On Normed Spaces and Modulared Semi-ordered Linear Spaces

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Abstract

To investigate the problems as to what normed spaces become modulared semi-ordered linear spaces, the present writer defines, as his first attempt, the constant modular of which the topology is equivalent to the norm topology in the original normed space R by the family of some operators in R.

§ 1. Intoduction. The theory of modulared semi-ordered linear spaces have been discussed by H. Nakano**, as the abstract theory of a function spaces including Orlicz spaces[6] and L_p-spaces, and also discussed by many others. The spaces are condisered as normed spaces, but it is not always true that normed spaces are modulared spaces. Accordingly, we have the problems as to what normed spaces become modulared spaces.

In this paper, to prove the above-mentioned problem we will consider the family of some operators, acting in the normed space R with a complete element, by which it becomes modulared space conforming to Orlicz space which is topologically equivalent to the original space.

From this point of view, the preliminaries and definitions shall be described in § 2. In § 3 we will give the family $\mathfrak{F}$ of the operators which answer our purpose, and investigate its properties. In § 4 we will construct the modular of which the topology is equivalent to the original topology in the space. In § 5 we will give the example of the space which has the family $\mathfrak{F}$.

Most of the same notations and terminologies as those in [MSLS] are used in this paper.

In conclusion, the present writer wishes to express his sincere thanks to Prof. S. Yamamuro for his kindly encouragement and advice.

§ 2. Preliminaries and definitions. Let R be universally continuous semi-ordered linear spaces, namely conditionally complete vector lattices in the sense of G. Birkhoff[2], which have a complete element* s.

Many important results on R, especially the spectral theory, are discussed by H. Nakano[3],[4].

First, preparatory to the next discussion, we will restate some definitions and results in [MSLS].

For a subset M of $\mathbf{R}$, the set

$$M^{\perp} = \{x \, ; \, |x| \frown |y| = 0^* \text{ for all } y \in M\}$$

is called the orthogonal complement of M.

Theorem 1. *For any subset M of $\mathbf{R}$, we have the following property: for any $a \in \mathbf{R}$, there exist x and y uniquely such that*

$$a = x + y, \quad x \in M^{\perp\perp} \text{ and } y \in \mathbf{M}^{\perp}$$

proof. Cf. Th. 4.10, 4.3 and 4.4 in [MSLS].

For any subset N, we define a *projection operator* $[N]$ by

$$[N]a = x, \quad a = x + y, \quad x \in N^{\perp\perp} \text{ and } y \in N^{\perp} \text{ for all } a \in \mathbf{R}.$$

In particular, when N consists of only an element p, $[p]$ is called a *projector*.

A set $\mathfrak{p}$ of projectors is called an *ideal*, if

1) $\mathfrak{p} \ni 0$,
2) $\mathfrak{p} \ni [p] \leq [q]$ implies $\mathfrak{p} \ni [q]$,
3) $\mathfrak{p} \ni [p]$, $[q]$ implies $\mathfrak{p} \ni [p][q]$.

An ideal $\mathfrak{p}$ is said to be *maximal*, if every ideal containing $\mathfrak{p}$ coincides with $\mathfrak{p}$. The set $\mathfrak{E}$ of all maximal ideals becomes a compact Hausdorff space with the neighborhood system $\{U_{[p]}\}$ defining by $U_{[p]} = \{\mathfrak{p} \, ; \, [p] \ni \mathfrak{p} \ni \mathfrak{E}\}$, and each $U_{[p]}$ is open and closed.

For a set N in $\mathbf{R}$, we put

$$U_{[N]} = \sum_{x \in N^{\perp\perp}} U_{[x]}$$

Then we have

Theorem 2. *$U_{[N]}$ is also open and closed.*

proof. Cf. Th. 8.11 in [MSLS].

For any elements a and b, we define the so-called *relative spectra*[10],[11]

$$\left(\frac{b}{a}, \mathfrak{p}\right) = \begin{cases} \lambda & \text{if } \mathfrak{p} \in \prod_{\varepsilon > 0} (A_{\lambda+\varepsilon} - A_{\lambda-\varepsilon}) \\ +\infty & \text{if } \mathfrak{p} \in \prod_{-\infty < \lambda < +\infty} (U_{[a]} - A_{\lambda}) \\ -\infty & \text{if } \mathfrak{p} \in \sum_{-\infty < \lambda < +\infty} A_{\lambda} \end{cases}$$

where

$$A_{\lambda} = U_{[(\lambda a - b)^+][a^+]} + U_{[(\lambda a - b)^-][a^-]} \quad (-\infty < \lambda < +\infty).$$

Then the following properties become known well.

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* $x^+ = x \smile 0$, $x^- = (-x) \smile 0$ and $|x| = x^+ + x^-$.
** $U_{[s]}$ for a complete element $s \in \mathbf{R}$ coincides with $\mathfrak{E}$.

Theorem 3. $\left(\dfrac{b}{a},\ \mathfrak{p}\right)$ *is almost finite in* $U_{[a]}$ *and continuous in the extended sense*. Moreover, we have the following relations:*

$$\left(\frac{\alpha x + \beta y}{a},\ \mathfrak{p}\right) = \alpha\left(\frac{x}{a},\ \mathfrak{p}\right) + \beta\left(\frac{y}{a},\ \mathfrak{p}\right),$$

$$\left(\frac{x \smile y}{a},\ \mathfrak{p}\right) = \text{Max}\left\{\left(\frac{x}{a},\ \mathfrak{p}\right),\ \left(\frac{y}{a},\ \mathfrak{p}\right)\right\},$$

$$\left(\frac{x \frown y}{a},\ \mathfrak{p}\right) = \text{Min}\left\{\left(\frac{x}{a},\ \mathfrak{p}\right),\ \left(\frac{y}{a},\ \mathfrak{p}\right)\right\}.$$

if those on the right hand have sense, and

$$[p]x \leqq [p]y,\quad \text{if}\ \ \left(\frac{x}{a},\ \mathfrak{p}\right) \leqq \left(\frac{y}{a},\ \mathfrak{p}\right)\ \text{for all}\ \ \mathfrak{p} \in U_{[p]} \subseteqq U_{[a]}.$$

For any almost finite and continuous function $\varphi(\mathfrak{p})$ on $U_{[N]}$, we can find a sequence of projectors $[p_n]\uparrow_{n=1}^{\infty}[N]$ such that $\varphi(\mathfrak{p})$ is bounded on each $U_{[p_n]}$ since $\boldsymbol{R}$ has the complete element s.

The integral

$$\int_{[p_n]} \varphi(\mathfrak{p})\,d\mathfrak{p}s$$

is defined by the limit of the following partial sums:

$$\sum_{i=1}^{\kappa} \varphi(\mathfrak{p}_i)\,[p_{n,i}]s$$

where $\mathfrak{p}_i \in U_{[p_{n,i}]}$, $\ \sum_{i=1}^{\kappa} U_{[p_{n,i}]} = U_{[p_n]}$ and $|\varphi(\mathfrak{p}) - \varphi(\mathfrak{p}')| < \varepsilon$ for $\mathfrak{p},\ \mathfrak{p}' \in U_{[p_{i,n}]}$.

If

$$\lim_{n \to \infty}\ \int_{[p_n]} \varphi(\mathfrak{p})\,d\mathfrak{p}s$$

exists, the limit is denoted by

$$\int_{[N]} \varphi(\mathfrak{p})\,d\mathfrak{p}s\ .$$

Furthermore the following relation:

$x = \int_{[a]} \varphi(\mathfrak{p})\,d\mathfrak{p}a$ is equivalent to $\varphi(\mathfrak{p}) = \left(\dfrac{x}{a},\ \mathfrak{p}\right)$ on $U_{[a]}$, is known in [MSLS].

$\boldsymbol{R}$ is said to be a *modulared space,* if it has a functional on $m(x)$ (called a modular) satisfying the following conditions:

M 1) $0 \leqq m(x) \leqq +\infty \qquad (x \in \boldsymbol{R})$,

M 2) if $m(\xi x) = 0$ for all $\xi \geqq 0$, then $x = 0$,

M 3) for any $x \in \boldsymbol{R}$ there exists $\alpha > 0$ such that $m(\alpha x) < +\infty$,

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* $\varphi(\mathfrak{p})$ is said to be almost finite, if $\varphi(\mathfrak{p})$ is finite on some dense set D in $U_{[a]}$. Moreover, $\varphi(\mathfrak{p})$ is said to be continuous in the extended sense, if $\varphi(\mathfrak{p})$ is continuous in D.

M 4) $m(\xi x)$ is a convex function of $\xi \geq 0$,

M 5) $|x| \leq |y|$ implies $m(x) \leq m(y)$,

M 6) $x \frown y = 0$ implies $m(x+y) = m(x) + m(y)$,

M 7) $0 \leq x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies $m(x) = \sup_{\lambda \in \Lambda} m(x_\lambda)$

The modulared space is a *normed space*, namely, we can define two kinds of functionals :

> *the first norm* $\|x\|_1 = \inf_{\xi > 0} \dfrac{1 + m(\xi x)}{\xi}$ and
>
> *the second norm* $\|x\|_2 = \inf_{m(\xi x) \leq 1} 1/|\xi|$

which satisfy the relations $\|x\|_2 \leq \|x\|_1 \leq 2\|x\|_2$ $(x \in R)$ and the norm conditions

N 1) $0 \leq \|x\|_i < +\infty$,

N 2) $\|x\|_i = 0$ if and only if $x = 0$,

N 3) $\|\alpha x\|_i = |\alpha| \|x\|_i$ for a real number α ,

N 4) $\|x + y\|_i \leq \|x\|_i + \|y\|_i$,

N 5) $|x| \leq |y|$ implies $\|x\|_i \leq \|y\|_i$.

A normed space $\boldsymbol{R}$ is called a *continuous normed space*, if it has a continuous norm, i.e., $x_n \downarrow_{n=1}^{\infty} 0$ implies $\lim_{n \to \infty} \|x_n\| = 0$.

$\boldsymbol{R}$ is said to be *regular*, if it has a complete linear functional $\phi \in \bar{\boldsymbol{R}}$ (*conjugate space* of $\boldsymbol{R}$*), i.e., $\phi(x) = 0$ implies $x = 0$ $(x \in \boldsymbol{R})$.

Theorem 4.   *If a continuous normed space $\boldsymbol{R}$ is semi-regular,** then it is superuniversally continuous.*

Proof.   Cf. Th. 30. 7 in [MSLS].

In the sequel, let $\boldsymbol{R}$ be a continuous normed space with a norm $\|\cdot\|$, and be regular.

Let $f(\xi) \geq 0$ be non-decreasing and continuous in $\xi \geq 0$ with $f(0) = 0$, and U be a complete system of enumerable number of orthogonal elements of $\boldsymbol{R}$, i.e., for any $0 \leq x \in \boldsymbol{R}$ there exists a $u \ni U$ such that $u \frown x \neq 0$.

$\boldsymbol{R}$ is called the $\boldsymbol{O}_*$-*space*, if there exists a *family $\mathfrak{H}$ of operators*, H_J acting from U into $\boldsymbol{R}^+ = \{x \,;\, 0 \leq x \in \boldsymbol{R}\}$, which satisfy the following conditions :

(I)   *for some fixed constants $0 < A \leq 1 \leq B$, if*

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*   A conjugate space $\bar{\boldsymbol{R}}$ of $\boldsymbol{R}$ is a set of universally continuous linear functionals $\bar{x}$ for which $\inf_{\lambda \in \Lambda} |\bar{x}(x_\lambda)| = 0$ if $x_\lambda \downarrow_{\lambda \in \Lambda} 0$.

** $\boldsymbol{R}$ is said to be *semi-regular*, if, for any $p \in \boldsymbol{R}$, $\bar{a}(p) = 0$ for all $\bar{a} \in \bar{\boldsymbol{R}}$ implies $p = 0$. Also $\boldsymbol{R}$ is said to be *superuniversally continuous*, if, for any orthogonal system $0 \leq a_\lambda$ $(\lambda \in \Lambda)$ and for any $p \geq 0$, such relations as $a_\lambda \frown p = 0$ are obtained except for at most enumerable λ.

$$c = \sum_{\nu=1}^{m} \sum_{k=1}^{n_\nu} \alpha_{\nu,k} [N_{\nu,k}] u_\nu \quad where \quad \alpha_{\nu,k} > 0, \ [N_{\nu,k}][N_{\nu,j}]^* = 0 \ (k \neq j) \ and \ \|c\| = 1, \ then$$

we have

$$0 \leqq H_J u_\nu \in B_{u_\nu}^{**} \qquad where \qquad J = \{u_1, u_2, \cdots, u_m\} \ (u_\nu \in U)$$

and

$$A \leqq \sum_{\nu=1}^{m} \sum_{k=1}^{n_\nu} \alpha_{\nu,k} f(\alpha_{\nu,k}) \phi([N_{\nu,k}] H_J u_\nu) \leqq B,$$

(II) *for* $H_{J_1}, H_{J_2} \in \mathfrak{H}$ *there exists a* $H_{J_3} \in \mathfrak{H}$ *such that*

$$H_{J_1} u \smile H_{J_2} u \leqq H_{J_3} u \qquad for \quad u \in U,$$

where $J_i \ (i = 1, 2)$ *are finite subsets of* U *and* $J_3 = J_1 + J_2$.

§3. The family $\mathfrak{F}$ of the operators in R.

In this section, we will deal with the family $\mathfrak{F}$ of the operarors which are constructed by H_J in §2.

For each finite subset $J = \{u_1, u_2, \cdots, u_m\}$ of U, we put

$$\mathfrak{D}(S) = \{x : |x| \frown u_\nu = 0 \text{ for all } \nu = 1, 2, \cdots, m\}$$
$$+ \left\{ \sum_{\nu=1}^{m} \sum_{i=1}^{n_\nu} \beta_{\nu,i} [P_{\nu,i}] u_\nu \ ; n_\nu, \ \beta_{\nu,i} \text{ and } [P_{\nu,i}] \text{ are any} \right\}.$$

Then, we will make the *operator* S on $\mathfrak{D}(S)$ in the following manner :

(1) *for* $$y = \sum_{\nu=1}^{m} \sum_{i=1}^{k_\nu} \alpha_{\nu,i} [N_{\nu,i}] u_\nu,$$

$$Sy = \sum_{\nu=1}^{m} \sum_{i=1}^{k_\nu} (\pm 1) f(|\beta_{\nu,i}|) \int_{[N_{\nu,i}]} \left(\frac{H_J u_\nu}{u_\nu}, \ \mathfrak{p} \right) d\mathfrak{p} s$$

where the sign $\pm$ *coincides with the sign of* $\beta_{\nu,i}$,

(2) $$S(-y) = -Sy \qquad for \quad y \in \mathfrak{D}(S),$$

(3) $$Sx = 0, \ if \ |x| \frown u_\nu = 0 \qquad for \ all \quad \nu = 1, 2, \cdots, m.$$

We will denote by $\mathfrak{F}$ the family of such operators S corresponding to every element of $\mathfrak{H}$.

Here after, we will give the properties of $\mathfrak{F}$.

[1]. *For any* $S \in \mathfrak{F}$, $\mathfrak{D}(S)$ *is dense in* R *with respect to the order topology.*

proof. First, for fixed $u \in U$, we will show that if $0 \leqq a \in Bu$, i.e., $0 \leqq a \leqq Ku$ for some constant K, then there exists a sequence $\{x_n\}$ such that $0 \leqq x_n \uparrow_{n=1}^{\infty} a$.

putting

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* $[N_{\nu,k}]$ are projection operators.

** $B_{u_\nu} = \{x; \ |x| \leqq \alpha_x u_\nu \text{ for some } \alpha_x > 0\}$ is called the *relative segment* of u_ν.

$$(\sharp\!\!\!\sharp) \qquad U_{[N_i^{(n)}]} = \left\{ \mathfrak{p} \; ; \; \frac{i-1}{n}\,K < \left(\frac{a}{u},\,\mathfrak{p} \right) < \frac{i}{n}\,K \right\}^{-*},$$

we have

$$\sum_{i=1}^{n} [N_i^{(n)}] \uparrow_{n=1}^{\infty} [u] \qquad\qquad (\text{cf. } \S\,8 \text{ in [MSLS]}).$$

Now, we put

$$x_n = \sum_{i=1}^{n} \frac{i-1}{n}\,K[N_i^{(n)}]u\,.$$

Then we have, from ($\sharp\!\!\!\sharp$),

$$\lim_{n\to\infty} \left(\frac{x_n}{u},\,\mathfrak{p} \right) = \sup_{n=1} \left(\frac{x_n}{u},\,\mathfrak{p} \right) = \left(\frac{a}{u},\,\mathfrak{p} \right)$$

for some σ-open dense set in $U_{[u]}$.

Furthermore we can find x such that $x_n \uparrow_{n=1}^{\infty} x_0$, because $0 \le x_n \le a$ by Theorem 3.

On the other hand, since $\boldsymbol{R}$ is totally continuous** by Theorem 4, we have

$$\lim_{n\to\infty} \left(\frac{x_n}{u},\,\mathfrak{p} \right) = \left(\frac{x_0}{u},\,\mathfrak{p} \right)$$

for some σ-open dense set in $U_{[u]}$. (cf. Th. 16.7 in [MSLS])

Thus we have

$$\left(\frac{a}{u},\,\mathfrak{p} \right) = \left(\frac{x_0}{u},\,\mathfrak{p} \right) \qquad \text{for some } \sigma\text{-open dense set in } U_{[u]}.$$

Therefore, by Theorem 3, we have $x_0 = a$ and hence $x_n \uparrow_{n=1}^{\infty} a$.

Next, for any $y \in [u]\boldsymbol{R}^+ = \{[u]x \; ; \; x \in \boldsymbol{R}_+\}$, putting

$$U_{[N_j]} = \left\{ \mathfrak{p} \; ; \; \left(\frac{y}{u},\,\mathfrak{p} \right) < j \right\}^{-},$$

we have

$$\lim_{j\to\infty} y_j = y \quad \text{where} \quad y_j = \int_{[N^j]} \left(\frac{y}{u},\,\mathfrak{p} \right) d\mathfrak{p} u \in B_u \frown \boldsymbol{R}^+\,.$$

Finally, when S corresponds to $H_J \in \mathfrak{H}$ where $J = \{u_1, u_2, \cdots, u_m\}$, $[J]\boldsymbol{R}^+ \ni x$ is uniquely expressed by the form: $x = \sum_{\nu=1}^{m} x_\nu$ where $x_\nu \in [u_\nu]\,\boldsymbol{R}^+$

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* M^- means a ordered-closure of a set M.

The existence of a projection operator $[N]$ satisfying $U_{[N]} = \left\{ \mathfrak{p} \; ; \; \alpha < \left(\frac{a}{u},\,\mathfrak{p} \right) < \beta \right\}^{-}$ has been shown by Th. 11.6 in [MSLS].

** $\boldsymbol{R}$ is said to be *totally continuous*, if for any double sequence of projectors $[p_{\nu,\mu}] \uparrow_{\mu=1}^{\infty} [p]$ ($\nu = 1, 2, \cdots$) there exist sequences $[p_\rho] \uparrow_{\rho=1}^{\infty} [p]$ and $\mu_{\nu,\rho}$ ($\nu = 1, 2, \cdots$) such that $[p_\rho] \le [p_\nu, \mu_{\nu,\rho}]$ ($\nu, \rho = 1, 2, \cdots$). If $\boldsymbol{R}$ is superuniversally continuous, then it is also totally continuous. (cf. Th. 30.11 in [MSLS])

Therefore, the positive part of $\mathfrak{D}(S)$ is dense in $\boldsymbol{R}^+$ and consequently by (2), the proof is completed.

[2]. *For any $S \in \mathfrak{F}$, we have the properties*:

(i) $D(S) \ni x, y$ *implies* $x \smile y \in \mathfrak{D}(S)$,

(ii) $0 \leq x \leq y \in \mathfrak{D}(S)$ *implies* $0 \leq Sx \leq Sy$.

Proof. (i) is easily seen. To show (ii), expressed

$$x = \sum_{\nu=1}^{m} \sum_{k=1}^{n_\nu} \alpha_{\nu,k} [N_{\nu,k}] u_\nu \qquad (\alpha_{\nu,k} \geq 0)$$

and

$$y = \sum_{\nu=1}^{m} \sum_{k=1}^{n_\nu} \beta_{\nu,k} [N_{\nu,k}] u_\nu \qquad (\beta_{\nu,k} \geq 0),$$

we have $\alpha_{\nu,k} \leq \beta_{\nu,k}$ and hence the required relation is obtained from the fact that $f(\xi)$ is non-decreasing.

[3]. *For any $S \in \mathfrak{F}$, we have*

$$S[N]x = [N]Sx, \quad Sx \in B_s \quad and \quad Sx = Sx^+ - Sx^-$$

for all $x \in \mathfrak{D}(S)$ and all projection operators $[N]$.

Proof. Let S be an element of $\mathfrak{F}$, corresponding to H_J where $T = \{u_1, u_2, \cdots, u_m\}$, and x be expressed with the form:

$$x = \sum_{\nu=1}^{m} \sum_{i=1}^{n_\nu} \alpha_{\nu,i} [N_{\nu,i}] u_\nu.$$

Then, by (I), there exists a number γ such that

$$H_J u_\nu \leq \gamma u_\nu \quad \text{for all} \quad \nu = 1, 2, \cdots, m.$$

Therefore, from the definition of the relative spectra, we have

$$|Sx| \leq \gamma_1 s \quad \text{where} \quad \gamma_1 = \text{Max} \{\gamma, f(|\alpha_{\nu,i}|)\}.$$

Other relations are obvious from the definition of S.

[4]. *For any $S \in \mathfrak{F}$, we have*

$$S\overset{)}{O} = 0 \quad and \quad S(a \smile b) = Sa \smile Sb.$$

Furthermore, $|a| \frown |b| = 0 \ (a, b \in \mathfrak{D}(S))$ implies

$$|Sa| \frown |Sb| = 0 \quad and \ hence \quad S(a+b) = Sa + Sb.$$

Proof. $SO = 0$ is obvious. Putting $p = (a-b)^+$, we have

$$[p](a \smile b) = [p]a \smile [p]b = [p]a \text{ and } (1-[p])(a \smile b) = (1-[p])b$$

and hence

$$S(a \smile b) = [p]Sa + (1-[p])Sb.$$

On the other hand, we have

$$[p](Sa \smile Sb) = [p]Sa \text{ and } (1-[p])(Sa \smile Sb) = (1-[p])Sb.$$

Therefore we have $S(a \smile b) = Sa \smile Sb$.

The relation: $S(a \frown b) = Sa \frown Sb$ is proved similarly.

　　[5]. *For any* $S \in \mathfrak{F}$, $a_\lambda, a \in \mathfrak{D}(S)$ *and*

$$0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda} a \quad \text{implies} \quad \sup_{\lambda \in \Lambda} Sa_\lambda = Sa.$$

　　Proof. By Theorem 4, there exists a sequence $\{a_i\}$ such that

$$0 \leqq a_i \uparrow_{i=1}^\infty a.$$

From the monotony of S, it is enough to show

$$\sup_{i \geqq 1} Sa_i = Sa.$$

Let $J = \{u_1, u_2, \cdots, u_{n_0}\}$ be a finite set of U corresponding to S.
We put, on account of $0 \leqq a_i$, $a \in \mathfrak{D}(S)$,

$$a = \sum_{\nu=1}^{n_0} \sum_{k=1}^{n_\nu} \alpha_{\nu,k} [N_{\nu,k}] u_\nu, \quad [N_{\nu,k}] a_i = b_i, \quad [N_{\nu,k}] a = b$$

and

$$b_{i+1} = \sum_{\lambda=1}^{l_{i,1}} \beta_{i+1,\lambda} [P_{i+1,\lambda}] u_\nu + \sum_{\lambda=l_{i,1}+1}^{l_{i,2}} \beta_{i+1,\lambda} [P_{i+1,\lambda}] u_\nu + \cdots + \sum_{\lambda=l_{i,i}+1}^{l_{i+1,0}} \beta_{i+1,\lambda} [P_{i+1,\lambda}] u_\nu$$

where

$$\sum_{\lambda=l_{i,k}+1}^{l_{i,k+1}} [P_{i+1,\lambda}] = [b_{k+1}] - [b_k] \qquad (k = 1, 2, \cdots, i-1),$$

$$\sum_{\lambda=l_{i,i}+1}^{l_{i+1,0}} [P_{i+1,\lambda}] = [b_{i+1}] - [b_i] \text{ and } \beta_{i\lambda} \leqq \beta_{i+1,\lambda} \qquad (\lambda = 1, 2, \cdots, l_{i,i}).$$

Putting $\lim_{i \to \infty} Sb_i = c$, if $c < Sb$ then there exists a $U_{[p]}$ such that

$$\left(\frac{Sb_i}{u_\nu}, \mathfrak{p} \right) \leqq \left(\frac{c}{u_\nu}, \mathfrak{p} \right) < \inf_{\mathfrak{p} \in U_{[p]}} \left(\frac{Sb}{u_\nu}, \mathfrak{p} \right) \qquad (i = 1, 2, \cdots \text{ and } \nu = 1, 2, \cdots, n_0).$$

Therefore, for some small enough $\varepsilon > 0$ we have

$$[p]Sb_i < [p]Sb - \varepsilon[p]u_\nu \qquad \text{(by Theorem 3)}$$

and hence by the construction of S

　　($\sharp\sharp$)　　$\{f(\alpha_{\nu,k}) - f(\beta_{i,\lambda})\}[p]h_\nu > \varepsilon[p]h_\nu \quad \text{for} \quad [P_{i,\lambda}][p] \neq 0 \qquad (i = 1, 2, \cdots),$

where

$$h_\nu = \int_{[s]} \left(\frac{H_J u_\nu}{u_\nu}, \mathfrak{p} \right) d\mathfrak{p}s.$$

On the other hand, from $b_i \uparrow_{i=1}^\infty b$, for any $\delta > 0$ we can find a $U_{[q]}$ and i such that

$$U_{[q]} \subseteqq U_{[p]} \quad \text{and} \quad \left(\frac{b_{i_0}}{u_\nu}, \mathfrak{p}\right) + \delta < \left(\frac{b}{u_\nu}, \mathfrak{p}\right) \quad \text{for all} \quad \mathfrak{p} \in U_{[q]}$$

and hence

$$0 \leqq \{\alpha_{\nu,k} - \beta_{i_0,\lambda}\}[q]u_\nu < \delta[q]u_\nu \quad \text{for all} \quad [P_{i_0,\lambda}][q] \neq 0$$

and consequently

$$0 \leqq \alpha_{\nu,k} - \beta_{i_0,\lambda} < \delta \quad \text{for all} \quad \lambda \text{ which } [P_{i_0,\lambda}][q] \neq 0 .$$

Therefore (♯♯) contradict the continuity of $f(\xi)$.

Thus we have

$$\lim_{i \to \infty} [N_{\nu,k}]Sa_i = [N_{\nu,k}]Sa \quad \text{for each } \nu \text{ and } k ,$$

and the required result

$$\lim_{i \to \infty} Sa_i = Sa .$$

[6]. $\mathfrak{F}$ *is a directed set with respect to the usual order.*

Proof. The so-called usual order in $\mathfrak{F}$ is defined by the relation: $S_1 \leqq S_2$, if and only if $\mathfrak{D}(S_1) \leqq \mathfrak{D}(S_2)$ and $S_1 x \leqq S_2 x$ $(x \in \mathfrak{D}(S))$ for $S_1, S_2 \in \mathfrak{F}$.

Let S_i be operators corresponding to H_{J_i} $(i = 1, 2)$.

Putting $J_3 = J_1 + J_2$, for S corresponding to H_{J_3} we have

$$\mathfrak{D}(S) = \mathfrak{D}(S_1) \smile \mathfrak{D}(S_2)^* + \mathfrak{D}(S_1) + \mathfrak{D}(S_2)$$

and hence $S_i \leqq S$ $(i = 1, 2)$, because H_{J_1}, $H_{J_2} \leqq H_{J_3}$ by (II).

[7]. *We have, for any $S \in \mathfrak{F}$ and for ϕ in §2,*

$$\phi(S\xi u_\nu) \leqq f(\xi) \quad \text{for} \quad u_\nu \in U, \ \nu = 1, 2, \cdots \quad \text{and} \quad \xi \geqq 0 .$$

Moreover,

$$x \in \mathfrak{D}(S) \text{ and } \|x\| \leqq 1 \text{ implies } \int_{[s]}\left(\frac{Sx}{s}, \mathfrak{p}\right)\phi(d\mathfrak{p}x)^{**} \leqq B .$$

Proof. This relation is obvious from (I), the definition of S and Theorem 3.

§4. **The construction of the modular on R.**

In this section, we will construct a modular of which the topology is equivalent to the norm topology of R. For this purpose, we consider a functional:

$$(1) \qquad \Lambda_S(a) = \int_0^1 d\xi \int_{[s]}\left(\frac{S\xi|a|}{s}, \mathfrak{p}\right)\left(\frac{|a|}{s}, \mathfrak{p}\right)\phi(d\mathfrak{p}s)$$

for $S \in \mathfrak{F}$ and $a \in \mathfrak{D}(S)$.

The existence of the integral of (1) is shown in the following process. From $S\xi a \in B_s$, we have

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* $M \smile N = \{x \smile y; \ x \in M, \ y \in N\}$ for sets M and N.

** This is a *Radon integral* with a measure $\phi([p]x)$ to $U_{[p]}$.

$$0 \leq \left(\frac{S\xi|a|}{s}, \ \mathfrak{p} \right) \leq K \quad (\mathfrak{p} \in \mathfrak{E} \ \text{and} \ 0 \leq \xi \leq 1) \quad \text{for some constant number } K.$$

Then we get a bounded* linear functional $\tilde{a}_\xi$ on $\boldsymbol{R}$ as

$$\tilde{a}_\xi(x) = \int_{[x]} \left(\frac{S\xi|a|}{s}, \ \mathfrak{p} \right) \phi(d\mathfrak{p}x)$$

for $x \in \boldsymbol{R}$ and $0 \leq \xi \leq 1$, and denote it

$$\tilde{a}_\xi = \int \left(\frac{S\xi|a|}{s}, \ \mathfrak{p} \right) d\mathfrak{p}\phi .$$

Furthermore, we get

$$0 \leq \sum_{\nu=1}^{\kappa} (\tilde{a}_{\xi_\nu} - \tilde{a}_{\xi_{\nu-1}}) (|a|) (\xi_\nu - \xi_{\nu-\kappa})$$

$$\leq 2K \sum_{\nu=1}^{\kappa} \phi(|a|) (\xi_\nu - \xi_{\nu-1}) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

where

$$0 = \xi_0 < \xi_1 < \cdots < \xi_\kappa = 1 \quad \text{and} \quad 0 < \xi_\nu - \xi_{\nu-1} < \varepsilon .$$

Therefore, we have

$$\int_0^1 \tilde{a}_\xi(|a|) d\xi = \inf \sum_{\nu=1}^{\kappa} \tilde{a}_{\xi_\nu}(|a|) (\xi_\nu - \xi_{\nu-1})$$

$$= \sup \sum_{\nu=1}^{\kappa} \tilde{a}_{\xi_{\nu-1}}(|a|) (\xi_\nu - \xi_{\nu-1})$$

for all partitions $0 = \xi_0 < \xi_1 < \cdots < \xi_\kappa = 1$.

On the other hand, it is obvious that

$$\int_0^1 \tilde{a}_\xi(|a|) d\xi = \Lambda_S(a) ,$$

because

$$\int_{[s]} \left(\frac{S\xi|a|}{s}, \ \mathfrak{p} \right) \left(\frac{|a|}{s}, \ \mathfrak{p} \right) \phi(d\mathfrak{p}s) = \int_{[s]} \left(\frac{S\xi|a|}{s}, \ \mathfrak{p} \right) \phi(d\mathfrak{p}|a|) = \tilde{a}_\xi(|a|) .$$

Next, we will show that for any $S \in \mathfrak{F}$ and $x > 0$, there exists a sequence $0 \leq x_n \in \mathfrak{D}(S)$ such that

(2) $x_n \uparrow_{n=1}^{\infty} x .$

Let S be the operator corresponding to $J = \{u_1, u_2, \cdots, u_\kappa\}$. Putting

$$U_{[N_l^\nu]} = \left\{ \mathfrak{p} ; \ \left(\frac{[u_\nu]x}{u_\nu}, \ \mathfrak{p} \right) < l \right\}^{-},$$

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* $\sup_{0 \leq x \leq a} |\tilde{a}_\xi(x)| < +\infty$ for each $a \in \boldsymbol{R}$.

$$U_{[N^{\nu}_{l,\mu,i}]} = \left\{\mathfrak{p} ; \frac{i-1}{\mu}l < \left(\frac{[u][N^{\nu}_l]x}{u_\nu}, \mathfrak{p}\right) < \frac{i}{\mu}l\right\}^-$$

and

$$y^{\nu}_{l,\mu} = \sum_{i=1}^{\mu}\frac{i-1}{\mu}l[N^{\nu}_{l,\mu,i}]u_\nu \qquad (\nu=1, 2, \cdots, \kappa \text{ and } l, \mu=1, 2, \cdots.)$$

we have

$$y^{\nu}_{l,\mu} \uparrow_{\mu=1}^{\infty} [N^{\nu}_l][u_\nu]x \qquad (\nu=1, 2, \cdots, \kappa \text{ and } l=1, 2, \cdots.)$$

and

$$[N^{\nu}_l][u_\nu]x \uparrow_{l=1}^{\infty} [u_\nu]x \qquad (\nu=1, 2, \cdots, \kappa.) .$$

Therefore, by the total continuity of $\boldsymbol{R}$, there exists a subsequence $\{\mu_l\}$ of $\{\mu\}$ such that

$$\mu_1 < \mu_2 < \cdots, \lim_{l\to\infty}\mu_l = +\infty \text{ and } y^{\nu}_{l,\mu_l} \uparrow_{l=1}^{\infty}[u_\nu]x \qquad (\nu=1, 2, \cdots, \kappa) .$$

Thus $x_n = \sum_{\nu=1}^{\kappa}y^{\nu}_{n,\mu_n} + \left[\sum_{\nu=\kappa+1}^{\infty}u_\nu\right]^* x$ are the requirements.

Now, we will consider a functional on $\boldsymbol{R}$:

(3) $$\Lambda_S(x) = \sup_{\substack{x_n\in\mathfrak{D}(S)\\0\leq x_n\uparrow|x|}} \sup_{n\geq 1} \Lambda_S(x_n) \qquad (x\in\boldsymbol{R}) .$$

Then we have

Theorem 5. *The functional $\Lambda_S(x)$ $(x\in\boldsymbol{R})$ defined by* (2) *satisfies the modular conditions except for* M2).

Proof. M1) is obvious. For any $a\in\boldsymbol{R}$, putting $\alpha=1/\|a\|$, M3) is obtained by the property [7] and the definition of S. M4) is obtained by the monotony of S. M5) is obvious from the property [2]. M6) is shown in the following process.

For $x \frown y = 0$, we have, by the property [4],

$$\begin{aligned}\Lambda_S(x+y) &= \sup_{\substack{z_n\uparrow(x+y)\\0\leq z_n\in\mathfrak{D}(S)}} \sup_{n\geq 1} \Lambda_S(z_n)\\ &= \sup_{\substack{z_n\uparrow(x+y)\\0\leq z_n\in\mathfrak{D}(S)}} \lim_{n\to\infty}\{\Lambda_S[x]z_n) + \Lambda_S([y]z_n)\}\\ &\leq \Lambda_S(x) + \Lambda_S(y) .\end{aligned}$$

On the other hand, if $0\leq x_n\uparrow_{n=1}^{\infty}x$, $0\leq y_n\uparrow_{n=1}^{\infty}y$ and $x_n, y_n\in\mathfrak{D}(S)$, then we have $z_n = x_n+y_n\in\mathfrak{D}(S)$ and $z_n\uparrow_{n=1}^{\infty}(x+y)$, and hence

$$\Lambda_S(x) + \Lambda_S(y) \leq \Lambda_S(x+y) .$$

Thus the orthogonal additivity of $\Lambda_S(x)$ is proved. M7) is obtained from the pro-

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* $\left[\sum_{\nu=\kappa+1}^{\infty}u_\nu\right]$ means the projection operaior $[\{u_{\kappa+1}, u_{\kappa+2}, \cdots\}]$.

perty [5]. The theorem is proved.

Moreover, considering a functional $m(x)$:

$$(4) \qquad m(x) = \sup_{S \in \mathfrak{F}} \Lambda_S(x) \qquad (x \in \mathbf{R}),$$

we have

Theorem 6. $m(x)$ *is a modular on* $\mathbf{R}$, *of which the topology is equivalent to the original topology, namely, the norm topology on* $\mathbf{R}$.

Proof. From the previous theorem and the definition of $m(x)$, it is obvious that $m(x)$ satisfies the modular conditions except for M2) and M6). But M6) is easily derived from the orthogonal additivity of $\Lambda_S(x)$ and the property [6].

Now, To show M2), if, for all $\xi > 0$, $m(\xi x) = 0$ and $x > 0$, then we have $\Lambda_S(\xi x) = 0$ for all $\xi > 0$ and all $S \in \mathfrak{F}$.

And also there exists a $u \in U$ such that $[u]x \neq 0$. By (2) and (3), for $S \in \mathfrak{F}$ corresponding to $J = \{u\}$ we can find a sequence $\{y_n\}$ such that

$$0 \leq y_n \uparrow_{n=1}^{\infty} y = [u]x \quad \text{and} \quad y_n = \sum_{i=1}^{\mu_n} \frac{i-1}{\mu_n} n [N_{n,\mu_n,i}] u \in \mathfrak{D}(S)^*.$$

putting $z_n = t_n x_n$ where $t_n = 1/\|x_n\|$, we have, on the assumption (I),

$$0 < A \leq \sum_{i=1}^{\mu_n} t_n \frac{i-1}{\mu_n} nf\left(t_n \frac{i-1}{\mu_n} n\right) \phi([N_{n,\mu_n,i}] H_J u)$$

$$\leq \sum_{i=1}^{\mu_n} \int_1^2 d\xi \int_{[N_{\nu,\mu_n,\xi}]} f\left(\xi t_n \frac{i-1}{\mu_n} n\right) \left(\frac{H_J u}{u}, \, \mathfrak{p}\right) \left(\frac{z_n}{s}, \, \mathfrak{p}\right) \phi(d\mathfrak{p}s)$$

$$\leq \Lambda_S(2z_n) \leq \Lambda_S(2x) = 0.$$

This is a contradiction. Thus M2) is proved.

Finally, the topological equivalence is shown in the following process. We have already described the two norms:

$$\|x\|_1 = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi}, \qquad \|x\|_2 = \inf_{m(\xi x) \leq 1} \frac{1}{|\xi|} \qquad (x \in \mathbf{R})$$

and the relations of the equivalence: $\|x\|_2 \leq \|x\|_1 \leq 2\|x\|_2$.

On account of the property [7] and (4), we get easily that if $\|x\| \leq 1$ then $m(x) \leq B$ for B in (I) and hence $\|x\|_2 \leq 1/C^{**}$ and consequently $\|x\|_2 \leq \|x\|/C$ for all $x \in \mathbf{R}$.

We put $y_\nu = [u_\nu]x$ for all $u_\nu \in U$ $(\nu = 1, 2, \cdots)$. Then we can find $0 \leq y_{\nu,k} \uparrow_{k=1}^{\infty} y_\nu$ which satisfy $y_{\nu,k} \in \mathfrak{D}(S_\nu)$ where $S_\nu \in \mathfrak{F}$ correspond to u_ν for each ν. Furthermore, putting $x_{n,k} = \sum_{\nu=1}^{n} y_{\nu,k}$, we have

$$\lim_{k \to \infty} x_{n,k} = \sum_{\nu=1}^{n} y_\nu \quad \text{and} \quad \lim_{n \to \infty} \sum_{\nu=1}^{n} y_\nu = x.$$

---

* Projection operators $[N_{n,\mu_n,i}]$ are those used in the proof of (2).

** $m(x/B) \leq C = \text{Max}\{1/B, 1\}$ by the convexity of $m(\xi x)$ in ξ.

Moreover, there exists a subsequence $\{k_n\}$ of $\{k\}$ such that

$$k_n \uparrow_{n=1}^{\infty} + \infty \quad \text{and} \quad x_{n,k_n} \uparrow_{n=1}^{\infty} x$$

by the total continuity of $\boldsymbol{R}$.

Now, using x_n instead of x_{n,k_n} for the sake of the symbols, and considering $S^{(n)} \in \mathfrak{F}$ which correspond to $J_n = \{u_1, u_2, \cdots, u_n\}$, we put

$$(5) \qquad g_n(\xi) = \frac{\displaystyle\sum_{\nu=1}^{n} \sum_{k=1}^{l} \alpha_{\nu,k} f(\xi \alpha_{\nu,k}) \phi([N_{k,\nu}] H_{J_n} u_\nu)}{\displaystyle\sum_{\nu=1}^{n} \sum_{k=1}^{l} \alpha_{\nu,k} f(\alpha_{\nu,k}) \phi([N_{\nu,k}] H_{J_n} u_\nu)}$$

where $x_n/\|x_n\| = \sum_{\nu=1}^{n} \sum_{k=1}^{l} \alpha_{\nu,k} [N_{\nu,k}] u_\nu$ and $\alpha_{\nu,k} > 0$.

Then $g_n(\xi)$ are non-decreasing and continuous in $\xi \geqq 0$ with $g_n(1) = 1$. And, for $F_n(\xi) = \int_0^\xi g_n(t) dt$, we have

$$1 + F_n(\xi) A \geqq \xi A \qquad \text{for all} \quad \xi \geqq 1.$$

Because, putting

$$G_n(\xi) = 1 + \{F_n(\xi) - \xi\} A$$

we have

$$G_n(1) = 1 - A \geqq 0 \qquad (\because \quad 0 \leqq A \leqq 1)$$

and

$$\frac{d}{d\xi} G_n(\xi) = \{g_n(\xi) - 1\} A \geqq 0 \qquad \text{for} \quad \xi \geqq 1.$$

Accordingly, by (3), (4), (5) and (I) we have

$$m(\xi x_n/\|x_n\|) \geqq \varLambda_{S^{(n)}}(\xi x_n/\|x_n\|) = G_n(\xi) \left\{ \sum_{\nu=1}^{n} \sum_{k=1}^{l} \alpha_{\nu,k} f(\alpha_{\nu,k}) \phi([N_{\nu,k}] H_{J_n} u_\nu) \right\}$$

$$\geqq G_n(\xi) A \qquad\qquad\qquad \text{for} \quad \xi > 0.$$

And it is also obvious that

$$1 + m(\xi x_n/\|x_n\|) \geqq 1 \geqq \xi A \qquad \text{for} \quad 0 < \xi < 1.$$

Therefore, we have

$$(6) \qquad \frac{\|x_n\|_1}{\|x_n\|} = \inf_{\xi>0} \frac{1 + m(\xi x_n/\|x_n\|)}{\xi} \geqq A$$

i.e., $\|x_n\|_1 \geqq A \|x_n\|$.

Furthermore, by the continuity of the norm $\|\cdot\|$ on $\boldsymbol{R}$ we have

$$\|x\|_1 \geqq A \cdot \|x\|.$$

Thus we get the inequalities:

$$\|x\|_2 \leqq \|x\|/C \leqq A \cdot \|x\|_1/C \leqq 2A\|x\|_2/C$$

which show the equivalence. The proof is completed.

Theorem 7.   *The modular $m(x)$ defined by (4) is a constant modular*.*

Proof.   By the definition of $m(x)$, we have

$$m(\xi[N]u) = \sup_{S \in \mathfrak{F}} \varLambda_S(\xi[N]u)$$

$$= \sup_{S \in \mathfrak{A}} \int_0^\xi dt \int_{[N]} \left(\frac{Stu}{s}, \mathfrak{p}\right)\left(\frac{u}{s}, \mathfrak{p}\right)\phi(d\mathfrak{p}s)$$

$$= \sup_J F(\xi)\phi([N]H_J u) \qquad \text{for } \xi > 0,$$

where

$$F(\xi) = \int_0^\xi f(t)\,dt.$$

Therefore we have

$$\frac{m(\xi[N]u)}{m([N]u)} = F(\xi) \qquad \text{for all } [N][u] \neq 0 \text{ and } \xi > 0.$$

Since $U = \{u\text{'s}\}$ is the complete manifold of $\boldsymbol{R}$, Theorem 7 is proved.

§ 5. The examples.

In this section, we give the examples of the spaces which satisfy the assumptions (I) and (II), i.e., $\boldsymbol{O}_*$-spaces.

Let $\boldsymbol{R}$ be a conjugately similar space, with its conjugately similar correspondence T between a universally continuous semi-ordered linear space $\boldsymbol{R}$ and its conjugate space $\overline{\boldsymbol{R}}$, namely, the space satisfying the following conditions:

T 1) $\boldsymbol{R} = \overline{\overline{\boldsymbol{R}}}$, i.e., reflexive space,

T 2) $T(-a) = -Ta$,

T 3) $Ta \leqq Tb$, if and only if $a \leqq b$,

T 4) $(Ta, a) \overset{**}{=} 0$ is equivalent to $a = 0$.

Then we can define a modular $m(x)$ by T as

$$(7) \qquad m(x) = \int_0^\xi (T\xi x, x)\,d\xi \qquad (x \in \boldsymbol{R}).$$

The following theorems are well known in [MSLS].

Theorem 8.***   *The modular $m(\xi x)$ by (7) is finite and strictly convex function of ξ. Furthermore, it is simple and monotone complete.*

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* $m(x)$ is said to be *constant*, if there exists a complete manifold U in $\boldsymbol{R}$ such that $m(\xi[N]a)/m([N]a) = m(\xi a)/m(a)$ for all $\xi > 0$, $a \in U$ and $[N][a] \neq 0$.

** (Ta, a) means the value of Ta at a. And the definition is made in § 60, of [MSLS].

*** $m(x)$ is said to be *simple*, if $m(x) = 0$ implies $x = 0$. It is said to be *monotone complete*, if $a_\lambda \uparrow_{\lambda \in \varLambda}$ and $\sup_\lambda m(a_\lambda) < +\infty$ there exists a such that $a_\lambda \uparrow_{\lambda \in \varLambda} a$. (cf. Th. 60.10 in [MSLS])

Theorem 9.   *If $m(x)$ is finite, then the first and second norm by m are continuous norms on R.*

Now, we assume that the modular by T is a constant modular, i.e., there exists a complete system U of orthogonal elements in R such that

$$(8) \qquad \frac{m(\xi [N] u)}{m([N] u)} = \frac{m(\xi u)}{m(u)}$$

for all $\xi \geqq 0$ and projection operators $[N][u] \neq 0$.   And the element u holding equality (8) is called a constant element in R.

Conjugately similar space R is semi-regular by T1), because *correspondences* $R \ni a \to \bar{a} \in \bar{\bar{R}} : \bar{a}(a) = \bar{\bar{a}}(\bar{a})$ for all $\bar{a} \in \bar{R}$: is one-to-one.   Therefore, by Theorem 4 and the footnote in p. 6 we get the superuniversal continuity and total continuity of R and hence we can set $U = \{u_1, u_2, \cdots, u_n, \cdots\}$.   Accordingly, when we put

$$s = \sum_{\nu=1}^{\infty} \frac{u_\nu}{2^\nu \|u_\nu\|_2}$$

where $\|\cdot\|_2$ is the second norm by the modular m, s is a complete element in R by the monotone completeness of m, because $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_\nu \|x_\nu\|_2 < +\infty$ implies $\sup_\nu m(x_\nu) < +\infty$.   (cf. Th. 40.7 in [MSLS])

Theorem 10.   *Conjugately similar spaces R which have the constant modular by the conjugately similar correspondence are O_*-spaces.*

To prove this theorem. we set the three lemmas.

Lemma 1.   *If u is a constant element in R, then*

$$(9) \qquad \frac{(T\xi [N] u, [N] u)}{(T[N] u, [N] u)} = \frac{(T\xi u, u)}{(Tu, u)}$$

for all $\xi \geqq 0$, $[N][u] \neq 0$.

Proof.   By the definition, we have

$$\frac{m(\xi [N] u)}{m([N] u)} = \frac{m(\xi u)}{m(u)} \qquad \text{for } \xi > 0 \text{ and } [N] u \neq 0,$$

where

$$m(\xi u) = \int_0^\xi (T\xi u, u) d\xi.$$

Since $(T\xi x, x)$ is a continuous function of ξ for any x, we have

$$\frac{d}{d\xi} m(\xi x) = (T\xi x, x)$$

and hence

$$\frac{1}{m(u)} \frac{d}{d\xi} m(\xi u) = \frac{1}{m([N] u)} \frac{d}{d\xi} m(\xi [N] u)$$

and consequently

$$\frac{1}{m(u)}(T\xi u, u) = \frac{(T\xi[N]u, [N]u)}{m([N]u)} \qquad \text{for all } \xi > 0 \text{ and } [N]u \neq 0.$$

Especially, for $\xi = 1$ we have

$$\frac{(Tu, u)}{m(u)} = \frac{(T[N]u, [N]u)}{m([N]u)} \qquad \text{for } [N]u \neq 0.$$

Accordingly we get the equality (9).

Lemma 2. *If R satisfies the assumptions in Theorem 10, then there exists a complete orthogonal system U of constant elements such that*

$$(10) \qquad \frac{(T\xi u_\nu, u_\nu)}{(Tu_\nu, u_\nu)} = \frac{(T\xi[N]u_\nu, [N]u_\nu)}{(T[N]u_\nu, [N]u_\nu)} = \frac{(T\xi u_1, u_1)}{(Tu_1, u_1)}$$

for all $\xi > 0$, $\nu = 1, 2, \cdots$ and $[N]u_\nu \neq 0$.

Proof. Let V be a complete orthogonal system of constant elements in R. By Theorem 55.5 in [MSLS], there exist $\alpha > 0$ such that

$$\frac{m(\alpha_\nu \xi v_\nu)}{m(\alpha_\nu v_\nu)} = \frac{m(\xi v_1)}{m(v_\nu)} \qquad \text{for all } \xi > 0, \ v_\nu \in V \text{ and } \nu = 1, 2, \cdots.$$

Therefore Lemma 2 is proved by Lemma 1.

Lemma 3. *On the assumptions in Theorem 10, we have*

$$1 \leq D = \inf_{\|x\|_2 = 1} (Tx, x) \leq B = \sup_{\|x\|_2 = 1} (Tx, x) < +\infty$$

where T is the conjugately similar correspondence and $\|\cdot\|_2$ is the second norm by the modular which is defined by T.

Proof. If $m(x)$ is finite, simple and monotone complete, then it is uniformly simple[5], and hence one has $\sup_{\|x\|_2 = 1} m(\xi_0 x) < +\infty$ for some $\xi_0 > 1$[1]. Accordingly we get the inequalities in Lemma 3, by virtue of Theorem 8, the convexity of $m(\xi x)$ and

$$(Tx, x) = \frac{d}{d\xi} m(x) \geq 1 \text{ for } \|x\|_2 = 1.$$

The proof of Theorem 10. For U in Lemma 2, we can put

$$f(\xi) = \frac{(T\xi u, u)}{(Tu, u)} \qquad \text{for all } u \in U,$$

Obviously, $f(\xi)$ is non-decreasing and continuous in $\xi \geq 0$.
Taking $Ts \in \bar{R}$ as ϕ, we have

$$\phi = \sum_{\nu=1}^{\infty} f(1/2^\nu \|u_\nu\|_2) Tu_\nu,$$

because

$$s = \sum_{\nu=1}^{\infty} u_\nu/2^\nu \|u_\nu\|_2.$$

Then R is the continuous normed space with the norm $\|x\|_2$ and is regular.

Now, for any finite subset $J = \{u_1, u_2, \cdots, u_m\}$ in U, we put

$$H_J u_\nu = \int_{[s]} \left(\frac{T(u_1 + u_2 + \cdots + u_m)}{\phi}, \ \mathfrak{p} \right) d\mathfrak{p} u_\nu \qquad \text{for all } \nu = 1, 2, \cdots.$$

Obviously, from

$$0 \leqq H_J u_\nu \leqq u_\nu / f(1/2^\nu \|u_\nu\|_2) \qquad \text{for } \nu = 1, 2, \cdots, m$$

and

$$H_J u_\mu = 0 \qquad \text{for } \mu \neq 1, 2, \cdots, m,$$

we have

$$H_J u_\nu \in B u_\nu.$$

If

$$c = \sum_{\nu=1}^{m} \sum_{k=1}^{n_\nu} \alpha_{\nu,k} [N_{\nu,k}] u_\nu, \ \|c\|_2 = 1 \quad \text{and} \quad \alpha_{\nu,k} > 0,$$

then

$$B \geqq \sum_{\nu=1}^{m} \sum_{k=1}^{n_\nu} \alpha_{\nu,k} f(\alpha_{\nu,k}) \phi([N_{\nu,k}] H_J u_\nu)$$

$$= (Tc, c) \geqq 1.$$

Therefore, putting $A = 1$, (I) is satisfied. (II) is obviously satisfied by the construction of H_J. Thus the proof of Theorem 10 is completed.

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