Some Extensional Constitutions of Integral

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Abstract

As the a priori measure is an extension of the Lebesgue measure, the Lebesgue integral is naturally extended by means of the a priori measure. Notions of 'integral remainder' and 'integral density' are introduced and discussed on some interesting cases.

1. Introduction

In the previous paper, the present author presented that collections of sets may, in the empiricist pragmatism[1], be assorted into two patterns, say, *summable*[2] and *non-summable* ones. Even if a family of disjoint sets (A_ι) ($\iota \in I$, I being a simply ordered set of indices) in a euclidean space E is non-summable, if

$$A = \cup A_\iota\,, \tag{1.1}$$

(A_ι) is regarded as a partition of A, though A cannot be considered as the limit of the sets $A_{(\kappa)} = \cup_{\iota \leqq \kappa} A_\iota$ ($\kappa \in I$). It is notable that, even if $A_{(\kappa)}$ is non-summable*), the aggregate A is considered as a determinate set since

$$(\forall p \in E)(\exists . \vee . \nexists \iota \in I)(p \in A_\iota)\,.$$

Eventually, the right side of (1.1) gives either a summable union (or, briefly a summation) or a non-summable union of A.

$\tilde{m}A$ indicates the a priori measure (value) of a set A. If mA is the Lebesgue measure of A, then

$$\tilde{m}A = mA\,.$$

Moreover, even if A is Lebesgue non-measurable, A can be $\tilde{m}$-measurable. In the empiricist pragmatism, any determinate set X (i.e., $(\forall p \in E)(p \in X . \vee . p \notin X)$) is proved to be $\tilde{m}$-measurable, so that A of (1.1) may be taken as $\tilde{m}$-measurable whenever all A_ι are determinate.

In our view, $\tilde{m}A$ is claimed to be written in the from

$$\tilde{m}A = \mu \cdot \nu(A)\,,$$

when all points of A are regarded to be uniformly of the same size μ. In case of (1.1), if $I = \{1, 2, \cdots\}$ and

$$\forall i, \mathrm{k} \in I: \ \tilde{m}A_i / \tilde{m}A_k = \nu(A_i)/\nu(A_k) = 1 \tag{1.2}$$

and if

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*) I.e., $\tilde{m}(A_{(\kappa)} - A_{(\lambda)}) \cup (A_{(\lambda)} - A_{(\kappa)}) \not\to 0$, when κ, $\lambda \to \infty$.

$$0<\tilde{m}A<\infty\,,$$

then it must be that

$$\forall i:\ \tilde{m}A_i=0$$

so that $\tilde{m}A_{(k)}=0$ and $\tilde{m}(A-A_{(k)})>0$ for any finite integer k. Hence (A_i) cannot be summable*).

In this paper, we limit the functions being considered to be real-valued, one-valued and to be bounded in its domain which is a bounded set in E (therefore, of finite *$\tilde{m}$-measure*). If D is the domain of a function $f(p)$, we define such that

$$D_x=\left\{p\in D\,|\,f(p)\leqq x\right\}$$

for a real number x. In this case, if

$$\forall p\in D:\ -\infty<a\leqq f(p)\leqq b<\infty$$

and

$$l=b-a\,,$$

we define as

$$\overset{(n)}{D}_k=D_{\left(a+\frac{k}{2^n}l\right)}-D_{\left(a+\frac{k-1}{2^n}l\right)}\quad(k=1,2,\cdots,2^n)\,,$$

$$\mathfrak{l}J_{(n)}=\sum_{k=1}^{2^n}\left(a+\frac{k-1}{2^n}l\right)\tilde{m}\overset{(n)}{D}_k\quad\text{and}\quad\mathfrak{u}J_{(n)}=\sum_{k=1}^{2^n}\left(a+\frac{k}{2^n}l\right)\tilde{m}\overset{(n)}{D}_k\,.$$

Then it is readily seen that

$$\mathfrak{u}J_{(1)}\geqq\mathfrak{u}J_{(2)}\geqq\cdots\geqq\mathfrak{l}J_{(2)}\geqq\mathfrak{l}J_{(1)}\,,$$

and moreover that

$$0\leqq\mathfrak{u}J_{(n)}-\mathfrak{l}J_{(n)}=\frac{l}{2^n}\Sigma\tilde{m}\overset{(n)}{D}_k=\frac{l}{2^n}\tilde{m}D\rightarrow0\quad\text{as }n\rightarrow\infty\,,$$

so that

$$\lim\mathfrak{u}J_{(n)}=\lim\mathfrak{l}J_{(n)}\,. \tag{1.3}$$

We define the integral

$$\int_D f(p)\,dp \tag{1.4}$$

to represent the value of (1. 3). Such is the same way of definition with that of the Lebesgue integral except that we use the a priori measure $\tilde{m}$ instead of the Lebesgu measure m. On the other hand, if we denote by $E(f, D)$ the (algebraic) expectation of f over D, we easily see that we may have

$$\int_D f(p)\,dp=E(f,D)\,\tilde{m}D\,. \tag{1.5}$$

*) An example of such a case is shown in 2).

As $\tilde{m}A$ has been extended beyond mA, the integral (1. 4) is naturally expected to be an extension beyound Lebesgue integrability. In the following, we will present seversl results obtained through the researches of the integral.

2. Integral Remainder

When a family of disjoint sets (D_ι) $(\iota\in I, I$ being a simply ordered set of indices) gives a partition of a set A, if $D_{(\kappa)}=\bigcup_{\iota\leqq\kappa}D_\iota$, we naturally have the relation

$$\int_D f(p)\,dp=\int_{D_{(\kappa)}} f(p)\,dp+\int_{D-D_{(\kappa)}} f(p)\,dp\,.$$

So it follows that

$$\int_D f(p)\,dp=\lim\int_{D_{(\kappa)}} f(p)\,dp+R$$

where

$$R=\lim\int_{D-D_{(\kappa)}} f(p)\,dp\,.$$

If the limitation

$$\lim\int_{D_{(\kappa)}} f(p)\,dp \tag{2. 1}$$

is convergent, then R gives a unique value because, as proved in Sect. 1, f is integrable over D, $D_{(\kappa)}$ and $D-D_{(\kappa)}$ for every $\kappa\in I$. If the limitation (2. 1) is not convergent, then R cannot give a unique value. The process (2. 1) is called an *inferior approximation* of the integral in respect to $(D_{(\kappa)})\,(\kappa\in I)$ and R is the *integral remainder* of the integral in respect to $(D_{(\kappa)})$.

As (D_ι) is a partition of D, we have

$$D=\bigcup D_\iota=\bigcup D_{(\iota)}\,,$$

but it is not asserted that

$$\lim\tilde{m}(D-D_{(\iota)})=0$$

when (D_ι) is not summable. Thus the integral remainder very often does not vanish.

Let an additive function $\pi(A)$ of a set A of real numbers be defined such that

$$\pi(A)=\pi(B)$$

whenever $\tilde{m}A=\tilde{m}B$ and

$$\pi(I_\infty)=1$$

I_∞ being the set $\{x|-\infty<x<\infty\}$. In this case, denoting by I_k the interval $\{x|-k<x<k\}$, we have for every positive integer k

$$\pi(I_k)=0$$

whereas
$$\pi(I-I_k)=1\,.$$
Hence we say $\pi(A)$ has an *unvanishing atmosphere* ($]\infty[$) with respect to the approximation sequence (I_k) $(k=1, 2, \cdots)$ by reason of the fact that
$$\lim \pi(I_\infty-I_k)=1\neq 0\,.$$
Thus, the cass of an unvanishing remainder may be observed as exactly similar to that of an unvanishing atmosphere.

In case of (1. 2), if there exists a real number β such that
$$\beta=\overline{\lim_k}\sup_{p\in A_k} f(p)=\underline{\lim_k}\inf_{p\in A_k} f(p)\,,$$
we conclude that
$$\int_A f(p)\,dp=\beta\cdot\lim \tilde{m}(A-A_{(k)})=\beta\cdot\tilde{m}A\,.$$
So, in regard to (1. 5), we have
$$E(f, A)=\beta\,.$$

3. Principal Part

Let us define a subset D_x of D by
$$D_x=\left\{p\in D|f(p)=x\right\}.$$
Then the value x may be reckoned as a set-function of D_x, so let this function be written as $\lambda_f(D_x)$. Since
$$x=y.\Longleftrightarrow .D_x=D_y\,,$$
λ_f is one-valued, and since f is bounded so is λ_f. Extensively, let us define $D(V)$ and $D(a, b)$ by
$$D(V)=\left\{p\in D|f(p)\in V\right\} \quad\text{and}\quad D(a, b)=\left\{p|a<f(p)<b\right\}$$
respectively. If $\tilde{m}D(a, b)=0$, the interval (a, b) is said to be involved in the *negligible part* or, briefly, to be *negligible.*

Let us remove from the set of real numbers every interval (a, b) which is negligible and for which there is no positive real number ε such that
$$\tilde{m}D(a-\varepsilon, b+\varepsilon)=0\,.$$
The rest part left after this process of removal is called the *principal part* of f and of λ_f and is denoted by $P(f)$. Let us denote as
$$(a, b)=\left\{x|a<x<b\right\}$$
and
$$[a, b)=\left\{x|a\leqq x<b\right\}.$$
If an interval is either $[a, b)$ or (a, b), then it is denoted as $\{a, b)$. If an

interval J is contained in $P(f)$, *then* J is called a *principal interval.* It is easily seen that the principal part $P(f)$ is at most an enumerable union of intervals or isolated singletons.

When $V=\{a, b)$ is a principal interval, then, for any $a<c<d\leqq b$, $\{c, d)$ is also a principal interval and

$$\tilde{m}D(c, d)>0$$

and it is obviously observed that

$$\tilde{m}D(V)=\tilde{m}D(\{a, c))+\tilde{m}D([c, d))+\tilde{m}D([d, b)).$$

Therefore $\tilde{m}D(\{a, x))$ gives a strictly increasing function of $x(\in(a, b))$. So it must be continuous except for points of at most an enumerable set and its inverse function also continuous wherever it is continuous. Let the discontinuous points of $\tilde{m}D(\{a, x))$ be $x_1, x_2, \cdots$, and let us define as

$$\tilde{m}D(\{a, x))=\mu, \qquad x=\phi_V(\mu)=\lambda_f(D_x)$$

where $x\in V$ and $\mu\in(0, \tilde{m}D(V))$. Then it is readily seen that there correspond intervals $J_1, J_2, \cdots$ of μ to the points $x_1, x_2, \cdots$ such that

$$\tilde{m}J_k=\tilde{m}D_{x_k} \qquad (k=1, 2, \cdots)$$

and if we define as

$$J(V)=(0, \tilde{m}D(V))$$

and

$$J^*(V)=J(V)-\cup J_k,$$

then ϕ_V is found to be continuous in $J^*(V)$. Thus we may have

$$\int_{D(V)} f(p)\, dp=\int_{J^*(V)} \phi_V(\mu)\, d\mu+\Sigma x_k l_k \tag{3. 1}$$

where $l_k=\tilde{m}D_{x_k}$ $(k=1, 2, \cdots)$.

However, for the consistence of (3. 1), it must be assumed that the union $\cup J_k$ is summable. The function ϕ_V is called the *measure interpretation* of λ_f.

4. Integral Density

If two functions $f(p)$ and $g(p)$ are both bounded in the same doman D, and if

$$0<\int_D |f(p)|dp\Big/\int_D |g(p)|dp<\infty, \tag{4. 1}$$

f and g are said to have *integral densities* of the *same level,* and if

$$\int_D |f(p)|dp\Big/\int_D |g(p)|dp=0 \tag{4. 2}$$

f is said to have an integral densiuy of *less level* than that of g. If the supports of f and g are D_f and D_g respectively, we may, by grace of (1. 5), prove that (4. 1) and (4. 2) coincide with the relations

$$0<\tilde{m}D_f\cap D/\tilde{m}D_g\cap D<\infty \tag{4. 1'}$$

and

$$\tilde{m}D_f\cap D/\tilde{m}D_g\cap D=0 \tag{4. 2'}$$

respectively on condition that both $E(|f|, D)$ and $E(|g|, D)$ are positive.

The a priori measure $\tilde{m}A$ of a set A in E can, as stated in Sect. 1, be written in the form

$$\tilde{m}A=\mu\cdot\nu(A). \tag{4. 3}$$

So then, if the domain D is an enumerable set of points $\{x_1, x_2, \cdots\}$, we may write the integral in the form

$$\int_D f(p)\,dp=\mu\sum f(x_k),$$

so that

$$\int_D f(p)\,dp\Big/\int_D g(p)\,dp=\sum f(x_k)/\sum g(x_k). \tag{4. 4}$$

If the two series are both convergent and $\sum f(x_k)=a$ and $\sum g(x_k)=b$, then we may have the ratio of (4. 4) to be equal to a/b. In this convergent case, it should however be noted that

$$E(f, D)=E(g, D)=0.$$

Thus, in this case, it is observed that the ratio of the integral densities cannot be simulated by

$$\tilde{m}D_f\cap D/\tilde{m}D_g\cap D. \tag{4. 5}$$

On the other hand, if $f(x_k)=1$ for all $k=1, 2, \cdots$ and

$$\begin{aligned} g(x_k)&=0 \quad \text{when} \quad k=1, 2, 4, 5, 7, 8, \cdots, \\ &=1 \quad \text{when} \quad k=3, 6, 9, \cdots, \end{aligned}$$

then we may count as

$$E(f, D)=1 \quad \text{and} \quad E(g, D)=1/3,$$

so that the ratio of the left side of (4. 4) is counted as equal to 3. In this case we observe that the ratio of the integral densities can again be simulated by (4. 5) because

$$\tilde{m}D_f/\tilde{m}D_g=\nu(D_f)/\nu(D_g)=3/1.$$

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References

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