Decompositional Study of Set Functions and Sketch-integral

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Yoshio Kinokuniya*

Abstract

When we aim to fix up any structural aspect on some events, there may be found several theoretical noises obscuring our view. How to eleminate these noises is put forth as a problem and some cases are, for this problem, solved in respective ways. The principle of *trans-induction*, *uniform decomposition* of a set with respect to measure assignment and a device to make an integral of the form

$$\mathfrak{S}\gamma_P^a$$

definable are explained in some details.

1. Probabilistic Translation

To make sure of the existence of a set function π on condition that

$$(\forall P,\ Q\in \boldsymbol{M})(\pi_P=\pi_Q)$$
$$\&\quad \pi(\boldsymbol{M})(=\mathfrak{S}\pi_P)=1 \tag{1. 1}$$

is not easy, even though we restrict the object to a set in a euclidian space **E** of finite dimension. In probabilism, we shall define this function by

$$\pi(\boldsymbol{N})=\Pr(P\in \boldsymbol{N}\subset \boldsymbol{M}), \tag{1. 2}$$

so that (1. 1) and (1. 2) may likely be taken as equivalent descriptions of the same function π. But, by some investigation, we may find that (1. 2) gives a more practical and really wider extent of definability. In a strictly ultra case (1. 2) is explained to be wholly available, whereas (1. 1) therein unavoidably implies a part in **E** to be inconsistent. On this observation, there may be put forth two cases to make up theoretical noises about the existence of $\pi(\boldsymbol{N})$, viz. :

i) For a certain subset $\boldsymbol{N}$ of $\boldsymbol{M}$, no value of

$$p_N=\Pr(P\in \boldsymbol{N})$$

can be found out ;

ii) For a certain subset $\boldsymbol{N}$, there is found more than one value possible to be of p_N.

The case (i) may be wholly absorbed by the theory of measure, when we

* 紀国谷芳雄

compel the relation

$$p_N = \pi(\boldsymbol{N})$$

to be really available for any subset $\boldsymbol{N}$ of $\boldsymbol{M}$. The case (ii) emerges when the essential measure value of p_N does not coincide with the limiting value of $\pi(\boldsymbol{N}_k)$ in case $\boldsymbol{N}_k \rightarrow \boldsymbol{N}$ as $k \rightarrow \infty$, i.e. in a strictly ultra case. In an ultra case the only relation relatively demanded is that

$$\boldsymbol{M} \supset \boldsymbol{N},\ \boldsymbol{F} \triangleright \pi(\boldsymbol{N} \cup \boldsymbol{F}) = \pi(\boldsymbol{N}) + \pi(\boldsymbol{F}) - \pi(\boldsymbol{N} \cap \boldsymbol{F}). \qquad (1.3)$$

So, the limiting value of $\pi(\boldsymbol{N}_k)$ is therein turned out to be possibly different from the value $\pi(\boldsymbol{N})$. This situation just makes the characteristic property of an ultra function π.

If two vaules might be assigned to the same subset $\boldsymbol{N}$, each reasoning on the two assignments must give each construction to the set $\boldsymbol{N}$. Then, if the two values mean a contradiction, the corresponding two constructions must mean a contradiction, too. However, in the above case, the difference between the measure value $\pi(\boldsymbol{N})$ and the limiting value of $\pi(\boldsymbol{N}_k)$ is not unavoidably thought to mean a contradiction. In effect, if

$$\boldsymbol{N}_1 \subset \boldsymbol{N}_2 \subset \cdots \subset \boldsymbol{N} \quad \& \quad \cup \boldsymbol{N}_k = \boldsymbol{N}$$

and

$$\lim \pi(\boldsymbol{N}_k) = p_N \neq \pi(\boldsymbol{N}),$$

it must be that

$$p_N < \pi(\boldsymbol{N}),$$

so that

$$\delta_N \equiv \lim \pi(\boldsymbol{N} - \boldsymbol{N}_k) = \pi(\boldsymbol{N}) - p_N > 0.$$

Then, on writing as

$$(\,]\boldsymbol{N}[\,) \equiv \lim (\boldsymbol{N} - \boldsymbol{N}_k) \qquad (1.4)$$

we may have the description

$$\pi((\,]\boldsymbol{N}[\,) = \delta_N.$$

What $(\,]\boldsymbol{N}[\,)$ of (1.4) defines, is called an *atmosphere* of $\boldsymbol{N}$ w.r.t. the function π. Thus, the noises (ii) is transmigrated and the meaning of 'noise' itself is thereby thought to vanish. By some reflection, however, it may readily be seen that, what essentially releases the ultra case from the noise is that, no other condition is imposed on π than (1.2) and (1.3).

If any empiricist assumption gives us a standpoint from which to regard some theoretical noises to be deleted, it is called a *reflective axiom* or a *reflection,* in the meaning that any argument to support the noises is by this axiom reflectively rejected. When a reflection is put forth, it must be accompanied with some illustrating reasoning. As a result of the foregoing discussion, we may here put forth the following reflection on π.

Reflection 1.1. *For any set of points in* $\boldsymbol{E}$, *there exists a set function* $(\equiv \pi_M(\boldsymbol{N}))$ $(\boldsymbol{N} \subseteq \boldsymbol{M})$ *satisfying* (1.2) *and* (1.3).

2. On Trans-induction

Previously, the notion of *trans-induction* was introduced and to some extent elucidated by the present author[1),2)]. He is going to take up this notion again toward a renovated version standing on the noise-theoretical viewpoint.

Principle of Trans-induction. *$\boldsymbol{F}$ being a subset of the given set $\boldsymbol{M}$, let a descriptive relation $\mathfrak{p}(x, \boldsymbol{F})$ be tested between any element x of $\boldsymbol{M}-\boldsymbol{F}$ and $\boldsymbol{F}$*(viz. $(\forall x \in \boldsymbol{M}-\boldsymbol{F})$ $(\mathfrak{p}(x, \boldsymbol{F}) \vee \overline{\mathfrak{p}}(x, \boldsymbol{F}))$). *Whenever there is a subset $\boldsymbol{N}$ of $\boldsymbol{M}$ such that*

$$(\forall x \in \boldsymbol{N})(\mathfrak{p}(x, \boldsymbol{N}-\{x\})),$$

then does exist a supremum $\tilde{\boldsymbol{F}}$ of such subsets $\boldsymbol{F}$ that

$$(\forall x \in \boldsymbol{F})(\mathfrak{p}(x, \boldsymbol{F}-\{x\})),$$

and this $\tilde{\boldsymbol{F}}$ can be considered as a simple-ordered set.

To tell the truth, this principle cannot be considered to be generally available. In the way of accepting this principle, the following two noises may be pointed out to be discussed:

(i) Any such supremum $\tilde{\boldsymbol{F}}$ as above-mentioned does not exist;

(ii) $\tilde{\boldsymbol{F}}$ cannot be simple-ordered.

If we resort to the principle of transfinite induction, the supremum $\tilde{\boldsymbol{F}}$ may be concluded as existent. On the empiricist ground, however, we may not apply the principle of transfinite induction beyond the 2nd class of ordinal numbers. But, it may be thought still reasonable, if we regard its process as a formal abstract mode and take it as a support of persisting in the existence of the suprememe $\tilde{\boldsymbol{F}}$. This reasoning may give a tolerable ground for elimination of the noise (i). While, as for (ii), it seems impossible to encounter any reasonable pretext to generally delete this noise. So, if it cannot be denied in any way that the supremum $\tilde{\boldsymbol{F}}$ is simple-ordered, we say that the relation $\mathfrak{p}$ is *of a trans-inductive mode.* In fine, the principle of trans-induction is considered applicable when and only when $\mathfrak{p}$ is proved to be of a trans-inductive mode. It may hereupon be noted that the principle of trans-induction can also be made well-established by the cut-process on a logical stand point of view.

3. Uniformizations

Let f be a powdery set function in $\mathbf{E}$, i.e.

$$(\forall P \in \mathbf{E})\ (\circledcirc \leqslant f(P) \leqslant \textcircled{\scriptsize$\triangle$}\ \text{(infinitesimal)}),$$

and $\boldsymbol{M}$ be a set in $\mathbf{E}$ for which

$$f(\boldsymbol{M}) = \infty .$$

Then, for a given positive real number c, we may have a c-cut[3)] $\boldsymbol{M}_1$ of $\boldsymbol{M}$ w.r.t. f, and then a c-cut $\boldsymbol{M}_2$ of $\boldsymbol{M}-\boldsymbol{M}_1$, and so on, to reach a sequence of disjoint subsets $(\boldsymbol{M}_k)$ such that

$$f(\boldsymbol{M}_1) = f(\boldsymbol{M}_2) = \cdots = c .$$

In this case, if $f(\boldsymbol{M} - \cup \boldsymbol{M}_k) \geqslant c$, we may continue again the c-cut process. Therefore, if the trans-induction is applicable, we have a disjoint family of sets

$$(\boldsymbol{M}_\lambda)_{\lambda \in \Lambda}$$

Λ being a simple-ordered set, for which

$$(\forall \lambda \in \Lambda)\ (f(\boldsymbol{M}_\lambda) = c)$$

and

$$0 \leqslant f(\boldsymbol{M} - \cup \boldsymbol{M}_\lambda) < c .$$

The trans-inductive mode of this process, i.e. that Λ can be simple-ordered, is readily seen if we use the axiom of choice* for $(\boldsymbol{M}_\lambda)_{\lambda \in \Lambda}$ and the fact that $\mathbf{E}$ can be simple-ordered as a set of points.

In case $f(\boldsymbol{M} - \cup \boldsymbol{M}_\lambda) \neq 0$, writing

$$\boldsymbol{R} = \boldsymbol{M} - \cup \boldsymbol{M}_\lambda ,$$

$\boldsymbol{R}$ may, by means of the c-cut process, have its partition $(\boldsymbol{R}_k)$ such that

$$f(\boldsymbol{R}_k) = f(\boldsymbol{R})/2^k \qquad (k = 1, 2, \cdots) .$$

On the other hand, extracting an enumerable sequence $(\boldsymbol{M}_{\lambda_k})$ from $(\boldsymbol{M}_\lambda)$, let $\boldsymbol{M}_{\lambda_k}$ be written as $\boldsymbol{M}_{(k)}$. Then, using again the c-cut process, we may gain disjoint sets $\boldsymbol{M}'_{(1)}$, $\boldsymbol{M}''_{(1)}$, $\boldsymbol{M}'_{(2)}$, $\boldsymbol{M}''_{(2)}$, $\boldsymbol{M}'''_{(2)}$, $\boldsymbol{M}'_{(3)}$, $\cdots$ such that:

$$\boldsymbol{M}'_{(1)} \cup \boldsymbol{M}''_{(1)} = \boldsymbol{M}_{(1)} \text{ and } f(\boldsymbol{M}''_{(1)}) = \frac{1}{2} f(\boldsymbol{R});$$

$$\boldsymbol{M}'_{(2)} \cup \boldsymbol{M}''_{(2)} \cup \boldsymbol{M}'''_{(2)} = \boldsymbol{M}_{(2)},\quad f(\boldsymbol{M}''_{(2)}) = \frac{1}{2} f(\boldsymbol{R}) \text{ and } f(\boldsymbol{M}'''_{(2)}) = \frac{1}{4} f(\boldsymbol{R});$$

$$\boldsymbol{M}'_{(3)} \cup \boldsymbol{M}''_{(3)} \cup \boldsymbol{M}'''_{(3)} = \boldsymbol{M}_{(3)},\quad f(\boldsymbol{M}''_{(3)}) = \frac{3}{4} f(\boldsymbol{R}) \text{ and } f(\boldsymbol{M}'''_{(3)}) = \frac{1}{8} f(\boldsymbol{R});$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\boldsymbol{M}'_{(k)} \cup \boldsymbol{M}''_{(k)} \cup \boldsymbol{M}'''_{(k)} = \boldsymbol{M}_{(k)},\quad f(\boldsymbol{M}''_{(k)}) = \frac{2^{k-1}-1}{2^{k-1}} f(\boldsymbol{R}) \text{ and } f(\boldsymbol{M}'''_{(k)}) = \frac{1}{2^k} f(\boldsymbol{R}) \qquad (k = 1, 2, \cdots) .$$

By means of this sequence, let a sequence of sets $(\boldsymbol{N}_k)$ be defined such that:

* In the empiricist theory of sets, the axiom of choice is thought to be independent of well-ordering theorem.

$$\begin{aligned}
\boldsymbol{N}_1 &= \boldsymbol{M}'_{(1)} \cup \boldsymbol{R}_1 ; \\
\boldsymbol{N}_2 &= \boldsymbol{M}'_{(2)} \cup \boldsymbol{M}''_{(1)} \cup \boldsymbol{R}_2 ; \\
\boldsymbol{N}_3 &= \boldsymbol{M}'_{(3)} \cup \boldsymbol{M}''_{(2)} \cup \boldsymbol{M}'''_{(2)} \cup \boldsymbol{R}_3 ; \\
\cdots & \quad \cdots \quad \cdots \\
\boldsymbol{N}_k &= \boldsymbol{M}'_{(k)} \cup \boldsymbol{M}''_{(k-1)} \cup \boldsymbol{M}'''_{(k-1)} \cup \boldsymbol{R}_k ; \quad (k=3, 4, \cdots).
\end{aligned}$$

We then have:

$$f(\boldsymbol{N}_1) = f(\boldsymbol{N}_2) = \cdots = f(\boldsymbol{M}_1) \ (= f(\boldsymbol{M}_2) = \cdots) = c ,$$

$$\boldsymbol{N}_k \cap \boldsymbol{N}_j = \text{void} \ (k \neq j)$$

and

$$\cup \boldsymbol{N}_k = (\cup \boldsymbol{M}_{(k)}) \cup \boldsymbol{R} .$$

Therefore, if we designate a family of sets $(\boldsymbol{F}_\lambda)$ by

$$\boldsymbol{F}_{\lambda_k} = \boldsymbol{N}_k \quad (k = 1, 2, \cdots),$$

and

$$\boldsymbol{F}_\lambda = \boldsymbol{M}_\lambda \text{ when } \lambda \neq \lambda_1, \lambda_2, \cdots ,$$

then we have

$$\boldsymbol{M} = \cup \boldsymbol{F}_\lambda , \quad (\forall \lambda, \ \mu \in \Lambda) \ (\lambda \neq \mu \triangleright \boldsymbol{F}_\lambda \cap \boldsymbol{F}_\mu = \text{void})$$

and

$$(\forall \lambda \in \Lambda) \ (f(\boldsymbol{F}_\lambda) = c) .$$

Consequently we conclude:

Proposition 3. 1 (*Principle of Uniformization of the* 1*st Kind*). *If f is a powdery set function and*

$$f(\boldsymbol{M}) = \infty ,$$

there is, for any positive real number c, a partition $(\boldsymbol{M}_\lambda)_{\lambda \in \Lambda}$ of $\boldsymbol{M}$ with a simple-ordered indication Λ such that

$$(\forall \lambda \in \Lambda) \ (f(\boldsymbol{M}_\lambda) = c) .$$

This uniformization is also called a *cut-uniformization*. If, for any positive real number c, the above-stated indication Λ is always found to be enumerable, then we say '*$\boldsymbol{M}$ has a simple lumping structure w.r.t. f*' or '*$\boldsymbol{M}(f)$ is a simple lumping set*'.

If $\boldsymbol{M}$ is a non-enumerable infinite set of points in **E**, then applying the trans-induction on the successive continuation of extracting an enumerable subset, we may attain a partition $(\boldsymbol{N}_\lambda)_{\lambda \in \Lambda}$ of $\boldsymbol{M}$ with a simple-ordered indication Λ, for which

$$\boldsymbol{N}_\lambda = \{P_{\lambda 2}, P_{\lambda 2}, \cdots\}$$

$$P_{\lambda k} \text{ being a point in } \boldsymbol{M} (k = 1, 2, \cdots).$$

Then, designating sets $\boldsymbol{M}_k$ by

$$\boldsymbol{M}_k = \bigcup_{\lambda \in \Lambda} \{P_{\lambda k}\} ,$$

we may have

$$\overline{\overline{\boldsymbol{M}}}_1 = \overline{\overline{\boldsymbol{M}}}_2 = \cdots .$$

This relation will become perfect if we may have

$$\mathfrak{n}(\boldsymbol{M}_1) = \mathfrak{n}(\boldsymbol{M}_2) = \cdots$$

$\mathfrak{n}(\boldsymbol{M}_k)$ being the inversion number of $\boldsymbol{M}_k$, i.e.

$$\widetilde{m}(\boldsymbol{M}_1) = \widetilde{m}(\boldsymbol{M}_2) = \cdots \tag{3. 1}$$

$\widetilde{m}(\boldsymbol{M}_k)$ being the a priori measure of $\boldsymbol{M}_k$. Besides, in case of $\boldsymbol{M}=\mathrm{E}$, the relation (3. 1) is realized by a partition composed of mutually homologous rectangles $(\boldsymbol{M}_k)$. Therefore, in case of $f=\widetilde{m}$ on Proposition 3. 1, if we may transport the points of $\boldsymbol{M}$ precisely into any rectangle of measure c, the indication Λ should be exhausted within enumerability. Thus, we may put forth the following assertion.

Reflection 3. 2 *(Principle of Uniformization of the 2nd Kind). If $\boldsymbol{M}$ is an infinite set of points in* E, *there is an enumerable partition $(\boldsymbol{M}_k)$ of $\boldsymbol{M}$ such that*

$$\mathfrak{n}(\boldsymbol{M}_1) = \mathfrak{n}(\boldsymbol{M}_2) = \cdots .$$

This uniformization is also called an *inversional uniformization*. In respect to the above-stated partition $(\boldsymbol{M}_k)$, defining a function φ by

$$\varphi(\boldsymbol{N}) = \sum \pi_{M_k}(\boldsymbol{N} \cap \boldsymbol{M}_k)^* ,$$

we have

$$\varphi(\boldsymbol{M}) = \infty$$

with $\varphi(\boldsymbol{M}_k)=1$ for each $k=1, 2, \cdots$. Hence we have:

Proposition 3. 2 *If $\boldsymbol{M}$ is an infinite set in* E, *there is a powdery set function φ for which*

$$\varphi(\boldsymbol{M}) = \infty$$

and $\boldsymbol{M}(\varphi)$ is found as a simple lumping set.

4. Regularity-wise Decomposition

In the following, we will show a development of the idea of Lebesgue decomposition. When f is a powdery set function, though it is restricted as

$$\circledcirc \leqslant f(P) \leqslant \triangle ,$$

the retio of mass-values of two points P, Q

$$f(P)/f(Q)$$

may possibly vary and especially the values

* As for the definition of a function $\pi_M(\boldsymbol{N})$, vid. Reflection 1. 1.

$$0 \text{ or } \infty$$

are not essentially therefrom excepted. If we classify points by the relation

$$0<f(P)/f(Q)<\infty\,,$$

it is the classification based on 'relative regularity'. On this classification, let the classes be written in signs $\overset{(\iota)}{\mathbf{E}}(\iota\in \boldsymbol{I})$, then, if $\iota,\ \kappa\in\boldsymbol{I}$ and $\iota\neq\kappa$, it must be either

$$(\forall P\in \overset{(\iota)}{\mathbf{E}})\ (\forall Q\in \overset{(\kappa)}{\mathbf{E}})\ (f(P)/f(Q)=0)$$

or

$$(\forall P\in \overset{(\iota)}{\mathbf{E}})\ (\forall Q\in \overset{(\kappa)}{\mathbf{E}})\ (f(P)/f(Q)=\infty)\,.$$

So, the indication $\boldsymbol{I}$ may be made simple-ordered by the stipulation that

$$\left.\begin{array}{l} P\in \overset{(\iota)}{\mathbf{E}},\ Q\in \overset{(\kappa)}{\mathbf{E}}\ \&\ f(P)/f(Q)=0 \quad \bowtie\ \iota<\kappa \\ \text{and} \\ P\in \overset{(\iota)}{\mathbf{E}},\ Q\in \overset{(\kappa)}{\mathbf{E}}\ \&\ f(P)/f(Q)=\infty \quad \bowtie\ \iota>\kappa \end{array}\right\} \tag{4.1}$$

On the above-stated construction, $\boldsymbol{I}$ is called *f-indication* and $\iota(\in\boldsymbol{I})$ an *f-index*.

Lemma 4.1. *When f is powdery set function and $0<f(\boldsymbol{N})<\infty$, there exists, for any positive real number $\varepsilon<1$, a point $P\in\boldsymbol{N}$ such that*

$$f(P)\,\mathfrak{n}(\boldsymbol{N})>f(\boldsymbol{N})\ (1-\varepsilon)\ (\textit{or}\ <f(\boldsymbol{N})\ (1+\varepsilon))\,.$$

$\mathfrak{n}(\boldsymbol{N})$ being the inversion number of $\boldsymbol{N}$.

Demostration. If there is no such point, it must be that

$$(\forall P\in\boldsymbol{N})\ (f(P)\,\mathfrak{n}(\boldsymbol{N})\leqslant f(\boldsymbol{N})\ (1-\varepsilon)\ (\text{or}\ \geqslant f(\boldsymbol{N})\ (1+\varepsilon))\,,$$

so that, on summing it for all points (say, P) of $\boldsymbol{N}$, we may have

$$f(\boldsymbol{N})\,\mathfrak{n}(\boldsymbol{N})\leqslant\mathfrak{n}(\boldsymbol{N})f(\boldsymbol{N})\ (1-\varepsilon)\ (\text{or}\ \geqslant\mathfrak{n}(\boldsymbol{N})f(\boldsymbol{N})\ (1+\varepsilon))\,.$$

Hence

$$f(\boldsymbol{N})\leqslant f(\boldsymbol{N})\ (1-\varepsilon)\ (\text{or}\ \geqslant f(\boldsymbol{N})\ (1+\varepsilon))\,,$$

which cannot be true when

$$0<f(\boldsymbol{N})<\infty\,.$$

Now, we take up a case in which, for a certain f-index

$$0<f(\boldsymbol{M}\cap\overset{(\iota)}{\mathbf{E}})$$

and

$$0<f(\boldsymbol{M})<\infty\,.$$

Writing herein as

$$\overset{(\kappa)}{\boldsymbol{M}}\equiv\boldsymbol{M}\cap\overset{(\kappa)}{\mathbf{E}}\,,$$

let $\boldsymbol{U}$ be defined by

$$\boldsymbol{U} \equiv \bigcup_{\kappa>\iota} \overset{(\kappa)}{\boldsymbol{M}}. \tag{4. 2}$$

Then, if $f(\boldsymbol{U})>0$, we have

$$0<f(\boldsymbol{U})\leqslant f(\boldsymbol{M})<\infty ,$$

so that we may, for any $0<\varepsilon<1$, have

$$0<\frac{f(\overset{(\iota)}{\boldsymbol{M}})\ (1-\varepsilon)}{f(\boldsymbol{U})\ (1+\varepsilon)}<\infty .$$

Let such a number ε be fixed and let k be defined by

$$k=\frac{f(\overset{(\iota)}{\boldsymbol{M}})\ (1-\varepsilon)}{f(\boldsymbol{U})\ (1+\varepsilon)} .$$

Then, by grace of Lemma 4.1, there are two points $P\in\overset{(\iota)}{\boldsymbol{M}}$ and $Q\in\boldsymbol{U}$ such that

$$\frac{f(P)\,\mathfrak{n}(\overset{(\iota)}{\boldsymbol{M}})}{f(Q)\,\mathfrak{n}(\boldsymbol{U})}>k . \tag{4. 3}$$

So, according to (4. 1) and (4. 2), we readily see that (4. 3) implies the relation

$$0=\frac{f(P)}{f(Q)}=\frac{\mathfrak{n}(\boldsymbol{U})}{\mathfrak{n}(\overset{(\iota)}{\boldsymbol{M}})}\cdot k ,$$

so that

$$\mathfrak{n}(\boldsymbol{U})/\mathfrak{n}(\overset{(\iota)}{\boldsymbol{M}})=0 .$$

As for the case $f(\boldsymbol{U})=0$,

$$P\in\overset{(\iota)}{\boldsymbol{M}}\ \triangleright\ f(P)\,\mathfrak{n}(\boldsymbol{U})\leqslant f(\boldsymbol{U}) .$$

So, for a point P of Lemma 4. 1, we have

$$\frac{\mathfrak{n}(\boldsymbol{U})}{\mathfrak{n}(\overset{(\iota)}{\boldsymbol{M}})}=\frac{f(P)\,\mathfrak{n}(\boldsymbol{U})}{f(P)\,\mathfrak{n}(\overset{(\iota)}{\boldsymbol{M}})}\leqslant\frac{f(\boldsymbol{U})}{f(\overset{(\iota)}{\boldsymbol{M}})\ (1-\varepsilon)}=0 ,$$

hence

$$\mathfrak{n}(\boldsymbol{U})/\mathfrak{n}(\overset{(\iota)}{\boldsymbol{M}})=0 .$$

Consequently we habe:

Proposition 4. 2. *When f is a powdery set function, if $0<f(\boldsymbol{M})<\infty$ and for a certain f-index*

$$f(\overset{(\iota)}{\boldsymbol{M}})>0 ,$$

then, for the set $\boldsymbol{U}=\bigcup_{\kappa>\iota}\boldsymbol{M}$, it must be that

$$\mathfrak{n}(\boldsymbol{U})/\mathfrak{n}(\overset{(\iota)}{\boldsymbol{M}})=0 . \tag{4. 4}$$

(4. 4) suggests to our intuition the relation

$$\Pr(P \in \boldsymbol{U})/\Pr(P \in \overset{(\varepsilon)}{\boldsymbol{M}}) = 0 .$$

On looking back the above-stated investigation there will be pointed out two fundamental conceptions, viz.

$$f(P)/f(Q) \text{ and } \mathfrak{n}(\boldsymbol{F})/\mathfrak{n}(\boldsymbol{M}) .$$

which cannot always be empirically evident, but generally of abstract and formal use except in the case in a space comprizing only a finite number of points. When $\tilde{m}(\boldsymbol{M})=0$ and $\boldsymbol{F}\subset\boldsymbol{M}$, the equivalence

$$\mathfrak{n}(\boldsymbol{F})/\mathfrak{n}(\boldsymbol{M}) \equiv \pi_M(\boldsymbol{F})$$

may furnish our theory with a useful medium. As for $f(P)/f(Q)$ or $f(P)/g(Q)$, some critical facts will be explained in the next section.

5. Cribble-sketch

It is not an easy event that a set function may stand on the relation

$$f(P) = \gamma_P^\alpha \tag{5. 1}$$

γ_P being point applications given to make up a powdery application $\tilde{\gamma}(\boldsymbol{M})$ in the form

$$\tilde{\gamma}(\boldsymbol{M}) = \underset{P \in M}{\mathfrak{S}} \gamma_P . \tag{5. 2}$$

To be exact, the notion of γ_P is, as it is, to be abstracted depending on the total concept $\tilde{\gamma}(\boldsymbol{M})$ by the relation (5. 2). Hence, γ_P cannot always be a unique concrete assignment to be simply contradistinguished from any other assignment. Moreover, even when we take up an apriori measure $\tilde{m}$ in place of $\tilde{\gamma}$, the difficulty in practically constructing the relation (5. 1) does not diminish.

For instance, let Γ be the Cantor set dwelling in the interval $\boldsymbol{M}=(0, 1)$. Then Γ is thought as the limiting set of the following decreasing sequence of sets:

$$\Gamma_1 = \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right) ;$$

$$\Gamma_2 = \left(0, \frac{1}{3^2}\right) \cup \left(\frac{2}{3^2}, \frac{1}{3}\right) \cup \left(\frac{2}{3}, \frac{7}{3^2}\right) \cup \left(\frac{8}{3^2}, 1\right) ;$$

$$\cdots \qquad \cdots \qquad \cdots .$$

In this sequence, the k-th set is composed as a sum of 2^k disjoint intervals having the uniform length $1/3^k$. Let these intervals be called *component intervals.* And, writing as

$$\alpha = \log_3 2 \,(= \log 2/\log 3) ,$$

let us assign to each component intervals of Γ_k the uniform value

$$(1/3^k)^\alpha$$

and designate it as σ_k, so that we may have

$$\sigma_k \Gamma_k = 2^k (1/3^k)^\alpha = (2/3^\alpha)^k = (2/2)^k = 1 \tag{5.3}$$

i.e.

$$\sigma_k \Gamma_k = 1 \text{ for all } k = 1, 2, \cdots .$$

Now, letting the component intervals of Γ_k be written in signs

$$\overset{(1)}{\varDelta}_k, \overset{(2)}{\varDelta}_k, \cdots, \overset{(2^k)}{\varDelta}_k,$$

we may construe (5. 3) in the form

$$\sigma_k \Gamma_k = \sum_{j=1}^{2k} (\tilde{m} \overset{(j)}{\varDelta}_k)^\alpha \tag{5.4}$$

where $\tilde{m}$ is an a priori measure so that

$$\tilde{m} \overset{(j)}{\varDelta}_k = 1/3^k .$$

In that we may then take it as

$$\lim \tilde{m} \overset{(j)}{\varDelta}_k = dt \ (t \in \Gamma),$$

it seems presumable that the relation (5. 4) has its limiting state to be expressed in the form

$$\underset{t \in \Gamma}{\mathfrak{S}} (dt)^\alpha = 1 . \tag{5.5}$$

By some inspection, however, it is proved that the formula (5. 5) is not yet well-formed. In effect, if we divide $\overset{(j)}{\varDelta}_k$ into two intervals $\overset{(j,1)}{\varDelta}_k$ and $\overset{(j,2)}{\varDelta}_k$ of equal length, we may take the assignment

$$\sigma_k \overset{(j)}{\varDelta}_k = \sigma_k \overset{(j,1)}{\varDelta}_k + \sigma_k \overset{(j,2)}{\varDelta}_k = 2\sigma_k \overset{(j,1)}{\varDelta}_k$$

so that, if we aim to approach to the element $(dt)^\alpha$, it shall be counted such as

$$2\sigma_k \overset{(j,1)}{\varDelta}_k = 2\left(\frac{1}{2}\frac{1}{3^k}\right)^\alpha = 2^{1-\alpha}\frac{1}{2^k},$$

hence

$$\sigma_k \Gamma_k = 2^k \sigma_k \overset{(j)}{\varDelta}_k = 2^{1-\alpha} . \tag{5.6}$$

Then, by analogy to the case of (5. 5), it may be conjectured that

$$\underset{t \in \Gamma}{\mathfrak{S}} (dt)^\alpha = \lim \sigma_k \Gamma_k = 2^{1-\alpha} \neq 1 .$$

This shows that the value of $\underset{t \in \Gamma}{\mathfrak{S}} (dt)^\alpha$ cannot be simply conjectured to be a unique one. In the following will be introduced a device which is named '*cribble sketching*' and is expected to construct a precise approach toward the form of integration

$$\mathfrak{S}\,\gamma_P^{\alpha}.$$

Starting with a partition

$$\overset{(1)}{\mathfrak{L}} = (\overset{(1)}{U_k})_{k=1,2,\cdots}$$

of the space **E**, let each $\overset{(1)}{U_k}$ be divided into n_1 disjoint subsets, then the reunion of these subsets for all $k=1, 2, \cdots$ will give a finer partition of **E**. Let this second partition be arranged and denoted as

$$\overset{(2)}{\mathfrak{L}} = (\overset{(2)}{U_k})_{k=1,2,\cdots}.$$

Then, there are, for each k, n_1 sets $\overset{(2)}{U}_{k_{j_1}}, \cdots, \overset{(2)}{U}_{k_{j_{n_1}}}$ such that

$$\overset{(1)}{U_k} = \bigcup_{i=1}^{n_1} \overset{(2)}{U}_{k_{j_i}}.$$

By succession of similar methods, each $\overset{(2)}{U_k}$ is divided into n_2 disjoint subsets by which the third partition

$$\overset{(3)}{\mathfrak{L}} = (\overset{(3)}{U_k})_{k=1,2,\cdots}$$

is constituted, $\cdots$, and each $\overset{(m)}{U_k}$ is then again divided into n_m subsets by which the $(m+1)$-th partition

$$\overset{(m+1)}{\mathfrak{L}} = (\overset{(m+1)}{U_k})_{k=1,2,\cdots}$$

is constituted. Thus we have a sequence of partitions (of **E**)

$$\mathfrak{L} = (\overset{(m)}{\mathfrak{L}})_{m=1,2,\cdots}.$$

Besides, we will hereto assign the stipulation

$$\lim_{n} (\operatorname{Max}_{k} (\text{diameter of } \overset{(n)}{U_k})) = 0. \tag{5.7}$$

When (5.7) is satisfied by the system of $(\overset{(n)}{U_k})$, $\mathfrak{L}$ is called a *cribble foundation* or simply a *foundation*, each $\overset{(n)}{U_k}$ a k-th cell $(n=1, 2, \cdots)$ and $\overset{(n)}{\mathfrak{L}}$ the *n-th cribble* of the system.

When $\mathfrak{L}$ is a foundation in **E**, defining such that

$$\overset{(n)}{N}(M) = \{k :\ M \cap \overset{(n)}{U_k} \neq \text{void}\}$$

and

$$\overset{(n)}{M_{\mathfrak{L}}} = \sum_{k \in \overset{(n)}{N}(M)} \overset{(n)}{U_k},$$

we readily have

$$\overset{(1)}{M_{\mathfrak{L}}} \supseteq \overset{(2)}{M_{\mathfrak{L}}} \supseteq \cdots.$$

So, there exists the limiting set

$$\boldsymbol{M}_{\mathfrak{L}} = \cap \overset{(k)}{\boldsymbol{M}_{\mathfrak{L}}}$$

and

$$\boldsymbol{M} \subseteq \boldsymbol{M}_{\mathfrak{L}} .$$

$\boldsymbol{M}_{\mathfrak{L}}$ is called the $\mathfrak{L}$-*sketch* of $\boldsymbol{M}$ or the *cribble sketch* of $\boldsymbol{M}$ i.r.t. $\mathfrak{L}$. When $\boldsymbol{M}_{\mathfrak{L}}$ and $\boldsymbol{N}_{\mathfrak{L}}$ are $\mathfrak{L}$-sketches of $\boldsymbol{M}$ and $\boldsymbol{N}$ respectively and

$$\boldsymbol{M}_{\mathfrak{L}} \cap \boldsymbol{N}_{\mathfrak{L}} = \text{void} ,$$

then $\boldsymbol{M}$ and $\boldsymbol{N}$ are said to be $\mathfrak{L}$-*disjoint*. If

$$\boldsymbol{M} = \boldsymbol{M}_{\mathfrak{L}} ,$$

$\boldsymbol{M}$ is called a $\mathfrak{L}$-*set*.

When P is an isolated point of $\boldsymbol{M}$, by virtue of (5. 7) is found, for any real number $\varepsilon > 0$, an integer N_ε such that

$$(\forall n)\ (n > N_\varepsilon)\ (\overset{(n)}{\boldsymbol{U}_k} \ni P \rhd \overset{(n)}{\boldsymbol{U}_k} \subseteq \{Q :\ |Q-P| < \varepsilon\})$$

$|Q-P|$ meaning the distance between Q and P. Therefore, if ε is sufficiently small, we have

$$n > N_\varepsilon\ \rhd (\{Q :\ |Q-P| < \varepsilon\} - \overset{(n)}{\boldsymbol{U}_k}) \cap \boldsymbol{M}_{\mathfrak{L}} = \text{void} .$$

Consequently :

Proposition 5. 1. *Let $\overline{\boldsymbol{M}}$ be the closure of $\boldsymbol{M}$ and $\boldsymbol{M}_{\mathfrak{L}}$ the $\mathfrak{L}$-sketch of $\boldsymbol{M}$, then we have*

$$\boldsymbol{M} \subseteq \boldsymbol{M}_{\mathfrak{L}} \subseteq \overline{\boldsymbol{M}} .$$

The quantity $d(\boldsymbol{M}, \boldsymbol{N}) = \inf |Q-P| (Q \in \boldsymbol{N}, P \in \boldsymbol{M})$ is the distance between sets $\boldsymbol{M}$ and $\boldsymbol{N}$. When $d(\boldsymbol{M}, \boldsymbol{N}) > 0$, $\boldsymbol{M}$ is said to be *distant from* $\boldsymbol{N}$. By Proposition 5. 1 it is then directly seen that :

Corollary 5. 1. ($\boldsymbol{M}$ *is distant from* $\boldsymbol{N}$) $\bowtie$ ($\boldsymbol{M}_{\mathfrak{L}}$ *is distant from* $\boldsymbol{N}_{\mathfrak{L}}$).

The measure defined by

$$\gamma^\alpha(\boldsymbol{M}_{\mathfrak{L}}) = \lim_{n\to\infty} \sum_k (\gamma(\overset{(n)}{\boldsymbol{U}_k}))^\alpha$$

is called the *sketch-integral of* $\boldsymbol{M}$ i.r.t. $\mathfrak{L}$, $\mathfrak{L}$ being a foundation in $\mathbf{E}$. About this measure, the computations in (5. 3) through (5. 6) shows that the relation

$$\gamma^\alpha(\boldsymbol{M}_{\mathfrak{L}_1}) = \gamma^\alpha(\boldsymbol{M}_{\mathfrak{L}_2})$$

does not always be realized even if

$$\boldsymbol{M} = \boldsymbol{M}_{\mathfrak{L}_1} = \boldsymbol{M}_{\mathfrak{L}_2} .$$

So, in the calculus of sketch-integral, the foundation $\mathfrak{L}$ should not be changed by another one among the procedures belonging to the same course of computation.

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