On Ky Fan's Theorem and its Application to the Free Vector Lattice

On Ky Fan's Theorem and its Application to the Free Vector Lattice

Kazuo Iwata*

Abstract

In this paper the author characterizes the linear dependency with positive coefficients in a real linear space. Using this, he proves the countable decomposabilyty** of free vector lattices.

Introduction. D. M. Topping's expectation[8)] that *it appears highly likely that free vector lattices are countably debomposable*** has been already justified by E. C. Weinberg[9)] and I. Amemiya[10)] independently. One of the purposes of this paper is to deal with Ky Fan's theorems[6)] (*generalizations of Stiemke-Carver***-Dines theorems*, etc.) somewhat strictly in case of inner product, and the other is to give another justification to the expectation above. We shall do the former in § 1, and the latter in § 2 by the use of § 1 and by the aid of the separability of l^2.

Here the author wishes to express his sincere thanks to Prof. I. Amemiya (Tokyo Woman's Christian College) and Prof. T. Itô (Wayne State Univ. Detroit) for their helpful guidances, and to Asst. Prof. K. Honda (Muroran Inst, Tech.) for his useful conversations.

§ 1. Linear Dependence with Positive Coefficients in a Real Linear Space

Let R be a real linear space. Let R be a real inner product space since any R can be given at least an inner product on it. Let us denote by $(\,,\,)$ the fixed inner product on R unless otherwise specified. Let a system of elements (not

* 岩 田 一 男

** A vector lattice is called *countably decomposable* if every positive orthogonal family is at most countable. (l. c, 8), p. 423 & p. 425)

*** *Carver's theorem: A necessary and sufficient condition that a given system of m linear inequalities in n variables,*

$$\sum_{j=1}^{n} \alpha_{ij} x_j + \beta_i > 0 \qquad (i = 1, 2, \cdots, m)$$

be inconsistent (cf. Theorem 4, foot note, this paper.) *is that there should exist a set of $m+1$ constants, $k, k, \cdots, k_{m+1}$, such that*

$$\sum_{i=1}^{m} k_i \alpha_{ir} = 0 \text{ and } \sum_{i=1}^{m} k_i \beta_i + k_{m+1} = 0 \qquad (r = 1, 2, \cdots, n)$$

where at least one of the k's being positive and non of them being negative. (2). p. 217) necessarily distinct) $a_1, a_2, \cdots, a_n \in R (n \geqq 1)$ be given. Let us define the *properties with respect to* $\{a_\nu : \nu = 1, 2, \cdots, n\}$ as follows:

(P): There exist $\mu_\nu > 0$ $(\nu = 1, 2, \cdots, n)$ with

$$\left(a_i, \sum_1^n \mu_\nu a_\nu\right) > 0 \qquad (i = 1, 2, \cdots, n);$$

(P.O): There exist $\mu_\nu > 0$ $(\nu = 1, 2, \cdots, n)$ with

$$\left(a_i, \sum_1^n \mu_\nu a_\nu\right) \geqq 0 \qquad (i = 1, 2, \cdots, n) \text{ and not all zero};$$

(P.N): Non-(P.O).

Furthermore let a system of real linear functionals $f_1, f_2, \cdots, f_n$ $(n \geqq 1)$ on R be given. In the above definitions, considering every $x \in R$ in places of $\sum_1^n \mu_\nu a_\nu$, $\mu_\nu > 0$ $(\nu = 1, 2, \cdots, n)$, we define analogously the *properties* (P), (P.O), and (P.N) *with respect to* $\{f_\nu : \nu = 1, 2, \cdots, n\}$ too. Here we note that the property (P.N) means that whenever there exist $x \in R$ with $\sum_1^n \{f_\nu(x)\}^2 > 0$, there exist i, j always such that $f_i(x) > 0$, $f_j(x) < 0$. Let us now denote by A_n a system $\{a_\nu : a_\nu \in R, \nu = 1, 2, \cdots, n\}$, $(n \geqq 1)$.

Theorem 1. (*Generalization of Stiemke-Dines Theorem*)

A necessary and sufficient condition that a given system A_n *has the property* (P.N) *is that there exist* $\lambda_\nu > 0$ $(\nu = 1, 2, \cdots, n)$ *such that* $\sum_1^n \lambda_\nu a_\nu = 0$.

Proof. The sufficiency of the condition is evident. We shall prove the necessity of the condition by the mathematical induction.

In case $n = 1$, we should have $a_1 = 0$, and it is clear. Next we suppose that it is true for a positive integer n. If two distinct subsystems with n members of $A_{n+1} = \{a_\nu : \nu = 1, 2, \cdots, n+1\}$ have (P.N) together, we come to the conclusion at once by the induction hypothesis (summing by two equalities). For the remaining cases $(n + 1 \geqq 3)$ we assume that A_n and $\{a_1, a_2, \cdots, a_{n-1}, a_{n+1}\}$ have (P.O). Here if we have both:

$$\left(a_i, \sum_{\nu \neq n+1} \mu_\nu a_\nu\right) \geqq 0 \ (i = 1, 2, \cdots, n, n+1) \text{ and not all zero for } \mu_\nu > 0,$$

$$\left(a_i, \sum_{\nu \neq n} \mu'_\nu a_\nu\right) \geqq 0 \ (i = 1, 2, \cdots, n, n+1) \text{ and not all zero for } \mu'_\nu > 0,$$

these immediately contradict the assumption. In the case $n + 1 = 2$, we get the same by the first half. That is, it suffices to verify the case where there exist $\mu_\nu > 0$ $(\nu = 1, 2, \cdots, n)$ and $r (1 \leqq r \leqq n)$ such that

$$\left(a_i, \sum_1^n \mu_\nu a_\nu\right) > 0 \qquad (i = 1, 2, \cdots, r),$$

$$\left(a_i,\ \sum_1^n \mu_\nu a_\nu\right)=0 \qquad (i=r+1, \cdots, n),$$

with

$$\left(a_{n+1},\ \sum_1^n \mu_\nu a_\nu\right)=-\alpha<0\,.$$

Then, setting $\sum_1^n \mu_\nu a_\nu = x$, let us determine $\xi_i>0$ such that

$$(\xi_i a_i + a_{n+1},\ x)=0 \qquad (i=1, 2, \cdots, r),$$
$$(a_i,\ x)=0 \qquad (i=r+1, \cdots, n).$$

Now suppose that n members:

$$\left.\begin{array}{ll} \xi_i a_i + a_{n+1} & (i=1, 2, \cdots, r) \\ a_i & (i=r+1, \cdots, n) \end{array}\right\} \qquad \text{(i)}$$

have (P.O). Then, by definition, there exists

$$y=\sum_1^r \eta_\nu(\xi_\nu a_\nu + a_{n+1}) + \sum_{r+1}^n \eta_\nu a_\nu \quad \text{with} \quad \eta_\nu>0\ (\nu=1,2,\cdots,n)$$

for which

$$(\xi_i a_i + a_{n+1},\ y)\geqslant 0 \qquad (i=1, 2, \cdots, r),$$
$$(a_i,\ y)\geqslant 0 \qquad (i=r+1, \cdots, n),$$

and not all zero. Here putting $(a_{n+1},\ y)=\beta$, if $\beta=0$, that is obviously contradictory to the assumption. If $\beta<0$, then it follows

$$(\xi_i a_i,\ y)\geqslant -(a_{n+1},\ y)>0 \qquad (i=1, 2, \cdots, r)$$

and so

$$(x,\ y)=\left(\sum_1^n \mu_\nu a_\nu,\ y\right)>0\,,$$

a contradiction to $(x,\ y)=0$. Therefore we have $\beta>0$.

We have

$$0\leqslant\left(\xi_i a_i + a_{n+1},\ \frac{\alpha}{\beta}\, y\right)=\left(\xi_i a_i,\ \frac{\alpha}{\beta}\, y\right)+\alpha$$
$$=\left(\xi_i a_i,\ \frac{\alpha}{\beta}\, y\right)+(\xi_i a_i,\ x)$$
$$=\left(\xi_i a_i,\ \frac{\alpha}{\beta}\, y + x\right) \qquad (i=1, 2, \cdots, r),$$
$$0\leqslant\left(a_i,\ \frac{\alpha}{\beta}\, y + x\right) \qquad (i=r+1, \cdots, n),$$

and not all zero. While we have

$$0=\left(a_{n+1},\ \frac{\alpha}{\beta}\, y + x\right).$$

These mean that A_{n+1} has (P.O) and hence contradictory to the assumption. Therefore n members of (i) should have (P.N). Therefore there exist $\tau_\nu>0$ ($\nu=1, 2, \cdots, n$) such that

$$\sum_1^r \tau_\nu(\xi_\nu a_\nu+a_{n+1})+\sum_{r+1}^n \tau_\nu a_\nu=0\,.$$

This completes the proof.

Here in this proof, replacing $\sum_1^n \mu_\nu a_\nu$ and $\sum_1^r \eta_\nu(\xi_\nu a_\nu+a_{n+1})+\sum_{r+1}^n \eta_\nu a_\nu$ by mere $x\in R$ and mere $y\in R$ respectively and using β as $\beta\neq 0$, we get the following result due to Ky Fan*.

Theorem 1′. *A necessary and sufficient condition that a given system* $\{f_\nu : \nu=1, 2, \cdots, n\}$, $(n\geqq 1)$ *has the property* (P.N) *is that there exist* $\lambda_\nu>0$ ($\nu=1, 2, \cdots, n$) *such that* $\sum_1^n \lambda_\nu f_\nu=0$.

Lemma 1. A_n *has* (P) *if and only if any* (*proper or improper, non-void*) *subsystem of* A_n *has* (P.O).

Proof. For the "if" part we proceed by the induction on n. In case $n=1$, it is trivial. Suppose that it is true for a positive integer n.

By the assumption, there exist $\xi_\nu>0$ ($\nu=1, 2, \cdots, n, n+1$) and i_0 with

$$\left(a_\nu, \sum_1^{n+1}\xi_\nu a_\nu\right)\geqq 0 \quad (i=1, 2, \cdots, n, n+1)\,, \quad \left(a_{i_0}, \sum_1^{n+1}\xi_\nu a_\nu\right)>0\,.$$

While by the induction hypothesis, there exist

$$\eta_\nu>0 \qquad (\nu=1, 2, \cdots, i_0-1, i_0+1, \cdots, n+1) \qquad \text{with}$$

$$\left(a_i, \sum_{\nu\neq i_0}\eta_\nu a_\nu\right)>0 \qquad (i=1, 2, \cdots, i_0-1, i_0+1, \cdots, n+1)\,.$$

Hence, choosing $\varepsilon>0$ so small that

$$\left|\left(a_{i_\nu}, \varepsilon\sum_{\nu\neq i_0}\eta_\nu a_\nu\right)\right|<\left(a_{i_0}, \sum_1^{n+1}\xi_\nu a_\nu\right),$$

and putting

$$\sum_1^{n+1}\xi_\nu a_\nu+\varepsilon\sum_{\nu\neq i_0}\eta_\nu a_\nu=\sum_1^{n+1}\mu_\nu a_\nu$$

we obtain $\mu_\nu>0$ ($\nu=1, 2, \cdots, n, n+1$) with

$$\left(a_i, \sum_1^{n+1}\mu_\nu a_\nu\right)>0 \qquad (i=1, 2, \cdots, n, n+1)\,.$$

The converse follows immediately from Theorem 1.

Definition 1. If A_n has not (P), by Lemma 1 and Theorem 1, there exists in A_n the largest subsystem which has (P.N). We call it the *maximum* (P.N) *system of* A_n. And we denote it as $A_m^*=\{a_\nu : \nu=1, 2, \cdots, m\}$ without loss of

* 6), Part I. Corollary 4.

generality. If A_n has (P), the *maximum* (P.N) *system of* A_n is empty.

Lemma 2. *Let* $A_m^*(0 \leqslant m \leqslant n)$ *be the maximum* (P.N) *system of* A_n, *where* $A_0^* = \emptyset$. *Then there exist* $\lambda_\nu > 0$ $(\nu = 1, 2, \cdots, m)$ *and* $\mu_\nu > 0$ $(\nu = 1, 2, \cdots, n)$ *such that*

$$\sum_1^m \lambda_\nu a_\nu = 0\,,$$

$$\left(a_i,\ \sum_1^n \mu_\nu a_\nu\right) = 0 \qquad (i = 1, 2, \cdots, m)\,,$$

$$\left(a_i,\ \sum_1^n \mu_\nu a_\nu\right) > 0 \qquad (i = m+1, \cdots, n)\,.$$

The converse is also true.

Proof. In case either $m = n$ or $m = 0$, it is trivial by Theorem 1 and Lemma 1. In the remaining cases, the first formula is clear. And since A_n has (P.O), there exist $\mu_\nu > 0$ $(\nu = 1, 2, \cdots, n)$ with

$$\left(a_i,\ \sum_1^n \mu_\nu a_\nu\right) = 0 \qquad (i = 1, 2, \cdots, m)\,,$$

$$\left(a_i,\ \sum_1^n \mu_\nu a_\nu\right) \geqslant 0 \qquad (i = m+1, \cdots, n)$$

and not all zero. Therefore we can complete the proof by means of Lemma 1. The converse is evident.

By Lemma 1, we have:

Theorem 2. (*Generalization of Carver-Dines Therem*)

A necessary and sufficient condition that a given system A_n *has not property* (P) *is that there exist* $\lambda_\nu \geqslant 0$ $(\nu = 1, 2, \cdots, n)$ *such that* $\sum_\nu^n \lambda_\nu a_\nu = 0$, *where at least one of the* λ*'s being positive.*

Definition 2. Let us call a system A_n *linearly dependent with positive coefficients*, if A_n satisfies the necessary condition of Theorem 2.

As a result including Theorems 1, and 2 we have:

Theorem 3. (*On Tucker's Theorem**)

Two systems:

$$\text{i)}\quad \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = 0\,, \qquad \lambda_\nu \geqslant 0 \qquad (\nu = 1, 2, \cdots, n)\,;$$

$$\text{ii)}\quad \begin{cases} (a_1, x) \geqslant 0 \\ \quad\vdots \\ (a_n, x) \geqslant 0\,; \end{cases}$$

possess solutions λ_ν $(\nu = 1, 2, \cdots, n)$ *and* x *such that*

$$\text{iii)}\quad \begin{cases} (a_1, x) + \lambda_1 > 0 \\ \quad\vdots \\ (a_n, x) + \lambda_n > 0\,. \end{cases}$$

* 5), l.c. Theorem 1.

Proof. If A_n has (P), take suitable $x=\sum_1^n \mu_\nu a_\nu$, $\mu_\nu>0$ $(\nu=1,2,\cdots,n)$ and $\lambda_\nu=0$ $(\nu=1,2,\cdots,n)$. If A_n has (P.O) but not (P), take x and λ_ν $(\nu=1,2,\cdots,m)$ just as in Lemma 2 and take $\lambda_\nu=0$ $(\nu=m+1,\cdots,n)$. If A_n has (P.N), take suitable $\lambda_\nu>0$ $(\nu=1,2,\cdots,n)$ and $x=0$.

Next let us deal with the non-homogeneous cases.

Theorem 4. *A necessary and sufficient condition that the following two systems of inequalities*:

$$\text{i)}\quad \begin{cases} (a_1,\ x)\geqslant 0 \\ \quad\vdots \\ (a_n,\ x)\geqslant 0 \end{cases} \qquad \textit{where we interpret at least one is positive};$$

$$\text{ii)}\quad \begin{cases} (a_1,\ y)+\alpha_1\geqslant 0 \\ \quad\vdots \\ (a_n,\ y)+\alpha_n\geqslant 0 \end{cases} \qquad \textit{where we mean as usual};$$

are both inconsistent is that there exist* $\lambda_\nu>0$ $(\nu=0,1,2,\cdots,n)$ *such that* $\sum_1^n \lambda_\nu a_\nu=0$ and $\lambda_0+\sum_1^n \lambda_\nu \alpha_\nu=0$.

Poof. To prove the necessity of the condition we think of a linear space $R\oplus V^1$. Taking

$$\begin{aligned} R\oplus V^1\in(0,\quad 1)&=b_0, \\ (a_1,\ \alpha_1)&=b_1, \\ \vdots\quad &\quad\ \ , \\ (a_n,\ \alpha_n)&=b_n; \end{aligned}$$

we consider the system $B=\{b_\nu:\ \nu=0,1,\cdots,n\}$. Suppose B has (P.O), then there exist $\mu_\nu>0$ $(\nu=0,1,\cdots,n)$ with

$$\left(b_i,\ \sum_0^n \mu_\nu b_\nu\right)\geqslant 0 \qquad (i=0,1,\cdots,n)$$

and not all zero. Now if

$$\left(b_0,\ \sum_0^n \mu_\nu b_\nu\right)>0,$$

then we have

$$0+\sum_0^n \mu_\nu \alpha_\nu>0$$

and

$$\left(a_i,\ \sum_0^n \mu_\nu a_\nu\right)+\alpha_i\sum_0^n \mu_\nu \alpha_\nu\geqslant 0 \qquad (i=1,2,\cdots,n),$$

* $(a_i,x)+\alpha_i\geqslant 0$ $(i=1,2,\cdots,n)$ is said to be *inconsistent*, if the system has not solution x in R, otherwise it is said to be *consistent*.

In this paper, unless otherwise specified, inequality signs $\geqslant$ and $>$ are used as usual.

where $a_0=0,\ \alpha_0=1$.

This yields that (ii) is consistent. Otherwise, since

$$\left(b_0,\ \sum_1^n \mu_\nu b_\nu\right)=0\,, \quad \text{and so} \quad \sum_0^n \mu_\nu \alpha_\nu = 0\,.$$

Hence

$$\left(a_i,\ \sum_0^n \mu_\nu a_\nu\right)+0\geqslant 0 \qquad (i=1, 2, \cdots, n)\,,$$

where not all zero.

Therefore (i) is consistent, and so B should have (P.N). Thus we have proved the necessity by Theorem 1. For the converse, it is clear that (i) is inconsistent. If (ii) is consistent under our assumptions, we have

$$\lambda_\nu(a_\nu,\ y)+\lambda_\nu\alpha_\nu\geqslant 0 \qquad (\nu=1, 2, \cdots, n)\,,$$

hence

$$\left(\sum_1^n \lambda_\nu a_\nu,\ y\right)+\sum_1^n \lambda_\nu\alpha_\nu\geqslant 0\,, \quad \text{and so} \quad \sum_1^n \lambda_\nu\alpha_\nu\geqslant 0\,.$$

This is contradictory to $\sum_1^n \lambda_\nu\alpha_\nu<0$.

Corollary 1*. *A necessary and sufficient condition that a system*

$$\begin{cases} (a_1,\ x)+\alpha_1\geqslant 0 \\ \quad\vdots \\ (a_n,\ x)+\alpha_n\geqslant 0 \end{cases}$$

is consistent is that $\sum_1^m \lambda_\nu\alpha_\nu\geqslant 0$ *holds for any* $\lambda_\nu>0$ $(\nu=1, 2, \cdots, m)$ *such that* $\sum_1^m \lambda_\nu a_\nu=0$, *where* A_m^* *is the maximum* (P.N) *system of* A_n.

And whenever the system is consistent, there exist solutions such that $x=\sum_1^n \mu_\nu a_\nu$ *with* $\mu_\nu>0$ $(\nu=1, 2, \cdots, n)$.

Proof. Necessity is evident. For the converse, we consider the system $B=\{b_\nu:\ \nu=0, 1, \cdots, n\}$ just as in the case of Theorem 4. We at first see that B should have (P.O) by Theorem 1 and assumption. Suppose there exists a (P.N) system which contains $b_0=(0, 1)$. Let the maximum (P.N) system of B be $B_m^*=\{b_0, b_1, \cdots, b_m\}$, $(m\geqslant 1)$. Then by Lemma 2, there exist $\lambda_\nu>0$ $(\nu=0, 1, \cdots, m)$ and $\xi_\nu>0$ $(\nu=0, 1, \cdots, n)$ such that

$$\sum_1^m \lambda_\nu a_\nu=0\,,$$

$$\lambda_0+\sum_1^m \lambda_\nu\alpha_\nu=0\,,$$

$$\left(b_i,\ \left(\sum_1^n \xi_\nu a_\nu,\ \xi_0+\sum_1^n \xi_\nu\alpha_\nu\right)\right)=0 \qquad (i=0, 1, \cdots, m)\,,$$

* 6), l.c. Part I, Theorem I & Part III, Theorem 14.

$$\left(b_i, \left(\sum_1^n \xi_\nu a_\nu, \xi_0 + \sum_1^n \xi_\nu \alpha_\nu\right)\right) > 0 \qquad (i = m+1, \cdots, n).$$

Hence, by the second half

$$\xi_0 + \sum_1^n \xi_\nu \alpha_\nu = 0$$

and

$$\left(a_i, \sum_1^n \xi_\nu a_\nu\right) = 0 \qquad (i = 1, 2, \cdots, m),$$

$$\left(a_i, \sum_1^n \xi_\nu a_\nu\right) > 0 \qquad (i = m+1, \cdots, n).$$

Applying Lemma 2, that is a contradiction to the assumption.

Therefore there exist $\mu_\nu > 0$ $(\nu = 0, 1, \cdots, n)$ with both

$$\left(b_0, \left(\sum_1^n \mu_\nu a_\nu, \mu_0 + \sum_1^n \mu_\nu \alpha_\nu\right)\right) > 0,$$

$$\left(b_i, \left(\sum_1^n \mu_\nu a_\nu, \mu_0 + \sum_1^n \mu_\nu \alpha_\nu\right)\right) \geqslant 0 \qquad (i = 1, 2, \cdots, n).$$

The rest of the proof follows at once from this.

Theorem 5. (*Generalization of Carver's Theorem*)

A necessary and sufficient condition that a given system of n linear inequalities:

$$\begin{cases} (a_1, x) + \alpha_1 > 0 \\ \vdots \\ (a_n, x) + \alpha_n > 0 \end{cases}$$

is inconsistent is that there exist $\lambda_\nu \geqslant 0$ $(\nu = 0, 1, \cdots, n)$ such that $\sum_1^n \lambda_\nu a_\nu = 0$ and $\lambda_0 + \sum_\nu^n \lambda_\nu \alpha_\nu = 0$, where at least one of the λ_ν $(\nu = 1, 2, \cdots, n)$ being positive.

Proof. Taking B as well as in case of Theorem 4, we see that some subsystems of B should have (P.N). This is the necessity of the condition. Sufficiency of the condition is evident.

In the sequel, applying Theorem 3 to the case:

$$R \oplus V^1 \ni (0, 1), (a_1, \alpha_1), \cdots, (a_n, \alpha_n);$$

we get the following.

Teorem 6. *Two systems*:

$$\text{i)} \quad \begin{cases} \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = 0 \\ \lambda_0 + \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \cdots + \lambda_n \alpha_n = 0, \quad \lambda_\nu \geqslant 0 \qquad (\nu = 0, 1, \cdots, n); \end{cases}$$

$$\text{ii)} \quad \begin{cases} \xi \geqslant 0 \\ (a_1, x) + \alpha_1 \xi \geqslant 0 \\ \vdots \\ (a_n, x) + \alpha_n \xi \geqslant 0; \end{cases}$$

possess solutions λ_ν $(\nu=0, 1, \cdots, n)$, x, *and* ξ *such that*

$$\text{iii)}\quad \begin{cases} \xi+\lambda_0>0 \\ (a_1, x)+\alpha_1\xi+\lambda_1>0 \\ \vdots \\ (a_n, x)+\alpha_n\xi+\lambda_n>0\,. \end{cases}$$

(We omit the proof.)

We can also describe Theorems 2, 3, 4, 5, 6, and Corollary 1 (except for the latter proposition) as the ones* with respect to linear functionals f_ν $(\nu=1,2,\cdots,n)$ on R. (We call them Theorems 2′, 3′, 4′, 5′, 6′, and Corollary 1′ respectively.) We can prove them by means of (P), (P.O), and (P.N) with respect to linear functionals. In particular Theorems 4′, 5′, 6′, and Corollary 1′ are done by the uses of linear functionals $f_\nu\oplus\alpha_\nu$ $(\nu=0, 1, \cdots, n)$ on $R\oplus V^1$ defined as $f_\nu\oplus\alpha_\nu(X)=f_\nu(x)+\alpha_\nu\xi$ for $X=(x, \xi)\in R\oplus V^1$.

We deal with the cases of inner product in the sequel.

Corollary 2. *Let a system* $A_n=\{a_\nu : a_\nu\in R,\ \nu=1, 2, \cdots, n\}$ *be given. A necessary and sufficient condition that* $\min\{a_\nu : \nu=1,2,\cdots,n\}>0$ *under some total order which orders all elements of a real linear space* R *as a vector lattice*** *is that there exist* $\mu_\nu>0$ $(\nu=1, 2, \cdots, n)$ *such that*

$$\left(a_i,\ \sum_{\nu=1}^{n}\mu_\nu a_\nu\right)>0 \qquad (i=1, 2, \cdots, n)\,.$$

Proof. The necessity follows from Theorem 2. The sufficiency follows from Corollary 3 in § 2.

Now if we are concerned with the linear independency of n members of R, we have obviously by Theorem 2:

Corollary 3. *Given a system* A_n, *it is linearly independent if and only if every system* $\{\delta_\nu a_\nu : \nu=1,2,\cdots,n\}$ *has* (P), *where each* δ_ν $(\nu=1,2,\cdots,n)$ *stands for* 1 *or* -1 *individually.*

By means of Corollary 3 we get at once the Gram's theorem (using the Laplace's expansion). Using the Gram's theorem we note here:

Note 1. *Given a system* $A_n (n\geqq 2)$ *whose* $n-1$ *members* $a_1, a_2, \cdots, a_{n-1}$ *are linearly independent, letting*

$$b_n=\begin{vmatrix} (a_1, a_1)\cdots\cdots\cdots(a_1, a_n) \\ \vdots \\ (a_{n-1}, a_1)\cdots\cdots(a_{n-1}, a_n) \\ a_1\cdots\cdots\cdots\cdots\cdots a_n \end{vmatrix}$$

and

* Theorems 2′, 5′, and Corollary 1′ are due to Ky Fan: 6), Part I.

** Cf. Proof of Lemma 3 in § 2.

$$\varDelta_\nu = \begin{vmatrix} \overbrace{(a_1, a_1) \cdots (a_1, a_n)}^{\nu} \cdots (a_1, a_{n-1}) \\ \vdots \\ (a_{n-1}, a_1) \cdots (a_{n-1}, a_n) \cdots (a_{n-1}, a_{n-1}) \end{vmatrix} \qquad (\nu = 1, 2, \cdots, n-1),$$

we have:

A_n *has not* (P) *if and only if* $b_n = 0$ *and* $\varDelta_\nu \leqslant 0$ $\quad (\nu = 1, 2, \cdots, n-1)$,

A_n *has* (P.N) *if and only if* $b_n = 0$ *and* $\varDelta_\nu < 0$ $\quad (\nu = 1, 2, \cdots, n-1)$.

As a special case:

Note 2. *For a system* $A_n (n \geqslant 2)$, *with the notations*:

$$b_\nu = \begin{vmatrix} (a_1, a_1) \cdots\cdots (a_1, a_n) \\ \vdots \\ (a_{\nu-1}, a_1) \cdots\cdots (a_{\nu-1}, a_n) \\ a_1 \cdots\cdots\cdots\cdots a_n \\ (a_{\nu+1}, a_1) \cdots\cdots (a_{\nu+1}, a_n) \\ \vdots \\ (a_n, a_1) \cdots\cdots (a_n, a_n) \end{vmatrix} \qquad (\nu = 1, 2, \cdots, n),$$

the following conditions are equivalent.

(1): $a_1, a_2, \cdots, a_n$ *are linearly independent.*

(2): $b_1, b_2, \cdots, b_n$ *are linearly independent.*

(3): $b_1, b_2, \cdots, b_n$ *have* (P).

§ 2. Countable Decomposability of Free Vector Lattices

Here we deal with this problem from the standpoint and with the notations of 8)* except that we write $A(\mathfrak{M})$ for A if necessary.

Let A be a real linear space in § 2.

Lemma 1. *A typical element* $Z \in \mathrm{FVL}(\mathfrak{M})$ *has the form*:

$$Z = \alpha X + \beta Y$$

being

* D. M. Topping gave in his 8) p. 418 as follows:

Let $\mathfrak{M}$ be any cardinal number and let S be a set of cardinality $\mathfrak{M}$, we construct the real vector space A of dimension $\mathfrak{M}$ having S as basis. Let $A_i (i \in I)$ denote the totality of totally ordered A as a vector lattice. We form the direct product $V = \prod_{i \in I} A_i$. Under the co-ordinate-wise ordering, V is a vector lattice. We denote D the subset of V such that

$$V \supset D = \{\{x_i\} : x_i = x,\ i \in I,\ x \in A\}.$$

We let $\mathrm{FVL}(\mathfrak{M})$ denote the distributive sublattice of V generated by D. Then his Theorem 6: $\mathrm{FVL}(\mathfrak{M})$ *is the free vector lattice on* $\mathfrak{M}$ *genarators.*

$$X=\bigcap_{\nu=1}^{p}\left(\{x_{\nu1}\}^{+}+\cdots+\{x_{\nu m}\}^{+}\right), \qquad x_{\nu\mu}\in A\,;$$

$$Y=\bigcap_{\nu=1}^{q}\left(\{y_{\nu1}\}^{+}+\cdots+\{y_{\nu n}\}^{+}\right), \qquad y_{\nu\mu}\in A\,;$$

where α, β are real numbers and

$$\{x\}^{+}=\begin{cases} x & \text{for } x\geqslant 0 \\ 0 & \text{for } x<0 \quad \text{in } V. \end{cases}$$

Proof. Let L be the totality of elements in above forms. We have obviously $\mathrm{FVL}(\mathfrak{M})\supset L$. In the right hand side of $\xi_1 Z_1+\xi_2 Z_2$, distinguishing the terms with positive coefficients from the others, we have the linearlity of L. For the lattice, in cases where $\alpha\beta\geqslant 0$ for $\alpha X+\beta Y\in L$, we have obviously $(\alpha X+\beta Y)^{+}\in L$. In case where $\alpha>0$ and $\beta<0$, since we have

$$(\alpha X+\beta Y)^{+}=\alpha X-(\alpha X\frown(-\beta)Y),$$

here $\alpha X\cap(-\beta)Y$ constitues the same form as Y, we have $(\alpha X+\beta Y)^{+}\in L$. Thus L is a lattice.

From Schwarz's inequality (Gram's theorem), we have easily:

Lemma 2. *Let A be an inner product space. If*

$$(x, a)>0 \text{ and } \|a-t\|<\frac{(x, a)}{\|x\|} \text{ for } a, x, t\in A\,;$$

then

$$0<(x, t)<2(x, a).$$

Lemma 3. *Let a total order* as a vector lattice for an inner product space A be given. Then another (in general) total order for A with respect to $0\neq t\in A$ can be introduced by*

for $(x-y, t)=0$, let the ordering between x and y be just as before;
for $(x-y, t)>0$, let $x>y$;
for $(x-y, t)<0$, let $x<y$.

Proof. Because the definition just mentioned satisfies the following order relations. 1) reflexivity, 2) asymmetry, 3) transitivity, 4) comparableness, 5) $\alpha\geqslant 0$ and $x\geqslant 0$ together imply $\alpha x\geqslant 0$, 6) $x\geqslant y$ implies $x+z\geqslant y+z$.

For brevity, we call this total order for A *a total order for t.*

Theorem. *Every free vector lattice* $\mathrm{FVL}(\mathfrak{M})$ *is countably decomposable.*

Proof. At first we will prove the fact "*if $O<Z^{\cup}O$ for $Z\in\mathrm{FVL}(\mathfrak{M})$, there exist $0\neq a\in A(\mathfrak{M})$ and $\varepsilon>0$ such that $Z>0$ by every total order for t satisfying $\|a-t\|<\varepsilon$; where a, norm, and ε altogether are the ones derived from the given*

* It is true that a real linear space A has at least a total order as a vector lattice. Hereafter we simply say of it "*total order for A*".

inner product on $A(\mathfrak{M})$". To prove this, let us represent $Z\in \mathrm{FVL}(\mathfrak{M})$ by means of Lemma 1 and call it expression (i). Here, since we can do the cases where $\alpha>0$ and $\beta\geqslant 0$ more simply rather than the case where $\alpha>0$ and $\beta<0$, we handle the latter case. Besides we may assume $\alpha=1$ and $\beta=-1$ without loss of generality. We fix a total order T (T-coordinate) for which $Z(T)>0$. Although the terms $\{x_{\nu\mu}\}^+$ for which $x_{\nu\mu}(T)\leqslant 0$, vanish at this T, we use the same expression (i) for the sake of simplicity. We assume

$$Y(T)=\{y_{11}\}^+(T)+\cdots+\{y_{1n}\}^+(T).$$

Then we may think that there exists $r(0\leqslant r\leqslant n)$ such that

$$\begin{aligned}
&y_{1\nu}<0 \qquad (\nu=1,2,\cdots,r),\\
&y_{1\nu}>0 \qquad (\nu=r+1,\cdots,n).\\
&(x_{11}+\cdots+x_{1m})-(y_{1r+1}+\cdots+y_{1n})=z_1>0,\\
&\vdots\\
&(x_{p1}+\cdots+x_{pm})-(y_{1r+1}+\cdots+y_{1n})=z_p>0,\\
&x_{\nu\mu}>0 \qquad (\nu=1,2,\cdots,p;\ \mu=1,2,\cdots,m),
\end{aligned}$$

at this T. Therefore, by Corollary 2 in § 1, under the given inner product on $A(\mathfrak{M})$, there exists $a\in A(\mathfrak{M})$ such that

$$\begin{aligned}
&(-y_{1\nu},a)>0 \qquad (\nu=1,2,\cdots,r),\\
&(y_{1\nu},a)>0 \qquad (\nu=r+1,\cdots,n),\\
&(z_\nu,\ a)>0 \qquad (\nu=1,2,\cdots,p),\\
&(x_{\nu\mu},a)>0 \qquad (\nu=1,2,\cdots,p;\ \mu=1,2,\cdots,m).
\end{aligned}$$

Now putting ε_1, ε_2, ε_3, ε_4, and ε as follows respectively by the derived norm:

$$\min\left\{\frac{(-y_{1\nu},a)}{\|y_{1\nu}\|}\right\}=\varepsilon_1>0,\qquad \min\left\{\frac{(y_{1\nu},a)}{\|y_{1\nu}\|}\right\}=\varepsilon_2>0,$$

$$\min\left\{\frac{(z_\nu,a)}{\|z_\nu\|}\right\}=\varepsilon_3>0,\qquad \min\left\{\frac{(x_{\nu\mu},a)}{\|x_{\nu\mu}\|}\right\}=\varepsilon_4>0,$$

$$\min\{\varepsilon_{,1}\ \varepsilon_2,\ \varepsilon_3,\ \varepsilon_4\}=\varepsilon;$$

we obtain the fact above mentioned by Lemmas 2 and 3.

Now suppose that for a certain $\mathrm{FVL}(\mathfrak{M})$ there exist uncountable number of mutually orthogonal positive elements. Here, if necessary, we extract $Z_\gamma(\gamma\in\Gamma)$ out of them so as to Γ satisfies $\aleph_0<\overline{\overline{\Gamma}}\leqslant\aleph$. The total number of members which take part in expression (i) of each $Z_\gamma(\gamma\in\Gamma)$ is not more than $\aleph$. Therefore letting $S_0(S_0\subset S)$ be the union of corresponding generators whose linear combination expresses each member above, we have $\overline{\overline{S}}_0\leqslant\aleph$.

Then taking $Z'_\gamma\in\mathrm{FVL}(\overline{\overline{S}}_0)(\gamma\in\Gamma)$ just as the forms $Z_\gamma\in\mathrm{FVL}(\mathfrak{M})(\gamma\in\Gamma)$ respectively, we can conclude at first that $Z'_\gamma(\gamma\in\Gamma)$ constitutes a positive orthogonal family in $\mathrm{FVL}(\overline{\overline{S}}_0)$ by virtue of Theorem 2 and Corollary 2 in § 1.

Next, introducing an inner product onto $A(\overline{\overline{S}}_0)$ by the uses of Hamel bases for both l^2 (or its suitable inner product subspace) and $A(\bar{\bar{S}}_0)$ we can make an $A(\bar{\bar{S}}_0)$ to be separable by the derived norm, and so we get the countability of mutually disjoint open sets (non-void) in $A(\bar{\bar{S}}_0)$. Let us now observe the positive orthogonal family $Z'_\gamma(\gamma\in\Gamma)$ from the viewpoint of total order for the just introduced inner product. Then, since Theorem 2 and Corollary 2 in §1 and the fact above proved hold good for any inner product, we see that the mutually orthogonal positive elements $Z'_\gamma(\gamma\in\Gamma)$ must be at most countable, a contradiction. This completes the proof.

Note 1. Here we shall show that there exist actually countably infinite number of mutually orthogonal positive elements in FVL($\aleph_0$).

Taking

$$A(\aleph_0)\ni x_n=(1,\ -n+1/2,0,0,\cdots), \qquad (n=1,2,\cdots);$$
$$A(\aleph_0)\ni y_n=(-1,n,0,0,\cdots), \qquad (n=1,2,\cdots);$$

we consider

$$Z_n=\{x_n\}^+\frown\{y_n\}^+\in \mathrm{FVL}(\aleph_0) \qquad (n=1,2,\cdots).$$

Then, by Corollary 2 in §1, we have $Z_n>O$ $(n=1,2,\cdots)$ if and only if there exist $(\alpha_n,\beta_n,\cdots)\in A(\aleph_0)$ respectively such that

$$\big(x_n,(\alpha_n,\beta_n,\cdots)\big)=\alpha_n-n\beta_n+1/2\beta_n>0$$

and

$$\big(y_n,(\alpha_n,\beta_n,\cdots)\big)=-\alpha_n+n\beta_n>0,$$

that is $n-1/2<\alpha_n/\beta_n<n$ $(n=1,2,\cdots)$. Accordingly we see also $Z_n\cap Z_m=O$ $(n\neq m)$ by the same grounds. Here in addition, as Theorem 2 in §1 says, *it holds that* x_n, y_n, *and* x_m $(n<m)$ *are linearly dependent with positive coefficients*:

$$\{2(m-n)-1\}x_n+2(m-n)\,y_n+x_m=0.$$

Of course, *the above is ananlogously true in* FVL (2).

Note 2. *In case* $\mathfrak{M}=n$ *(finite), any total order for* $A(\mathfrak{M})$ *can be given as a general lexicographical order which depends upon a regular matrix (exceping the uniqueness) of order* n.

(Received May. 19, 1970)

References

1) E. Stiemke: Über positive Lösungen homogener linearer Gleichungen. Math. Ann. 76 (1915), 340–342.
2) W. B. Carver: Systems of linear inequalities. Annals of Math. (2), 23 (1922), 212–220.
3) L. L. Dines: Definite linear dependence. Annals of Math. 27 (1925), 57–64.
4) L. L. Dines: Note on certain associated systems of linear equalities and inequalities.

Annals of Math. 28 (1926-27), 41–42.
5) A. W. Tucker: Dual system of homogeneous linear relations. Annals of Math. Study. 38 (1956), 3–18.
6) Ky Fan: On systems of linear inequalities. Annals of Math. Study. 38 (1956), 99–156.
7) E. C. Weinberg: Free lattice-ordered groups. Math. Ann. 151 (1963), 189–199.
8) D. M. Topping: Some homological pathology in vector lattices. Can. J. Math. 17 (1965), 411–428.
9) E. C. Weinberg: Free lattice-ordered Abelian groups II. Math. Ann. 159 (1965), 217–222.
10) I. Amemiya: Countable decomposability of vector lattices. J. Fac. Sci. Hokkaido Univ. Ser. I, Vol. XIX, Nos. 3, 4 (1966).