On a Three Dimensional Stress Analysis of an Annular Cylindrical Body Subjected by Non-axisymmetrical Loading

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Abstract

The three dimensional stress problem written in the cylindrical co-ordinate system is solved using the finite Fourier-Hankel transform method. The boundary conditions are so given as to produce anti-symmetrical stress distribution with respect to a diameter. The detail discussion is focused on the case when a thick hollow cylinder is subjected by a partially distributed load acting in the radial direction with the numerical results.

1. Introduction

Though the recent progress of the electric digital computer makes it possible for us to handle the three dimensional stress problems by means of "Finite Element Method" or "Finite Difference Method", etc., many problems still remain untouched in the field of the three dimensional elasticity.

It is because that the convergency of F. E. M. seems to be uncertain, and the capacity of the computer still is not large enough for the thorough treatment of the three dimensional stress state. The analytical solutions so far presented, seems to be confined to the case of infinite, semi-infinite elastic body and the thick plate.

In this paper, the nonaxial stress problem of a hollow cylinder is dealt with, by means of the Finite Fourier-Hankel transform. Specifically, the paper considers the case when the thich hollow cylinder is subjected by bending.

2. The Fundamental Differential Eguation

Let σ_r, σ_θ and σ_z be the normal stresses in the r, θ and z directions, and $\tau_{r\theta}$, $\tau_{\theta z}$ and τ_{zr} be the corresponding shearing stresses respectivery, the equilibrium of forces are expressed by

$$\begin{bmatrix} \dfrac{\partial}{\partial r}+\dfrac{1}{r} & -\dfrac{1}{r} & 0 \\ 0 & \dfrac{\partial}{r\partial\theta} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} + \begin{bmatrix} \dfrac{\partial}{r\partial\theta} & 0 & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial z}+\dfrac{2}{r} & \dfrac{\partial}{\partial z} & 0 \\ 0 & \dfrac{\partial}{r\partial\theta} & \dfrac{\partial}{\partial r}+\dfrac{1}{r} \end{bmatrix} \begin{bmatrix} \tau_{r\theta} \\ \tau_{\theta z} \\ \tau_{zr} \end{bmatrix} = 0 \quad (1)$$

Denoting the components of displacement in the r, θ and z directions by u, v and w, the Hook's law is written as follows

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda \\ \lambda & 2\mu+\lambda & \lambda \\ \lambda & \lambda & 2\mu+\lambda \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r}+\dfrac{\partial v}{r\partial\theta} \\ \dfrac{\partial w}{\partial z} \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} \tau_{r\theta} \\ \tau_{\theta z} \\ \tau_{zr} \end{bmatrix} = \mu \begin{bmatrix} \dfrac{\partial}{r\partial\theta} & \dfrac{\partial}{\partial r}-\dfrac{1}{r} & 0 \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{r\partial\theta} \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{3}$$

where μ, λ are the Lamé's elastic constants.

3. Finite Fourier-Hankel Transformation and Their Inversion Formulas

a) Fourier Transforms

If $f(x)$ satisfies Dirichlet's conditions in the interval $(0, a)$ and if its finite Fourier transforms in that range are defined to be

$$\boldsymbol{S}_n\Big[f(x)\Big] = \int_0^a f(x)\sin\frac{n\pi}{a}x dx,$$

$$\boldsymbol{C}_n\Big[f(x)\Big] = \int_0^a f(x)\cos\frac{n\pi}{a}x dx,$$

where $n=1, 2, \cdots,$

then at any point of $(0, a)$ at which the function $f(x)$ is continuous

$$f(x) = \frac{2}{a}\sum_{n=1}^{\infty}\boldsymbol{S}_n\Big[f(x)\Big]\sin\frac{n\pi}{a}x,$$

$$f(x) = \frac{2}{a}\left\{\sum_{n=1}^{\infty}\boldsymbol{C}_n\Big[f(x)\Big]\cos\frac{n\pi}{a}x+\frac{1}{2}\boldsymbol{C}_0\Big[f(x)\Big]\right\}.$$

b) Hankel Transforms

If $f(x)$ satisfies Dirichlet's conditions in the interval (a, b) and if its finite Hankel transforms in that range are defined to be

$$\boldsymbol{H}_\nu\Big[f(x)\Big] = \int_a^b f(x)\, xH_\nu(\xi_i x)\, dx,$$

$$\boldsymbol{H}_{\nu\pm1}\Big[f(x)\Big] = \int_a^b f(x)\, xH_{\nu\pm1}(\xi_i x)\, dx,$$

then at any point of (a, b) at which the function $f(x)$ is continuous

$$f(x)=\frac{2\nu a^{2\nu}b^{2\nu}}{b^{2\nu}-a^{2\nu}}\cdot\frac{1}{x^{\nu+1}}\int_a^b f(x)\,x^{-\nu}dx+\frac{2}{b^2}\sum_{i=1}^{\infty}\boldsymbol{H}_{\nu+1}\Big[f(x)\Big]\frac{H_{\nu+1}(\xi_i x)}{\Theta_{i\nu}^2},$$

$$f(x)=\frac{2}{b^2}\sum_{i=1}^{\infty}\boldsymbol{H}_{\nu}\Big[f(x)\Big]\frac{H_{\nu}(\xi_i x)}{\Theta_{\nu i}^2},$$

$$f(x)=\frac{2\nu}{b^{2\nu}-a^{2\nu}}\cdot x^{\nu-1}\int_a^b f(x)\,x^{\nu}dx+\frac{2}{b^2}\sum_{i=1}^{\infty}\boldsymbol{H}_{\nu-1}\Big[f(x)\Big]\frac{H_{\nu-1}(\xi_i x)}{\Theta_{\nu i}^2},$$

where

$$H_j(\xi_i x)=J_j(\xi_i x)\,Y_\nu(\xi_i a)-J_\nu(\xi_i a)\,Y_j(\xi_i x),$$

$$j=\nu-1,\ \nu,\ \nu+1,$$

$$\Theta_{\nu i}^2=\Big\{H_{\nu+1}(\xi_i b)\Big\}^2-\left(\frac{a}{b}\right)^2\Big\{H_{\nu+1}(\xi_i a)\Big\}^2$$

and ξ_i is a root of transcendental equation

$$H_\nu(\xi_i b)=0.$$

4. Solution by Means of Finite Fourier-Hankel Transform

A method of solution was presented before, wew ill focus the discussion on the case when the hollow cylinder is of antiaxial stresss tate. The hollow cylinder of which inner radius, outer radius and depth are denoted by $b(=a_1)$, $a(=a_2)$ and c, respectively. Multiplying Eq. (1) by

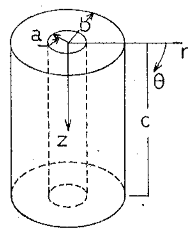

Fig. 1.

$$L=\Big\{\cos\nu\theta\cdot X(r,z)\ \ \sin\nu\theta\cdot X(r,z)\ \ \cos\nu\theta\cdot X(r,z)\Big\}$$

where

$$\nu=\frac{m\pi}{\varphi}=\frac{m}{2}\ (\because\ \ \varphi=2\pi)\qquad m=2,4,\cdots,$$

and integrating by parts, with the aid of Eqs. (2) and (3), we have the Fourier transformations with respect to θ as follows ;

$$\int_{A_\theta}\boldsymbol{K}_{1\nu}\boldsymbol{u}_\nu dA_\theta=\int_0^c\Big[\boldsymbol{K}_{2\nu}\boldsymbol{u}_\nu+X\cdot\boldsymbol{\sigma}_{1\nu}\Big]_a^b dz+\int_a^b\Big[\boldsymbol{K}_{3\nu}\boldsymbol{C}_\nu[w]+X\cdot\boldsymbol{\sigma}_{2\nu}\Big]_0^c dr \tag{4}$$

where

$$\boldsymbol{u}_\nu=\begin{bmatrix}\boldsymbol{C}_\nu[u]\\ \boldsymbol{S}_\nu[v]\\ \boldsymbol{C}_\nu[w]\end{bmatrix},\quad \boldsymbol{\sigma}_{1\nu}=\begin{bmatrix}\boldsymbol{C}_\nu[\sigma_r]\\ \boldsymbol{S}_\nu[\tau_{r\theta}]\\ \boldsymbol{C}_\nu[\tau_{zr}]\end{bmatrix},\quad \boldsymbol{\sigma}_{2\nu}=\begin{bmatrix}\boldsymbol{C}_\nu[\tau_{zr}]\\ \boldsymbol{S}_\nu[\tau_{\theta z}]\\ \boldsymbol{C}_\nu[\sigma_z]\end{bmatrix},$$

$$\boldsymbol{K}_{1\nu}=\begin{bmatrix} (2\mu+\lambda)r\dfrac{\partial}{\partial r}\left(\dfrac{\partial X}{r\partial r}\right)-\mu\left(\nu^2\dfrac{X}{r^2}+\dfrac{\partial X}{\partial z}\dfrac{\partial}{\partial z}\right) & (\mu+\lambda)\nu\dfrac{\partial X}{r\partial r}+2\nu\mu\dfrac{X}{r^2} & (\mu+\lambda)\dfrac{\partial^2 X}{\partial r\partial z} \\ (\mu+\lambda)\nu\dfrac{\partial X}{r\partial r}-2\nu(2\mu+\lambda)\dfrac{X}{r^2} & \mu\left\{r\dfrac{\partial}{\partial r}\left(\dfrac{\partial X}{r\partial r}\right)-\dfrac{\partial X}{\partial z}\dfrac{\partial}{\partial z}\right\}-(2\mu+\lambda)\nu^2\dfrac{X}{r^2} & \nu(\mu+\lambda)\dfrac{r\partial z}{\partial X} \\ \lambda r\dfrac{\partial}{\partial r}\left(\dfrac{\partial X}{r\partial z}\right)-\mu r\dfrac{\partial}{\partial r}\left(\dfrac{X}{r}\right)\dfrac{\partial}{\partial z} & -\nu\left(\lambda\dfrac{\partial X}{r\partial z}+\mu\dfrac{X}{r}\dfrac{\partial}{\partial z}\right) & \mu\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial}{\partial r}\left(\dfrac{X}{r}\right)\right\}-\mu\nu^2\dfrac{X}{r^2}+(2\mu+\lambda)\dfrac{\partial^2 X}{\partial z^2} \end{bmatrix},$$

$$\boldsymbol{K}_{2\nu}=\begin{bmatrix} -(2\mu+\lambda)\dfrac{\partial X}{\partial r}+2\mu\dfrac{X}{r} & \nu\mu\dfrac{X}{r} & -\mu\dfrac{\partial X}{\partial z} \\ \nu\lambda\dfrac{X}{r} & \mu\dfrac{\partial X}{\partial r}-2\mu\dfrac{X}{r} & 0 \\ -\lambda\dfrac{\partial X}{\partial z} & 0 & \mu r\dfrac{\partial}{\partial r}\left(\dfrac{X}{r}\right) \end{bmatrix},$$

$$\boldsymbol{K}_{3\nu}=\begin{bmatrix} \lambda\dfrac{\partial X}{\partial r} \\ -\nu\lambda\dfrac{X}{r} \\ (2\mu+\lambda)\dfrac{\partial X}{\partial z} \end{bmatrix},$$

and

$$\int_{A_\theta} f\cdot dA_\theta=\int_a^b\int_0^c f dr dz$$

Introducing $A_{\nu rz}$, $B_{\nu rz}$ as

$$\begin{bmatrix} A_{\nu rz} \\ B_{\nu rz} \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \boldsymbol{C}_\nu[u] \\ \boldsymbol{S}_\nu[v] \end{bmatrix}$$

and substituting then into Eq. (4), we find that

$$\int_{A_\theta}\boldsymbol{K}_{1\nu}'\boldsymbol{u}_\nu' dA_\theta=\int_0^c\Big[\boldsymbol{K}_{2\nu}'\boldsymbol{u}_\nu'+X\cdot\boldsymbol{\sigma}_{1\nu}'\Big]_a^b dz+\int_a^b\Big[\boldsymbol{K}_{3\nu}'\boldsymbol{C}_\nu[w]+X\cdot\boldsymbol{\sigma}_{2\nu}'\Big]_0^c dr \quad (5)$$

The 1st and 2nd rows of $\boldsymbol{K}_\nu'$, $\boldsymbol{u}_\nu'$ and $\boldsymbol{\sigma}_\nu'$ are replaced by the addition and subtraction with the 1st and 2nd rows of $\boldsymbol{K}_\nu$, $\boldsymbol{u}_\nu$ and $\boldsymbol{\sigma}_\nu$, respectively.

Then substituting $\sin Nz\cdot H_{\nu+1}(\xi_i r)$ for X in the 1st row of Eq. (5), $\sin Nz\cdot H_{\nu-1}(\xi_i r)$ for X in the 2nd row, and $\cos Nz\cdot H_\nu(\xi_i r)$ for X in the 3rd row, we finally have the equations for the Fourier-Hankel transforms of $A_{\nu rz}$, $B_{\nu rz}$ and w, as follows ;

$$\begin{bmatrix} \{(3\mu+\lambda)\xi_i^2+2\mu N^2\} & -(\mu+\lambda)\xi_i^2 & -(\mu+\lambda)\xi_i N \\ -(\mu+\lambda)\xi_i^2 & \{(3\mu+\lambda)\xi_i^2+2\mu N^2\} & (\mu+\lambda)\xi_i N \\ -(\mu+\lambda)\xi_i N & (\mu+\lambda)\xi_i N & \{\mu\xi_i^2+(2\mu+\lambda)N^2\} \end{bmatrix} \begin{bmatrix} \boldsymbol{H}_{\nu+1}\boldsymbol{S}_n[A_{\nu rz}] \\ \boldsymbol{H}_{\nu-1}\boldsymbol{S}_n[B_{\nu rz}] \\ \boldsymbol{H}_\nu \boldsymbol{C}_\nu \boldsymbol{C}_n[w] \end{bmatrix}$$

$$= \left[H_{\nu+1}(\xi_i r)\cdot \begin{bmatrix} 4\mu(\nu+1) & 0 & -\mu N r & r & r \\ 0 & -4\mu(\nu-1) & -\mu N r & -r & r \\ 0 & 0 & \mu r & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{S}_n[A_{\nu rz}] \\ \boldsymbol{S}_n[B_{\nu rz}] \\ \boldsymbol{C}_\nu \boldsymbol{C}_n[w] \\ \boldsymbol{C}_\nu \boldsymbol{S}_n[\sigma_r] \\ \boldsymbol{S}_\nu \boldsymbol{S}_n[\tau_{r\theta}] \end{bmatrix} \right]_{r=a}^{r=b}$$

$$+ \left[\cos Nz \cdot \begin{bmatrix} -2\mu N & 0 & 0 \\ 0 & -2\mu N & 0 \\ \mu\xi_i & -\mu\xi_i & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{H}_{\nu+1}[A_{\nu rz}] \\ \boldsymbol{H}_{\nu-1}[B_{\nu rz}] \\ \boldsymbol{H}_\nu \boldsymbol{C}_\nu[\sigma_z] \end{bmatrix} \right]_{z=0}^{z=c} \tag{6}$$

where

$$N=\frac{n\pi}{c}, \quad n=1, 2, \cdots$$

Solving Eq. (6) simultaneously, the inversion formulas lead to the compornents of displacement in the following forms:

$$\boldsymbol{u} = \frac{1}{\pi}\sum_{k=1}^{2}\sum_{\nu=0}^{\infty} \boldsymbol{T}_\theta \Big[\frac{1}{c}\boldsymbol{J}_{\nu 0}\cdot \boldsymbol{D}_{\nu 0k} + \frac{2}{c}\sum_{n=1}^{\infty} \boldsymbol{J}^z_{\nu n}\cdot \boldsymbol{D}_{\nu nk}\cdot \boldsymbol{T}_z$$
$$+\frac{2}{b^2}\sum_{i=1}^{\infty}\{\boldsymbol{J}^r_{i\nu 1}\cdot \boldsymbol{E}^1_{i\nu k}\cdot \boldsymbol{T}_{r1} + \boldsymbol{J}^r_{i\nu 2}\cdot \boldsymbol{E}^2_{i\nu k}\cdot \boldsymbol{T}_{r2}\}\Big] \tag{7}$$

where

$$\boldsymbol{T}_\theta = \begin{bmatrix} \cos\nu\theta \\ \sin\nu\theta \\ \cos\nu\theta \end{bmatrix}, \quad \boldsymbol{T}_z = \begin{bmatrix} \sin Nz \\ \sin Nz \\ \cos Nz \end{bmatrix}, \quad \boldsymbol{T}_{r1} = \frac{1}{\Theta^2_{\nu i}}\cdot \begin{bmatrix} H_{\nu+1}(\xi_i r)-H_{\nu-1}(\xi_i r) \\ H_{\nu+1}(\xi_i r)+H_{\nu-1}(\xi_i r) \\ H_\nu(\xi_i r) \end{bmatrix},$$

$$\boldsymbol{T}_{r2} = \frac{1}{\Theta^2_{i\nu}} \begin{bmatrix} H_{\nu+1}(\xi_i r) \\ H_{r-1}(\xi_i r) \\ H_\nu(\xi_i r) \end{bmatrix}, \quad \boldsymbol{D}_{\nu 0k} = \begin{bmatrix} E^a_{\nu 0k} \\ E^b_{\nu 0k} \\ D_{\nu 0k} \end{bmatrix}, \quad \boldsymbol{D}_{\nu nk} = \begin{bmatrix} \alpha_{\nu nk} \\ \beta_{\nu nk} \\ A_{\nu nk} \\ B_{\nu nk} \\ D_{\nu nk} \end{bmatrix}$$

$$\boldsymbol{E}^2_{i\nu k} = \begin{bmatrix} E^a_{\nu ik} \\ E^b_{\nu ik} \\ \gamma_{\nu ik} \end{bmatrix}, \quad \boldsymbol{J}_{\nu 0} = \begin{bmatrix} \dfrac{2\nu a^{2\nu} b^{2\nu}}{b^{2\nu}-a^{2\nu}} r^{-(\nu+1)} f^{(k)}(z) & \dfrac{2\nu}{b^{2\nu}-a^{2\nu}} r^{\nu-1} f^{(k)}(z) & 0 \\ \dfrac{2\nu a^{2\nu} b^{2\nu}}{b^{2\nu}-a^{2\nu}} r^{-(\nu+1)} f^{(k)}(z) & -\dfrac{2\nu}{b^{2\nu}-a^{2\nu}} r^{\nu-1} f^{(k)}(z) & 0 \\ 0 & 0 & g^{(k)}_\nu(r) \end{bmatrix}$$

$$\boldsymbol{J}^z_{\nu n}=\begin{bmatrix} \frac{1}{\mu N}\left\{PG^{(k)}_\nu(Nr)-\frac{\mu+\lambda}{2(2\mu+\lambda)}PF^{(k)}_\nu(Nr)\right\} & \frac{1}{\mu N}MG^{(k)}_\nu(Nr) \\ \frac{1}{\mu N}\left\{MG^{(k)}_\nu(Nr)-\frac{\mu+\lambda}{2(2\mu+\lambda)}MF^{(k)}_\nu(Nr)\right\} & \frac{1}{\mu N}PG^{(k)}_\nu(Nr) \\ -\frac{\mu+\lambda}{2\mu(2\mu+\lambda)}\frac{1}{N}F^{(k)}_\nu(Nr) & 0 \end{bmatrix}$$

$$\begin{matrix} \frac{\nu+1}{N}\left\{2\chi^{(k)}_{\nu p}(Nr)-\frac{\mu+\lambda}{2\mu+\lambda}PF^{(k)}_\nu(Nr)\right\} & \frac{\nu-1}{N}\left\{-2\chi^{(k)}_{\nu s}(Nr)+\frac{\mu+\lambda}{2\mu+\lambda}PF^{(k)}_\nu(Nr)\right\} \\ \frac{\nu+1}{N}\left\{2\chi^{(k)}_{\nu p}(Nr)-\frac{\mu+\lambda}{2\mu+\lambda}MF^{(k)}_\nu Nr)\right\} & \frac{\nu-1}{N}\left\{2\chi^{(k)}_{\nu s}(Nr)+\frac{\mu+\lambda}{2\mu+\lambda}MF^{(k)}_\nu(Nr)\right\} \\ -\frac{\nu+1}{N}\cdot\frac{\mu+\lambda}{2\mu+\lambda}F^{(k)}_\nu(Nr) & \frac{\nu-1}{N}\,\frac{\mu+\lambda}{2\mu+\lambda}F^{(k)}_\nu(Nr) \end{matrix}$$

$$\left.\begin{matrix} -\left\{PG^{(k)}_\nu(Nr)-\frac{\mu+\lambda}{2\mu+\lambda}PF^{(k)}_\nu(Nr)\right\} \\ -\left\{MG^{(k)}_\nu(Nr)-\frac{\mu+\lambda}{2\mu+\lambda}MF^{(k)}_\nu(Nr)\right\} \\ G^{(k)}_\nu(Nr)+\frac{\mu+\lambda}{2\mu+\lambda}F^{(k)}_\nu(Nr) \end{matrix}\right], \quad \boldsymbol{J}^r_{i\nu 2}=Q^{(k)}(\xi_i z)\cdot\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{J}^r_{i\nu 1}=\begin{bmatrix} -\frac{\mu+\lambda}{2(2\mu+\lambda)}P^{(k)}(\xi_i z) & -\frac{\mu+\lambda}{4\mu(2\mu+\lambda)}\frac{1}{\xi_i}P^{(k)}(\xi_i z) \\ -\frac{\mu+\lambda}{2(2\mu+\lambda)}P^{(k)}(\xi_i z) & -\frac{\mu+\lambda}{4\mu(2\mu+\lambda)}\frac{1}{\xi_i}P^{(k)}(\xi_i z) \\ -\left\{\phi^{(k)}(\xi_i z)-\frac{\mu+\lambda}{2\mu+\lambda}\phi^{(k)}(\xi_i z)\right\} & -\frac{1}{\mu\xi_i}\left\{\phi^{(k)}(\xi_i z)-\frac{\mu+\lambda}{2(2\mu+\lambda)}\phi^{(k)}(\xi_i z)\right\} \end{bmatrix},$$

$$\boldsymbol{E}^1_{i\nu k}=\begin{bmatrix} E^a_{\nu ik}-E^b_{\nu ik} \\ \gamma_{\nu ik} \end{bmatrix}, \quad \alpha_{rnk}=\Big[\boldsymbol{C}_\nu\boldsymbol{S}_n[\sigma_r]\Big]_{r=a_k}, \quad \beta_{\nu nk}=\Big[\boldsymbol{S}_\nu\boldsymbol{S}_n[\tau_{r\theta}]\Big]_{r=a_k}$$

$$A_{\nu nk}=\Big[\boldsymbol{S}_n[A_{\nu rz}]\frac{1}{r}\Big]_{r=a_k}, \quad B_{\nu nk}=\Big[\boldsymbol{S}_n[B_{\nu rz}]\frac{1}{r}\Big]_{r=a^\gamma}, \quad D_{\nu nk}=\Big[\boldsymbol{C}_\nu\boldsymbol{C}_n[w]\Big]_{r=a_k},$$

$$E^a_{\nu i\frac{1}{2}}=\boldsymbol{H}_{\nu+1}\Big[A_{\nu rz})_{z=0}\pm A_{\nu rz})_{z=c}\Big], \quad \gamma_{\nu i\frac{1}{2}}=\boldsymbol{H}_\nu\boldsymbol{C}_\nu\Big[\sigma_z)_{z=0}\pm\sigma_z)_{z=c}\Big], \quad \text{etc,}$$

and

$$f^{(1)}(z)=1\,, \quad f^{(2)}(z)=1-\frac{2z}{c}, \quad g^{(k)}_\nu(r)=\frac{(ab)^\nu}{b^{2\nu}-a^{2\nu}}\left\{\left(\frac{r}{a_{k-1}}\right)^\nu-\left(\frac{a_{k-1}}{r}\right)^\nu\right\},$$

$$G^{(k)}_\nu(Nr)=\frac{2}{b^2}\sum_i\frac{\xi_i}{N^2+\xi_i^2}\,\frac{H_\nu(\xi_i r)}{\Theta^2_{i\nu}}(-1)^{k-1}a_kH_{\nu+1}(\xi_i a_k)=\frac{R^{(k)}_{\nu,\nu}(Nr)}{R^{(k)}_{\nu,\nu}(Na_k)}$$

$$\chi^{(k)}_{\nu p}(Nr)=\frac{2\nu a_k^{2\nu}}{b^{2\nu}-a^{2\nu}}(-1)^{k-1}\frac{1}{Na_k}\left(\frac{a_k}{r}\right)^{\nu+1}$$
$$+\frac{2}{b^2}\sum_i\frac{N}{N^2+\xi_i^2}\,\frac{H_{\nu+1}(\xi_i r)}{\Theta^2_{i\nu}}(-1)^{k-1}a_kH_{\nu+1}(\xi_i a_k)=\frac{R^{(k)}_{\nu+1,\nu}(Nr)}{R^{(k)}_{r,\nu}(Na_k)},$$

$$\chi_{\nu s}^{(k)}(Nr)=\frac{2\nu a_k^{2\nu}}{b^{2\nu}-a^{2\nu}}(-1)^{k-1}\frac{1}{Na_k}\left(\frac{r}{a_k}\right)^{\nu-1}$$
$$+\frac{2}{b^2}\sum_i\frac{N}{N^2+\xi_i^2}\frac{H_{\nu-1}(\xi_i r)}{\Theta_{i\nu}^2}(-1)^{k-1}a_k H_{\nu+1}(\xi_i a_k)=\frac{R_{\nu-1,\nu}^{(k)}(Nr)}{R_{\nu,\nu}^{(k)}(Na_k)},$$

$$PG_\nu^{(k)}(Nr)=\frac{1}{2}\{\chi_{\nu p}+\chi_{\nu s}\},\quad MG_\nu^{(k)}(Nr)=\frac{1}{2}\{\chi_{\nu p}-\chi_{\nu s}\},$$

$$F_\nu^{(k)}(Nr)=\frac{4}{b_2^2}\sum_i\frac{N^2\xi_i}{(N^2+\xi_i^2)^2}\frac{H_\nu(\xi_i r)}{\Theta_{i\nu}^2}(-1)^k a_k H_{\nu+1}(\xi_i a_k)$$
$$=\frac{N}{\{R_{\nu,\nu}^{(k)}(Na_k)\}^2}\Big[R_{\nu,\nu}^{(k)}(Na_k)\Big\{rR_{\nu-1,\nu}^{(k)}(Nr)-a_{k-1}R_{\nu,\nu+1}^{(k)}(Nr)\Big\}$$
$$-R_{\nu,\nu}^{(k)}(Nr)\Big\{a_k R_{\nu-1,\nu-1}^{(k)}(Na_k)-a_{k-1}R_{\nu,\nu+1}^{(k)}(Na_k)\Big\}\Big],$$

$$\omega_{\nu p}^{(k)}(Nr)=\frac{4}{b^2}\sum_i\frac{N\xi_i^2}{(N^2+\xi_i^2)^2}\frac{H_{\nu+1}(\xi_i r)}{\Theta_{i\nu}^2}(-1)^{k-1}a_k H_{\nu+1}(\xi_i a_k)$$
$$=\frac{N}{\{R_{\nu,\nu}^{(k)}(Na_k)\}^2}\Big[R_{\nu,\nu}^{(k)}(Na_k)\Big\{rR_{\nu,\nu}^{(k)}(Nr)-a_{k-1}R_{\nu+1,\nu+1}^{(k)}(Nr)\Big\}$$
$$-R_{\nu+1,\nu+1}^{(k)}(Nr)\Big\{a_k R_{\nu-1,\nu-1}^{(k)}(Na_k)-a_{k-1}R_{\nu,\nu+1}^{(k)}(Na_k)\Big\}\Big],$$

$$\omega_{\nu s}^{(k)}(Nr)=\frac{4}{b^2}\sum_i\frac{N\xi_i^2}{(N^2+\xi_i^2)^2}\frac{H_{\nu-1}(\xi_i r)}{\Theta_{i\nu}^2}(-1)^k a_k H_{\nu+1}(\xi_i a_k)$$
$$=\frac{N}{\{R_{\nu,\nu}^{(k)}(Na_k)\}^2}\Big[R_{\nu,\nu}^{(k)}(Na_k)\Big\{rR_{\nu,\nu}^{(k)}(Nr)-a_{k-1}R_{\nu-1,\nu-1}^{(k)}(Nr)\Big\}$$
$$-R_{\nu-1,\nu-1}^{(k)}(Nr)\Big\{a_k R_{\nu+1,\nu+1}^{(k)}(Na_k)-a_{k-1}R_{\nu,\nu-1}^{(k)}(Na_k)\Big\}\Big],$$

$$PF_\nu^{(k)}(Nr)=\frac{1}{2}\{\omega_{\nu p}+\omega_{\nu s}\},\quad MF_\nu^{(k)}(Nr)=\frac{1}{2}\{\omega_{\nu p}-\omega_{\nu s}\},$$

$$R_{ij}^{(k)}(Nr)=I_i(Nr)\,K_j(Na_{k-1})-(-1)^{i+j}I_j(Na_{k-1})\,K_i(Nr),\quad a_0=a.$$

$$Q^{(k)}(\xi_i z)=\frac{2}{c}\sum_n\Big\{1+(-1)^k(-1)^n\Big\}\frac{N}{N^2+\xi_i^2}\sin Nz=\frac{\cosh\xi_i(c-z)\pm\cosh\xi_i z}{\cosh\xi_i c\pm 1},$$

$$P^{(k)}(\xi_i z)=\frac{4}{c}\sum_n\Big\{1+(-1)^k(-1)^n\Big\}\frac{N\xi_i^2}{(N^2+\xi_i^2)^2}\sin Nz$$
$$=\frac{\xi_i z\sinh\xi_i(c-z)\pm\xi_i(c-z)\sinh\xi_i z}{\cosh\xi_i c\pm 1},$$

$$\phi^{(k)}(\xi_i z)=\frac{2}{c}\sum_n\Big\{1+(-1)^k(-1)^n\Big\}\frac{\xi_i}{N^2+\xi_i^2}\cos Nz$$
$$+\frac{\{1+(-1)^k\}}{\xi_i c}=\frac{\sinh\xi_i(c-z)\mp\sinh\xi_i z}{\cosh\xi_i c\pm 1},$$

$$\psi^{(k)}(\xi_i z)=\frac{4}{c}\sum_n\Big\{1+(-1)^k(-1)^n\Big\}\frac{N^2\xi_i}{(N^2+\xi_i^2)^2}\cos Hz$$
$$=\phi^{(k)}(\xi_i z)-\frac{\xi_i z\cosh\xi_i(c-z)\mp\xi_i(c-z)\cosh\xi_i z}{\cosh\xi_i c\mp 1}.$$

$D_{\nu nk}$, $E_{\nu ik}$ in Eq. (7) can be determined so as to satisfy the boundary conditions. The stresses will be found by substituting Eq. (7) into Eqs. (2) and (3).

5. Numerical Example

When the thick hollow cylinder, with the supports at the points $z=0$ and $z=c$, is subjected by the load with the variation of $\cos\theta$, which locally distributes over the central part of the span. The boundary conditions are expressed by

(i) $\sigma_z=0$, $u=v=0$, for $z=0$, $z=c$,

which yields $\gamma_{\nu i}=E^a_{\nu i}=E^b_{\nu i}=0$.

(ii) $\sigma_r=\tau_{r\theta}=\tau_{zr}=0$, for $r=a$,

from which $\alpha_{\nu n2}=\beta_{\nu n2}=0$.

(iii) $\tau_{r\theta}=\tau_{zr}=0$, for $r=b$,

and

$$\sigma_r=\begin{cases} q_0\cos\theta & c/2-c_0<z>c/2-c_0 \\ 0 & c/2-c_0>z, \quad c/2+c_0<z, \end{cases}$$

from which

$$\alpha_{\nu n1}=0, \quad (\nu\neq 1); \; \alpha_{\nu n1}=\frac{q_0\pi}{N}\sin\frac{n\pi}{2}\sin NC_0, \quad (\nu=1); \; \beta_{\nu n1}=0.$$

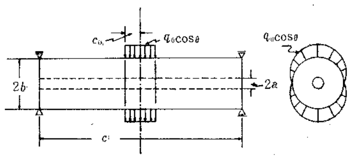

Fig. 2.

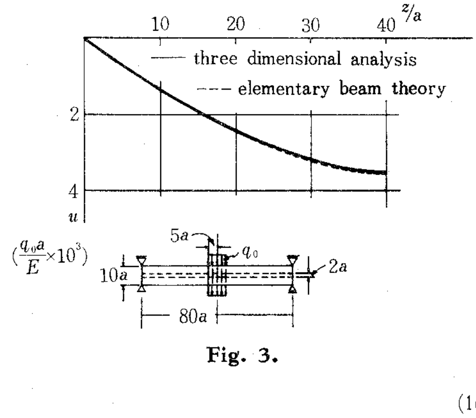

Fig. 3.

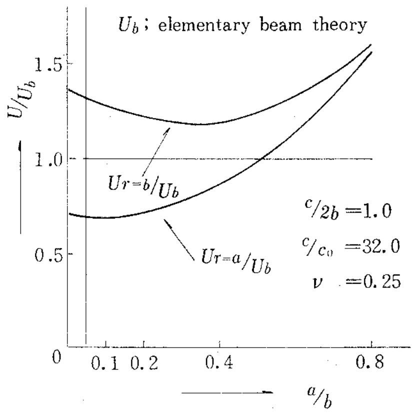

Fig. 4.

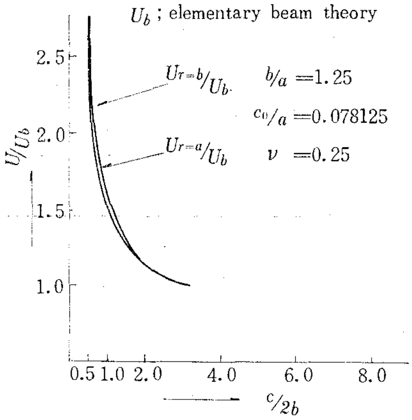

Fig. 5.

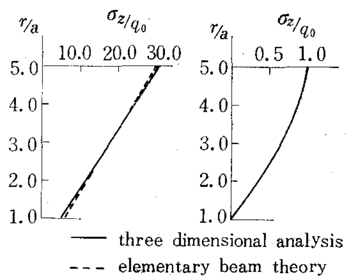

Fig. 6.

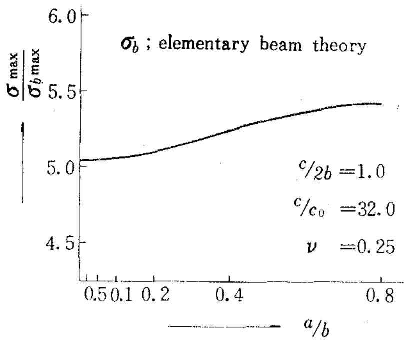

Fig. 7.

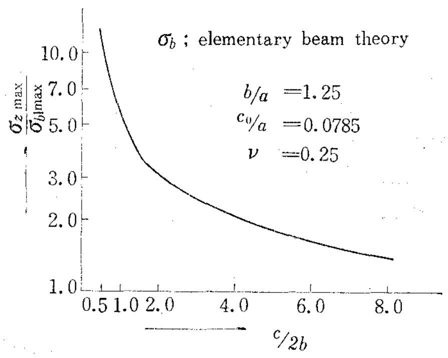

Fig. 8.

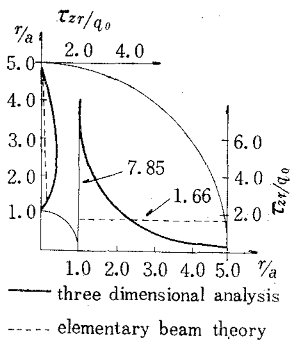

Fig. 9.

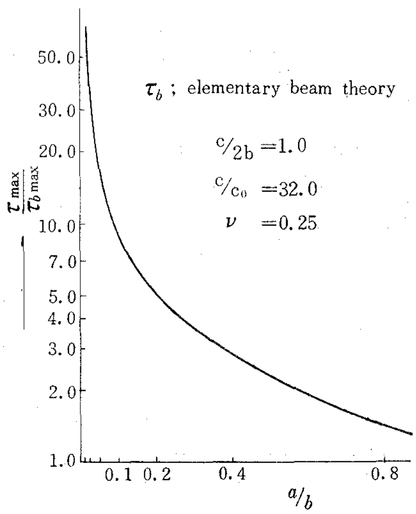

Fig. 10.

Fig. 3 shows how the radial displacement at the middle plane of the cylinder varys, and the figure also has the variation of deflection calculated by the elementary beam theory. The variations of the ratio between the outside and inside displacements u, for $z=c/2$ and $\theta=0$, and the deflection of beam, are shown in Figs. 4 and 5.

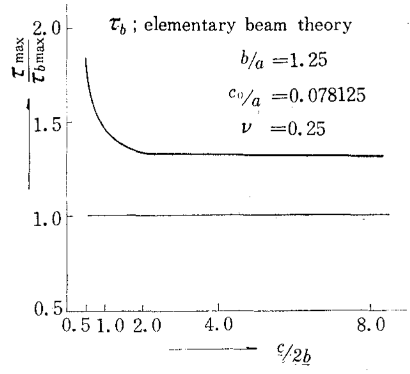

Fig. 11.

The distributions of σ_z and σ_r, for $z=c/2$ and $\theta=0$, are shown in Fig. 6. Figs. 7 and 8 show the variation of the ratio betweed the outside σ_z, for $z=c/2$ and $\theta=0$, and the maximum fiber stress due to the beam theory.

The distributions of the shearing stress for $r=0$ are shown in Fig. 9. The value of $\tau_{\theta z}$, for $z=0$, $\theta=\pi/2$ and $r=a$, is shown in Figs. 10 and 11, as the multiple with the maximum shearing stress due to elementary theory.

We find that the result, for $c/2b=1$, quite differ from that of the elementary theory. And in the particular case when $b/a=1.25$, the results also fairy differs from that obtained by the beam theory, except for the deflection. It shoud be noted that the shearing stress always takes larger values than that of the elementary theory, because of the stres concentration around the inside hole.

6. Final Remark

By making use of the finite Fourier-Hankel transform, the antiaxial symmetrical stress state concerning the thick hollow cylinder of finite length is expressed by the function involving the boundary values in it, so that we can easily handle the boundary conditions. The series of the function has good convergency. Hence, we may say that the method quite fit solving such the problem as this ; we need not use the similtaneous equations by large array and not to take a long run of digital computer. The solution can widels apply to the other boundary conditions.

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References

1) S. G. Nomachi : On One Method of Solving Stress Problems in Cylindrical Co-ordinates by Means of Finite Fourier Hankel Transforms (Part I & Part II), the Memoirs of the Muroran Institute of Technology, Vol. 3, No. 3 (1960) & Vol. 3, No. 4 (1961).