# Outside-standing Subsidiary Observations on Mathematical Logic

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**Abstract**

Axiomatics and logical inferences stand on the basic universe of objects and possibly have changes as the constructive assumptions on the universe change. Epistemological study may particularly play an important role in such cases.

## 0. General Introduction

It seems that any logical subject which is raised in philosophy may not always be directly transferred to mathematical logic. In effect, ethical or theological subjects very often have this character and so it may be difficult to put them forward in any mathematical logic unless we modify them on some conditions to restrict them exactly.

If logic is taken on some scientific subject, it shall have its available field to be found in some total set $U$ of scientific objects, say, a primitive universe of objects. So then our essential form of investigation shall naturally depend on the observational behavior toward the events given or defined in the universe. The events being defined in the universe are also naturally connected with a set theory in it. For purposes of mathematical development, predicates and relations may be interpreted as predicates of which loci are sets in $U$ and relations between sets respectively. The logic which conforms to such views is called an *analytic logic*.

If any subject is to be reconstructed to fit in with the analytic logic, we will then have an observational work of testing the possibility of such a reconstruction.

There are some statements which have hitherto been considered as of static state, but, when practicality is emphasized, are forced some chrono-logical reconstructions and are shifted to be of historical genre. The following well-known prima-facie paradoxical statement may also be reconstructed as a historical one:

*Epimenides the Cretan says "Nothing said by a Cretan is the case".* (0. 1)

The discussion on this satement will be shown in Sect. 2.

The assignation of truth on events taken in this paper is either "true" or "false", because the empiricist set theory adopted here as the ground for analytic

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logic is properly taken as of 2-valued system. In the empiricist set theory ordinals larger than the 3rd class are refrained from using unless with some special conditions. Besides, we have recently begun to refer to the following dogma.

**Pragmatist Dogma.** *A completly unfounded mere abstraction can give only a meaningless object.*

Under this dogma, for any bounded increase of sets in a euclidean space $(A_\iota)$ $(\iota\in I)$ $(\iota\leqslant\kappa.\Longleftrightarrow.\ A_\iota\subseteq A_\kappa)$ $(\forall\iota\in I)$ $(A_\iota\subset B)$ $(\widetilde{m}B<\infty)$ ($\widetilde{m}$ meaning the a priori measure), *if it is generally true that if*

$$(\forall_\iota\in I)\ (A_\iota \textit{ is } \widetilde{m}\textit{-measurable})$$

*and*

$$A=\bigcup A_\iota\,,$$

*then* $A$ *is* $\widetilde{m}$*-measurable and*

$$\widetilde{m}A=\sup\widetilde{m}A_\iota\,, \tag{0.2}$$

then the following important result is directly concluded[1]:

*There can exists no ordinal which may correspond to the continuum (in the empricist pragmatism).*

Traditionally, a euclidean space itself is the one accomplished by human considerations in line with the euclidean geometry and with the cartesian geometry, and moreover is thought to be connected with the general dynamics since more than twenty centuries ago. Therefore, if a set theory is posited to be applied in a euclidean space, it cannot only be composed by axioms simply arranged consistently, but each axiom of it must always be examined if it does not go counter to any traditional character expected in a euclidean space. Eventually, a theory of sets in a euclidean space cannot always, as hitherto taken, be equivalent to a one generated by a finite system of axioms, but it should be an observational course of study of the space whatever axioms are therefor chosen. Nuisances occurred in the classical set theory shall hence be considered as caused only by processes monopolized by the set thory, and therefore euclidean spaces themselves may have no ascription for them. The above-mentioned provisional proposition (0.2) in respect to the $\widetilde{m}$-measure may also issue from such observational discussions, the detail of which will be shown in Sect. 6.

## 1. Analytic Predicate

To a set $M$ in a universe $U$, the following predicate $\boldsymbol{p}$ may be defined to correspond:

$$x\in M.\Longleftrightarrow\cdot\boldsymbol{p}x\,;\ x\notin M.\Longleftrightarrow\cdot\sim\boldsymbol{p}x\,;$$
$$\boldsymbol{p}U\equiv\{x|x\in U.\ \&\ .\vdash\boldsymbol{p}x^{*)}\}=M\,.$$

*) $\vdash\boldsymbol{p}x$ renders "$x$ satisfies $\boldsymbol{p}$" or "$\boldsymbol{p}x$ is true".

Such a predicate $\boldsymbol{p}$ is an *analytic predicate* of the lst species (standing) on $U$.

In two universes $U_1$, $U_2$, two sets $M_1 \subseteq U_1$, $M_2 \subseteq U_2$ be respectively given, then if $\boldsymbol{f}$ is defined by

$$x \in M_1 \cdot \Longleftrightarrow \cdot \boldsymbol{f}x \in M_2, \tag{1.1}$$

$\boldsymbol{f}$ may be considered as a mapping from $M_1$ onto $M_2$, though it is, in our theory, called an *analytic predicate* of the 2nd species. In case of (1.1), we write

$$\boldsymbol{f}M_1 = M_2$$

and always assume that

$$\boldsymbol{f}\varnothing = \varnothing.$$

A predicate finitarily[2] composed by means of a finite number of analytic predicates is also called *analytic* on condition that it is meaningful. A definition of *meaningfulness* of an analytic predicate will be shown in Sect. 3. Incidentally, whether a given predicate is meaningless or not may not be decided without any observational examination.

If two predicates $\boldsymbol{p}$ and $\boldsymbol{q}$ are both possible (i. e., meaningful and their ranges are both non-void) and if

$$\boldsymbol{pq} = \boldsymbol{qp}, \tag{1.2}$$

then they are said to be (mutually) *homogenetic.* Homogenetic predicates may be considered to be of the same level, so that the relation (1.2) may be adopted as a definition of *equilevelness* of $\boldsymbol{p}$ and $\boldsymbol{q}$.

## 2. Analytic Modality

For a statement $\boldsymbol{p}$ describing a proposition, there may be referred to the following four modalities: (i) *It is possible that* $\boldsymbol{p}$; (ii) *It is impossible that* $\boldsymbol{p}$; (iii) *It is necessary that* $\boldsymbol{p}$; (iv) *It is not necessary that* $\boldsymbol{p}$. If $\vdash \boldsymbol{p}$ (i. e., that $\boldsymbol{p}$ is true) is proved under a certain *circumstance* (i. e., a set of conditions of the objects in the given universe), the circumstance is said to be *favorable* for $\boldsymbol{p}$. By

$$\vdash \boldsymbol{p}/\sigma$$

we mean that $\boldsymbol{p}$ is true under the circumstance $\sigma$. Then, if $\Omega_+$ is the collection of all favorable circumstances for $\boldsymbol{p}$, $\Omega_-$ the collection of all favorable circumstances for $\sim\boldsymbol{p}$, and if

$$\Omega = \Omega_+ \cup \Omega_-,$$

then we have

$$(\forall\sigma \in \Omega_+)(\vdash \boldsymbol{p}/\sigma);$$

$$(\forall\sigma \in \Omega_-)(\vdash \sim\boldsymbol{q}/\sigma);$$

hence

$$(\forall\sigma\in\Omega)(\vdash p/\sigma\cdot\vee\cdot\vdash\sim p/\sigma).$$

Elements of $\Omega$ are called *p-circumstances.*

The above-stated four modalities are found to be equivalent to the following four relations respectively:

$$\text{(i) } \Omega_+\neq\emptyset, \quad \text{(ii) } \Omega_+=\emptyset, \quad \text{(iii) } \Omega_-=\emptyset, \quad \text{(iv) } \Omega_-\neq\emptyset.$$

If $\Omega\neq\emptyset$, $\boldsymbol{p}$ is said to be *meaningful,* and if $\Omega=\emptyset$, *meaningless.* Such being the conditons, the four modalities may also be thought as *analytic relations.*

If a special condition $\gamma$ is to be emphasized in treating $p$, it may be done by only choosing circumstances which imply $\gamma$ from $\Omega$ or taking $p\wedge\gamma$ instead of $p$. In such cases we are to examine whether $p$ is under $\gamma$ meaningful or not. The pragmatist meaninglessness (in regard to the pragmatist dogma) should also be examined, and to rule out this kind of meaninglessness is always requisite to have a course of empiricist pragmatism.

### 3. Historical Observation

When any observation of facts has been needed, its procedure has been taken as non-logical and according to its bearings called *synthetic* or *empirical.* If a proposition has been considered neither necessarily true nor necessarily false, it has been said to be *factual*[3]. However, in our present theory, a factual event may be simply said to be a *possible event* if $\Omega_+\neq\emptyset$. A proof of possibility of an event (or a proposition) $e$ will be gained if an evidence or a (circumstance) $\sigma$ is really found such that $\vdash e/\sigma$, or if it is concluded that there should exist at least one such evidence $\sigma$. This process of proof may also be said to be an observation, and such an observation shall also be a logical observation.

For example of an event in contact with observation we may refer to a historical statement. Incidentally, since historical events essentially refer to chronological objects, they are very often transferred to stochastics. By the way, on examining the logical treatment of the prima-facie paradox (0.1), a historical inspection is reasonably found to be possible, so in the following we show a sketch of it.

Let $\boldsymbol{E}$ render "Epimenides", and $\boldsymbol{S}$ "nothing said by a Cretan is the case". Then, (0.1) may be resolved into the following two events:

$$\boldsymbol{E} \text{ is a Cretan;} \tag{3.1}$$

and

$$\boldsymbol{E} \text{ says } \boldsymbol{S}. \tag{3.2}$$

So we may reconstrue it as $(0.1)=(3.1)\wedge(3.2)$.

About the statement (0.1), some classical logicians asserted that since, by (3.1), (3.2) itself refers to the objects of $\boldsymbol{S}$, (0.1) is regarded as *self-referential* and this relation should be the cause of the paradoxy of (0.1). However, such a mere assertion cannot be said to have exhausted the observational materials related to (0.1). In effect, in inspecting ***S*** itself, we find that:

**S** is false if there is at least one true Cretan utterance[4]; (3. 3)

or

***S*** is up to now true if there, up to now, is no true Cretan utterance. (3. 4)

In case of (3.3) it must be that ***E*** says a falsehood and in case of (3.4) it must be pending whether ***E*** says a falsehood or (0.1) gives a paradoxical evidence, because, in future, (0.1) will be transferred to the case of (3.3) as soon as there will emerge a true Cretan utterance. Thus our inquiry is related to the historical observation. Incidentally, that ***S*** is only pendingly possible may be considered to force an observation referring to the aristotelian concept of "*future contingency*"[5].

Ultimately, the observational content of (0.1) may be decided either such that

***E*** said ***S***, but ***S*** is false, (3. 5)

or such that

(0.1) is a pending paradox unless any true Cretan utterance is found. (3. 6)

Hence, it may be said that (0.1) has a construction of *historical dilemma* in pending between (3.5) and (3.6).

## 4. On 3-valued System

That logic might admit a third intermediate truth value in addition to the values of truth and falsehood, might be taken as already implied in the aristotelian notion of "future contingency". On the other hand, for the introduction of such a value, it may give a clear mark to define it to indicate a truth-status that is "possible but not necessary". Noting the value of such a status as **I** (truth as **T** and falsehood as **F**), the truth-table of the 3-valued system of Lukasiewicz is found as follows[6]:

| $p$ | T | I | F |
|---|---|---|---|
| $\sim p$ | F | I | T. |

Prima facie it seems that we may take an event of pending state to be assigned the value **I**. But, in this case, if we do not cease to consider that a proceeding of observation may cause a shift of evaluation, the fixed meaning of the value **I** may possibly vanish away.

If we take **I** as indicating no pending state, we will then naturally have both cases (i) and (iv) cited in Sect. 2, i. e.,

$$\Omega_+ \neq \emptyset . \ \& \ . \Omega_- \neq \emptyset$$

on condition that **I** is the truth-value of $p$. Then, if we restrict circumstances within $\Omega_+$ we have $p$ as of **T** and $\sim p$ as of F, and if within $\Omega_-$ $p$ as of **F** and $\sim p$ as of **T**. Therefore in these relative cases, the calculus is sufficiently provided by the 2-valued systems. Thus, $p$ and $\sim p$ should always have their ranges as complemental, so it would ipso facto be unnatural to assign the same value **I** to both of $p$ and $\sim p$.

Such being the conditions, it will be rather rigorous if we assert only 2-valued system can generally refer to analytic logic. However, if logic is used on restriction that only some sort of physical phenomena is taken to make the primitive universe and **I** refers to a certain neutral state of phenomenon, where **T** refers to a certain positive state and **F** to a certain negative state, and no other value than **I**, **T**, **F** is taken possible, then the 3-valued system generated by **I**, **T** and **F** may be considered to be possible as a special system of inferences thereupon produced.

## 5. Euclidean Geometry

In history, the clarification of the relation between the euclidean geometry and the axiom of parallels made two geometries admitted as possible, though this problem might, in the early days (e. g., the days of G. Saccheri), possibly be regarded as a pendig one, say a historical dilemma. On being broken the state of dilemma, there emerged the above bifurcation of geometry—that would be said to be a result of the outside-standing observation afterward made. However, it is reflected, in our view, that there is yet left another way of observation on epistemological standpoint.

After extending the conception of a space which firstly was comprehended in an a priori form of intuition to what has been idealized as a space which is everywhere homogeneous and spreads unboundedly, there should exist no contradiction between the space itself and the human sight which may be regarded as the original one of the idealization. In this meaning we call *a priori space* the above-stated idealized one. Not in mathematics but instead in epistemology, the conception of the a priori space shall precede the system of axioms. Thus, the euclidean system of axioms may eventually be said to be a sort of protocol of human results of the epistemological work tried to embody the spatial construction of the a priori space.

The eulidean system has been thought to be ipso facto correct, and almost all of the scientists have admtted both of the euclidean and non-euclideain geometries*).

*) In our view, if the problem of consistency of the euclidean system is asked, it shall be transferred to the discussion of theoretical noises in connection with historical improvements.

However, there is no definition of a "plane" in the euclidean plane geometry. It seems to refer to the axiom of parallels again. In this respect, we will here posit one epistemological (or physical) course of conjecturing to reach an assertion of the axiom.

For a given triangle $\triangle$ABC, let it be that AB $\perp$ BC and BC $=\infty$. Then the following epistemo-physical verdict will be found to be rightly implemental.

**Postulate B** (*Bird's Eye Conformity*). *To compare* $\angle$A *and* $\angle$B *of the above-stated triangle is equivalent to compare them on alternative conditions* BC$=1$ *and* AB$=0$.

An illative ground of this assertion may straightforwardly be obtained by the relation

$$1/\infty = 0 .$$

If what is called a plane should be everywhere homogeneous and unboundedly spread with no bending, the bird's eye sight which today is possible for everyone to experience if he only emberk in an observation balloon will directly convince him of the above conformity. Though a finite system of axioms gives us a space thereby generatable, it may then leave no room for incorporation with additional convictions approached through human direct intuition. Incidentally, the following remark due to J. Wallis may be considered to bear the same assertion as Postulate B: *if there is a geometry lacking the axiom of prallels, two configurations of different sizes must always be non-homologous in it.*

## 6. Subsidiary Observation on $\tilde{m}$-Measure

In the following we will proceed our discussion under the presupposition that our a priori space completely conform to the 3-dimensional euclidean geometry and if any theory of sets or measures on this space comes accross a contradiction its cause must wholly be implied within the theory itself. When I. Kant presented the a priori form of space, it had not ipso facto to consist of points but to be only an extensive spread of the space, and after it was idealized and provided with homogeneity to establish the a priori space the first nextly requisite concept had to be the unit length and then the unit cube. That may be to say that in epistemology the concept of continuum itself precedes the other elements incorporated into the space. We may take the relations today considered to hold between points and the space (or, points and the continuum) as the results ultimately obtained through hands of Zenon, F. B. Cavalieri, G. Cantor, J. W. R. Dedekind etc. That the concept of $\tilde{m}$-measure may be thought to be essentially implied in the naive conception of the a priori space may now similarly induced as in the case of the notion of continuum.

In specification of $\tilde{m}$-measure, it must deservedly be taken into account that the $\tilde{m}$-value assigned to a geometric figure should coincide with the notion of size which is used by the spatial occupation of the figure. Promised such an epistemological stipulation, the mathematical notion of $\tilde{\mathfrak{m}}$-measure shall be called *a priori measure.* Thus the theories of sets and $\tilde{\mathfrak{m}}$-measure are considered to be through some out-standing observations scrutinized. Therefore, if neglecting such mutual stipulations they were simply formalized and transferred to symbolic logic, there might be left important omissions.

For all the mentions in the above, if we are inquired "What shall then be actually confirmed?", we may not easily answer, because no structural confirmation may be gained without oberving types of configuration, whereas such types must possibly exist infinitely variously. However, the following announcement shall yet deserve to be taken as a fundamental confirmation in the present course.

**Postulate $\mathbf{M}_1$** (*Size Conformity*). *$\tilde{\mathfrak{m}}$-measure of a set must be proportional to the size of the set so that, for any bounded increase**) $(A_\iota)$ $(\iota \in I)$, *it must be destined***) *that for any positive real number* $\varepsilon$

$$\tilde{\mathfrak{m}}(A - A_\iota) < \varepsilon$$

*with*

$$A = \bigcup A_\iota$$

*if $\iota$ is sufficiently large.*

We yet put forward two more subsidiary assertions.

**Postulate $\mathbf{M}_2$** (*Null Measure Assertion*). *Assume that*

$$(\forall N \subset M)(N \textit{ is } \tilde{\mathfrak{m}}\textit{-measurable} \cdot \Rightarrow \cdot \tilde{\mathfrak{m}}N = 0),$$

*then it must be that $M$ itself is $\tilde{\mathfrak{m}}$-measurable and*

$$\tilde{\mathfrak{m}}M = 0 .$$

**Postulate $\mathbf{M}_3$** (*Measure Pragmatism*). *If it is not destined that $\tilde{\mathfrak{m}}M \leqslant \alpha < \infty$, then there must be a set $N$ such that $N \subset M$, $N$ is $\tilde{\mathfrak{m}}$-measurable and $\tilde{\mathfrak{m}}N > \alpha$.*

As the reasonable ground for illating Postulates $\mathbf{M}_2$ and $\mathbf{M}_3$, the pragmatist dogma may be very powerful***). Thus we find it well-provided to conclude pragmatistly that *if in the bounded increase $(A_\iota)$ all $A_\iota$ are $\tilde{\mathfrak{m}}$-measurable then $A = \bigcup A_\iota$ is also $\tilde{\mathfrak{m}}$-measurable and*

$$\tilde{\mathfrak{m}}A = \sup \tilde{\mathfrak{m}}A_\iota .$$

Then, as auggested in Sect. 0, we may conclude that *there can exist no ordinal which may correspond to the continuum.*

## 7. Epistemo-physical Characterization

Since C. Huyghens, physics has ceased to take any part of the cosmic space to be vacuum. Hence, if mathematics intends to hold on in concert with physics,

*) i. e., $\iota \leqslant \kappa \cdot \Leftrightarrow \cdot A_\iota \subseteq A_\kappa$, and $(\forall \iota \in I)\ (A_\iota \subset B)\ (\tilde{\mathfrak{m}}B < \infty)$.

**) "It is destined that $\tilde{\mathfrak{m}}C < \alpha$" renders "If $C$ is $\tilde{\mathfrak{m}}$-measurable, then $\tilde{\mathfrak{m}}C < \alpha$".

***) However, we shall not abuse this dogma, for instance, automatically to deny the notion of "any real nuber".

it may not deny the hypothetical structure that the a priori space is everywhere homogeneously filled with a quantitative matter. The huyghensian structure of the space may not originally be the one that regards the space to consist of points. However, a set theory in a euclidean space cannot be without the notion of point as element. So, we ultimately may not have any other way than to assume each point has its point-weight and $\tilde{m}A$ is understood to be the total sum of point-weights contained in a set $A$. Such an assumption will give an epistemo-physical characterization to the a priori space. Let this assumption be called the postulate of *physical conformity*.

However, the point-weight must, as it is, be measured as $=0$. So, in order to keep harmony, it should be posited as an infinitesimal quantity. Then, $\tilde{m}A$ will turn to be meaningless if its integral construction with respect to the point-weights is unsolved. Such being the conditions, to proceed on the ground of the physical conformity is found to be more annoying than to proceed on the ground of the size conformity. So then to mathematics it will only be a burden to concert with the huyghensian physics. However, we may here find an alleviate course in avoiding the direct work on the assertion that all results derived from the gpound of the size conformity do not ipso facto contradict the physical conformity.

By the way, there is a case where can be induced a decision which is hesitated to make in pure mathematical bearings, by taking the standpoint of physical conformity. In effect, if a bounded set $A$ (i.e., $A \subset M$. & $.\tilde{m}M < \infty$) is not $\tilde{m}$-measurable, then there may be no other way than to consider it as having an oscillating weight. This should mean that $A$ is in an indeterminate state—hence $A$ should be taken as an indeterminate set.

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