A Renovated Course of Functional Analysis

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By

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Abstract

In making observations on the most primitive space of numerical functions, where two functions equal almost everywhere are not always regarded equivalent, the classical analysis is not a competent one. Renovations are needed in definition of orthogonality, effectuation of a linear operator, spectral methodization and so on. In this paper, two important methods are introduced to topologize the space, called "*reaxilization*" and "*orthogonal transmutation*". It is remarkable that the ideal-theoretical method is found useful in similar way to what is found in the classical analysis.

1. Introduction.

In this paper the space $\boldsymbol{X}$ is posited as the aggregation of all the complex-valued functions $x(\xi)$ that satisfy the conditions: (i) $|x(\xi)| < \infty$ for all $\xi \in \Xi$; (ii) $x(\xi) = x(\eta)$ whenever $\xi = \eta$. When $x(\xi) \in \boldsymbol{X}$, $x(\xi)$ is called a *vector* in $\boldsymbol{X}$ and is denoted by x. Ξ is a metric space provided with a normal system μ to measure a subset Γ of Ξ by the a priori measure

$$\widetilde{m}\Gamma = \mathfrak{S}_{\xi \in \Gamma} \mu_\xi = \mu \mathfrak{n}(\Gamma)^{1)} \tag{1}$$

($\mu_\xi = \mu$ for all $\xi \in \Xi$ and $\mathfrak{n}(\Gamma)$ is the inversion number of Γ w. r. t. $\widetilde{m}$) and it is posited that $\bar{\bar{\Xi}} > \mathfrak{a}$ (the cardinal of enumerability) and

$$\widetilde{m}\,\Xi = 1. \tag{1, 1'}$$

On the other hand, x may be originally expressed in the form[2)]

$$x = \mathfrak{S}\, x(\xi)\, \partial_\xi$$

where $\partial_\xi(\eta)$ is the characteristic function of the single point set $\{\xi\}$, so ∂_ξ is naturally regarded as a vector in $\boldsymbol{X}$. Next, let a scalar product be introduced by the formula

$$(x|y) = \mathfrak{S}\, x(\xi)\, \overline{y(\xi)}\, \mu_\xi$$

so that we have a norm measure $\|x\|$ such as

$$\begin{aligned} \|x\|^2 &= \mathfrak{S}\, |x(\xi)|^2 \mu_\xi \\ \|\partial_\xi\|^2 &= \mu_\xi. \end{aligned} \tag{1, 2}$$

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The calculus of $(x|y)$ is specially characteristic in point that

$$(x|y)=0 \rtimes\!\!\lhd \mathfrak{S}\, x(\xi)\, \overline{y(\xi)}=0.$$

In this meaning, if exactly, the symbol 0 should be altered by the symbol $\circledcirc$ which indicates the empty nully (= sheer void), and then any infinitesimal quantity will be indicated by the symbol ◬. In addition, for an integral of the form

$$\mathop{\mathfrak{S}}_{\lambda\in\Lambda} x(\lambda)\,\gamma_\lambda \tag{1, 3}$$

Λ being a metric space provided with a positive system γ_λ (say, $>\circledcirc$), is demanded the restriction

$$\mathop{\mathfrak{S}}_{\lambda\in\Lambda} |x(\lambda)|\gamma_\lambda<\infty$$

whenever the value (1,3) is posited as finitely determined (= convergent). This is that we conform to Riemann's viewpoint on convergence in a generalized form.

Hereafter, a space or a subspace will mean a vector space or a vector subspace respectively, generated on complex suffices. The minimun subspace wich contains any of subspaces $\boldsymbol{Y}_\iota$ $(\iota\in\boldsymbol{I})$ is the *span*[3] of the family $(\boldsymbol{Y}_\iota)$ and is denoted by

$$\vee\, \boldsymbol{Y}_\iota \quad \text{or} \quad \vee_{\iota\in I}\boldsymbol{Y}_\iota\,;$$

when $\boldsymbol{I}=(1,2)$ it is written as

$$\boldsymbol{Y}_1\vee\boldsymbol{Y}_2.$$

The orthogonal projection of a vector x on a subspace $\boldsymbol{Y}$ is denoted by

$$P_{\boldsymbol{Y}}(x).$$

A subspace generated by a single vector y will be denoted by $\ll y\gg$, and on such a subspace the projection $P_{\ll y\gg}(x)$ may be simply denoted by

$$P_y(x).$$

When
$$\|y\|>0$$

it must be remarked that $P_y(\partial_\xi)$ cannot be existent except the case of vanishing, because then it must be

$$\|P_y(\partial_\xi)\|\leqslant\|\partial_\xi\|=\sqrt{\mu_\xi}=\sqrt{\text{◬}}=\text{◬} \tag{1, 4}$$

while $P_y(\partial_\xi)$ is written in the form

$$cy\ (c:\ \text{a complex number})$$

if existent, so that

$$\|P_y(\partial_\xi)\|=|c|\cdot\|y\|\geqslant 0. \tag{1, 5}$$

As the quantity 0 should mean $\circledcirc$ from our standpoint of view, (1, 5) means either its left hand is strictly positive or vanishes to be equal to $\circledcirc$. So, in case of not vanishing $P_y(\partial_\xi)$, $(1, 5)$ is contradictory to $(1, 4)$.

In our theory, the function $\overline{x}|$ of x defined by

$$\overline{x}| = \sup|x(\xi)|$$

is called the *height* of x to be distinguished from $\|x\|$ defined by $(1, 2)$; and when $\overline{x}|$ or $\|x\|$ is finite, x is said to be of finite height or of finite norm respectively. By a certain reason we do not define a norm nor a height of a transformation. So as to be of finite norm, a vector may not necessarily be of finite height, but when a vector is of finite height it must be of finite norm (because of the restriction $(1, 1)$).

2. Reaxilization.

The set

$$\Xi_x = (\xi : \xi \in \Xi \text{ and } x(\xi) \neq 0)$$

corresponding to a fixed vector $x \in \boldsymbol{X}$, is called the *defining support* of x. When $\boldsymbol{Y}$ is a subspace of $\boldsymbol{X}$, the set

$$\hat{\Xi}_\xi = \hat{\Xi}_\xi(\boldsymbol{Y}) = \cap \Xi_x (x \in \boldsymbol{Y} \text{ and } x(\xi) \neq 0)$$

will be called a *supporting scale* or simply a *scale* of $\boldsymbol{Y}$ on condition $\hat{\Xi}_\xi \neq$ void, which, in other words, is the infimum of the defining supports of $x \in \boldsymbol{Y}$ for which $x(\xi) \neq 0$. Now, on denoting as

$$\boldsymbol{Y}_\xi = (x : x \in \boldsymbol{Y} \text{ and } x(\xi) \neq 0)$$

we have

$$\boldsymbol{Y}_\xi \subseteq \boldsymbol{Y}_\eta \text{ for all } \eta \in \hat{\Xi}_\xi \tag{2. 1}$$

because it is direct from the the definitions that

$$(x(\xi) \neq 0 \text{ and } \eta \in \hat{\Xi}_\xi) \rhd x(\eta) \neq 0. \tag{2. 2}$$

Next, let us suppose that there exists a vector $y \in \boldsymbol{Y}_\eta$ for a certain η such that

$$y(\eta) \neq 0 \text{ but } y(\xi) = 0,$$

then, on picking up an arbitrary vector $x \in \boldsymbol{Y}_\xi$, by (2. 1) it follows

$$x \in \boldsymbol{Y}_\eta,$$

so we have

$$z = x - \frac{x(\eta)}{y(\eta)} y \in \boldsymbol{Y}$$

for which

$$z(\xi) \ (= x(\xi)) \neq 0 \text{ but } z(\eta) = 0.$$

This is contradictory to $(2, 2)$. Hence, it must be

$$\boldsymbol{Y}_\xi \supseteq \boldsymbol{Y}_\eta. \tag{2. 3}$$

Then, by (2. 1) and (2. 3), we see

$$\boldsymbol{Y}_\xi = \boldsymbol{Y}_\eta \text{ for each } \eta \in \hat{\Xi}_\xi$$

so that, in regard to the definition of $\hat{\Xi}_\xi$, we may conclude:

Lemma 1. $$\hat{\Xi}_\xi \ni \eta \rhd \hat{\Xi}_\xi = \hat{\Xi}_\eta .$$

If there exist two vectors x and y in $\boldsymbol{Y}$, for which $x(\xi) \neq 0$ and $y(\xi) \neq 0$ but

$$\frac{x(\xi)}{y(\xi)} \neq \frac{x(\eta)}{y(\eta)}$$

for a certain $\eta \in \hat{\Xi}_\xi$,

$$z = x - \frac{x(\eta)}{y(\eta)} y$$

is a vector in $\boldsymbol{Y}$, because then $x,\ y \in \boldsymbol{Y}_\xi$ and therefore $y(\eta) \neq 0$ by Lemma 1, and then

$$z(\xi) = y(\xi)\left\{\frac{x(\xi)}{y(\xi)} - \frac{x(\eta)}{y(\eta)}\right\} \neq 0$$

but $$z(\eta) = 0.$$

This is a contradiction again to (2, 2). Hence, there must exist a vector ρ in $\boldsymbol{X}$ for which

$$\Xi_\rho = \hat{\Xi}_\xi$$

and such that

$$\boldsymbol{Y} \in y \rhd y = k\rho + \mathop{\mathfrak{S}}_{\zeta \in \Xi_\rho} y(\xi)\,\partial_\zeta \tag{2, 4}$$

k being a complex number.

From Lemma 1 it is direct that

$$\hat{\Xi}_\xi \cup \hat{\Xi}_\eta \neq \text{void} \rhd \hat{\Xi}_\xi = \hat{\Xi}_\eta$$

so that the family of the distinct scales of $\boldsymbol{Y}$ is a family of disjoint sets, i. e. a partition of the set

$$\Xi_{\boldsymbol{Y}} = \bigcup_{x \in \boldsymbol{Y}} \Xi_x$$

which is called the *support* of $\boldsymbol{Y}$. Now, let it be written as

$$(\Xi_\lambda)_{\lambda \in \Lambda}$$

then the set Λ of indices may not be generally expected as a set of ordinal numbers, though it is always found possibly existent. If $\Xi_\rho = \Xi_\lambda$ in (2, 4), let ρ be altered by the notation

$$\rho_\lambda$$

and let the subset of $\boldsymbol{X}$ generated by a single vector ρ_λ be denoted by $\boldsymbol{Y}_\lambda$, then it is direct from (2. 4) that

$$\boldsymbol{Y} \subseteq \vee_\lambda \boldsymbol{Y}_\lambda. \tag{2, 5}$$

It is very convenient if we may use the relation

$$\vee_\lambda \boldsymbol{Y}_\lambda = \vee_\lambda \ll \rho_\lambda \gg \tag{2, 6}$$

on the ground that

$$\boldsymbol{Y}_\lambda = \ll \rho_\lambda \gg, \tag{2, 7}$$

but if we try to induce (2. 6) from (2. 7) we shall come across a need of Zermelo's axiom. However, to avoid the controversial troubles about the axiom (of choice), we may posit (2. 6) as an intuitional representation of the structure of the subspace $\vee_\lambda \boldsymbol{Y}_\lambda$, caused by the reality of the span (say, the right hand of (2, 5)) and the destination (2, 7), while we shall then leave the set Λ free from well-ordering in general. Thus we posit the relation

$$\boldsymbol{Y} \subseteq \vee_\lambda \ll \rho_\lambda \gg \tag{2, 8}$$

instead of (2, 5).

To tell the truth, the inversive relation of (2, 8)

$$\boldsymbol{Y} \supseteq \vee_\lambda \ll \rho_\lambda \gg \tag{2, 9}$$

is not evident. In effect, on an arbitrary fiinite number of indices

$$\lambda_1,\ \lambda_2 \cdots,\ \lambda_n$$

it can be easily shown that there exists a vector x in $\boldsymbol{Y}$ such that

$$x = x_1 \rho_1 + x_2 \rho_2 + \cdots + x_n \rho_{\lambda_n} + x'$$

with
$$x'(\xi) = 0 \ \text{ for any } \xi \in \bigcup_{k=1}^{n} \Xi_{\lambda_k}$$

on an arbitrary choice of n complex numbers $x_1, x_2, \cdots, x_n$. But this may not be sufficient to assert that any vector x of the form

$$x = \mathfrak{S}\, x(\lambda) \rho_\lambda$$

belongs to $\boldsymbol{Y}$. The logical leap between the above-stated result and (2. 9) may be simply cleared if we adopt some appropriate system of logic, though then may be caused some new discussion about the construction of a subspace. However, we merely posit the destination (2. 9) as effectuated, in this paper, without getting in the logical details. Then, combining (2. 9) with (2. 8) we will have the formula

$$Y = \vee_\lambda \ll \rho_\lambda \gg \tag{2, 10}$$

as effectuated, at all events. It can be verified in itself that

$$\Xi_Y = \cup\, \Xi_\lambda.$$

The family of the vectors $(\rho_\lambda)_{\lambda\in\Lambda}$ will be called the *natural basis* of $\boldsymbol{Y}$, aud will be denoted by

$$B(\boldsymbol{Y});$$

and the formula (2, 10) is called the *law of reaxilization* or simply the *reaxilization* on the subspace $\boldsymbol{Y}$.

3. On Linear Operators.

A linear operator L is understood as

$$Lx \in \boldsymbol{X} \text{ for each } x \in \boldsymbol{X}.$$

The aggregation of the vectors Lx is called the *range* of L and is denoted by

$$\boldsymbol{R}_L.$$

A complex number ω is regarded as an operator in the meaning that we write it as

$$y = \omega x \; (x, y \in \boldsymbol{X})$$

when
$$y(\xi) = \omega x(\xi) \text{ for each } \xi \in \Xi.$$

As shown in § 1, the projection P_y on the subspace $\langle\!\langle y \rangle\!\rangle$ has no effective image of a vector ∂_ξ when $y(\xi) \neq 0$ and $\|y\| > 0$; therefore, in this case, P_y may not be regarded as a linear operator of $\boldsymbol{X}$. A linear operator in our theory must be first described by the formula

$$L\,\partial_\xi = \mathfrak{S}\, l(\xi, \eta)\, \partial_\eta \tag{3, 1}$$

so that for an arbitrary vector

$$x = \mathfrak{S}\, x(\xi)\, \partial_\xi$$

we may have

$$Lx = \mathfrak{S}_{\eta} \, (\mathfrak{S}_{\xi}\, x(\xi)\, l(\xi, \eta))\, \partial_\eta.$$

Since the right hand of (3, 1) must be a vector in $\boldsymbol{X}$, $l(\xi, \eta)$ take complex values. The alternation of the summation procedue $\mathfrak{S}_{\xi}\mathfrak{S}_{\eta}$ by $\mathfrak{S}_{\eta}\mathfrak{S}_{\xi}$ is always thought as possible in our theory, whenever L is given as an effective operator of $\boldsymbol{X}$. Next, when

$$B(\boldsymbol{R}_L) = (\rho_\lambda)_{\lambda\in\Lambda},$$

at least one vector x_λ is claimed to be existent for each $\lambda \in \Lambda$ such that

$$Lx_\lambda = \rho_\lambda. \tag{3, 2}$$

An operator thus defined by (3, 1) to (3, 2) is called an *intimate operator*, or simply an *operator* if there is no fear of confusion.

For an operator L, if exists an operator L_1 such that

$$L_1 L = 1,$$

L_1 is called the *left inverse operator* or simply the *inverse* of L (in this paper) and is denoted by

$$L^{(-1)}.$$

When $L^{(-1)}$ exists L is said *left-invertible.* The set Ω_L of the complex numbers ω such that $(L-\omega)$ are not invertible, is the *spectrum* of L. Besides, as it is well-known, the spectrum of L has been originally defined by several authors as the set of the proper values of L, i. e. the set of the complex numbers ω such that

$$(\exists x)\,(x \in \boldsymbol{X} \text{ and } Lx = \omega x).$$

When the linear operators L and $L^{(-1)}$ are not necessarily expected to be intimate ones, for that the above-stated two definitions of a spectrum perfectly accord, it is necessary and sufficient that

$$(\nexists\, L^{(-1)}) \bowtie (\exists\, y)\,(y \in \boldsymbol{X},\ y \neq 0 \text{ and } Ly = 0) \tag{3, 3}$$

though, when L and $L^{(-1)}$ are restricted within the intimate operators, to establish the assertion (3, 3) is not easy. It will be a nonsense under such conditions to proceed any analysis dispensing with (3, 3). So, it may be an opportune disposal if we restrict our analysis within the case where the intimate operators L and $L^{(-1)}$ conform to the criterion (3, 3).

It is remarkable that the vectors x cannot be restricted to be of finite height. In effect, for the operators defined as

$$S\partial_\xi = s(\xi)\partial_\xi,\ s(\xi) \neq 0 \text{ for each } \xi \in \Xi$$

and

$$s(\xi) \to 0 \text{ whenever } \xi_0 \neq \xi \to \xi_0,$$

$S^{(-1)}$ is existent and

$$S^{(-1)}\partial_\xi = \frac{1}{s(\xi)}\partial_\xi \,(\equiv y_\xi)$$

and then it is observed that

$$\overline{y_\xi|} = \overline{S^{(-1)}\partial_\xi|} = \left|\frac{1}{s(\xi)}\right| \to \infty \text{ as } \xi \to \xi_0,$$

so that apparently many vectors of infinite height may be found in the range of $S^{(-1)}$. In addition, the notion of the morm $\|L\|$ of an operator L will not be used by similar reasoning.

For an operator L defined as

$$L\partial_\xi = \underset{\eta}{\mathfrak{S}}\, l(\xi, \eta)\,\partial_\eta$$

the corresponding operator $\bar{L}$ by the difinition

$$\bar{L}\,\partial_\xi = \mathfrak{S}_\eta\, \overline{l(\eta, \xi)}\, \partial_\eta \tag{3, 4}$$

($\overline{l(\eta, \xi)}$: the conjugate number of $l(\eta, \xi)$) is called the *adjoint* of L (i. r. t. the basis $(\partial_\xi)_{\xi\in\Xi}$). An operator is called a *finite operator* if both of the set of such η that

$$l(\xi. \eta) \neq 0 \text{ for a fixed } \xi$$

and the set of such ξ that

$$l(\xi, \eta) \neq 0 \text{ for a fixed } \eta$$

are always found as finite sets.

Proposition 1. *When both of L and $\bar{L}$ are intimate operators, they are finite operators, too.*

Demonstration. In effect, if

$$y = Lx\,(x \in \boldsymbol{X})$$

then, in view of (3, 4)

$$y(\eta) = \mathfrak{S}_\xi\, x(\xi)\, l(\xi, \eta).$$

Besides, x may be such that

$$x(\xi) = \frac{1}{l(\xi, \eta)}$$

for a fixed η, so that

$$y(\eta) = 1 + 1 + \cdots.$$

Since $y(\eta) \neq \infty$ the points ξ for which

$$l(\xi, \eta) \neq 0$$

must be exhausted within a finite count.

Next, if we take $\bar{L}$ instead of L, according to the definition of $\bar{L}$ we similarly see the points η for which

$$\overline{l(\xi, \eta)} \neq 0$$

must be exhausted within a finite count. Then, since

$$\overline{l(\xi, \eta)} \neq 0 \rtimes\!\!\lhd\, l(\xi, \eta) \neq 0,$$

the verification is completed.

4. Orthogoual Transmutation.

Given a subspace $\boldsymbol{Y}$, if there exists another subspace $\boldsymbol{Z}$ such that any vector x in $\boldsymbol{X}$ may be uniquely expressed in the form

$$x=\alpha y+\beta z \quad (y \in \boldsymbol{Y},\ z \in \boldsymbol{Z})$$

(α, β : complex numbers), then $\boldsymbol{Z}$ is called a *linear supplement* of $\boldsymbol{Y}$. It is remarkable that there may be found many linear supplements possible for a fixed subspace. For an arbitrary vector ρ in $\boldsymbol{X}$, the following process to make up a supplement of $\ll\rho\gg$ is very important. First, let a point ξ for which $\rho(\xi)\neq 0$ be fixed in Ξ_ρ; next, let the subspace

$$\boldsymbol{Z}=\vee_{\eta\neq\xi}\ll\partial_\eta\gg$$

be taken up; then it is easily seen that $\boldsymbol{Z}$ is a linear supplement of $\ll\rho\gg$. In case of a subspace $\boldsymbol{Y}$ generated by an enumerable family of vectors $(\rho_k)_k=1, 2, \cdots$ the process may be simply generalized as follows. First, let a point ξ_λ for which $\rho_\lambda(\xi_\lambda)\neq 0$ be fixed in Ξ_λ for each $\lambda=1, 2, \cdots$; next, take up the set

$$\Gamma=(\eta:\ \eta\in\Xi \text{ but } \eta\neq\xi_\lambda \text{ for all } \lambda=1, 2, \cdots);$$

then the subspace

$$\boldsymbol{Z}=\vee_{\eta\in\Gamma}\ll\partial_\eta\gg$$

is found to be a linear supplement of $\boldsymbol{Y}$.

This process of supplementing is brought to an evident standstill when the power of $B(\boldsymbol{Y})$ is larger than enumerability, because then the axiom of choice will be needed again if we try to build up the set $(\xi_\lambda)_{\lambda\in\Lambda}$ in analogization. Nevertheless, there is still a way to throw light to our idea. If we cease to treat the total space $\boldsymbol{X}$ as perfectly given but prefer to test it to restrict within the construction of the meaning of a formal extension of the above-shown supplementing, then there may be left no objection in regard to the axiom of choice. Thus reasoning, we may posit a set of ξ_λ as given such that

$$\rho_\lambda(\xi_\lambda)\neq 0$$

whenever
$$B(\boldsymbol{Y})=(\rho_\lambda)_{\lambda\in\Lambda}.$$

Now, let us take a positive system of application $(\gamma_\lambda)_{\lambda\in\Lambda}+(\gamma_\eta)_{\eta\in\Gamma}$ such that

$$\mathfrak{S}\gamma_\lambda+\mathfrak{S}\gamma_\eta=1$$

where
$$\Gamma=\Gamma(\boldsymbol{Y})=(\eta:\ \eta\neq\xi_\lambda \text{ for all } \lambda\in\Lambda),$$

and introduce a new product described in the form

$$(x|y)^\circ=\mathfrak{S}\mathring{x}(\lambda)\overline{\mathring{y}(\lambda)}\gamma_\lambda+\mathfrak{S}\mathring{x}(\eta)\overline{\mathring{y}(\eta)}\gamma_\eta \tag{4, 1}$$

on condition that x and y are uniquely expressed as

$$x=\mathfrak{S}\mathring{x}(\lambda)\rho_\lambda+\mathfrak{S}\mathring{x}(\eta)\partial_\eta$$

ond
$$y=\mathfrak{S}\mathring{y}(\lambda)\rho_\lambda+\mathfrak{S}\mathring{y}(\eta)\partial_\eta$$

respectively; the universal possibility of such expressions is now evident. Then the subspace

$$Z = \vee_{\eta \in \Gamma} \ll \partial_\eta \gg$$

is found orthogonal to Y in respect to the product (4, 1,) and thus we will have a new topological structure of X. So, the formula (4, 1) is called an *orthogonal transmutation of X with respect to the subspace Y.*

The adjoint $\mathring{L}$ of an operator L i. r. t. (4, 1) may be defined by the relation

$$(\forall_X x)(\forall_X y) : (\mathring{L}x|y)^\circ = (x|Ly)^\circ. \tag{4, 2}$$

If it is described as

$$\mathring{L}\rho_\lambda = \mathfrak{S}_{\nu \in \Lambda} \mathring{g}_\lambda(\nu)\rho_\nu + \mathfrak{S}_{\zeta \in \Gamma} \mathring{f}_\lambda(\zeta)\partial_\zeta$$

and

$$L\partial_\eta = \mathfrak{S}_{\nu \in \Lambda} g_\eta(\nu)\rho_\nu + \mathfrak{S}_{\zeta \in \Gamma} f_\eta(\zeta)\partial_\zeta,$$

we have

$$(\mathring{L}\rho_\lambda|\partial_\lambda) = \mathring{f}_\lambda(\eta)\gamma_\eta$$

and

$$(\rho_\lambda|L\partial_\eta) = g_\eta(\lambda)\gamma_\lambda,$$

so that in view of (4, 2)

$$\mathring{f}_\lambda(\eta)\gamma_\eta = g_\eta(\lambda)\gamma_\lambda$$

which means

$$0 < \gamma_\eta/\gamma_\lambda < \infty$$

whenever $\mathring{f}_\lambda(\eta) \neq 0$, because it is evident that $\mathring{f}_\lambda(\eta) \neq 0$ implies $g_\eta(\lambda) \neq 0$ and conversely. Since L is essentially arbitrary, this result may force the system $(\gamma_\lambda)_{\lambda \in \Lambda} + (\partial_\eta)_{\eta \in \Gamma}$ to be given as a regular one, i. e. $0 < \gamma_\lambda/\gamma_{\lambda'} < \infty$, $0 < \gamma_\lambda/\gamma_\eta < \infty$ and $0 < \gamma_\eta/\gamma_{\eta'} < \infty$ for all λ, $\lambda' \in \Lambda$ and η, $\eta' \in \Gamma$.

Proposition 2. *For that each intimate operator has its adjoint effective with respect to the transmutation* (4, 1), *it is necessary and sufficient that the system* $(\gamma_\lambda)_{\lambda \in \Lambda} + (\gamma_\eta)_{\eta \in \Gamma}$ *is a regular one.*

5. On R_L.

When $B(R_L) = (\rho_\lambda)_{\lambda \in \Lambda}$ the set of the vectors y such that

$$Ly = \rho_\lambda$$

is denoted by* X_λ. For any vector x in X the vector Lx can be expressed in the from

$$Lx = \mathfrak{S}\, \tilde{x}(\lambda)\rho_\lambda. \tag{5, 1}$$

Now, let us take a vector y describable in the form

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* As is stated in § 3, it is claimed that $X_\lambda \neq$ void, in our theory.

$$y=\mathfrak{S}\,\bar{x}(\lambda)x_\lambda$$

on condition that

$$x_\lambda \in \boldsymbol{X}_\lambda \text{ for each } \lambda\in\Lambda,$$

and let x be composed as

$$x=y+z.$$

Then we have

$$Lx=Ly+Lz=\mathfrak{S}\,\bar{x}(\lambda)\rho_\lambda+Lz,$$

so that, in view of (5, 1), it must be

$$Lz=0.$$

If $z\neq 0$, $L^{(-1)}$ cannot exist, because then

$$L(x_\lambda+z)=L(x_\lambda)=\rho_\lambda$$

whereas $x_\lambda+z\neq x_\lambda$. Consequently we may have:

Proposition 3. *For existence of $L^{(-1)}$ it is necessary and sufficient that each $\boldsymbol{X}_\lambda$ $(\lambda\in\Lambda)$ consists of a single vector x_λ and any vector $x\in\boldsymbol{X}$ can be uniquely expressed in the form*

$$x=\mathfrak{S}\,\bar{x}(\lambda)x_\lambda.$$

It is remarkable that the original domain of definition of $L^{(-1)}$ is $\boldsymbol{R}_L$ but not $\boldsymbol{X}$ itself; in other words, $L^{(-1)}$ may not be regarded as an intimate operator of $\boldsymbol{X}$ except the case $\boldsymbol{R}_L=\boldsymbol{X}$. In case $\boldsymbol{R}_L=\boldsymbol{X}$, as it is evident that

$$B(\boldsymbol{R}_L)=(\partial_\xi)_{\xi\in\Xi},$$

the sets $\boldsymbol{X}_\lambda$ may be altered by the sets

$$\boldsymbol{X}_\xi=(x:\ x\in\boldsymbol{X} \text{ and } Lx=\partial_\xi),$$

and almost directly we may see:

Lemma 2. *In case $\boldsymbol{R}_L=\boldsymbol{X}$, if a family $(x_\xi)_{\xi\in\Xi}$ satisfies the condition $x_\xi\in\boldsymbol{X}_\xi$, then for the operators defined as*

$$S\partial_\xi=x_\xi \text{ for each } \xi\in\Xi$$

it is observed that $$L=S^{(-1)}.$$

By Prop. 3 ond Lemma 2 it is obtatained that:

Proposition 4. *In case $\boldsymbol{R}_L=\boldsymbol{X}$, if $L^{(-1)}$ exists, $L^{(-1)}$* is in *fact the both side inverse of L*, i. e.

$$LL^{(-1)}=L^{(-1)}L=1.$$

6. Left Ideal in the Ring of Operators.

Apparently the intimate operators of $\boldsymbol{X}$ make up a ring together; so let it be denoted by $\mathfrak{R}$. If an operator S is defined on a subspace $\boldsymbol{Y}$ of $\boldsymbol{X}$, on replenishing such as

$$Sx=0 \text{ for all } x\in \boldsymbol{X}-\boldsymbol{Y}$$

we may regard S as an element of $\mathfrak{R}$. In the following, we will deal with only thus replenished operators, so that the inverse $L^{(-1)}$ may have $\boldsymbol{X}$ always as its domain of definition instead of $\boldsymbol{R}_L$. If a subset $\mathfrak{I}$ of $\mathfrak{R}$ satisfies the following conditions, $\mathfrak{I}$ is a *left ideal*: (i) $(\mathfrak{I}\ni L,S) \triangleright aL+bS\in\mathfrak{I}$, by arbitrary complex numbers a and b; (ii) $\mathfrak{I}\supseteq{}^*\mathfrak{R}\mathfrak{I}$. When $\mathfrak{I}\neq\{0\}$, $\mathfrak{R}$, the ideal $\mathfrak{I}$ is said *non-trivial*. Given an operator L, the subset $\mathfrak{R}L$ is evidently the minimum of the ideals containing L, and is denoted by $\mathfrak{I}_L$. If there may be no non-trivial ideal $\mathfrak{I}'$ such that

$$\mathfrak{I}\neq\mathfrak{I}' \text{ and } \mathfrak{I}\subset\mathfrak{I}'\subset\mathfrak{R},$$

then the non-trivial ideal $\mathfrak{I}$ is a *maximal* (*left*) *ideal*.

The results that hnceforth follow are what have been gained by a study aimed for analogization of the theory of normed ring of operators. In our study it may be specially characteristic that all of the measures $\|x\|$, $\overline{x}|$, $\|L\|$ and $\overline{L}|$ are put out of use and the restriction (3, 3) is universally applied. Let the aggregation of the operators of $\boldsymbol{X}$ which are not left-invertible be denoted by $\mathfrak{N}$; this subset of $\mathfrak{R}$ plays an important role in our analysis. By some simple computations the following lemmas may be gained, where a left ideal is simply called an *ideal*.

Lemma 3. *If an ideal $\mathfrak{I}$ is non-trivial, then*

$$\{0\}\neq\mathfrak{I}\subseteq\mathfrak{N}.$$

Lemma 4. *If $0\neq L\in\mathfrak{N}$, the ideal $\mathfrak{I}_L$ is non-trivial.*

Lemma 5. *For that an ideal $\mathfrak{I}$ is non-trivial, it is necessary and sufficient that*

$$\{0\}\neq\mathfrak{I}\subseteq\mathfrak{N}.$$

We insert here the following axiom as a logical agent in our analysis:

Axiom 0. *Given a set $\mathfrak{A}$ fixed, by $\mathfrak{p}$ be denoted a property which is either possessed by a subset in $\mathfrak{A}$ or not, in relation to at most a finite number of fixed sets, and let it be assumed that: if there is a subset $\mathfrak{B}_1\subset\mathfrak{A}$ (i. e. $\mathfrak{A}-\mathfrak{B}_1\neq$ void) haviug $\mathfrak{p}$, there exists another subset $\mathfrak{B}_2$ having $\mathfrak{p}$ such that*

$$\mathfrak{B}_1\subset\mathfrak{B}_2\subseteq\mathfrak{A},$$

* $\mathfrak{R}\mathfrak{I}$ means the set of all SL such that $S\in\mathfrak{R}$ and $L\in\mathfrak{I}$.

and moreover it is certain that at least one subset exists in $\mathfrak{A}$ *having* $\mathfrak{p}$. *Then, there exists an increasing sequence of subsets having* $\mathfrak{p}$

$$\mathfrak{B}_1 \subseteq \mathfrak{B}_2 \subseteq \cdots \subseteq \mathfrak{A}$$

for which no subset $\mathfrak{B}$ *is found having* $\mathfrak{p}$ *such that*

$$\cup \mathfrak{B}_k \subset \mathfrak{B} \subseteq \mathfrak{A}.$$

Proposition 4. *For that an operstor L is left-invertible, it is necessary and sufficient that thers is no maximal ideal containing L.*

Demonstration. If L is left-invertible, in view of Lemma 3, no maximal ideal contains L, because a maximal ideal is a non-trivial one too. Therefore, for completion of the demonstration it is sufficient if the negative assumption for existence of a maximal ideal containing L is induced to a contradiction when L is not left-invertible. In this case, as $L \in \mathfrak{R}$, by Lemma 4 $\mathfrak{J}_L$ is a non-trivial ideal, and if $\mathfrak{J}_L$ is not a maximal ideal there exists by definition an increasing sequence of non-trivial ideals such as

$$\mathfrak{J}_L \subset \mathfrak{J}_1 \subset \mathfrak{J}_2 \subset \cdots \subseteq \mathfrak{R}.$$

We may denote by $\mathfrak{p}$ the property that a subset of $\mathfrak{R}$ be a non-trivial ideal containing $\mathfrak{J}_L$, so that by Axiom 0 we may assume that there exists no non-trivial ideal $\mathfrak{J}$ such that

$$\cup \mathfrak{J}_k \subset \mathfrak{J} \subset \mathfrak{R}. \tag{6, 1}$$

Now let it be denoted as

$$\tilde{\mathfrak{J}}_L = \cup \mathfrak{J}_k,$$

then $\tilde{\mathfrak{J}}_L$ is a non-trivial ideal. In effect,

$$\tilde{\mathfrak{J}}_L \ni S, S' \rhd (\exists k)(\exists k')(S \in \mathfrak{J}_k,\ S' \in \mathfrak{J}_{k'})$$

and on denoting $\quad \bar{k} = \max(k, k')$

we have $\quad S,\ S' \in \mathfrak{J}_{\bar{k}}$

because then $\quad \mathfrak{J}_k \subseteq \mathfrak{J}_{\bar{k}}$ and $\mathfrak{J}_{k'} \subseteq \mathfrak{J}_{\bar{k}}$,

so that $\quad \alpha S + \beta S' \in \mathfrak{J}_{\bar{k}} \subseteq \tilde{\mathfrak{J}}_L$

by arbitrary complex numbers α and β. Hence, we consequently have

$$\tilde{\mathfrak{J}}_L \ni S,\ S' \rhd \alpha S + \beta S' \in \tilde{\mathfrak{J}}_L.$$

Next, $\quad \mathfrak{R}\tilde{\mathfrak{J}}_L \ni S \rhd (\exists S_1)(\exists S_2)(\exists k)(S_1 \in \mathfrak{R},\ S_2 \in \mathfrak{J}_k \text{ and } S = S_1 S_2)$

and then $\quad S = S_1 S_2 \in \mathfrak{R}\mathfrak{J}_k \subseteq \mathfrak{J}_k \subseteq \tilde{\mathfrak{J}}_L.$

Besides, it is evident that

$$\tilde{\mathfrak{J}}_L \subseteq \mathfrak{R}$$

so that $\tilde{\mathfrak{J}}_L$ is *a* non-trivial ideal. Then, in regard to the indication about (6, 1), we may conclude that $\tilde{\mathfrak{J}}_L$ is a maximal ideal containing $\mathfrak{J}_L$, which gives the contradiction promised.

For any maximal ideal $\mathfrak{J}$ of $\mathfrak{R}$, let the residue class ring $\mathfrak{R}/\mathfrak{J}$ be described as

$$\mathfrak{R}/\mathfrak{J} = (\dot{S}(\mathfrak{J}))_{S \in \mathfrak{R}}$$

where
$$\dot{S}(\mathfrak{J}) \ni L \rtimes\!\!\lhd L - S \in \mathfrak{J}.$$

Then, since evidently

$$L - S \in \mathfrak{J} \rtimes\!\!\lhd S - L \in \mathfrak{J}$$

we have
$$L \in \dot{S}(\mathfrak{J}) \rtimes\!\!\lhd S \in \mathring{L}(\mathfrak{J})$$

which is written as
$$L \equiv S (\mathrm{mod}\, \mathfrak{J}).$$

When it is observed for a fixed complex number ω that

$$(\forall_X x)\,(Sx = \omega x),$$

we adopt the simple notation ω instead of S, and $\dot{\omega}(\mathfrak{J})$ instead of $\dot{S}(\mathfrak{J})$. By Ω_L is denoted the spectrum of an operator L. Then, the following theorem may be verified without any difficulty, which is an analogization of the classical one.[4]

Proposition 5. *When $\Omega_L \neq$ void, if*

$$L \in \dot{S}(\mathfrak{J})$$

for a maximal ideal $\mathfrak{J}$, then exists a number $\omega \in \Omega_L$ such that

$$S = \omega,$$

and conversely if $\omega \in \Omega_L$, exists a maximal ideal $\mathfrak{J}$ such that

$$L \in \dot{\omega}(\mathfrak{J}.)$$

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