Several Further Extension Criteria

Several Further Extension Criteria

Kazuo IWATA

Abstract

Influenced by Agnew-Morse 6), by modifying the recent work 29), the author furnishes several further extension criteria.

Introduction. As abstracted above, the present investigation is continued from the short note 29), etc. That is, in a word, in this paper, by means of the earlier [19), Lemmas 1-4], various extension theorems such as of the Hahn-Banach, Krein's, Agnew-Morse's etc. are simultaneously generalized.

For such a sake, to tell the truth, the problems with which we are concerned here amount to somewhat general setting (nevertheless the conclusions are not so complicated). Besides, this time, some pairs of our results are given to overlap each other respectively (see e.g. Theorem 1 and its Corollary 3). The reason is that the problems of an Abelian semigroup of linear transformations are treated more circumstantially than those of a semigroup of linear transformations.

For reference, it may safely be said that the present results are self-contained except the Zorn's lemma. Partly for this reason, it appears to me that the viewpoint [19), Lemmas 1-4] is somewhat suited to deal with these materials.

Preliminaries. Let $E(\neq\{0\})$ denote a linear space over the real field $\boldsymbol{R}$. Let L denote a product linear space $E\times\boldsymbol{R}$ or topological product $E\times\boldsymbol{R}$ ($\boldsymbol{R}$ being endowed with the usual topology) We need

DEFINITION 1. If on E there is defined a binary relation "$\leqslant$" satisfying all postulates in [19), Def. 1, c)] excepting perhaps 2) and 4), E is called a *preordered linear space* with respect to "$\leqslant$". Convex cone $C=\{c\geqslant 0\}$ is called the *associated cone* with "$\leqslant$". *Preordered linear topological space* may be analogized.

DEFINITION 2. a) Let q be a gauge function on a subspace $K\subset E$. By $\tilde{q}$ is meant a gague function on K with $\tilde{q}(y)=q(-y)$ $(y\in K)$.

The set $\{(y,\eta):y\in K, q(y)<\eta\}\subset L$, where q is a gauge function on K, is briefly termed the "epigraph" of q. With this terminology:

b) Let f be a linear form on a subspace $M\subset E$. B_f stands for the epigraph of f.

c) C_q is the epigraph of gauge function q on K.

d) C_{qC} is the quasi-epigraph of gauge function q with respect to C: $C_{qC}=\{(y,\eta)$: there exists $c\in C$ such that $y+c\in K$ with $q(y+c)<\eta\}$. In this way, $C_{\tilde{q}C}=\{(y,\eta)$: there exists $c\in C$ such that $y+c\in K$ with $\tilde{q}(y+c)<\eta\}$, i.e., $C_{\tilde{q}C}=\{(y,\eta)$:

there exists $c \in C$ such that $-y-c \in K$ with $q(-y-c)<\eta\}=\{(-y,\eta)$: there exists $c \in C$ such that $y-c \in K$ with $q(y-c)<\eta\}$.

e) $C_{qC\mathscr{L}}$ is the quasi-epigraph of gauge function q with respect to C and $\mathscr{L}$ (for $\mathscr{L}$, see Theorem 1 below) : $C_{qC\mathscr{L}}=\{(y,\eta)$: there exist $c \in C$ and $T_1, T_2, \cdots, T_m \in \mathscr{L}$ (m is finite) such that $y+c \in K$ and $\frac{1}{m}q(\sum_1^m T_\mu(y+c))<\eta\}$. Especially $C_{qC\{I\}}=C_{qC}$, where I is the identity map of E to E.

Besides, for convenience, let the notations and terminology employed in 19), 23), 27), 28), and 29) be available.

Statement of the results. Slight modifications of the preceding [29), Theorem 1] yield the following which includes the Agnew-Morse type extension theorems† etc.

THEOREM 1. *Let E be a preordered linear space with an associated cone C. Let K be a linear subspace of E, q a gauge function on K. Let M be a linear subspace of E, f a linear form on M. Suppose that $\mathscr{L}$ is a semigroup of linear transformations on E such that $T(K) \subset K$ and $q(T(y)) \leqslant q(y)$ $(y \in K)$ for all $T \in \mathscr{L}$. In order that*

(1. 0) *there exists an $F \in E^*$ extending f and satisfying*
 (b) $F(y) \leqslant q(y+c)$ *whenever $y+c \in K$ for $c \in C$,*
 (c) $F(T(y))=F(y)$ *for all $y \in K$ and $T \in \mathscr{L}$,*

one of the following two conditions is necessary and sufficient :

(1. 1) *There exists a t.o.l.s. $(L, \mathscr{R})$ such that*
 (i) $B_f \cup C_{qC\mathscr{L}} \subset (L, \mathscr{R})^+$,
 (ii) $(L, \mathscr{R})^+$ *is absorbing at* (0, 1) *for L.*

(1. 2) *There exists a convex absorbing set U in E such that $B_f \cup C_{qC\mathscr{L}} \cup (U \times \{1\})$ is positively independent in L.*

PROOF. We treat the cyclic scheme (1. 0)⇒(1. 2)⇒(1. 1)⇒(1. 0). (1. 0)⇒(1. 2): Hypothesis entails that $F(y) \leqslant F(y)+F(c)=\frac{1}{m}\sum_1^m F(T_\mu(y+c))=\frac{1}{m}F(\sum_1^m T_\mu(y+c)) \leqslant \frac{1}{m}q(\sum_1^m T_\mu(y+c))$ for $y+c \in K, y \in E, c \in C, T_\mu \in \mathscr{L}$, whereby $C_{qC\mathscr{L}} \subset B_F$ follows. Hence $B_F \cup C_{qC\mathscr{L}}$ is positively independent in L, whence one has $U=\{x \in E : F(x)<1\}$ as required. For (1. 2)⇒(1. 1), appeal to [27), Rem. 2] and [19), Lemma 1]. (1. 1)⇒(1. 0) : Likewise as in the case of [27), Th. 1 ("if" part)], anyway one obtains an $F_1 \in E^*$ such that extending f and satisfying $F_1(y) \leqslant \frac{1}{m}q(\sum_1^m T_\mu(y+c))$ $(y+c \in K, y \in E, c \in C, T_\mu \in \mathscr{L}, m$ is finite). This implies (b) of (1. 0) is clear. For (c), therewith, in the light of Agnew-Morse [6), Lemma 2.01]††, it follows that $F_1(y-T(y)) \leqslant \frac{1}{m}q(T(y-T(y))+T^2(y-T(y))+\cdots+T^m(y-T(y)))=\frac{1}{m}q(T(y)-T^{m+1}(y)) \leqslant \frac{1}{m}(q(y)+q(-y))$ $(y \in K, T \in \mathscr{L}, m=1,2,\cdots)$. This implies that $F_1(y-T(y)) \leqslant 0$, and $F_1(y-T(y)) \geqslant 0$ (replacing y with $-y$) which complete the proof.

† By this the author means Agnew-Morse [6), Lemma 2. 01] and Cotlar-Cignoli [24), III, § 2. 1. 5].
†† Cf. also Larsen [22), Sec. 4. 3].

COROLLARY 1. *Let in particular* $K=E$ *(in such case, in what follows,* q *is written by* p*) in Theorem* 1. *Then* (1.2) *is reduced to*

(1. 2)′ $B_f \cup C_{pc\mathscr{L}}$ *is positively independent in* L.

PROOF. For the sufficiency, since $C_{qc\mathscr{L}} \supset C_q$, one can take $U=\{y\in E : p(y)< 1\}$ for (1. 2).

In practice, conditions (1. 1), (1. 2), (1. 2)′ seem to be applicable. In fact, in view of this, if we are concerned with Agnew-Morse [6), Lemma 2.01], our proof† (that of the "if" part of Corollary 2) runs as follows. This proof seems to somewhat simplify the original. Before starting, we put the following lemma which is also needed in our subsequent discussion.

LEMMA. *Let* C, K, q, $\mathscr{L}$ *be as in Theorem* 1. *There holds*

$$(*)\qquad \frac{1}{M}q\Big(\sum_{i=1}^{M} h_i(\alpha_1(y_1+c_1))\Big)+\frac{1}{N}q\Big(\sum_{j=1}^{N} h_j'(\alpha_2(y_2+c_2))\Big)$$
$$\geqslant \frac{1}{MN}q\Big(\sum_{i,j=1}^{M,N}(h_j'h_i(\alpha_1(y_1+c_1))+h_ih_j'(\alpha_2(y_2+c_2)))\Big),$$

where $h_i,\ h_j'\in\mathscr{L}$; $y_1+c_1,\ y_2+c_2\in K$; $c_1,\ c_2\in C$.

PROOF. This is easily read in the proof of [6), Lemma 2.01] or in the proof of [22), Theorem 4.3.1].

COROLLARY 2. *Let in particular* $K=E$, $C=\{0\}$, *and* f *be invariant*†† *in Theorem* 1. *Then* (1. 0) *is reduced to*

(1. 0)′ *(Agnew-Morse-Klee type condition*†††*) There exists an* $F_1\in E^*$ *extending* f *and satisfying both* $F_1(x)\leqslant p(x)$ $(x\in E)$ *and* $F_1(h_1h_2(x))=F_1(h_2h_1(x))$ $(h_1,h_2\in\mathscr{L}\,;\ x\in E)$.

PROOF. Necessity of the condition is easily obtained. (Sufficiency) To begin with, let $x+\alpha_1y_1+\alpha_2y_2=0$ $(\alpha_1,\alpha_2>0)$ for $(x,\xi)\in B_f$; (y_1,η_1), $(y_2,\eta_2)\in C_{pc\mathscr{L}}$, where $\frac{1}{M}p(\sum_{i=1}^{M}h_i(y_1))<\eta_1$, $\frac{1}{N}p(\sum_{j=1}^{N}h_j'(y_2))<\eta_2$ for $h_i,\ h_j'\in\mathscr{L}$. Then, in view of the lemma and the hypothesis, it follows that

$$\xi+\alpha_1\eta_1+\alpha_2\eta_2>f(x)+\frac{1}{MN}\sum_{i,j=1}^{M,N}F_1(h_ih_j'(\alpha_1y_1+\alpha_2y_2))=f(x)+f(-x)=0.$$

To this end, generalize (*) by the induction. Therewith (1. 2)′ follows which completes the proof.

Now, as prefaced before, if $\mathscr{L}$ is *Abelian*, Theorem 1 is specified as follows.

† But, for the fact $F_1(g_1g_2(x))=F_1(g_2g_1(x))$ $(g_1,g_2\in H\,;\ x\in E)$, we owe to them.

†† C is called *invariant* if $T(c)\in C$ for all $c\in C$, $T\in\mathscr{L}$;and a functional f on M is called *invariant* if M is invariant and $f(T(x))=f(x)$ whenever $x\in M$, $T\in\mathscr{L}$.

††† This condition is introduced by referring to [6), Lemma 2.01] and [8), (2. 2) Theorem].

Corollary 3. *Let in particular* $\mathscr{L}$ *be an Abelian semigroup of linear transformations on* E *in Theorem* 1. *Then* (1.0) *of Theorem* 1 *becomes equivalent to each of the following conditions.*

(1.3) *There exists a convex absorbing set* U *in* E *such that* $\xi+\eta+1>0$ *whenever* $x+y+u=0$ *for* $(x,\xi)\in B_f$, $(y,\eta)\in C_{qC\mathscr{L}}$, $u\in U$.

(1.4)[†] *There exists a convex absorbing set* V *in* E *such that* $f(x)+\eta+1>0$ *whenever* $x=v-y$ *for* $x\in M$, $(y,\eta)\in C_{qC\mathscr{L}}$, $v\in V$.

(1.5) (*Anger-Lembcke type condition*[††]) *There exists a convex absorbing set* V *in* E *such that the set* $\{f(x)+\frac{1}{m}q(\sum_1^m T_\mu(y)) : x\in M, y\in K, x+y\in V+C, T_\mu\in\mathscr{L}, m$ *is finite*$\}$ *is bounded below.*

Proof. Now that $\mathscr{L}$ is commutative, by the lemma, $C_{qC\mathscr{L}}$ proves to be a convex cone in L. Therefore the proof of (1.2)$\Leftrightarrow$(1.3) parallels that of (4.3)$\Leftrightarrow$(4.4) of [29], Th. 4]. Equating V to $-U$ and U to $-V$, (1.3)$\Leftrightarrow$(1.4) is easily verified. (1.4)$\Rightarrow$(1.5): Let $x+y\in V+C$, say, $x+y=v+c$ for $x\in M, y\in K, c\in C, v\in V$. Then in view of $x=v-(y-c)$, it follows that $f(x)+\eta>-1$ for $\frac{1}{m}q(\sum_1^m T_\mu(y))$ $(=\delta)<\eta$, i.e. $f(x)+\delta\geqslant -1$. (1.4)$\Leftarrow$(1.5): Let the lower bound in question be -1 (without loss of generality). Let $x=v-y$ for $x\in M$, $(y,\eta)\in C_{qC\mathscr{L}}$ (i.e., there are $c\in C$, n, $S_\nu\in\mathscr{L}$ such as $\frac{1}{n}q(\sum_1^n S_\nu(y+c))$ $(=\theta)<\eta$), $v\in V$. Then in view of $x+(y+c)\in V+C$, it follows that $f(x)+\theta\geqslant -1$ implying $f(x)+\eta+1>0$.

Example 1 (Analogue of the Banach limit). Let E be the partially ordered linear space $m\times m$ endowed with the pointwise order, where m is the bounded sequence space with zero element θ. Let K be the linear subspace $m\times\{\theta\}$, and let $q(y)=\overline{\lim}\,\eta_\nu$ for $y=((\eta_1,\eta_2,\cdots),(\theta))\in K$. Letting M be the linear subspace $c\times\{\theta\}$, where c is the convergent sequence space, let $f(x)=\lim\xi_\nu$ for $x=((\xi_1,\xi_2,\cdots),(\theta))\in M$. And let $\mathscr{L}=\{T^n : T$ is the shift such that $T(z)=((\alpha_2,\alpha_3,\cdots),(\beta_2,\beta_3,\cdots))$ for $z=((\alpha_1,\alpha_2,\alpha_3,\cdots),(\beta_1,\beta_2,\beta_3,\cdots))\in E, n=1,2,\cdots\}$. Then in view of (1.3) we obtain an $F\in E^*$ satisfying (1.0).

Proof. For short, $\sup\{\alpha_1,\alpha_2,\cdots,\beta_1,\beta_2,\cdots\}$ (resp. inf thereof) is written by $\sup z$ (resp. $\inf z$) for $z=((\alpha_1,\alpha_2,\cdots),(\beta_1,\beta_2,\cdots))\in E$, besides $q(y)$ $(y\in K)$, $f(x)$ $(x\in M)$ are written by $\overline{\lim}\,y$, $\lim x$ respectively. Taking $U=\{u\in E : \sup u<1\}$, let $x+y+u=0$ for $(x,\xi)\in B_f$, $(y,\eta)\in C_{qC\mathscr{L}}$, $u\in U$. To this end, we have

$$\begin{aligned}1 &> \sup(-x-y)\\ &\geqslant -\inf(x+y)-\inf c \quad (\text{where } c\in E^+, y+c\in K)\end{aligned}$$

† Cf. Remark 2 below.

†† Cf. [26], Theorem 3.4].

$$\geqslant -\inf(x+y+c) \geqslant -\underline{\lim}(x+(y+c))$$
$$= -\lim x - \underline{\lim}(y+c) > -\xi - \frac{1}{m}\overline{\lim}\left(\sum_1^m T_\mu(y+c)\right)$$
$$> -\xi - \eta.$$

EXAMPLE 2. In Example 1, let $\mathscr{L}$ be replaced by $\mathscr{L}' = \{T : T(z) = ((\gamma_1, \gamma_2, \cdots), (\delta_1, \delta_2, \cdots))$, where $z = ((\alpha_1, \alpha_2, \cdots), (\beta_1, \beta_2, \cdots))$ for which $(\gamma_1, \gamma_2, \cdots), (\delta_1, \delta_2, \cdots)$ are resp. some (depending on T) subsequences of $(\alpha_1, \alpha_2, \cdots), (\beta_1, \beta_2, \cdots)\}$ (i. e., $\mathscr{L}'$ $(\supset \mathscr{L})$ is not *Abelian*). Then, notwithstanding (1. 3) remains true, (1. 0) fails to follow. This is made out by the fact that (1. 2) is impossible ($C_{qC\mathscr{L}'}$ itself is positively dependent).

COROLLARY 4. *If* $K=E$ *in Corollary* 3, *the condition* (1. 3) (*accordingly, so are also* (1. 1), (1. 2), (1. 4), (1. 5)) *is reduced to*

(1. 3)′ $f(x) \leqslant \frac{1}{m} p\left(\sum_1^m T_\mu(x+c)\right)$ *holds, where* $x \in M$, $c \in C$, $T_\mu \in \mathscr{L}$, m *is finite.*

PROOF. Now that $C_{pC\mathscr{L}}$ is a convex cone in L, in effect (1. 3)′ proves to be equivalent to (1. 2)′. (Naturally, the alternative direct proof can be made.)

As an application

COROLLARY 5. *Letting in particular* $K=E$; f, p *both invariant,* $f(x) \leqslant p(x)$ $(x \in M)$, *and* $p(-c) \leqslant 0$ $(c \in C)$ *in Corollary* 3, *Cotlar-Cignoli* [24), III, §2. 1. 5] *a fortiori follows.*

PROOF. By hypothesis, it follows that $f(x) \leqslant \frac{1}{m} p\left(\sum_1^m T_\mu(x)\right) - \frac{1}{m} p\left(\sum_1^m T_\mu(-c)\right)$ $\leqslant \frac{1}{m} p\left(\sum_1^m T_\mu(x) + \sum_1^m T_\mu(c)\right) = \frac{1}{m} p\left(\sum_1^m T_\mu(x+c)\right)$ $(x \in M, c \in C, T_\mu \in \mathscr{L})$ (if the invariance of C is assumed, it is immediate from $f(x) \leqslant p(x+c)$ $(x \in M, c \in C)$) which completes the proof.

For reference, the following is easily seen.

COROLLARY 6. *Suppose that* $\mathscr{T}$ *is a set of linear transformations of* E *into* E. *Taking* $\mathscr{L} = \{I\}$, $C = \{\sum_{i=1}^k T_i y_i : T_i \in \mathscr{T}, y_i \in E, k$ *is finite*$\}$, *Corollary* 4 *coincides with Klee* [8), (2. 1) *Lemma*].

Returning to the subject, we add the following remarks.

REMARK 1. In Corollary 3, invariances of C and f are not assumed. But the linearity of K thereof can not be dropped (of course, if $\mathscr{L} = \{I\}$, any proper pointed convex cone K may be applicable), i. e., otherwise none of (1. 1)-(1. 5) necessarily implies (1. 0). To see this, let E be the l_1 space with l_1 norm $\|\cdot\|$, and let $C = \{0\}$. Let $K \subset E$ be the pointed convex cone generated by $a = (-1, -\frac{1}{2}, -\frac{1}{4}, \cdots)$ and let $q(y) = \|y\|$ $(y \in K)$. Let $M \subset E$ be the linear subspace generated by $\{a\}$ and let $f(\alpha a) = -2\alpha$. And let $\mathscr{L} = \{T^n : T((\xi_1, \xi_2, \xi_3, \cdots)) = (\xi_2, \xi_3, \cdots)$ for $(\xi_1, \xi_2, \xi_3, \cdots) \in E$, $n = 1, 2, \cdots\}$. This answers the question, i. e., this satisfies (1. 3) but not (1.0)

(c) (a priori).

REMARK 2. Shifting the courses, there are two alternative ways to settle Corollary 3. One is concerned with the course (1. 0)⇔(1. 4), and the other is so with (1. 0)⇔(1. 5) (to the purpose, (1. 2)⇐(1. 1) holds directly).

SKETCH OF THE PROOF OF (1. 0)⇔(1. 4). Since (1. 4) is rephrased by

(1. 4)$^{\#}$ *there exists a convex absorbing set* V *is* E *such that* $-f(x)+\xi'<1$ *whenever* $(x, \xi')=(v, o)-(y, \eta)$ *for* $(x, \xi')\in M\times \boldsymbol{R}, (y, \eta)\in C_{qC\mathscr{L}}, v\in V$,

under L with $C_{qC\mathscr{L}}$, observe $\varphi\in(M\times \boldsymbol{R})^*$ defined by $\varphi(x, \xi')=-f(x)+\xi'$. (Necessity) Via (1. 4)$^{\#}$, the "only if" part of the Bauer-Namioka theorem† answers the purpose. (Sufficiency) Let $(x, \xi')=(\frac{v}{2}, \delta)-(y, \eta)$ for $(x, \xi')\in M\times \boldsymbol{R}$, $(y, \eta)\in C_{qC\mathscr{L}}$, $v\in V$, $|\delta|<\frac{1}{2}$. Then in view of $(2x, 2\xi'-2\delta)=(v, o)-(2y, 2\eta)$, $(2y, 2\eta)\in C_{qC\mathscr{L}}$, it follows that $-f(2x)+2\xi'-2\delta<1$ yielding $-f(x)+\xi'<1$. Therewith the "if" part of the theorem cited (cf. $C_{qC\mathscr{L}}\neq\emptyset$) leads up to the conclusion.

PROOF OF (1. 0)⇔(1. 5). By use of the new gauge $\hat{q}$ defined†† by

$$\hat{q}(y)=\inf\{\frac{1}{m}q(\sum_{1}^{m} T_\mu(y)) : T_\mu\in \mathscr{L}, m \text{ is finite}\} \quad (y\in K),$$

the assertion (1)⇔(9) of Anger-Lembcke [26), Theorem 3. 4] answers the purpose. Indeed the said conditions (1) and (9) with respect to $\hat{q}$ are respectively equivalent to (1. 0)††† and (1. 5).

REMARK 3. Replacing E, $F\in E^*$, U, etc. by preordered linear topological space E, $F\in E'$, 0-neighbourhood U, etc. respectively, we can state and prove the topological version of Theorem 1 (we call this Theorem 2 corresponding to [29), Th. 2]). The details are omitted.

We close this note with the following criterion. Non-Abelian version, non-topological version etc. thereof may be realized without difficulty.

CRITERION. *Let* E *be a linear topological space. Let* I, J *be disjoint index sets with* $\Lambda=I\cup J\neq\emptyset$. *For each* $\lambda\in\Lambda$, *let* C_λ *be a pointed convex cone in* E, K_λ *a linear subspace of* E, *and* q_λ *a gauge function on* K_λ. *Let* M *be a linear subspace of* E, f *a linear form on* M. *Suppose that* $\mathscr{L}$ *is an Abelian semigroup of linear transformations on* E *such that* $T(K_\lambda)\subset K_\lambda$ *and* $q_\lambda(T(y))\leqslant q_\lambda(y)$ $(y\in K_\lambda)$ *for all* $T\in\mathscr{L}, \lambda\in\Lambda$. *In order that*

(4. 0) *there exists an* $F\in E'$ *extending* f *and satisfying all of*
(a) $-q_i(y-c)\leqslant F(y)$ *whenever* $y-c\in K_i\,(c\in C_i)$ *for* $i\in I$,

† By this we here quote [16), (V, 5. 4)].

†† K being a subspace of E, the gauge $\hat{q}$ is well-defined.

††† For this we owe to Cotlar-Cignoli [24), III, 1. 2].

(b) $F(y) \leqslant q_j(y+c)$ *whenever* $y+c \in K_j$ $(c \in C_j)$ *for* $j \in J$,
(c) $F(T(y)) = F(y)$ *for all* $y \in \bigcup_{\lambda \in \Lambda} K_\lambda$, $T \in \mathscr{L}$,

one of the following five conditions is necessary and sufficient, where $\hat{q}_\lambda(y) = \inf\{\frac{1}{m} q_\lambda (\sum_1^m T_\mu(y)) : T_\mu \in \mathscr{L}, m$ *is fiuite*$\}$ $(y \in K_\lambda)$ *for each* $\lambda \in \Lambda$.

(4. 1) *Replace* $h_\lambda(\lambda \in I \cup J)$ *by* $\hat{q}_\lambda(\lambda \in \Lambda)$ *in* (4. 2) *of* [29), *Th.* 4].

(4. 2) *Replace so in* (4. 3) *op. cit.*

(4. 3) *Replace so in* (4. 4) *op. cit.*

(4. 4)# (*Bauer-Namioka type condition*) *There exists a convex* 0-*neighbourhood* V *in* E *such that* $-f(x)+\xi' < 1$ *whenever* $(x, \xi') = (v, o) - \sum_{\nu \in I \cap N} (-y_\nu, \eta_\nu) - \sum_{\nu \in J \cap N} (y_\nu, \eta_\nu)$ *for* $(x, \xi') \in M \times \boldsymbol{R}$, $(-y_\nu, \eta_\nu) \in C_{\tilde{\hat{q}}_\nu c_\nu}$ $(\nu \in I \cap N)$, $(y_\nu, \eta_\nu) \in C_{\hat{q}_\nu c_\nu}$ $(\nu \in J \cap N)$, $v \in V$, *where* N *in any finite subset of* Λ.

(4. 5) (*Anger-Lembcke type condition*†) *There exists a convex* 0-*neighbourhood* V *in* E *such that the set* $\{f(x) + \sum_{\nu \in N} \hat{q}_\nu(y_\nu) : x \in M, N$ *is any finite subset of* Λ, $y_\nu \in K_\nu$ $(\nu \in N)$, $x - \sum_{\nu \in I \cap N} y_\nu + \sum_{\nu \in J \cap N} y_\nu \in V + \sum_{\nu \in N} C_\nu\}$ *is bounded below.*

PROOF. In fact, the sets $C_{\tilde{\hat{q}}_i c_i}$ $(= C_{\hat{\tilde{q}}_i c_i})$ $(i \in I)$, $C_{\hat{q}_j c_j}$ $(j \in J)$ coincide with $C_{\tilde{q}_i c_i \mathscr{L}}$, $C_{q_j c_j \mathscr{L}}$ respectively. So that the proofs of (4. 0)⇒(4. 2)⇒(4. 1)⇒(4. 0) and of (4. 2)⇔(4. 3)⇔(4. 4)#⇔(4. 5) are given *mutatis mutandis* from those of Theorem 1 and Corollary 3 (cf. the proof of [29), Th. 4]).

To prove (4. 0)⇔(4. 4) directly, we employ the generated convex cone:

$$\mathrm{co}\left(\left(\bigcup_{i \in I} C_{\tilde{q}_i c_i \mathscr{L}}\right) \cup \left(\bigcup_{j \in J} C_{q_j c_j \mathscr{L}}\right)\right),$$

and apply the Bauer-Namioka theorem.

The analogue of Anger-Lembcke [26), Theorem 6. 3] also proves (4. 0)⇔(4. 5).

(Received May 18, 1978)

References

1) E. STIEMKE : Über positive Lüsungen homogener linearer Gleichungen. Math. Ann. 76 (1915), 340-342.
2) W. B. CARVER : System of linear inequalities. Annals of Math. (2) 23 (1922), 212-220.
3) L. L. DIENES : Definite linear dependence. Annals of Math. 27 (1925), 57-64.
4) S. BANACH : Sur les fonctionnelles linéaires II, Studia Math. 1 (1929), 223-239.
5) S. MAZUR : Über konvexe Mengen in linearen normierten Räumen. Studia Math. 4 (1933), 70-84.
6) R. P. AGNEW and A. P. MORSE : Extensions of linear functionals, with applications to limits, integrals, measures, and densities. Ann. of Math. (2) 39, 20-30 (1938).

† Cf. [26), Th. 6. 3].

7) V. L. KLEE : Convex sets in linear spaces. Duke Math. J. 18 (1951), 443-466.
8) ———— : Invariant extensions of linear functionals. Pacific J. Math. 4 (1954), 37-46.
9) P. C. HAMMER : Maximal convex sets. Duke Math. J. 22 (1955), 103-106.
10) I. NAMIOKA : Partially ordered linear topological spaces. Mem. Amer. Math. Soc. no. 24 (1957).
11) N. DUNFORD and J. T. SCHWARTZ : Linear operators, Part I, chaps. II and V. Wiley (Interscience Publishers), Inc., New York, 1958.
12) J. L. KELLEY, I. NAMIOKA and CO-AUTHORS : Linear topological spaces, chaps. 1 and 4. D. Van Nostrand Co. Inc., Princeton, 1963.
13) R. E. EDWARDS : Functional analysis, chap. 2. Holt, Rinehart and Winston, Inc., New York, 1965.
14) BOURBAKI, N : Espaces vectoriels topologiques, chap. I et II. Éléments de mathématique, livre V. Hermann, Paris, 1966.
15) G. KÖTHE : Topologische linear Räume, I, §16, §17. Springer-Verlag, Berlin, 1966.
16) H. H. SCHAEFER : Topological vector spaces, chaps II and V. The Macmillan Co., New York, 1966.
17) MATH. SOC. OF JAPAN : Iwanami Sūgaku-Ziten (Encyclopedic Dictionary of Mathematics), 2nd ed., p.598. Iwanami Shoten, Publishers, Tokyo, 1968.
18) V. L. KLEE : Separation and support properties of convex sets-a survey, control theory and the calculus of variations. Academic Press Inc. New York, (1969), 235-303.
19) K. IWATA : Totally ordered linear space structures and separation theorem. Hokkaido Math. Jour. (Sapporo), Vol. I, No.2 (1972), 211-217.
20) C. E. MOORE : Concrete semispaces and lexicographic separation of convex sets. Pacific J. Math. Vol. 44, No.2 (1973), 659-670.
21) S. K. BERBERIAN : Lectures in functional analysis and operator theory, chap. 3. Springer-Verlag, Berlin, 1973.
22) R. LARSEN : Functional analysis; an introduction, chaps. 4, 5. Marcel Dekker, Inc., New York, 1973.
23) K. IWATA : Totally ordered linear space structures and separation theorem in real linear topological spaces. Mem. Muroran Inst, Tech. Vol. 8, No.1 (1973), 43-48.
24) M. COTLAR and R. CIGNOLI : An introduction to functional analysis. chap. III, 1, 2. North-Holland Pub. Co., Amsterdam, The Netherlands, 1974.
25) B. ANGER and J. LEMBCKE : Hahn-Banach type extension theorems for hypolinear functionals. Math. Ann. 209 (1974), 127-151.
26) ———— : Hahn-Banach type theorems for hypolinear functionals on preordered topological vector spaces. Pacific J. Math. Vol. 54, No.1 (1974), 13-33.
27) K. IWATA: Totally ordered linear space structures and Hahn-Banach type extension theorem. Mem. Muroran Inst. Tech. Vol. 8, No.2 (1974), 429-434.
28) ———— : Totally ordered linear space structures and extension theorems. Mem. Muroran Inst. Tech. Vol. 8, No.3 (1976), 737-744.
29) ———— : A supplement on the paper "Totally ordered linear space structures and extension theorems". Mem. Muroran Inst. Tech. Vol. 9, No.1 (1976), 335-341.