transforms of stresses, and after making use of Hook's law, again integrating by parts we can obtain the integral transforms regarding three components of the displacement. Here if we choose an adequate function for L, we can easily find finite Fourier Hankel transforms of the components of displacement. The inversion theorems yield the components of displacement from which the six components of stresses are reduced.

2. Equations of Equilibrium of Forces and Green's Formulas

The stresses acting on six sides of a cubic element of an elastic medium are expressed by three normal stresses σ_r, σ_θ, and σ_z; and three shearing stresses $\tau_{\theta z}$, τ_{zr}, and $\tau_{r\theta}$. Then the equations of equilibrium of forces may be written in the well-known forms

$$\frac{\partial\sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}+\frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta}+\frac{\partial\tau_{zr}}{\partial z}=K_r\,, \qquad (1)$$

$$\frac{\partial \tau_{r\theta}}{\partial r}+\frac{2\tau_{r\theta}}{r}+\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta}+\frac{\partial \tau_{z\theta}}{\partial z}=K_\theta\,, \tag{2}$$

$$\frac{\partial \tau_{rz}}{\partial r}+\frac{\tau_{rz}}{r}+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \sigma_z}{\partial z}=K_z\,, \tag{3}$$

where K_r, K_θ, K_z denote components of body forces in r, θ, z directions, respectively.

If we let u, v, and w be the components of the displacement in r, θ, and z directions, respectively; and let ε_r, ε_θ, and ε_z be the direct strain, and $\gamma_{\theta z}$, γ_{zr}, and $\gamma_{r\theta}$ be the three tangential strains respectively, we have the relations

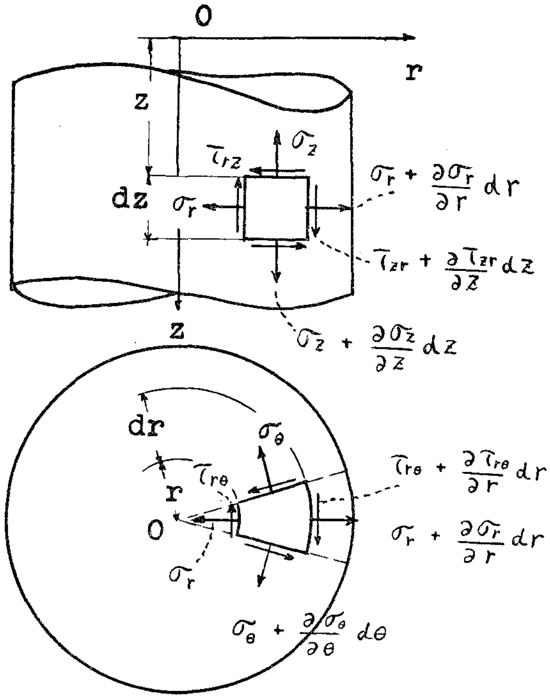

Fig. 1.

$$\left.\begin{aligned}
&\varepsilon_r=\frac{\partial u}{\partial r}\,, \qquad \varepsilon_\theta=\frac{u}{r}+\frac{\partial v}{r\partial \theta}\,, \qquad \varepsilon_z=\frac{\partial w}{\partial z}\,.\\
&\gamma_{r\theta}=\frac{\partial u}{r\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\,,\\
&\gamma_{zr}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\,,\\
&\gamma_{\theta z}=\frac{\partial v}{\partial z}+\frac{\partial w}{r\partial \theta}\,,
\end{aligned}\right\} \tag{4}$$

By makinguse of the above, Hook's law is expressed as

$$\left.\begin{aligned}
&\sigma_r=(2\mu+\lambda)\frac{\partial u}{\partial r}+\lambda\left(\frac{u}{r}+\frac{\partial v}{r\partial \theta}\right)+\lambda\frac{\partial w}{\partial z}\,,\\
&\sigma_\theta=\lambda\frac{\partial u}{\partial r}+(2\mu+\lambda)\left(\frac{u}{r}+\frac{\partial v}{r\partial \theta}\right)+\lambda\frac{\partial w}{\partial z}\,,\\
&\sigma_z=\lambda\frac{\partial u}{\partial r}+\lambda\left(\frac{u}{r}+\frac{\partial v}{r\partial \theta}\right)+(2\mu+\lambda)\frac{\partial w}{\partial z}\,,
\end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned}\tau_{\theta z} &= \mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{r\partial\theta}\right),\\ \tau_{zr} &= \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right),\\ \tau_{r\theta} &= \mu\left(\frac{\partial u}{r\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right),\end{aligned}\right\} \tag{9}$$

where μ, λ are the Lamé's elastic constants.

Multiplying the left side of Eq. (1) by L which is a function differentiatable two times with respect to r, θ, and z, and integrating by parts, we have

$$\begin{aligned}&\int[\sigma_r L_1]_a^b dA_r+\int[\tau_{r\theta}L_1]_0^\varphi dA_\theta+\int[\tau_{zr}L_1]_0^c dA_z\\ &\quad-\int\left\{\sigma_r\left(\frac{\partial L_1}{\partial r}-\frac{L_1}{r}\right)+\sigma_\theta\frac{L_1}{r}+\tau_{\theta r}\frac{\partial L_1}{r\partial\theta}+\tau_{zr}\frac{\partial L_1}{\partial z}\right\}dV=\int K_r L_1 dV,\end{aligned} \tag{7}$$

where

$$\begin{aligned}\int_a^b\int_0^\varphi\int_0^c f(r\theta z)\,drd\theta dz &= \int f(r\theta z)\,dV,\\ \int_0^\varphi\int_0^c f(r\theta z)\,d\theta dz &= \int f(r\theta z)\,dA_r,\\ \int_0^c\int_a^b f(r\theta z)\,dzdr &= \int f(r\theta z)\,dA_\theta,\\ \int_a^b\int_0^\varphi f(r\theta z)\,drd\theta &= \int f(r\theta z)\,dA_z.\end{aligned}$$

Putting herein the relations (4) and agin integrating by parts, we finally get

$$\left.\begin{aligned}&\int\Big[\sigma_r L_1\Big]_a^b dA_r+\int\Big[\tau_{r\theta}\frac{L_1}{r}\Big]_0^\varphi dA_\theta+\int\Big[\tau_{rz}L_1\Big]_0^c dA_z\\ &-\int\Big[u\left\{(2\mu+\lambda)\frac{\partial L_1}{\partial r}-2\mu\frac{L_1}{r}\right\}+\mu v\frac{\partial L_1}{r\partial\theta}+\mu w\frac{\partial L_1}{\partial z}\Big]_a^b dA_r\\ &-\int\Big[v\left\{\lambda\frac{\partial L_1}{r\partial r}+2\mu\frac{L_1}{r^2}\right\}+\mu u\frac{\partial L_1}{r\partial\theta}\Big]_0^\varphi dA_\theta\\ &-\int\Big[\lambda w\frac{\partial L_1}{\partial r}+\mu u\frac{\partial L_1}{\partial z}\Big]_0^c dA_z\\ &+\int u\left\{(2\mu+\lambda)\left(\frac{\partial^2 L_1}{\partial r^2}-\frac{\partial L_1}{r\partial r}\right)+\mu\left(\frac{\partial^2 L_1}{r^2\partial\theta^2}+\frac{\partial^2 L_1}{\partial z^2}\right)\right\}dV\\ &+\int v\left\{(\mu+\lambda)\frac{\partial^2 L_1}{r\partial r\partial\theta}+2\mu\frac{\partial L_1}{r^2\partial\theta}\right\}dV+\int(\mu+\lambda)w\frac{\partial^2 L_1}{\partial r\partial z}dV=\int K_r L_1 dV.\end{aligned}\right\} \tag{8}$$

Similary, Eqs. (2) and (3) yield the following formulas

$$\left.\begin{aligned}
&\iint\Big[\tau_{r\theta}L_2\Big]_a^b dA_r+\iint\Big[\sigma_\theta\frac{L_2}{r}\Big]_0^\varphi dA_\theta+\iint\Big[\tau_{\theta z}L_z\Big]_0^c dA_z\\
&\quad-\iint\Big[(2\mu+\lambda)v\frac{\partial L_2}{r^2\partial\theta}+\mu w\frac{\partial L_2}{r\partial z}+\mu u\Big(\frac{\partial L_2}{r\partial r}-2\frac{L_2}{r^2}\Big)\Big]_0^\varphi dA_\theta\\
&\quad-\iint\Big[\mu v\frac{\partial L_2}{\partial z}+\lambda w\frac{\partial L_2}{r\partial\theta}\Big]_0^c dA_z\\
&\quad-\iint\Big[\mu v\Big(\frac{\partial L_2}{\partial r}-2\frac{L_2}{r}\Big)+\lambda u\frac{\partial L_2}{r\partial\theta}\Big]_a^b dA_r\\
&\quad+\int u\Big\{(\mu+\lambda)\frac{\partial^2L_2}{r\partial r\partial\theta}-2(2\mu+\lambda)\frac{\partial L_2}{r^2\partial\theta}\Big\}dV\\
&\quad+\int v\Big\{\mu\Big(\frac{\partial^2L_2}{\partial r^2}-\frac{\partial L_2}{r\partial r}+\frac{\partial^2L_2}{\partial z^2}\Big)+(2\mu+\lambda)\frac{\partial^2L_2}{r^2\partial\theta^2}\Big\}dV\\
&\quad+\int(\mu+\lambda)w\frac{\partial^2L_2}{r\partial\theta\partial z}dV=\int K_\theta L_2 dV,
\end{aligned}\right\}\qquad(9)$$

and

$$\left.\begin{aligned}
&\iint\Big[\tau_{rz}L_3\Big]_a^b dA_r+\iint\Big[\tau_{\theta z}\frac{L_3}{r}\Big]_0^\varphi dA_\theta+\iint\Big[\sigma_zL_3\Big]_0^c dA_z\\
&\quad-\iint\Big[\mu w\Big(\frac{\partial L_3}{\partial r}-\frac{L_3}{r}\Big)+\lambda u\frac{\partial L_3}{\partial z}\Big]_a^b dA_r\\
&\quad-\iint\Big[\mu w\frac{\partial L_3}{r^2\partial\theta}+\lambda v\frac{\partial L_3}{r\partial z}\Big]_0^\varphi dA_\theta\\
&\quad-\iint\Big[\mu u\Big(\frac{\partial L_3}{\partial r}-\frac{L_3}{r}\Big)+\mu v\frac{\partial L}{r\partial\theta}+(2\mu+\lambda)w\frac{\partial L_3}{\partial z}\Big]_0^c dA_z\\
&\quad+\int(\mu+\lambda)\,ur\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial L_3}{\partial z}\Big)dV+\int(\mu+\lambda)v\frac{\partial^2L_3}{r\partial\theta\partial z}dV\\
&\quad+\int w\Big\{\mu\Big(\frac{\partial^2L_3}{\partial r^2}-\frac{\partial L_3}{r\partial r}+\frac{L_3}{r^2}+\frac{\partial^2L_3}{r^2\partial\theta^2}\Big)+(2\mu+\lambda)\frac{\partial^2L_3}{\partial z^2}\Big\}dV\\
&\quad=\int K_zL_3dV,
\end{aligned}\right\}\qquad(10)$$

where each of L_2 and L_3 are a function which can be differentiated two times with respect to r, θ, and z.

3. Symbolic Notations

A. Fourier Transforms and Inversion Theorem

If $f(x)$ satisfies Dirichlet's conditions in the intervla $(0, a)$ and if for that range its finite sine and cosine transform are defined to be

$$\left.\begin{aligned} \mathbf{S}_m[f(x)] &= \int_0^a f(x)\sin\frac{m\pi}{a}x\,dx\,, \\ \mathbf{C}_m[f(x)] &= \int_0^a f(x)\cos\frac{m\pi}{a}x\,dx\,, \end{aligned}\right\} \tag{11}$$

where $m = 1, 2, 3, \cdots,$

then, at each point of $(0, a)$ at which $f(x)$ is continuous

$$\left.\begin{aligned} f(x) &= \frac{2}{a}\sum_m \mathbf{S}_m[f(x)]\cdot\sin\frac{m\pi}{a}x\,, \\ f(x) &= \frac{2}{a}\left\{\frac{1}{2}\int_0^a f(x)\,dx + \sum_m \mathbf{C}_m[f(x)]\cdot\cos\frac{m\pi}{a}x\,,\right. \end{aligned}\right\} \tag{12}$$

which can be extended to functions of two or three variables.

B. Hankel Transforms and Inversion Theorems.

If $f(x)$ satisfies Dirichlet's condition in the interval $(0, a)$ and if for that range its finite Hankel transform is defined to be

$$\mathbf{J}_\nu[f(x)] = \int_0^a f(x)\,x J_\nu(\xi_i x)\,dx\,, \tag{13}$$

then, at each point of $(0, a)$ at which $f(x)$ is continuous

$$f(x) = \frac{2}{a^2}\sum_i \mathbf{J}_\nu[f(x)]\frac{J_\nu(x\xi_i)}{\{J'_\nu(a\xi_i)\}^2} \tag{14}$$

in which ξ_i is a root of the transcendental equation

$$J_\nu(a\xi_i) = 0\,, \tag{15}$$

or

$$f(x) = \frac{2}{a^2}\sum_i \mathbf{J}_\nu[f(x)]\frac{\xi_i^2}{h^2+\left(\xi_i^2-\frac{\nu_2}{a^2}\right)}\frac{J_\nu(x\xi_i)}{\{J_\nu(a\xi_i)\}^2} \tag{16}$$

in which ξ_i is a root of the transcendental equation

$$\xi_i J'_\nu(\xi_i a) + hJ_\nu(\xi_i a) = 0\,. \tag{17}$$

If $f(x)$ satisfies Dirichlet's condition in the interval (a, ∞) and if for that range its finite Hankel transform is defined as

$$\mathbf{Y}_\nu[f(x)] = \int_a^\infty f(x)\,x\,Y_\nu(\xi_i x)\,dx \tag{18}$$

where

$$Y_\nu(\xi_i x) = \frac{1}{\sin\nu\pi}\left(\cos\nu\pi\,J_\nu(\xi_i x) - J_{-\nu}(\xi_i x)\right) \tag{19}$$

then , at each point of (a, ∞) at which $f(x)$ is continuous

$$f(x) = \frac{2}{a^2} \sum_i \mathbf{Y}_\nu[f(x)] \frac{Y_\nu(\xi_i x)}{\{Y'_\nu(\xi_i a)\}^2} \tag{20}$$

in which ξ_i is a root of the transcendental equation

$$Y_\nu(\xi_i a) = 0\,, \tag{21}$$

or

$$f(x) = \frac{2}{a^2} \sum_i \frac{\xi_i^2 \mathbf{Y}_\nu[f(x)]}{h^2 + \left(\xi_i^2 - \frac{\nu^2}{a^2}\right)} \frac{Y_\nu(\xi_i x)}{\{Y_\nu(\xi_i a)\}^2} \tag{22}$$

in which ξ_i is a root of the transcendental equation

$$\xi_i Y'_\nu(\xi_i a) + h\, Y_\nu(\xi_i a) = 0\,. \tag{23}$$

If $f(x)$ satisfies Dirichlet's condition in the interval (a, b) and if for that range its finite Hankel transform is defined as

$$\mathbf{H}_\nu[f(x)] = \int_a^b f(x)\, x H_\nu(\xi_i x)\, dx\,, \tag{25}$$

where

$$H_\nu(\xi_i x) = J_\nu(\xi_i x)\, Y_\nu(\xi_i a) - J_\nu(\xi_i a)\, Y_\nu(\xi_i x) \tag{25}$$

then, at each point of (a, b) at which $f(x)$ is continuous

$$f(x) = \sum_i \frac{2\xi_i^2 J_\nu(\xi_i b)\, \mathbf{H}_\nu[f(x)]}{J_\nu^2(\xi_i a) - J_\nu^2(\xi_i b)} H_\nu(\xi_i x) \tag{26}$$

in which ξ_i is a root of the transcendental equation

$$J_\nu(\xi_i b)\, N_\nu(\xi_i a) - J_\nu(\xi_i a)\, Y_\nu(\xi_i b) = 0\,, \tag{27}$$

or

$$f(x) = \sum_i \frac{2\xi_i^2 \mathbf{H}_\nu[f(x)] \cdot H_\nu(\xi_i x)}{(h^2 + \xi_i^2)\, b^2 H_\nu'^2(\xi_i b) - \nu^2 H_\nu^2(\xi_i b)}\,, \tag{28}$$

where ξ_i is in this case a root of the transcendental equation

$$\xi_i H'_\nu(\xi_i b) + h\, H_\nu(\xi_i b) = 0\,. \tag{29}$$

4. Fourier Hankel Transforms of *u*, *v*, and *w*

Now we locate the cylindrical co-ordinates as shown in Fig. 2 and let

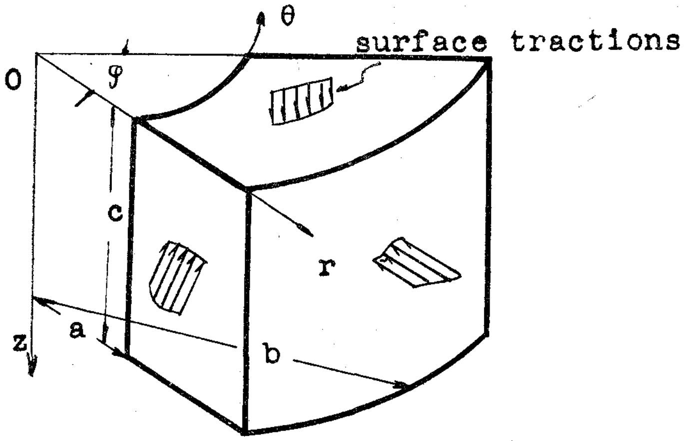

Fig. 2.

$$L_1 = \cos\frac{m\pi}{\varphi}\theta\cdot\cos\frac{n\pi}{c}z\cdot R\,,$$

$$L_2 = \sin\frac{m\pi}{\varphi}\theta\cdot\cos\frac{n\pi}{c}z\cdot R\,,$$

$$L_3 = \cos\frac{m\pi}{\varphi}\theta\cdot\sin\frac{n\pi}{c}z\cdot R_1\,.$$

$$m,\ n = 1,\ 2,\ 3, \cdots,$$

then with the abbreviations $M=\dfrac{m\pi}{\varphi}$, $N=\dfrac{n\pi}{c}$,
Eq. (8) yields
for $m=n=0$

$$\left.\begin{aligned}&\int\Big[\sigma_r R\Big]_a^b dA_r+\int\Big[\tau_{r\theta}\frac{R}{r}\Big]_0^\varphi dA_\theta+\int\Big[\tau_{rz}R\Big]_0^c dA_z\\&\quad-\int\Big[u\Big\{(2\mu+\lambda)\frac{dR}{dr}-2\mu\frac{R}{r}\Big\}\Big]_a^b dA_r\\&\quad-\int\Big(\lambda\frac{dR}{rdr}+2\mu\frac{R}{r}\Big)\Big\{v_{\theta=\varphi}-v_{\theta=0}\Big\}dA_\theta\\&\quad-\int\lambda\frac{dR}{dr}\Big\{w_{z=c}-w_{z=0}\Big\}dA_z+\int(2\mu+\lambda)u\Big(\frac{d^2R}{dr^2}-\frac{dR}{rdr}\Big)dr\\&\quad=\int K_r R dV,\end{aligned}\right\}\qquad(30)$$

for $m \neq n=0$

$$\left.\begin{aligned}
&\int\left\{\mathbf{C}_m[(\sigma_r)_{r=b}]R_{r=b}-\mathbf{C}_m[(\sigma_r)_{r=a}]R_{r=a}\right\}dA_r\\
&\quad+\int\frac{1}{r}\left\{(-1)^m[(\tau_{r\theta})_{\theta=\varphi}]-[(\tau_{r\theta})_{\theta=\theta}]\right\}dA_\theta+\int_a^b\left\{\mathbf{C}_m[(\tau_{rz})_{z=c}]-\mathbf{C}_m[(\tau_{rz})_{z=0}]\right\}dr\\
&\quad-\int_0^c\mathbf{C}_m[u_{r=b}]\left\{(2\mu+\lambda)\left(\frac{dR}{dr}\right)_{r=b}-2\mu\frac{R_{r=b}}{b}\right\}dz\\
&\quad+\int_0^c\mu M\cdot\mathbf{S}_m[v_{r=b}]\frac{R_{r=b}}{b}dz+\int_0^c\mathbf{C}_m[u_{r=a}]\left\{(2\mu+\lambda)\left(\frac{dR}{dr}\right)_{r=a}-2\mu\frac{R_{r=a}}{a}\right\}dz\\
&\quad-\int_0^c\mu M\cdot\mathbf{S}_m[v_{r=a}]\frac{R_{r=a}}{a}dz-\int\left\{\lambda\frac{dR}{rdr}+2\mu\frac{R}{r^2}\right\}\left\{(-1)^m[v_{\theta=\varphi}]\right.\\
&\quad\left.-[v_{\theta=0}]\right\}dA_\theta-\int_a^b\lambda\frac{dR}{dr}\left\{\mathbf{C}_m[w_{z=c}]-\mathbf{C}_m[w_{z=0}]\right\}dr\\
&\quad+\int\mathbf{C}_m[u]\left\{(2\mu+\lambda)\left(\frac{d^2R}{dr^2}-\frac{dR}{rdr}\right)-\mu M^2\frac{R}{r^2}\right\}dA_\theta\\
&\quad-\int M\mathbf{S}_m[v]\left\{(\mu+\lambda)\frac{dR}{rdr}+2\mu\frac{R}{r^2}\right\}dA_\theta=\int\mathbf{C}_m[K_r]\,RdA_\theta\,,
\end{aligned}\right\}\quad(31)$$

for $n \neq m = 0$

$$\left.\begin{aligned}
&\int_0^\varphi\left\{\mathbf{C}_n[(\sigma_r)_{r=b}]R_{r=b}-\mathbf{C}_n[(\sigma_r)_{r=a}]R_{r=a}\right\}d\theta+\int_a^b\frac{1}{r}\left\{\mathbf{C}_n[(\tau_{r\theta})_{\theta=\varphi}]\right.\\
&\quad\left.-\mathbf{C}_n[(\tau_{r\theta})_{\theta=0}]\right\}Rdr+\int\left[(-1)^n\left\{(\tau_{rz})_{z=c}\right\}-\left\{(\tau_{rz})_{z=0}\right\}\right]RdA_z\\
&\quad-\int_0^\varphi\left[\mathbf{C}_n[u]\left\{(2\mu+\lambda)\frac{dR}{dr}-2\mu\frac{R}{r}\right\}+\mu N\cdot\mathbf{S}_n[w]\cdot R\right]_a^b d\theta\\
&\quad-\int_a^b\left\{\lambda\frac{dR}{rdr}+2\mu\frac{R}{r}\right\}\left\{\mathbf{C}_n[v_{\theta=\varphi}]-\mathbf{C}_n[v_{\theta=0}]\right\}dr\\
&\quad-\int\lambda\frac{dR}{dr}\left\{(-1)^n[w_{z=c}]-[w_{z=0}]\right\}dA_z\\
&\quad+\int\mathbf{C}_n[u]\left\{(2\mu+\lambda)\left(\frac{d^2R}{dr^2}-\frac{dR}{rdr}\right)-\mu N^2R\right\}dA_z\\
&\quad-\int\mathbf{S}_n[w](\mu+\lambda)N\frac{dR}{dr}dA_z=\int\mathbf{C}_n[K_r]RdA_z\,,
\end{aligned}\right\}\quad(32)$$

for $m,\ n \neq 0$

$$\left.\begin{aligned}
&\mathbf{C}_m\mathbf{C}_n[(\sigma_r)_{r=b}]R_{r=b}-\mathbf{C}_m\mathbf{C}_n[(\sigma_r)_{r=a}]R_{r=a}+\int_a^b\frac{R}{r}\left\{(-1)^m\mathbf{C}_n[(\tau_{r\theta})_{\theta=\varphi}]\right.\\
&\quad\left.-\mathbf{C}_n[(\tau_{r\theta})_{\theta=0}]\right\}dr+\int_a^b R\left\{(-1)^n\mathbf{C}_m[(\tau_{rz})_{z=c}]-\mathbf{C}_m[(\tau_{rz})_{z=0}]\right\}dr\\
&\quad-\left[\mathbf{C}_m\mathbf{C}_n[u]\left\{(2\mu+\lambda)\frac{dR}{dr}-2\mu\frac{R}{r}\right\}-\mu\frac{M}{r}\mathbf{S}_m\mathbf{C}_n[v]R-\mu N\mathbf{C}_m\mathbf{S}_n[w]R\right]_a^b
\end{aligned}\right|$$

$$\left.\begin{aligned}
&-\int_a^b\left(\lambda\frac{dR}{rdr}+2\mu\frac{R}{r^2}\right)\left\{(-1)^m\mathbf{C}_n[v_{\theta=\varphi}]-\mathbf{C}_n[v_{\theta=0}]\right\}dr\\
&-\int_a^b\lambda\frac{dR}{dr}\left\{(-1)^n\mathbf{C}_m[w_{z=c}]-\mathbf{C}_m[w_{z=0}]\right\}dr\\
&+\int_a^b\mathbf{C}_m\mathbf{C}_n[u]\left\{(2\mu+\lambda)\left(\frac{d^2R}{dr^2}-\frac{dR}{rdr}\right)-\mu\frac{M^2R}{r^2}-\mu N^2R\right\}dr\\
&-\int_a^b\mathbf{S}_m\mathbf{C}_n[v]\left\{(\mu+\lambda)\frac{dR}{rdr}+2\mu\frac{R}{r^2}\right\}dr\\
&-\int_a^b\mathbf{C}_m\mathbf{S}_n[w](\mu+\lambda)N\frac{dR}{dr}dr=\int_a^b\mathbf{C}_m\mathbf{C}_n[K_r]Rdr\,.
\end{aligned}\right\}\quad(33)$$

Eq. (9) yields
for $n=0$

$$\left.\begin{aligned}
&\int_0^c\left\{\mathbf{S}_m[(\tau_{r\theta})_{r=b}]R_{r=b}-\mathbf{S}_m[(\tau_{r\theta})_{r=a}]R_{r=a}\right\}dz\\
&\quad+\int_a^b R\left\{\mathbf{S}_m[(\tau_{\theta z})_{z=c}]-\mathbf{S}_m[(\tau_{\theta z})_{z=0}]\right\}dr\\
&\quad-\int_a^b(2\mu+\lambda)\frac{R}{r^2}M\left\{(-1)^m v_{\theta=\varphi}-v_{\theta=0}\right\}dr\\
&\quad-\int_a^b\lambda\frac{R}{r}M\left\{\mathbf{C}_m[w_{z=c}]-\mathbf{C}_m[w_{z=0}]\right\}dr\\
&\quad-\int_0^c\left[\mu\mathbf{S}_m[v]\left(\frac{dR}{dr}-2\frac{R}{r}\right)+\lambda M\mathbf{C}_m[u]\frac{R}{r}\right]_a^b dz\\
&\quad+\int\mathbf{C}_m[u]\cdot M\left\{(\mu+\lambda)\frac{dR}{rdr}-2(2\mu+\lambda)\frac{R}{r^2}\right\}dA_\theta\\
&\quad+\int\mathbf{S}_m[v]\left\{\mu\left(\frac{d^2R}{dr^2}-\frac{dR}{rdr}\right)-(2\mu+\lambda)M^2\frac{R}{r^2}\right\}dA_\theta\\
&\quad=\int\mathbf{C}_m[K_\theta]R\,dA_\theta\,.
\end{aligned}\right\}\quad(34)$$

for $n\neq 0$

$$\left.\begin{aligned}
&\mathbf{S}_m\mathbf{C}_n[(\tau_{r\theta})_{r=b}]R_{r=b}-\mathbf{S}_m\mathbf{C}_n[(\tau_{r\theta})_{r=a}]R_{r=a}\\
&\quad+\int_a^b R\left\{(-1)^n\mathbf{S}_m[(\tau_{\theta z})_{z=c}]-\mathbf{S}_m[(\tau_{\theta z})_{z=b}]\right\}dr\\
&\quad-\int_a^b(2\mu+\lambda)\frac{R}{r^2}M\left\{(-1)^m\mathbf{C}_n[v_{\theta=\varphi}]-\mathbf{C}_n[v_{\theta=0}]\right\}dr\\
&\quad-\int_a^b\lambda\frac{R}{r}M\left\{(-1)^n\mathbf{C}_m[w_{z=c}]-\mathbf{C}_m[w_{z=0}]\right\}dr\\
&\quad-\left[\mu\mathbf{S}_m\mathbf{C}_n[v]\left(\frac{dR}{pr}-2\frac{R}{r}\right)+\lambda M\mathbf{C}_m\mathbf{C}_n[u]\frac{R}{r}\right]_a^b
\end{aligned}\right\}\quad(35)$$

$$
\left.
\begin{aligned}
&+\int_a^b \mathbf{C}_m\mathbf{C}_n[u]\left\{(\mu+\lambda)M\frac{dR}{rdr}-2(2\mu+\lambda)\frac{R}{r^2}M\right\}dr\\
&+\int_a^b \mathbf{S}_m\mathbf{C}_n[v]\left\{\mu\left(\frac{d^2R}{dr^2}-\frac{dR}{rdr}-N^2R\right)-(2\mu+\lambda)M^2\frac{R}{r^2}\right\}dr\\
&-\int_a^b \mathbf{C}_m\mathbf{S}_n[w](\mu+\lambda)NM\frac{R}{r}dr=\int_a^b \mathbf{S}_m\mathbf{C}_n[K_\theta]R\,dr\,.
\end{aligned}
\right|
$$

Eq. (10) yields
for $m=0$

$$
\left.
\begin{aligned}
&\int_0^\varphi\left\{\mathbf{S}_n[(\tau_{rz})_{r=b}]R_{r=b}-\mathbf{S}_n[(\tau_{rz})_{r=a}]R_{r=a}\right\}d\theta\\
&\quad+\int_a^b\frac{R}{r}\left\{\mathbf{S}_n[(\tau_{\theta z})_{\theta=\varphi}]-\mathbf{S}_n[(\tau_{\theta z})_{\theta=0}]\right\}dr\\
&\quad-\int_0^\varphi\left[\mu\mathbf{S}_n[w]\left\{\frac{dR}{dr}-\frac{R}{r}\right\}\right]_{r=a}^{r=b}d\theta-\int_0^\varphi\Big[\lambda N\mathbf{C}_n[u]R\Big]_{r=a}^{r=b}d\theta\\
&\quad-\int_a^b\lambda N\frac{R}{r}\left\{\mathbf{C}_n[v_{\theta=\varphi}]-\mathbf{C}_n[v_{\theta=0}]\right\}dr\\
&\quad-\int(2\mu+\lambda)RN\left\{(-1)^n w_{z=c}-w_{z=0}\right\}dA_z\\
&\quad+\int(\mu+\lambda)N\mathbf{C}_n[u]\left\{r\frac{d}{rd}\left(\frac{R}{r}\right)\right\}dA_z\\
&\quad+\int\mathbf{S}_n[w]\left\{\mu\left(\frac{d^2R}{dr^2}-\frac{dR}{rd^{\perp}}+\frac{R}{r^2}\right)-(2\mu+\lambda)N^2R\right\}dA_z\\
&\quad=\int\mathbf{S}_n[K_z]R\,dA_z\,.
\end{aligned}
\right\}\qquad(36)
$$

for $m\neq0$

$$
\left.
\begin{aligned}
&\mathbf{C}_m\mathbf{S}_n[(\tau_{rz})_{r=b}]R_{r=b}-\mathbf{C}_m\mathbf{S}_n[(\tau_{rz})_{r=a}]R_{r=a}\\
&\quad+\int_a^b\frac{R}{r}\left\{(-1)^m\mathbf{S}_n[(\tau_{\theta z})_{\theta=\varphi}]-\mathbf{S}_n[(\tau_{\theta z})_{\theta=0}]\right\}dr\\
&\quad-\left[\mu\mathbf{C}_m\mathbf{S}_n[w]\left\{\frac{dR}{dr}-\frac{R}{r}\right\}\right]_{r=a}^{r=b}-\Big[\lambda N\mathbf{C}_m\mathbf{C}_n[u]R\Big]_{r=a}^{r=b}\\
&\quad-\int_a^b\lambda N\frac{R}{r}\left\{(-1)^m\mathbf{C}_n[v_{\theta=\varphi}]-\mathbf{C}_n[v_{\theta=0}]\right\}dr\\
&\quad-\int_a^b(2\mu+\lambda)RN\left\{(-1)^n\mathbf{C}_m[w_{z=c}]-\mathbf{C}_m[w_{z=0}]\right\}dr\\
&\quad+(\mu+\lambda)\int_a^b N\mathbf{C}_m\mathbf{C}_n[u]\left\{r\frac{d}{dr}\left(\frac{R}{r}\right)\right\}dr\\
&\quad-(\mu+\lambda)\int_a^b NM\mathbf{S}_m\mathbf{C}_n[v]\frac{R}{r}dr+\int_a^b\mathbf{C}_m\mathbf{S}_n[w]\left\{\mu\frac{d^2R}{dr^2}-\frac{dR}{rdr}\right.\\
&\quad\left.+\frac{R}{r^2}-M^2\frac{R}{r^2}\right)-(2\mu+\lambda)N^2R\Big\}dr=\int_a^b\mathbf{C}_m\mathbf{S}_n[K_z]R\,dr\,.
\end{aligned}
\right\}\qquad(37)
$$

5. Axially symmetrical Stress Distribution in a Solid of Revolution

In this case, the components of the displacement vector and of the stress tensor will all be independent of the angle θ, and the operator $\partial/\partial\theta$ will be a null operator. The formulas corresponding to the above are Eq. (30), Eq. (32), and Eq. (36). Eq. (30) presents the two-dimensional axially symmetrical stress distribution, and tribution, and Eqs. (32) and (36) show the three-dimensional axially symmetrical stress distribution.

A. The Two-dimensional Stress Distribution

The stress distribution is, in this case, not only independent of the angle θ but also of the length z, so we have

$$\tau_{r\theta} = \tau_{zr} = 0\,,$$

$$v_{\theta=\varphi} = v_{\theta=0}\,,$$

$$w_{z=c} = w_{z=0}\,,$$

Eq. (30) accordingly yields

$$\left.\begin{aligned}&(R_{r=b})\cdot(\sigma_r)_{r=b}-(R_{r=a})\cdot(\sigma_r)_{r=a}\\&\quad-\left[u(2\mu+\lambda)\frac{dR}{dr}-2\mu\frac{R}{r}\Big\}\right]_a^b\\&\quad+(2\mu+\lambda)\int_a^b u\left(\frac{d^2R}{dr^2}-\frac{dR}{rdr}\right)dr=\int_a^b K_r R\,dr\,.\end{aligned}\right\}\qquad(38)$$

Choosing R as

$$R = rH_1(\xi_i r) = r\{J_1(\xi_i r)\cdot Y_1(\xi_i a) - J_1(\xi_i a)\,Y_1(\xi_i r)\}\,,$$

in which ξ_i is a root of the transcendental equation

$$J_1(\xi_i b)\,Y_1(\xi_i a) - J_1(\xi_i a)\,Y_1(\xi_i b) = 0\,,$$

then we have

$$\mathbf{H}_1[u] = \frac{u_b H_1'(\xi_i b)}{\xi_i} - \frac{u_a H_1'(\xi_i a)}{\xi_i} + \frac{\mathbf{H}_1[K_r]}{(2\mu+\lambda)\xi_i^2} \qquad (39)$$

where

$$\mathbf{H}_1[u] = \int_a^b u\cdot H_1(\xi_i r)\,r\,dr\,.$$

By making use of the relations

$$\left.\begin{aligned} &\frac{a_\nu b_\nu}{a^{2\nu}-b^{2\nu}}\cdot\mathbf{H}_\nu\left[\frac{x^\nu}{a^\nu}-\frac{a^\nu}{x^\nu}\right]=-\frac{H'_\nu(\xi_i b)}{\xi_i}\,,\\ &\frac{a^\nu b^\nu}{a^{2\nu}-b^{2\nu}}\cdot\mathbf{H}_\nu\left[\frac{x^\nu}{b^\nu}-\frac{b^\nu}{x^\nu}\right]=-\frac{H'_\nu(\xi_i a)}{\xi_i}\,, \end{aligned}\right\}\quad(40)$$

where

$$H_\nu(\xi_i b)=0\,,$$

the displacement is expressed by

$$\left.\begin{aligned} u=&\left\{-u_b\left(\frac{x}{a}-\frac{a}{x}\right)+u_a\left(\frac{x}{b}-\frac{b}{x}\right)\right\}\frac{ab}{a^2-b^2}\\ &+\sum_i\frac{2\mathbf{H}[K_r]}{b^2(2\mu+\lambda)\,\xi_i^2}H_1(\xi_i r)/\Theta_i^2\,,\quad \Theta_i^2=H_0^2(\xi_i b)-\frac{a^2}{b^2}H_0^2(\xi a)\,. \end{aligned}\right\}\quad(41)$$

This result may be adopted to represent the stress distribution in a hollow cylinder submitted to uniform pressure on the inner and outer surfaces (Fig. 3). Let p_i and p_0 be the uniform internal and external pressures, then the boundary conditions are

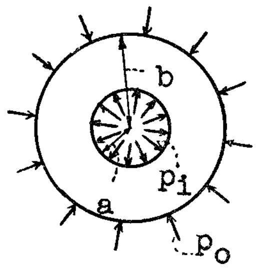

Fig. 3.

$$\left.\begin{aligned} \sigma_r)_{r=a}&=(2\mu+\lambda)\frac{\partial u}{\partial r}+\lambda\frac{u}{r}\Big)_{r=a}=-p_i\,,\\ \sigma_r)_{r=b}&=(2\mu+\lambda)\frac{\partial u}{\partial r}+\lambda\frac{u}{r}\Big)_{r=b}=-p_0\,, \end{aligned}\right\}\quad(42)$$

When the body force is absent, that is $K_r=0$, Eq. (42) yields

$$\left.\begin{aligned} &u_b\left\{\frac{2}{a}(\mu+\lambda)+2\mu\frac{a}{b^2}\right\}+2u_a\frac{1}{b}(2\mu+\lambda)=-p_0\frac{a^2-b^2}{ab}\,,\\ &2u_b\frac{1}{a}(2\mu+\lambda)+u_a\left\{\frac{2}{b}(\mu+\lambda)+2\mu\frac{b}{a^2}\right\}=-p_i\frac{a^2-b^2}{ab}\,, \end{aligned}\right\}\quad(43)$$

from which we have

$$\left.\begin{aligned} \sigma_r&=\frac{a^2b^2(p_0-p_i)}{b^2-a^2}\frac{1}{r^2}+\frac{p_ia^2-p_0b^2}{b^2-a^2}\,,\\ \sigma_\theta&=-\frac{a^2b^2(p_0-p_i)}{b^2-a^2}\frac{1}{r^2}+\frac{p_ia^2-p_0b^2}{b^2-a^2}\,. \end{aligned}\right\}\quad(44)$$

The above mentioned solution is the same as the solution due to Lamé[3].

B. Three-dimensional Stress Distribution

In this case, the stress distributions are independent of the angle θ, so

$$\tau_{\theta r}=0\,,$$

$$v_{\theta=\varphi} = v_{\theta=0}\,.$$

Eq. (32) and Eq. (36) are accordingly expressed by

$$\left.\begin{aligned}
&\mathbf{C}_n[(\sigma_r)_{r=b}]R_{r=b}-\mathbf{C}_n[(\sigma_r)_{r=a}]R_{r=a}+\int_a^b\Big[(-1)^n\{(\tau_{rz})_{z=c}\}\\
&\quad-\{(\tau_{rz})_{z=c}\}\Big]R\,dr-\Big[\mathbf{C}_n[u]\Big\{(2\mu+\lambda)\frac{dR}{dr}-2\mu\frac{R}{r}\Big\}\\
&\quad+\mu N\mathbf{S}_n[w]R\Big]_a^b-\int_a^b\lambda\frac{dR}{dr}\Big\{(-1)^n[w_{z=c}]-[w_{z=0}]\Big\}\,dr\\
&\quad+\int\mathbf{C}_n[u]\Big\{(2\mu+\lambda)\Big(\frac{d^2R}{dr^2}-\frac{dR}{rdr}\Big)-\mu N^2R\Big\}\,dA_z\\
&\quad-\int\mathbf{S}_n[w](\mu+\lambda)N\frac{dR}{dr}\,dA_z=\int\mathbf{C}_n[K_r]R\,dA_z\,,
\end{aligned}\right\}\tag{45}$$

and

$$\left.\begin{aligned}
&\mathbf{S}_n[(\tau_{rz})_{r=b}]R_{r=b}-\mathbf{S}_n[(\tau_{rz})_{r=a}]R_{r=a}-\Big[\mu\,\mathbf{S}_n[w]\Big\{\frac{dR}{rd}-\frac{R}{r}\Big\}\Big]_a^b\\
&\quad-\Big[\lambda N\mathbf{C}_n[u]\cdot R\Big]_a^b-\int_a^b(2\mu+\lambda)N\cdot R\{(-1)^n w_{z=c}-w_{z=0}\}\,dA_z\\
&\quad+\int_a^b(\mu+\lambda)N\mathbf{C}_n[u]\Big\{r\frac{d}{dr}\Big(\frac{R}{r}\Big)\Big\}\,dr\\
&\quad+\int_a^b\mathbf{S}_n[w]\Big\{\mu\Big(\frac{d^2R}{dr_2}-\frac{dR}{rdr}+\frac{R}{r^2}\Big)+(2\mu+\lambda)N^2R\Big\}\,dr\\
&\quad=\int_a^b\mathbf{S}_n[K_z]R\,dr\,.
\end{aligned}\right\}\tag{46}$$

Letting

$$\mathbf{H}_1[f(r)]=\int_a^b f(r)H_1(\xi_i r)\,rdr\,,\qquad \mathbf{H}_0[f(r)]=\int_a^b f(r)H_0(\xi_i r)\,rdr\,,$$

$$H_1(\xi_i r)=J_1(\xi_i r)Y_1)(\xi_i a)-J_1(\xi_i a)\,Y_1(\xi_i r)$$

$$H_0(\xi_i r)=J_0(\xi_i r)\,Y_1(\xi_i a)-J_1(\xi_i a)\,Y_0(\xi_i r)$$

in which ξ_i is a root of the transcendental equation

$$H_1(\xi_i b)=0\,,$$

we find, from Eq. (45) and Eq. (46), that

$$\left.\begin{aligned}
&(-1)^n\mathbf{H}_1[(\tau_{rz})_{z=c}]-\mathbf{H}_1[(\tau_{rz})_{z=0}]\\
&\quad-(2\mu+\lambda)\xi_i\{H_0(\xi_i b)\cdot\mathbf{C}_n[u_{r=b}]-H_0(\xi_i a)\cdot\mathbf{C}_n[u_{r=a}]\}\\
&\quad-\lambda(-1)^n\mathbf{H}_0[w_{z=c}]-\mathbf{H}_0[w_{z=0}]\\
&\quad-\mathbf{H}_1\mathbf{C}_n[u]\{(2\mu+\lambda)\xi_i^2+\mu N^2\}\\
&\quad-\mathbf{H}_0\mathbf{S}_n[w](\mu+\lambda)N\xi_i=\mathbf{H}_1\mathbf{C}_n[K_r]\,,
\end{aligned}\right\}\tag{48}$$

and

$$\left.\begin{aligned}&\mathbf{S}_n[(\tau_{rz})_{r=b}]H_0(\xi_1 b)-\mathbf{S}_n[(\tau_{rz})_{r=a}]H_0(\xi_i a)\\&\quad-\lambda N\mathbf{C}_n[u_{r=b}]H_0(\xi_i b)+\lambda N\mathbf{C}_n[u_{r=a}]H_0(\xi_i a)\\&\quad-(2\mu+\lambda)N\{(-1)^n\mathbf{H}_0[w_{z=c}]-\mathbf{H}_0[w_{z=0}]\}\\&\quad-\mathbf{H}_1\mathbf{C}_n[u](\mu+\lambda)N\xi_i\\&\quad-\mathbf{H}_0\mathbf{S}_n[w]\mu\xi_i^2+(2\mu+\lambda)N^2\}=\mathbf{H}_0\mathbf{S}_n[K_z]\,.\end{aligned}\right\}\qquad(48)$$

Putting for convenience sake

$$\begin{aligned}&\mathbf{C}_n[u_{r=b}]=\frac{c}{2}A_n\,, &&\mathbf{C}_n[u_{r=a}]=\frac{c}{2}A'_n\\&\mathbf{S}_n[(\tau_{rz})_{r=b}]=\frac{c}{2}NB_n\,, &&\mathbf{S}_n[(\tau_{rz})_{r=a}]=\frac{c}{2}NB'_n\\&\mathbf{H}_0[w_{z=c}]=\frac{b^2}{2}(D_i+D'_i) &&\mathbf{H}_0[w_{z=0}]=\frac{b^2}{2}(D_i-D'_i)\\&\mathbf{H}_1[(\tau_{rz})_{z=c}]=\frac{b^2}{2}\xi_i(E_i+E'_i) &&\mathbf{H}_1[(\tau_{rz})_{z=0}]=\frac{b^2}{2}\xi_i(E_i-E'_i)\end{aligned}$$

into Eqs. (47) and (48), and solving them simultaneously, we find the Fourier Hankel transformations regarding u and w as

$$\left.\begin{aligned}&\mathbf{H}_1\mathbf{C}_n[u]=-\frac{c}{2}\Big\{bA_nH_0(\xi_i b)-aA'_nH_0(\xi_i a)\Big\}\Big\{\frac{\xi_i}{N^2+\xi_i^2}+\frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{\xi_iN^2}{(N^2+\xi_i^2)^2}\\&\quad-\frac{c}{2}\Big\{bB_nH_0(\xi_i b)-aB'_nH_0(\xi_i a)\Big\}\Big\{\frac{N\xi_i(\mu+\lambda)}{\mu(2\mu+\lambda)(N^2+\xi_i^2)^2}\\&\quad-\frac{b^2}{2}\Big\{D_i(1-(-1)^n)+D'_i)1+(-1)^n)\Big\}\Big\{\frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{N^2\xi_i}{(N^2+\xi_i^2)^2}\\&\quad-\frac{\lambda}{2\mu+\lambda}\frac{\xi_i}{N^2+\xi_i^2}\Big\}-\frac{b^2}{2}\Big\{E_i(1-(-1)^n+E'_i(1+(-1)^n)\frac{1}{2\mu+\lambda}\frac{1}{N^2+\xi_i^2}\\&\quad+\frac{\mu+\lambda}{\mu(2\mu+\lambda)}\frac{N^2}{(N^2+\xi_i^2)^2}\Big\}\end{aligned}\right\}\qquad(49)$$

$$\left.\begin{aligned}&\mathbf{H}_0\mathbf{S}_n[w]=-\frac{c}{2}\Big\{bA_nH_0(\xi_i b)-aA'_nH_0(\xi_i a)\Big\}\Big\{\frac{\lambda}{2\mu+\lambda}\frac{N}{N^2+\xi_i^2}\\&\quad-\frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{N\xi_i^2}{(N^2+\xi_i^2)^2}\Big\}+\frac{c}{2}\Big\{(bB_nH_0(\xi_i b)-aB'_nH_0(\xi_i a)\Big\}\\&\quad\times\Big\{\frac{1}{2\mu+\lambda}\frac{1}{N^2+\xi_i^2}+\frac{\mu+\lambda}{\mu(2\mu+\lambda)}\frac{\xi_i^2}{(N^2+\xi_i^2)^2}\Big\}\\&\quad+\frac{b^2}{2}\Big\{D_i(1-(-1)^n)+D'_i(1+(-1)^n)\Big\}\Big\{\frac{N}{N^2+\xi_i^2}+\frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{\xi_iN}{(N^2+\xi_i^2)^2}\\&\quad+\frac{b^2}{2}\Big\{E_i(1-(-1)^n)+E'_i(1+(-1)^n)\Big\}\frac{(\mu+\lambda)N\xi_i}{\mu(2\mu+\lambda)(N^2+\xi_i^2)^2}\end{aligned}\right\}\qquad(50)$$

According to the inversion theorem

$$u=\sum_i \frac{4}{cb^2}\frac{H_1(\xi_i r)}{\{\Theta_i\}^2}\left\{\frac{1}{2}\int_0^c \mathbf{H}_1[u]\,dz+\sum_n \cos Nz\cdot\mathbf{H}_1\mathbf{C}_n[u]\right\},$$

$$w=\sum_i\sum_n \frac{4}{cb^2}\sin Nz\frac{H_0(\xi_i r)}{\{\Theta_i\}^2}\mathbf{H}_0\mathbf{S}_n[w]+\sum_n \frac{4}{c^2}\sin Nz\int_a^b \frac{\mathbf{S}_n[w]rdr}{b^2-a^2},$$

u and w are written as

$$\left.\begin{aligned}
u=&\sum_i \frac{1}{b}\frac{H_1(\xi_i r)}{\{\Theta_i\}^2\xi_i}\left\{H_0(\xi_i b)A_0-\frac{a}{b}A_0'H_0(\xi_i a)\right\}\\
&+\sum_i\sum_n\frac{H_1(\xi_i r)}{\{\Theta_i\}^2}\cos Nz\left[\frac{2}{b}\left\{-A_nH_0(\xi_i b)+\frac{a}{b}A_n'H_0(\xi_i a)\right\}\right.\\
&\times\left\{\frac{\xi_i}{N^2+\xi_i^2}+\frac{2(\mu+\lambda)\xi_iN^2}{(2\mu+\lambda)(N^2+\xi_i^2)^2}\right\}-\frac{2}{b}\left\{B_nH_0(\xi_i b)-\frac{a}{b}B_n'H_0(\xi_i a)\right\}\\
&\times\frac{(\mu+\lambda)N^2\xi_i}{\mu(2\mu+\lambda)(N^2+\xi_i^2)^2}\Big\}-\frac{2}{c}\left\{D_i(1-(-1)^n)+D_i'(1+(-1)^n)\right\}\\
&\times\frac{2(\mu+\lambda)N^2\xi_i}{(2\mu+\lambda)(N^2+\xi_i^2)^2}-\frac{\lambda\xi_i}{(2\mu+\lambda)(N^2+\xi_i^2)}\Big\}-\frac{2}{c}\Big\{E_i(1-(-1)^n\\
&\left.+E_i'(1+(-1)^n)\right\}\left\{\frac{\xi_i}{(2\mu+\lambda)(N^2+\xi_i^2)}+\frac{(\mu+\lambda)N\xi_i}{\mu(2\mu+\lambda)(N^2+\xi_i^2)^2}\right\}\Bigg]
\end{aligned}\right\}\quad(51)$$

and

$$\left.\begin{aligned}
w=&\sum_i\sum_n \sin Nz\frac{H_0(\xi_i r)}{\{\Theta_i\}^2}\left[\frac{2}{b}\left\{-A_nH_0(\xi_i b)+\frac{a}{b}A_n'H_0(\xi_i a)\right\}\right.\\
&\times\left\{\frac{\lambda N}{(2\mu+\lambda)(N^2+\xi_i^2)}-\frac{2(\mu+\lambda)N\xi_i^2}{(2\mu+\lambda)(N^2+\xi_i^2)^2}\right\}+\frac{2}{b}\Big\{B_nH_0(\xi_i b)\\
&-\frac{a}{b}B_n'H_0(\xi_i a)\Big\}\left\{\frac{N}{(2\mu+\lambda)(N^2+\xi_i^2)}+\frac{(\mu+\lambda)N\xi_i^2}{\mu(2\mu+\lambda)(N^2+\xi_i^2)^2}\right\}\\
&+\frac{2}{c}\left\{D_i(1-(-1)^n)+D_i'(1+(-1)^n)\right\}\left\{\frac{N}{N^2+\xi_i^2}+\frac{2(\mu+\lambda)N\xi_i^2}{(2\mu+\lambda)(N^2+\xi_i^2)^2}\right.\\
&+\frac{2}{c}\left\{E_i(1-(-1)^n)+E_i'(1+(-1)^n)\right\}\left\{-\frac{2ab}{(a^2-b^2)}\sum_n \sin Nz\right.\\
&\times\left\{\frac{\lambda}{2\mu+\lambda}\left(\frac{A_n}{Na}-\frac{A_n'}{Nb}\right)+\frac{1}{2\mu+\lambda}\left(\frac{B_n}{Na}-\frac{B_n'}{Nb}\right)\right\}.
\end{aligned}\right\}\quad(52)$$

By the aid of the following formulas:

$$\left.\begin{aligned}
&\sum_i \frac{2}{b}H_0(\xi_i b)\frac{H_i(\xi_i r)}{\Theta_i^2}\frac{\xi_i}{N^2+\xi_i^2}=-\frac{R_{11}^{(1)}(Nr)}{R_{11}^{(1)}(Nb)},\\
&\sum_i \frac{2a}{b^2}\frac{H_0(\xi_i a)H_1(\xi_i r)}{\{\Theta_i\}^2}\frac{\xi_i}{N^2+\xi_i^2}=-\frac{R_{11}^{(2)}(Nr)}{R_{11}^{(2)}(Na)},
\end{aligned}\right\}\quad(53)$$

$$\left.\begin{aligned}
&\sum_i \frac{4}{b} H_0(\xi_i b)\frac{H_1(\xi_i r)}{\Theta_i^2}\frac{N\xi_i}{(N^2+\xi_i^2)^2} = \frac{R_{11}^{(1)}(Nr)\{bR_{01}^{(1)}(Nb)-aR_{10}^{(1)}(Nb)\}}{\{R_{11}^{(1)}(Nb)\}^2} \\
&\qquad -\frac{R_{11}^{(1)}(Nb)\{rR_{01}^{(1)}(Nr)-aR_{10}^{(1)}(Nr)\}}{\{R_{11}^{(1)}(Nb)\}^2}, \\
&\sum_i \frac{4a}{b^2}\frac{H_0(\xi_i a)H_1(\xi_i r)}{\{\Theta_i\}^2}\frac{N\xi_i}{(N^2+\xi_i^2)^2} = \frac{R_{11}^{(2)}(Nr)\{aR_{01}^{(2)}(Na)-bR_{10}^{(2)}(Na)\}}{\{R_{11}^{(2)}(Na)\}^2} \\
&\qquad -\frac{R_{11}^{(2)}(Na)\{rR_{01}^{(2)}(Nr)-bR_{10}^{(2)}(Nr)\}}{\{R_{11}^{(2)}(Na)\}^2},
\end{aligned}\right\} \tag{54}$$

where

$$\left.\begin{aligned}
R_{11}^{(1)}(Nr) &= I_1(Nr)K_1(Na)-I_1(Na)K_1(Nr)^*, \\
R_{01}^{(1)}(Nr) &= I_0(Nr)K_1(Na)+I_1(Na)K_0(Nr), \\
R_{10}^{(1)}(Nr) &= I_1(Nr)K_0(Na)+I_0(Na)K_1(Nr), \\
R_{11}^{(2)}(Nr) &= I_1(Nr)K_1(Nb)-I_1(Nb)K_1(Nr), \\
R_{01}^{(2)}(Nr) &= I_0(Nr)K_1(Nb)+I_1(Nb)K_0(Nr), \\
R_{10}^{(2)}(Nr) &= I_1(Nr)K_0(Nb)+I_0(Nb)K_1(Nr).
\end{aligned}\right\} \tag{55}$$

$$\left.\begin{aligned}
&\sum \frac{2}{b} H_0(\xi_i b)\frac{H_0(\xi_i r)}{\Theta_i^2}\frac{N}{N^2+\xi_i^2} = \frac{R_{01}^{(1)}(Nr)}{R_{11}^{(1)}(Nb)} - \frac{2b}{(a^2-b^2)N}, \\
&\sum \frac{2a}{b^2}\frac{H_0(\xi_i a)H_1(\xi_i r)}{\{\Theta_i\}^2}\frac{N}{N^2+\xi_i^2} = -\frac{R_{01}^{(2)}(Nr)}{R_{11}^{(2)}(Na)} - \frac{2a}{(a^2-b^2)N},
\end{aligned}\right\} \tag{56}$$

$$\left.\begin{aligned}
&\sum \frac{4}{b} H_0(\xi_i b)\frac{H_0(\xi_i r)}{\Theta_i^2}\frac{\xi_i^2}{(N^2+\xi_i^2)^2} = \frac{2R_{01}^{(1)}(Nr)}{NR_{11}(Nb)} \\
&\quad -\frac{R_{01}^{(1)}(Nr)\{bR_{01}^{(1)}(Nb)-aR_{10}^{(1)}(Nb)\}-R_{11}^{(1)}(Nb)\{rR_{11}^{(1)}(Nr)-aR_{00}^{(1)}(Nr)\}}{\{R_{11}^{(1)}(Nb)\}^2}, \\
&\sum \frac{4a}{b^2}\frac{H_0(\xi_i a)H_0(\xi_i r)}{\{\Theta_i\}^2}\frac{\xi_i^2}{(N^2+\xi_i^2)^2} = -\frac{2R_{01}^{(2)}(Nr)}{NR_{11}^{(2)}(Na)} \\
&\quad +\frac{R_{01}^{(2)}(Nr)\{aR_{01}^{(2)}(Na)-bR_{10}^{(2)})Na)\}-R_{11}^{(2)}(Na)\{rR_{11}^{(2)}(Nr)-bR_{00}^{(2)}(Nr)\}}{\{R_{11}^{(2}(Na)\}^2},
\end{aligned}\right\} \tag{57}$$

where

$$\left.\begin{aligned}
R_{00}^{(1)}(Nr) &= I_0(Nr)K_0(Na)-I_0(Na)K_0(Nr), \\
R_{00}^{(2)}(Nr) &= I_0(Nr)K_0(Nb)-I_0(Nb)K_0(Nr).
\end{aligned}\right\} \tag{58}$$

$$\left.\begin{aligned}
&\sum_n \begin{matrix}(1-(-1)^n)\\(1+(-1)^n)\end{matrix}\ \frac{\cos Nz}{N^2+\xi_i^2} = \frac{c}{2\xi_i}\begin{cases}\phi^{(1)}(\xi_i z)\\ \phi^{(2)}(\xi_i z)-\dfrac{2}{\xi_i c},\end{cases} \\
&\sum_n \begin{matrix}(1-(-1)^n)\\(1+(-1)^n)\end{matrix}\ \frac{N^2}{(N^2+\xi_i^2)^2}\cos Nz = \frac{c}{4\xi_i}\begin{cases}\phi^{(1)}(\xi_i z)-\Psi^{(1)}(\xi_i z)\\ \phi^{(2)}(\xi_i z)-\Psi^{(2)}(\xi_i z),\end{cases}
\end{aligned}\right\} \tag{59}$$

* K_1 (Na) and I_1 (Na) denote the modified Bessel functions.

$$\left.\begin{aligned}
&\left.\begin{array}{l}\phi^{(1)}(\xi_i z)\\ \phi^{(2)}(\xi_i z)\end{array}\right\} = \frac{\operatorname{sh}\xi_i(c-z)\mp\operatorname{sh}\xi_i z}{\operatorname{ch}\xi_i c\pm 1},\\
&\left.\begin{array}{l}\Psi^{(1)}(\xi_i z)\\ \Psi^{(2)}(\xi_i z)\end{array}\right\} = -\frac{\xi_i\{(c-z)\operatorname{ch}\xi_i z\mp z\operatorname{ch}\xi_i(c-z)\}}{\operatorname{ch}\xi_i c\pm 1},
\end{aligned}\right\} \quad (60)$$

$$\left.\begin{aligned}
&\sum_n \begin{matrix}(1-(-1)^n)\\(1+(-1)^n)\end{matrix}\frac{N}{N^2+\xi_i^2}\sin Nz = \frac{c}{2}\begin{cases}Q^{(1)}(\xi_i z)\\ Q^{(2)}(\xi_i z),\end{cases}\\
&\sum_n \begin{matrix}(1-(-1)^n)\\(1+(-1)^n)\end{matrix}\frac{N}{(N^2+\xi_i^2)^2}\sin Nz = \frac{c}{4\xi_i^2}\begin{cases}P^{(1)}(\xi_i z)\\ P^{(2)}(\xi_i z),\end{cases}
\end{aligned}\right\} \quad (61)$$

$$\left.\begin{aligned}
&\left.\begin{array}{l}Q^{(1)}(\xi_i z)\\ Q^{(2)}(\xi_i z)\end{array}\right\} = \frac{\operatorname{ch}\xi_i(c-z)\pm\operatorname{ch}\xi_i z}{\operatorname{ch}\xi_i c\pm 1},\\
&\left.\begin{array}{l}P^{(1)}(\xi_i z)\\ P^{(2)}(\xi_i z)\end{array}\right\} = \frac{\xi_i\{z\operatorname{sh}\xi_i(c-z)\pm(c-z)\operatorname{sh}\xi_i z\}}{\operatorname{ch}\xi_i c\pm 1},
\end{aligned}\right\} \quad (62)$$

$$\left.\begin{aligned}
u = &-A_0\left(\frac{r}{a}-\frac{a}{r}\right)\frac{ab}{a^2-b_2}+A_0'\left(\frac{r}{b}-\frac{b}{r}\right)\frac{ab}{a^2-b^2}\\
&+\sum_n \cos Nz\left[A_n\left\{G^{(1)}(Nr)+\frac{\mu+\lambda}{2\mu+\lambda}F^{(1)}(Nr)\right\}+A_n'\left\{G^{(2)}(Nr)\right.\right.\\
&\left.\left.+\frac{\mu+\lambda}{2\mu+\lambda}F^{(2)}(Nr)\right\}-\frac{\mu+\lambda}{2\mu(2\mu+\lambda)}\left\{B_n\cdot F^{(1)}(Nr)+B_n'F^{(2)}(Nr)\right\}\right]\\
&-\sum_i\frac{H_1(\xi_i r)}{\Theta_i^2}\left[D_i\left\{\frac{\mu+\lambda}{2\mu+\lambda}\left(\phi^{(1)}(\xi_i z)-\Psi^{(1)}(\xi_i z)\right)-\frac{\lambda}{2\mu+\lambda}\phi^{(1)}(\xi_i z)\right\}\right.\\
&+D_i'\left\{\frac{\mu+\lambda}{2\mu+\lambda}\left(\phi^{(2)}(\xi_i z)-\Psi^{(2)}(\xi_i z)\right)-\frac{\lambda}{2\mu+\lambda}\phi^{(2)}(\xi_i z)\right\}\\
&-E_i\left\{\frac{1}{2\mu+\lambda}\phi^{(1)}(\xi_i z)+\frac{\mu+\lambda}{2\mu(2\mu+\lambda)}\left(\phi^{(1)}(\xi_i z)-\Psi^{(1)}(\xi_i z)\right)\right\}\\
&\left.-E_i'\left\{\frac{1}{2\mu+\lambda}\phi^{(2)}(\xi_i z)+\frac{\mu+\lambda}{2\mu(2\mu+\lambda)}\left(\phi^{(2)}(\xi_i z)-\Psi^{(2)}(\xi_i z)\right)\right\}\right],
\end{aligned}\right\} \quad (63)$$

$$\left.\begin{aligned}
w = &\sum_n \sin Nz\left[-A_n\left\{\frac{\lambda}{2\mu+\lambda}\chi^{(1)}(Nr)-\frac{\mu+\lambda}{2\mu+\lambda}\left(\chi^{(1)}(Nr)-\omega^{(1)}(Nr)\right)\right\}\right.\\
&-A_n'\left\{\frac{\lambda}{2\mu+\lambda}\chi^{(2)}(Nr)-\frac{\mu+\lambda}{2\mu+\lambda}\left(\chi^{(2)}(Nr)-\omega^{(2)}(Nr)\right)\right\}\\
&+B_n\left\{\frac{1}{2\mu+\lambda}\chi^{(1)}(Nr)-\frac{\mu+\lambda}{2\mu(2\mu+\lambda)}\left(\chi^{(1)}(Nr)-\omega^{(1)}(Nr)\right)\right\}\\
&\left.+B_n'\left\{\frac{1}{2\mu+\lambda}\chi^{(2)}(Nr)-\frac{\mu+\lambda}{2\mu(2\mu+\lambda)}\left(\chi^{(2)}(Nr)-\omega^{(2)}(Nr)\right)\right\}\right]\\
&+\sum_i\frac{H_0(\xi_i r)}{\Theta_i^2}\left[D_i\left\{Q^{(1)}(\xi_i z)+\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\xi_i z)\right\}+D_i'\left\{Q^{(2)}(\xi_i z)\right.\right.\\
&\left.\left.+\frac{\mu+\lambda}{2\mu+\lambda}P^{(2)}(\xi_i z)\right\}+\frac{\mu+\lambda}{2\mu(2\mu+\lambda)}\left\{E_iP^{(1)}(\xi_i z)+E_i'P^{(2)}(\xi_i z)\right\}\right],
\end{aligned}\right\} \quad (64)$$

where

$$\left.\begin{aligned}
&G^{(1)}(Nr)=\frac{R_{11}^{(1)}(Nr)}{R_{11}^{(1)}(Nb)}\,, \qquad G^{(2)}(Nr)=\frac{R_{11}^{(2)}(Nr)}{R_{11}^{(2)}(Na)}\,,\\
&\frac{F^{(1)}(Nr)}{N}\\
&\quad=\frac{R^{(1)}(Nr)\{bR_{01}^{(1)}(Nb)-aR_{10}^{(1)}(Nb)\}-R_{11}^{(1)}(Nb)\{rR_{01}^{(1)}(Nr)-aR_{10}^{(1)}(Nr)\}}{\{R^{(1)}(Nb)\}^2}\,,\\
&\frac{F^{(1)}(Nr)}{N}\\
&\quad=\frac{R_{11}^{(2)}(Nr)\{aR_{01}^{(2)}(Na)-bR_{10}^{(1)}(Na)\}-R_{11}^{(2)}(Na)\{rR_{01}^{(2)}(Nr)-bR_{10}^{(2)}(Nr)\}}{\{R_{11}^{(2)}(Na)\}^2}\,,\\
&\chi^{(1)}(Nr)=\frac{R_{01}^{(1)}(Nr)}{R_{11}^{(1)}(Nb)}\,, \qquad \chi^{(2)}(Nr)=\frac{R_{01}^{(2)}(Nr)}{R_{11}^{(2)}(Na)}\,,\\
&\frac{\chi^{(1)}(Nr)+\omega^{(1)}(Nr)}{N}\\
&\quad=\frac{R_{01}^{(1)}(Nr)\{bR_{01}^{(1)}(Nb)-aR_{10}^{(1)}(Nb)\}-R_{11}^{(1)}(Nb)\{rR_{11}^{(1)}(Nr)-aR_{00}^{(1)}(Nr)\}}{\{R_{11}^{(1)}(Nb)^2\}^2}\,,\\
&\frac{\chi^{(2)}(Nr)+\omega^{(2)}(Nr)}{N}\\
&\quad=\frac{R_{01}^{(2)}(Nr)\{aR_{01}^{(2)}(Na)-bR_{10}^{(2)}(Na)\}-R_{11}^{(2)}(Na)\{rR_{11}^{(2)}(Nr)-bR_{00}^{(2)}(Nr)\}}{\{R_{11}^{(2)}(Na)\}^2}\,.
\end{aligned}\right\}\tag{65}$$

So that the dilatation is expressed by

$$\left.\begin{aligned}
\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}&=-A_0\frac{2b}{a^2-b^2}+A_0'\frac{2a}{a^2-b^2}\\
&+\frac{2\mu}{2\mu+\lambda}\sum_n\cos Nz\left\{NA_n\chi^{(1)}(Nr)+NA_n'\chi^{(2)}(Nr)\right\}\\
&+\frac{1}{2\mu+\lambda}\sum_n\cos Nz\left\{NB_n\chi^{(1)}(Nr)+NB_n'\chi^{(2)}(Nr)\right\}\\
&-\frac{2\mu}{2\mu+\lambda}\sum_i H_0(\xi_i r)\left\{\xi_i D_i\phi^{(1)}(\xi_i z)+\xi_i D_i'\phi^{(2)}(\xi_i z)\right\}\\
&-\frac{1}{2\mu+\lambda}\sum_i H_0(\xi_i r)\left\{\xi_i E_i\phi^{(1)}(\xi_i z)+\xi_i E_i'\phi^{(2)}(\xi_i z)\right\}.
\end{aligned}\right\}\tag{66}$$

The components of the stress are now obtained as follows:

$$\begin{aligned}
\sigma_r&=(2\mu+\lambda)\frac{2ab}{b^2-a^2}\left(\frac{A_0}{a}-\frac{A_0'}{b}\right)-2\mu\frac{ab}{b^2-a^2}\left\{A_0\left(\frac{1}{a}-\frac{a}{r^2}\right)\right.\\
&\left.-A_0'\left(\frac{1}{b}-\frac{b}{r_2}\right)\right\}+\sum_n\cos Nz\left[NA_n\left\{\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\left(\chi^{(1)}(Nr)+\omega^{(1)}(Nr)\right)\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{2\mu G^{(1)}(Nr)}{Nr}-\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\,\frac{F^{(1)}(Nr)}{Nr}\Big\}+NA'_n\Big\{\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\Big(\chi^{(2)}(Nr)\\
&+\omega^{(2)}(Nr)\Big)-\frac{2\mu G^{(2)}(Nr)}{Nr}-\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\,\frac{F^{(2)}(Nr)}{Nr}\Big\}-NB_n\Big\{\frac{\mu+\lambda}{2\mu+\lambda}\\
&\times\Big(\chi^{(1)}(Nr)-\omega^{(1)}(Nr)-\frac{F^{(1)}(Nr)}{Nr}\Big)+\frac{\lambda(\mu+\lambda)}{\mu(2\mu+\lambda)}\,\chi^{(1)}(Nr)\Big\}\\
&-NB'_n\Big\{\frac{\mu+\lambda}{2\mu+\lambda}\Big(\chi^{(2)}(Nr)-\omega^{(2)}(Nr)-\frac{F^{(2)}(Nr)}{Nr}\Big)+\frac{\lambda(\mu+\lambda)}{\mu(2\mu+\lambda)}\,\chi^{(2)}(Nr)\Big\}\Big]\\
&-\sum_i \xi_i\,\frac{H'_1(\xi_i r)}{\Theta_i^2}\Big[D_i\Big\{\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\Big(\phi^{(1)}(\xi_i z)-\Psi^{(1)}(\xi_i z)\Big)-\frac{2\mu\lambda}{2\mu+\lambda}\,\phi^{(1)}(\xi_i z)\Big\}\\
&+D'_i\Big\{\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\Big(\phi^{(2)}(\xi_i z)-\Psi^{(2)}(\xi_i z)\Big)-\frac{2\mu\lambda}{2\mu+\lambda}\,\phi^{(2)}(\xi_i z)\Big\}\\
&+E_i\Big\{\frac{\mu+\lambda}{2\mu+\lambda}\Big(\phi^{(1)}(\xi_i z)-\Psi^{(1)}(\xi_i z)\Big)+\frac{2\mu}{2\mu+\lambda}\,\phi^{(1)}(\xi_i z)\Big\}+E'_i\Big\{\frac{\mu+\lambda}{2\mu+\lambda}\\
&\times\Big(\phi^{(2)}(\xi_i z)-\Psi^{(2)}(\xi_i z)\Big)+\frac{2\mu}{2\mu+\lambda}\,\phi^{(2)}(\xi_i z)\Big\}\Big]+\sum_i \xi_i\,\frac{H_0(\xi_i r)}{\Theta_i^2}\Big[\frac{2\mu\lambda}{2\mu+\lambda}\\
&\times\Big(D_i\phi^{(1)}(\xi_i z)+D'_i\phi(\xi_i z)\Big)-\frac{\lambda}{2\mu+\lambda}\Big(E_i\phi^{(1)}(\xi_i z)+E'_i\phi^{(2)}(\xi_i z)\Big)\Big]
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
\sigma_\theta &= 2\mu\,\frac{ab}{b^2-a^2}\Big\{A_0\Big(\frac{1}{a}-\frac{a}{r^2}\Big)-A'_0\Big(\frac{1}{b}-\frac{b}{r^2}\Big)\Big\}\\
&+\lambda\Big\{\frac{A_0 2b}{b_2-a_2}+\frac{A'_0 2a}{a^2-b^2}\Big\}+\sum_n\frac{\cos Nz}{r}\Big[A_n\Big\{2\mu G^{(1)}(Nr)\\
&+\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}F^{(1)}(Nr)-\frac{2\mu\lambda}{2\mu+\lambda}Nr\cdot\chi^{(1)}(Nr)\Big\}+A'_n\Big\{2\mu G^{(2)}(Nr)\\
&+\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}F^{(2)}(Nr)-\frac{2\mu\lambda}{2\mu+\lambda}Nr\cdot\chi^{(2)}(Nr)\Big\}-B_n\Big\{\frac{\mu+\lambda}{2\mu+\lambda}F^{(1)}(Nr)\\
&+\frac{\lambda}{2\mu+\lambda}Nr\cdot\chi^{(1)}(Nr)\Big\}-B'_n\Big\{\frac{\mu+\lambda}{2\mu+\lambda}F^{(2)}(Nr)+\frac{\lambda}{2\mu+\lambda}Nr\cdot\chi^{(2)}(Nr)\Big\}\Big]\\
&-\sum_i\frac{H_1(\xi r)}{\Theta_i^2\,r}\Big[D_i\Big\{\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\Big(\phi^{(1)}(\xi_i z)-\Psi^{(1)}(\xi_i z)\Big)-\frac{2\mu\lambda}{2\mu+\lambda}\phi^{(1)}(\xi_i z)\Big\}\\
&+D'_i\Big\{\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\Big(\phi^{(2)}(\xi_i z)-\Psi^{(2)}(\xi_i z)\Big)-\frac{2\mu\lambda}{2\mu+\lambda}\,\phi^{(2)}(\xi_i z)\Big\}\\
&+E_i\Big\{\frac{2\mu}{2\mu+\lambda}\,\phi^{(1)}(\xi_i z)+\frac{\mu+\lambda}{2\mu+\lambda}\Big(\phi^{(1)}(\xi_i z)-\Psi^{(1)}(\xi_i z)\Big)\Big\}\\
&+E'_i\Big\{\frac{2\mu}{2\mu+\lambda}\,\phi^{(2)}(\xi_i z)+\frac{\mu+\lambda}{2\mu+\lambda}\Big(\phi^{(2)}(\xi_i z)-\Psi^{(2)}(\xi_i z)\Big)\Big\}\Big]\\
&-\sum_i \xi_i\,\frac{H_0(\xi_i r)}{\Theta_i^2}\Big[\frac{2\mu\lambda}{2\mu+\lambda}\Big\{D_i\phi^{(1)}(\xi_i z)+D'_i\phi^{(2)}(\xi_i z)\Big\}\\
&+\frac{\lambda}{2\mu+\lambda}\Big\{E_i\phi^{(1)}(\xi_i z)+E'_i\phi^{(2)}(\xi_i z)\Big\}\Big],
\end{aligned}
\tag{68}
$$

$$
\left.
\begin{aligned}
\sigma_z = {} & \lambda\left\{\frac{A_0 2b}{b^2-a^2}+\frac{A_0' 2a}{a_2-b_2}\right\} \\
& +\sum_n \cos Nz\Big[\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\Big\{NA_n\Big(\chi^{(1)}(Nr)-\omega^{(1)}(Nr)\Big)+NA_n'\Big(\chi^{(2)}(Nr) \\
& -\omega^{(2)}(Nr)\Big)\Big\}+NB_n\Big\{\frac{\mu+\lambda}{2\mu+\lambda}\Big(\omega^{(1)}(Nr)-\chi^{(1)}(Nr)\Big)+\chi^{(1)}(Nr)\Big\} \\
& +NB_n'\Big\{\frac{\mu+\lambda}{2\mu+\lambda}\Big(\omega^{(2)}(Nr)-\chi^{(2)}(Nr)\Big)+\chi^{(2)}(Nr)\Big\}\Big] \\
& -\sum_i \frac{H_0(\xi_i r)}{\Theta_i^2}\Big[\frac{2\mu)\mu+\lambda)}{2\mu+\lambda}\Big\{\xi_i D_i\Big(\phi^{(1)}(\xi_i z)+\Psi^{(1)}(\xi_i z)\Big)+\xi_i D_i'\Big(\phi^{(2)}(\xi_i z) \\
& +\Psi^{(2)}(\xi_i z)\Big)\Big\}+\xi_i E_i\Big\{\frac{\lambda}{2\mu+\lambda}\phi^{(1)}(\xi_i z)-\frac{\mu+\lambda}{2\mu+\lambda}\Big(\phi^{(1)}(\xi_i z)-\Psi^{(1)}(\xi_i z)\Big)\Big\} \\
& +\xi_i E_i'\Big\{\frac{\lambda}{2\mu+\lambda}\phi^{(2)}(\xi_i z)-\frac{\mu+\lambda}{2\mu+\lambda}\Big(\phi^{(2)}(\xi_i z)-\Psi^{(2)}(\xi_i z)\Big)\Big\}\Big],
\end{aligned}
\right\} \quad (69)
$$

$$
\left.
\begin{aligned}
\tau_{rz} = {} & \sum_n N\sin Nz\Big[\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\Big\{A_nF^{(1)}(Nr)+A_n'F^{(2)}(Nr)\Big\}+B_n\Big\{G^{(1)}(Nr) \\
& -\frac{\mu+\lambda}{2\mu+\lambda}F^{(1)}(Nr)\Big\}+B_n'\Big\{G^{(2)}(Nr)-\frac{\mu+\lambda}{2\mu+\lambda}F^{(2)}(Nr)\Big\}\Big] \\
& -\sum_i \xi_i\frac{H_1(\xi_i r)}{\Theta_i^2}\Big[\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\Big\{D_iP^{(1)}(\xi_i z)+D_i'P^{(2)}(\xi_i z)\Big\}+E_i\Big\{Q^{(1)}(\xi_i z) \\
& -\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\xi_i z)\Big\}+E_i'\Big\{Q^{(2)}(\xi_i z)-\frac{\mu+\lambda}{2\mu+\lambda}P^{(2)}(\xi_i z)\Big\}\Big].
\end{aligned}
\right\} \quad (70)
$$

6. Annular thick Plate having unequal Displacement between inner and outer fixed Boundaries

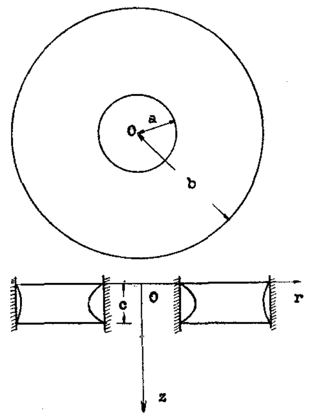

Fig. 4.

To solve the stress problem, three conditions* must be always satisfied at each boundary belonging to the elastic medium under consideration. The condition that a circumference is fixed, is denoted by the three cases as follows:

(i) $w=0$, $\quad u=0$;

(ii) $w=0$, $\quad \dfrac{\partial w}{\partial r}=0$;

(iii) $u=0$, $\quad \dfrac{\partial u}{\partial r}=0$.

---

* Because of the axially symmetrical stress distribution, a condition among the three which denote that a circumference is fixed: that is $v=0$, can be satisfied in itself. So in each condition of the three: (i), (ii), and (iii), there is given two conditions.

The first case denotes all the points at the boundary can not displace, and the second denotes the well-known condition, the so called "fixed boundary". We will choose the condition of the first case for satisfying that the inner and outer boundaries are fixed, and one more condition we must consider is that there act no force on both of the surfaces $z=0$ and $z=c$. Hence

$$u_{r=b}=0\,,\qquad u_{r=a}=0\,,$$
$$(\tau_{rz})_{z=0}=0\,,\qquad (\tau_{rz})_{z=c}=0\,,$$

which yield

$$A_n=A'_n=0\,,\qquad E_i=E'_i=0\,.$$

The difference of the deflections between inner and outer circumferences relate to a pair of shearing forces which act along the inner and outer circumferences, so u and w can be described by the boundary shear T. Accordingly the shearing forces per unit of length along both circumferences become $T/2a\pi$ and $T/2b\pi$. If the distributions of the shearing tractions applied to the edges are regarded as a parabolic surface, we may assume

$$\left.\begin{aligned}(\tau_{rz})_{r=a}&=\frac{3T}{\pi ac}\,\frac{z(c-z)}{c^2}\,,\\(\tau_{rz})_{r=b}&=\frac{3T}{\pi bc}\,\frac{z(c-z)}{c^2}\,.\end{aligned}\right\}\tag{71}$$

Hence

$$\left.\begin{aligned}S_n\left[(\tau_{rz})_{z=a}\right]&=\frac{3T}{\pi ac^3}\,\frac{(1-(-1)^n)}{N^3}\,,\\S_n\left[(\tau_{rz})_{r=b}\right]&=\frac{3T}{\pi bc^3}\,\frac{(1-(-1)^n)}{N^3}\,,\end{aligned}\right\}\tag{72}$$

$$\therefore\quad B_nN=\frac{6T}{\pi abc^4}\,\frac{(1-(-1)^n)}{N^3}\,,\qquad B'_nN=\frac{6T}{\pi b^2c^4}\,\frac{(1-(-1)^n)}{N^3}\,.\tag{73}$$

We therefore have

$$\left.\begin{aligned}u=&-\sum_n\cos Nz\,\frac{6T(\mu+\lambda)(1-(-1)_n)}{2\pi c^4N^4b(2\mu+\lambda)\mu}\left\{\frac{F^{(1)}Nr}{b}-\frac{F^{(2)}(Nr)}{a}\right\}\\&-\sum_i\frac{H_1(\xi_ir)}{\Theta_i^2}\cdot D_i\left\{\frac{\mu+\lambda}{2\mu+\lambda}\Big(\phi^{(1)}(\xi_iz)-\Psi^{(1)}(\xi_iz)\Big)-\frac{\lambda}{2\mu+\lambda}\,\phi^{(1)}(\xi_iz)\right\},\end{aligned}\right\}\tag{74}$$

$$\left.\begin{aligned}w=&-\sum_n\sin Nz\,\frac{6T(1-(-1)^n)}{\pi bc^4N^4}\left[\left\{\frac{1}{b(2\mu+\lambda)}\chi^{(1)}(Nr)-\frac{\mu+\lambda}{2b\mu(2\mu+\lambda)}\right.\right.\\&\times\left.\Big(\omega^{(1)}(Nr)-\chi^{(1)}(Nr)\Big)\right\}+\left\{\frac{1}{a(2\mu+\lambda)}\chi^{(2)}(Nr)-\frac{\mu+\lambda}{2a\mu(2\mu+\lambda)}\right.\\&\times\left.\left.\Big(\omega^{(2)}(Nr)-\chi^{(2)}(Nr)\Big)\right\}\right]+\sum_i\frac{H_0(\xi_ir)}{\Theta_i^2}\cdot D_i\left\{Q^{(1)}(\xi_iz)+\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\xi_iz)\,,\right.\end{aligned}\right\}\tag{75}$$

$$\left.\begin{aligned}\sigma_r = \sum_n \cos Nz \frac{6T(1-(-1)^n)}{\pi bc^4N^4} \frac{\mu+\lambda}{b(2\mu+\lambda)}\left(\omega^{(1)}(Nr)-\chi^{(1)}(Nr)-\frac{F^{(2)}(Nr)}{Nr}\right) \\ +\frac{\lambda(\mu+\lambda)}{b\mu(2\mu+\lambda)}\chi^{(1)}(Nr)-\frac{\mu+\lambda}{a(2\mu+\lambda)}\left(\omega^{(2)}(Nr)-\chi^{(2)}(Nr)-\frac{F^{(2)}(Nr)}{Nr}\right) \\ +\frac{\lambda(\mu+\lambda)}{a\mu(2\mu+\lambda)}\chi^{(2)}(Nr)\Big\}-\sum_i \frac{H_i'(\xi_i r)}{\Theta_i^2}\cdot\xi_i D_i\left\{\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}\left(\phi^{(1)}(\xi_i z)-\Psi^{(1)}(\xi_i z)\right)\right. \\ \left.-\frac{2\mu\lambda}{2\mu+\lambda}\phi^{(1)}(\xi_i z)\right\}-\sum_i \frac{H_0(\xi_i r)}{\Theta_i^2}\frac{2\mu\lambda}{2\mu+\lambda}\xi_i D_i\phi^{(1)}(\xi_i z)\,,\end{aligned}\right\} \quad (76)$$

$$\left.\begin{aligned}\sigma_z = -\sum_n \cos Nz \frac{6T(1-(-1)^n)}{\pi bc^4N_3}\left\{\frac{\mu+\lambda}{b(2\mu+\lambda)}\left(\omega^{(1)}(Nr)-\chi^{(1)}(Nr)\right)\right. \\ \left.+\frac{1}{b}\chi^{(1)}(Nr)-\frac{\mu+\lambda}{a(2\mu+\lambda)}\left(\omega^{(2)}(Nr)-\chi^{(2)}(Nr)\right)-\frac{1}{a}\chi^{(2)}(Nr)\right\} \\ -\sum_i \frac{H_0(\xi_i r)}{\Theta_i^2}\frac{2\mu(\xi+\lambda)}{2\mu+\lambda}\xi_i D_i\left(\phi^{(1)}(\xi_i z)+\Psi^{(1)}(\xi_i z)\right),\end{aligned}\right\} \quad (77)$$

$$\left.\begin{aligned}\tau_{rz} = \sum_n \sin Nz \frac{24T(1-(-1)^n)}{\pi bc_4N^3}\left\{\frac{1}{b}G^{(1)}(Nr)-\frac{\mu+\lambda}{b(2\mu+\lambda)}F^{(1)}(Nr)\right. \\ \left.-\frac{1}{a}G^{(2)}(Nr)+\frac{\mu+\lambda}{a(2\mu+\lambda)}F^{(2)}(Nr)\right\}+\sum_i \frac{H_1(\xi_i r)}{\Theta_i^2}\cdot\xi_i D_i\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}P^{(1)}(\xi_i z).\end{aligned}\right\} \quad (78)$$

The constant of integration D_i will now be calculated from the boundary condition. As the plate is free from any force on the surface $z=0$ and $z=c$, then we have

$$\sigma_z)_{z=0} = 0\,. \quad (79)$$

Before the above calculation can be performed, it is better to transform the first term in Eq. (52), as follows:

The first Term

$$= -\sum_n \cos Nz \frac{6T(1-(-1)^n)}{\pi bc^4N^3}\left\{\frac{\mu+\lambda}{b(2\mu+\lambda)}\left(\omega^{(1)}(Nr)-\chi^{(1)}(Nr)\right)\right.$$
$$\left.+\frac{1}{b}\chi^{(1)}(Nr)-\frac{\mu+\lambda}{a(2\mu+\lambda)}\left(\omega^{(2)}(Nr)-\chi^{(2)}(Nr)\right)-\frac{1}{a}\chi^{(2)}(Nr)\right\},$$

with the aid of Eqs. (66), (56), and (57)

The first term

$$= -\sum_n\sum_i \cos Nz \frac{6T(1-(-1)^n)}{\pi bc^4N^3}\frac{H_0(\xi_i r)}{\Theta_i^2}\Big\{H_0(\xi_i b)$$
$$+H_0(\xi_i a)\Big\}\left\{\frac{\mu+\lambda}{2\mu+\lambda}\frac{4\xi_i^2 N}{(N^2+\xi_i^2)^2}+\frac{2N}{N^2+\xi_i^2}\right\}.$$

$\therefore$ The first term

$$= -\sum_i \frac{6TH_0(\xi_i r)}{\pi b^2 \{\Theta_i\}^2}\left\{H_0(\xi_i b)+H_0(\xi_i a)\right\}$$

$$\left\{\frac{1}{\xi_i^2 c^2}\frac{4\mu+3\lambda}{2\mu+\lambda}\rho^{(1)}(\xi_i z)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{1}{\xi_i^3 c^3}\Omega^{(1)}(\xi_i z)\right\},$$

where

$$\left.\begin{aligned}\rho^{(1)}(\xi_i z) &= \frac{c-2z}{2c}-\frac{1}{\xi_i c}\phi^{(1)}(\xi_i z)\\ \Omega^{(1)}(\xi_i z) &= \phi^{(1)}(\xi_i z)+\Psi^{(1)}(\xi_i z)\,.\end{aligned}\right\} \tag{80}$$

Accordingly, Eq. (80) yields

$$D_i = \frac{6T}{\pi b^2 \xi_i}\cdot\frac{H_0(\xi_i b)-H_0(\xi_i a)\,b/a}{\mu\{\Theta_i\}^2}\left\{\frac{4\mu+3\lambda}{2\mu+\lambda}\,\frac{\rho^{(1)}(0)}{(\xi_i c)^2\,\Omega^{(1)}(0)}-\frac{1}{2(\xi_i c)^3}\right\}. \tag{81}$$

7. Numerical Example

As an example, we consider a very thick plate the radius of the inner and outer portion of which are $a=c$ and $b=2c$, where c denotes the thickness of the plate. In this case we have the values of the coefficients as shown in the following tables:

Table 1.

i	$\xi_i c$	$\mathrm{sh}\,\xi_i c$	$\mathrm{ch}\,\xi_i c$
1	3.19	12.1236	12.1648
2	6.31	274.557	274.559
3	9.44	6290.86	6290.86
4	12.58	145343.	145343.

Table 2.

i	$\phi(0)$	$\Psi(0)$	$\rho(0)/(\xi_i c)^2$
1	0.92091	0.24231	0.02077
2	0.99634	0.02309	0.00859
3	0.99984	0.00150	0.00441
4	0.99999	0.00009	0.00266

Table 3.

i	$\Omega(0)$	$\rho(0)/(\xi_i c)^2\cdot\Omega(0)$	$1/(\xi_i c)^3$
1	0.68860	0.03016	0.03081
2	0.97325	0.00882	0.00398
3	0.99834	0.00442	0.00119
4	0.99990	0.00266	0.00050

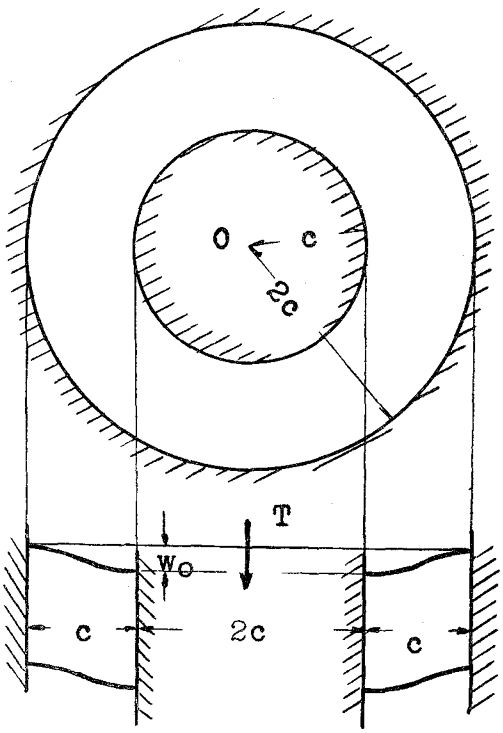

Fig. 5.

Let the Poisson's ratio be 0.3, then

$$\lambda = 1.5\,\mu\,, \qquad \frac{4\mu+3\lambda}{2\mu+\lambda} = 1.7\,.$$

Hence the values of D_i are calculated from Eq. (58) as follows:

Table 4.

i	$\xi_i a \cdot D_i H_0(\xi_i b)$	$D_i H_0(\xi_i b)$
1	0.04685	0.00937
2	0.03275	0.00330
3	0.01877	0.00127
4	0.01205	0.00060
5	0.00801	0.00032

On the other hand the values of $H_0(\xi_i r)/H_0(\xi_i b)$ are given in the following table

Table 5.
Values of $H_0(\xi_i r)/H_0(\xi_i b)$

i	$r=a$	$r=(b+4a)/5$	$r=(2b+3a)/5$	$r=(3b+2a)/5$	$r=(4b+a)/5$	$r=b$
1	−1.3360	−1.0943	−0.4779	0.2148	0.7557	1.0000
2	1.4105	0.4818	−0.9203	−0.9315	0.2793	1.0000
3	−1.4100	0.3410	1.0000	−0.8744	−0.3619	1.0000
4	1.4134	−1.0195	0.3324	−0.4022	−0.6676	1.0000
5	−1.3964	1.2759	−1.1812	1.1044	0.0105	1.0000

By virtue of the above results we can obtain the values of w and σ_r; the variation of w and σ_r with r and $z=0$, and the variation of σ_r with z and $r=a$, b are shown in the following tables and figure.

Table 6.
The variation of w with r $(z=0)$. Tb^2/Ec^3

$r=a$	$r=1.2\,a$	$r=1.4\,a$	$r=1.6\,a$	$r=1.8\,a$	$r=2\,a$
0.0214	0.0165	0.0111	0.0020	0.0030	0.0000

The variation of σ_r with r $(z=0)$. T/c^2

$r=a$	$r=1.2\,a$	$r=1.4\,a$	$r=1.6\,a$	$r=1.8\,a$	$r=2\,a$
−0.407	−0.159	−0.046	0.024	0.122	0.245

Table 7.

The variation of σ_r with z. T/c^2

r	$z=0$	$z=c/8$	$z=c/4$	$z=3c/4$	$z=7c/8$	$z=c$
a	−0.407	−0.228	−0.140	0.140	0.228	0.407
b	0.245	0.136	0.083	−0.083	−0.136	−0.245

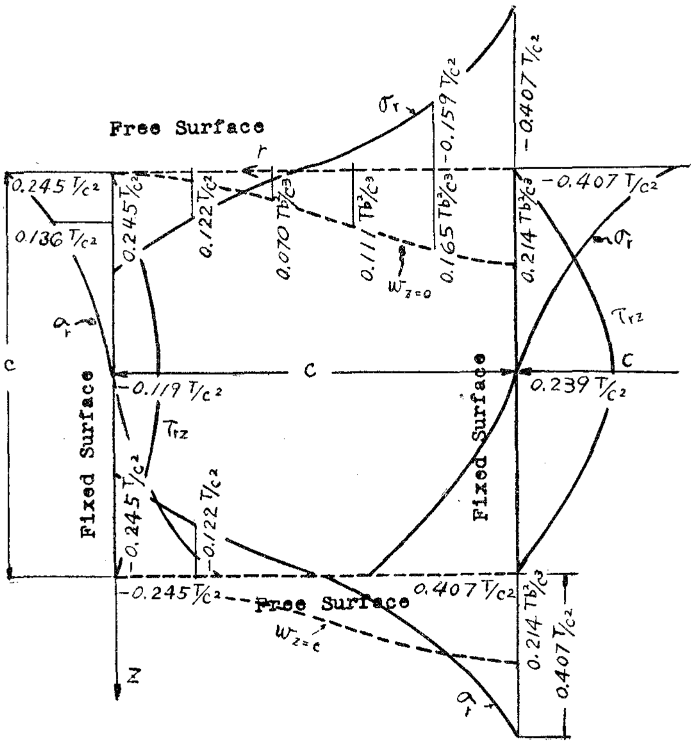

Fig. 6.

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