Congruence Primes of the Kim-Ramakrishnan-Shahidi Lift

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Abstract

For a primitive form f of weight k for $SL_2(\mathbb{Z})$, let $\mathrm{KS}(f)$ be the Kim-Ramakrishnan-Shahidi lift (K-R-S lift) of f to the space of cusp forms of weight $\det^{k+1} \otimes \mathrm{Sym}^{k-2}$ for $Sp_2(\mathbb{Z})$. Based on some working hypothesis, we propose a conjecture, which relates the ratio $\dfrac{\langle \mathrm{KS}(f), \mathrm{KS}(f) \rangle}{\langle f, f \rangle^3}$ of the periods (Petersson norms) to the symmetric 6-th L-value $L(3k-2, f, \mathrm{Sym}^6)$ of f. From this, we also propose that a prime ideal dividing the (conjectural) algebraic part $\mathbf{L}(3k-2, f, \mathrm{Sym}^6)$ of $L(3k-2, f, \mathrm{Sym}^6)$ gives a congruence between the K-R-S lift and non-K-R-S lift, and test this conjecture numerically.

1 Introduction

For a primitive form, that is, a normalized Hecke eigenform f for $SL_2(\mathbb{Z})$ let $\hat{f}$ be a lift of f to a space of (scalar valued or vector valued) modular forms for another group. Here, the lift $\hat{f}$ of f means a Hecke eigenform whose certain L-function is expressed in terms of certain L-functions of f. As examples of lifts, we can take the Doi-Naganuma lift, the Saito-Kurokawa lift, the Duke-Imamoglu-Ikeda lift, the Ikeda-Miyawaki lift, the Yoshida lift, the Kim-Ramakrishnan-Shahidi lift(K-R-S lift), and so on. It is an interesting problem to consider the relation between the period (or the Petersson norm) $\langle \widehat{f}, \widehat{f} \rangle$ of $\widehat{f}$ and the period $\langle f, f \rangle$ of f. Therefore we propose the following problem.

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Problem A. *Express the ratio* $\dfrac{\langle \widehat{f}, \widehat{f} \rangle}{\langle f, f \rangle^e}$ *with some* e *in terms of the special values of certain* L*-functions of* f *(e.g.* $L(s, f, \chi)$, $L(s, f, \mathrm{St})$*), and prove that the above ratio is algebraic.*

In the case where the above problem is solved affirmatively, a prime ideal dividing $\dfrac{\langle \hat{f}, \hat{f} \rangle}{\langle f, f \rangle^e}$ sometimes gives a congruence between $\hat{f}$ and a Hecke eigenform not coming from the lift. Here we say that a prime ideal $\mathfrak{P}$ gives a congruence between two cuspidal Hecke eigenforms if the corresponding Hecke eigenvalues are congruent mod $\mathfrak{P}$. So the following problem seems also interesting:

Problem B. *Characterize primes giving congruence between* $\widehat{f}$ *and another Hecke eigenform not coming from the lifting, in terms of the* L*-values of* f *appearing in Problem A.*

Zagier [45] solved Problem A for the Doi-Naganuma lift $\widehat{f}$ of a primitive form f. Based on this period relation, Doi, Hida, and Ishii [10] proposed a conjecture on the congruence between the Doi-Naganuma lift and non-Doi-Naganuma lift. Kohnen and Skoruppa [29] solved Problem A in the case where f is the Saito-Kurokawa lift of a primitive form f. By using this period relation, Brown [7] and Katsurada [24] independently proved a modification of Harder's conjecture on congruences occurring between Saito-Kurokawa lifts and non-Saito-Kurokawa lifts under mild conditions. Böecherer, Dummigan, and Schulze-Pillot [4] proved the period relation for the Yoshida lift and gave a similar result on the congruence between the Yoshida lift and non-Yoshida lift. Katsurada and Kawamura [26] proved Ikeda's conjecture on the period of the Duke-Imamoglu-Ikeda lift proposed in [23], and by using this period relation Katsurada proved Problem B for the Duke-Imamoglu-Ikeda lift in [25] (see also [8]). Based on the conjectural period relation in [23], Ibukiyama, Katsurada, Poor, and Yuen [18] proposed a conjecture on the congruence of the Ikeda-Miyawaki lift and tested it numerically. Now, for a primitive form f of weight k for $SL_2(\mathbb{Z})$ let us consider the lift $\mathrm{KS}(f)$ of f to the space $S_{k+1,k-2}(Sp_2(\mathbb{Z}))$ of cusp forms of weight $\det^{k+1} \otimes \mathrm{Sym}^{k-2}$ for $Sp_2(\mathbb{Z})$ whose spinor L function $L(s, \mathrm{KS}(f), \mathrm{Sp})$ is the symmetric third L-function $L(s, f, \mathrm{Sym}^3)$ of f. The existence of this type of lifting from modular forms of one variable to Siegel modular forms appeared first in [28] for generic Siegel modular forms and then was proved for holomorphic vector valued Siegel modular forms by [35]. Therefore we call the above $\mathrm{KS}(f)$ the Kim-Ramakrishnan-Shahidi lift or K-R-S lift for short. Ibukiyama and Katsurada [19], among other things, proposed a conjecture on the algebraicity of the period relation of the K-R-S lift and proved the congruence between the K-R-S lift and non-K-R-S lift in some case.

In this paper, we propose a more precise conjecture on the period relation of the K-R-S lift and the congruence between the K-R-S lift and non-K-R-S lift. Moreover we test the congruence between the K-R-S lift and non-K-R-S lift numerically. One of main conjectures in this paper can be stated, roughly speaking, as follows:

"Let $\mathrm{KS}(f)$ be the Kim-Ramakrishnan-Shahidi lift of a primitive form f of weight k for $SL_2(\mathbb{Z})$ and let $\mathbf{L}(3k-2, f, \mathrm{Sym}^6)$ be the (conjectural) algebraic part of the symmetric 6-th L-function of f at $s = 3k-2$. Then a prime ideal dividing $\mathbf{L}(3k-2, f, \mathrm{Sym}^6)$ gives a congruence between $\mathrm{KS}(f)$ and non-Kim-Ramakrishnan-Shahidi lift."

We note that $3k-2$ is immediately to the right of the central point for the functional equation of $L(s, f, \mathrm{Sym}^6)$. We also give numerical examples which support our conjecture in case where $k = 16, 18$ and 20. Let us explain more precisely in the case $k = 16$. In this case, for the unique primitive form f in $S_{16}(SL_2(\mathbb{Z}))$, the K-R-S lift of f to the space $S_{17,14}(Sp_2(\mathbb{Z}))$ is uniquely determined up to a constant multiple, and take a $\mathrm{KS}(f)$ appropriately. Then we can prove that there is a Hecke eigenform G in $S_{17,14}(Sp_2(\mathbb{Z}))$ such that

$$c_{\mathrm{KS}(f)}(T) \equiv c_G(T) \bmod \mathfrak{P},$$

for any positive definite half-integral symmetric matrix T of degree two, where $c_*(T)$ is the T-th Fourier coefficient of a Siegel modular form, and $\mathfrak{P}$ is a prime ideal in the Hecke field of G lying above 92467. From this, we easily see that $\mathfrak{P}$ gives a congruence between $\mathrm{KS}(f)$ and G. On the other hand, at present there is no way of computing the algebraic part $\mathbf{L}(46, f, \mathrm{Sym}^6)$ rigorously. However, by using Dokchitser's L-calculator [11], we can numerically check that it is divisible by 92467. In other cases, we can also test the above conjecture numerically.

This paper is organized as follows. In Section 2, we review on the vector valued modular forms. In Section 3, we review on the critical values of the higher symmetric power L-functions of elliptic modular forms, and in Section 4, we review on the critical values of the standard L-functions of Siegel modular forms. In Section 5, we review the conjecture on the algebraicity of the period relation of the Kim-Ramakrishnan-Shahidi lift in [19]. In Section 6, we propose a more precise conjecture on the period of the Kim-Ramakrishnan-Shahidi lift based on some heuristic argument, and also propose a conjecture on the congruence of the Kim-Ramakrishnan-Shahidi lift. Finally, in Section 7, we give numerical examples.

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Notation. Let R be a commutative ring. We denote by $R^{\times}$ and R^* the semigroup of non-zero elements of R and the unit group of R, respectively. We also put $S^{\square} = \{a^2 \mid a \in S\}$ for a subset S of R. We denote by $M_{mn}(R)$ the set of $m \times n$-matrices with entries in R. In particular put $M_n(R) = M_{nn}(\mathbb{R})$. Put $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, where $\det A$ denotes the determinant of a square matrix A. For an $m \times n$-matrix X and an $m \times m$-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Let $S_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, if R is an integral domain of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathcal{H}_n(R)$ is the subset of symmetric matrices of degree n with entries in the field of fractions of R whose (i,j)-component belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. In particular, we put $\mathcal{H}_n = \mathcal{H}_n(\mathbb{Z})$. For a subset S of $M_n(R)$ we denote by $S^{\times}$ the subset of S consisting of non-degenerate matrices, that is, matrices with non-zero determinant. If S is a subset of $S_n(\mathbb{R})$ with $\mathbb{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. The group $GL_n(R)$ acts on the set $S_n(R)$ in the following way:

$$GL_n(R) \times S_n(R) \ni (g, A) \longmapsto A[g] \in S_n(R).$$

Let G be a subgroup of $GL_n(R)$. For a G-stable subset $\mathcal{B}$ of $S_n(R)$ we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ under the action of G. We sometimes use the same symbol $\mathcal{B}/G$ to denote a complete set of representatives of $\mathcal{B}/G$. We abbreviate $\mathcal{B}/GL_n(R)$ as $\mathcal{B}/\sim$ if there is no fear of confusion. Let R' be a subring of R. Then two symmetric matrices A and A' with entries in R are said to be equivalent over R' with each other and write $A \sim_{R'} A'$ if there is an element X of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X \perp Y =$

$\begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

For an integer $D \in \mathbb{Z}$ such that $D \equiv 0$ or $\equiv 1 \bmod 4$, let $\mathfrak{d}_D$ be the discriminant of $\mathbb{Q}(\sqrt{D})$, and put $\mathfrak{f}_D = \sqrt{\frac{D}{\mathfrak{d}_D}}$. We call an integer D a fundamental discriminant if it is either 1 or the discriminant of some quadratic extension of $\mathbb{Q}$. For a fundamental discriminant D, let $\left(\frac{D}{*}\right)$ be the character corresponding to $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. Here we make the convention that $\left(\frac{D}{*}\right) = 1$ if $D = 1$.

We put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbb{C}$. For a prime ideal $\mathfrak{P}$ of the ring of integers of an algebraic number field K we denote by $\mathrm{ord}_{\mathfrak{P}}(*)$ the additive valuation of the $\mathfrak{P}$-adic field $K_{\mathfrak{P}}$ normalized so that $\nu_{\mathfrak{P}}(\varpi) = 1$, where ϖ is a prime element of $K_{\mathfrak{P}}$.

2 Siegel modular forms

For any natural number n, we denote by $\mathbb{H}_n$ the Siegel upper half space of degree n.

$$\mathbb{H}_n = \{Z \in M_n(\mathbb{C}); Z = {}^tZ = X + \sqrt{-1}Y,\ X, Y \in M_n(\mathbb{R}), Y > 0\}.$$

For any ring R and any natural integer n, we define the symplectic group $GSp_n(R)$ over R with symplectic similitudes by

$$GSp_n(R) = \{g \in M_{2n}(R);\ {}^tgJ_ng = \nu(g)J_n \text{ with some } \nu(g) \in R^\times\},$$

and

$$Sp_n(R) = \{g \in M_{2n}(R);\ {}^tgJ_ng = J_n\},$$

where $J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. In particular, if R is the field $\mathbb{R}$ of real numbers, we put

$$GSp_n(\mathbb{R})^+ = \{g \in GSp_n(\mathbb{R}) \mid \nu(g) > 0\}.$$

We put $\Gamma^{(n)} = Sp_n(\mathbb{Z})$ for the sake of simplicity. For a positive integer d, we define the paramodular group $\Gamma^{\mathrm{para}}(d)$ of degree two of level d as

$$\Gamma^{\mathrm{para}}(d) = \{g \in M_4(\mathbb{Z}) \mid {}^tgJ_2(d)g = J_2(d)\},$$

where $J_2(d) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d \\ -1 & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \end{pmatrix}$. A modular form for the paramodular group is called a paramodular form. For any irreducible representation (ρ, V) of $GL(n, \mathbb{C})$, for any V-valued function F on $\mathbb{H}_n$, and for any $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_n(\mathbb{R})$, we write

$$F|_\rho[\gamma] = \rho(CZ+D)^{-1}F(\gamma Z).$$

Let Γ be an arithmetic subgroup of $Sp_n(\mathbb{Q})$ commensurable with $\Gamma^{(n)}$, and χ be a character of Γ. We say that F is a holomorphic Siegel modular form of weight ρ and character χ for Γ if F is holomorphic on $\mathbb{H}_n$ and $F|_\rho[\gamma] = \chi(\gamma)F$ for any $\gamma \in \Gamma$ (with the extra condition of holomorphy at cusps if $n = 1$). We denote by $\mathbb{C}[u_1, \cdots, u_n]_m$ the vector space of homogeneous polynomials of degree m in $u_1, \cdots, u_n$. Then the m-th symmetric tensor representation Sym^m of $GL(n, \mathbb{C})$ is defined by

$$\mathrm{Sym}^m(g)(P(u_1, \cdots, u_n)) = P((u_1, \cdots, u_n)g)$$

for any $g \in GL(n, \mathbb{C})$ and $P \in \mathbb{C}[u_1, \cdots, u_n]_m$. For any integer k, we denote by $\det^k$ the representation of $GL(n, \mathbb{C})$ given by $\det^k(g) = (\det(g))^k$ for any $g \in GL(n, \mathbb{C})$. We put $\rho_{k,m} = \det^k \otimes \mathrm{Sym}^m$. We denote by $M_{k,m}(\Gamma, \chi)$ the vector space of holomorphic Siegel modular forms of weight $\rho_{k,m}$ and character χ for Γ. Let F be an element of $M_{k,m}(\Gamma, \chi)$. Then for any $g \in Sp_n(\mathbb{Q})$, F has the following Fourier expansion.

$$F|_{\rho_{k,m}}[g](Z) = \sum_{A \in S_n(\mathbb{Q})_{\geq 0}} c_{F,g}(A; u) \exp(2\pi\sqrt{-1}\mathrm{tr}(AZ)),$$

where $c_{F,g}(A)$ is a homogeneous polynomial of degree m in $u = (u_1, \cdots, u_n)$ with coefficients in $\mathbb{C}$ for any $A \in S_n(\mathbb{Q})_{\geq 0}$, Therefore, we often write $c_{F,g}(A)$ as $c_{F,g}(A; u)$. If $g = 1_{2n}$, we write $c_{F,g}(A)$ and $c_{F,g}(A; u)$ as $c_F(A)$ and $c_F(A; u)$, respectively. We say that F is a cusp form if $c_{F,g}(A; u) = 0$ unless $A \in S_n(\mathbb{Q})_{>0}$. We denote by $S_{k,m}(\Gamma, \chi)$ the space of cusp forms in $M_{k,m}(\Gamma, \chi)$. If χ is the trivial character, we simply write $M_{k,m}(\Gamma, \chi)$ and $S_{k,m}(\Gamma, \chi)$ as $M_{k,m}(\Gamma)$ and $S_{k,m}(\Gamma)$, respectively. Let F be an element of $S_{k,m}(\Gamma)$ with the following Fourier expansion:

$$F(Z) = \sum_{A \in \mathcal{H}_n(\mathbb{Z})_{>0}} c_F(A; u) \exp(2\pi\sqrt{-1}\mathrm{tr}(AZ)).$$

For a fundamental discriminant $D < 0$ we then define the $|D|$-th Bessel function of F as

$$B_F(|D|) = \frac{\sqrt{|D|}}{2} \sum_{\substack{A \in \mathcal{H}_2(\mathbb{Z})_{>0}/SL_2(\mathbb{Z}) \\ 4\det A = |D|}} \frac{1}{\#SO(A)} \int_{A[u] \leq 1} c_F(A; u) du,$$

where $SO(A)$ is the special orthogonal group of A. In the case where $m=0$ we have

$$B_F(|D|) = \frac{1}{\sqrt{|D|}} \sum_{\substack{A \in \mathcal{H}_2(\mathbb{Z})_{>0}/SL_2(\mathbb{Z}) \\ 4\det A = |D|}} \frac{c_F(A)}{\#SO(A)}.$$

We note that any element F of $M_{k,m}(\Gamma^{(n)})$ has the following Fourier expansion.

$$F(Z) = \sum_{A \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} c_F(A;u) \exp(2\pi\sqrt{-1}\mathrm{tr}(AZ)),$$

and in particular, if F is a cusp form then

$$F(Z) = \sum_{A \in \mathcal{H}_n(\mathbb{Z})_{>0}} c_F(A;u) \exp(2\pi\sqrt{-1}\mathrm{tr}(AZ)).$$

We also note that a paramodular form has also a Fourier expansion similar to above.

For any ring $R \subset \mathbb{C}$, we denote by $M_{k,m}(\Gamma^{(n)})(R)$ the R-submodule of $M_{k,m}(\Gamma^{(n)})$ consisting of modular forms whose A-th Fourier coefficient belongs to $R[u_1, \cdots, u_n]_m$ for all $A \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}$. We consider Siegel modular forms of genus 2. We note that any rational irreducible representation of $GL(2,\mathbb{C})$ is given by $\rho_{k,m}$ up to equivalence. In particular, we have $M_{k,m}(\Gamma^{(2)}) = 0$ if m is odd and $M_{k,m}(\Gamma^{(2)}) = S_{k,m}(\Gamma^{(2)})$ if k is odd.

For any non-negative integers $a_1, \cdots, a_n, b_1, \cdots, b_n$ with $a_1 + \cdots + a_n = b_1 + \cdots + b_n = m$, we put

$$\langle u_1^{a_1} \cdots u_n^{a_n}, u_1^{b_1} \cdots u_n^{b_n} \rangle = \frac{a_1! \cdots a_{n-1}! b_1! \cdots b_{n-1}!}{m!} \delta_{a_1 b_1} \cdots \delta_{a_{n-1}, b_{n-1}},$$

where δ_{ij} is Kronecker's delta. We define the hermitian inner product on $\mathbb{C}[u_1, \cdots, u_n]_m$ by extending this linearly. Then we have

$$\langle \rho_{k,m}(A)x, y \rangle = \langle x, \rho_{k,m}({}^t\overline{A})y \rangle$$

for any x, $y \in \mathbb{C}[u_1, \cdots, u_n]_m$ and $A \in GL_n(\mathbb{C})$. Let Γ be a subgroup of $Sp_n(\mathbb{Q})$ commensurable with $\Gamma^{(n)}$. We define the volume $\mathrm{vol}(\Gamma)$ by

$$\mathrm{vol}(\Gamma) = \int_{\Gamma \backslash \mathbb{H}_n} \det Y^{-n-1} dX dY.$$

For two vector valued Siegel cusp forms $F, G \in S_{k,m}(\mathit{\Gamma})$, we define the inner product $\langle F, G\rangle$ of F and G by

$$\langle F, G\rangle = \mathrm{vol}(\mathit{\Gamma}^{(n)})\mathrm{vol}(\mathit{\Gamma})^{-1}$$
$$\times \int_{\mathit{\Gamma}\backslash\mathbb{H}_n} \langle \rho_{k,m}(\sqrt{Y})F(Z), \rho_{k,m}(\sqrt{Y})G(Z)\rangle \det(Y)^{-n-1}\, dX\, dY$$

where $Z = X + \sqrt{-1}Y$ and $dX = \wedge_{1\le i\le j\le n} dx_{ij}$, $dY = \wedge_{1\le i\le j\le n} dy_{ij}$ for $X = (x_{ij})$, $Y = (y_{ij}) \in M_n(\mathbb{R})$. In particular, we call $\langle F, F\rangle$ the period or the Petersson norm of F. We denote by $\widetilde{\mathbf{L}}_n$ the $\mathbb{Q}$-vector space whose genenerators over $\mathbb{Q}$ are the symbols $\mathit{\Gamma}\alpha\mathit{\Gamma}$ ($\alpha \in GSp_n(\mathbb{Q})^+$). The vector space $\widetilde{\mathbf{L}}_n$ becomes a commutative ring with a certain multiplication. Any $T \in \widetilde{\mathbf{L}}_n$ acts on $M_{k,m}(\mathit{\Gamma}^{(n)})$. We write the action of T on $F \in M_{k,m}(\mathit{\Gamma}^{(n)})$ as $F|T$. An element F of $S_{l,j}(\mathit{\Gamma}^{(n)})$ is called a Hecke eigenform (for $\widetilde{\mathbf{L}}_n$) if $F|T = \lambda_F(T)F$ with $\lambda_F(T) \in \mathbb{C}$ for any $T \in \widetilde{\mathbf{L}}_n$. We call $\lambda_F(T)$ the Hecke eigenvalue of T for F. For a Hecke eigenform $F \in S_{k,j}(\mathit{\Gamma}^{(n)})$, we denote by $\mathbb{Q}(F)$, the field generated over $\mathbb{Q}$ by all the Hecke eigenvalues of F. We also have a Hecke theory for any congruence subgroup of $\mathit{\Gamma}^{(n)}$ and any paramaodular group.

3 Symmetric power L-functions of elliptic modular forms

Let

$$f = \sum_{n=1}^{\infty} a(n) \exp(2\pi\sqrt{-1}nz)$$

be a primitive form in $S_k(SL_2(\mathbb{Z}))$. For a prime number p let α_p, β_p be complex numbers such that $\alpha_p + \beta_p = a(p)$ and $\alpha_p\beta_p = p^{k-1}$. For a Dirichlet character χ define the symmetric j-th L function of f twisted by χ as

$$L(s, f, \mathrm{Sym}^j, \chi) = \prod_{p:prime} \prod_{i=0}^{j} (1 - \alpha_p^i \beta_p^{j-i} p^{-s}\chi(p))^{-1}.$$

In particular, put $L(s, f) = L(s, f, \mathrm{Sym}^1)$. It is proved that $L(s, \mathrm{Sym}^j, \chi)$ is continued meromorphically to the whole s plane and that for small j it satisfies the functional equation for $s \to (k-1)j + 1 - s$ (cf. [42] for example.) To consider the algebraicity of $L(s, f, \mathrm{Sym}^j, \chi)$ let

$$\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$$

and put

$$\Lambda(s,\, f) = \Gamma_{\mathbb{C}}(s)L(s,\, f).$$

Then there exist $\Omega_{\pm} = \Omega_{\pm}(f) \in \mathbb{R}^{\times}$ so that

$$\frac{\Lambda(j, f)}{\Omega_{(-1)^j}} \in \mathbb{Q}(f)$$

for $1 \le j \le k-1$ (cf. [38]). Here, $\Omega_{(-1)^j}$ is Ω_+ or Ω_- according as $(-1)^j$ is 1 or -1. To consider the algebraicity of the higher symmetric power L-functions, for an odd integer $m = 2r-1$ and an integer $(k-1)(r-1)/2 < l \le (k-1)r/2$ put

$$\mathbf{L}(l, f, \mathrm{Sym}^m) = \frac{L(l, f, \mathrm{Sym}^m)}{(2\pi)^{rl-r(r-1)(k-1)/2}\Omega^r_{(-1)^l}\langle f, f\rangle^{r(r-1)/2}}$$

and for an even integer $m = 2r$ and an even integer $(k-1)r < l \le (k-1)(r+1)$ put

$$\mathbf{L}(l, f, \mathrm{Sym}^m) = \frac{L(l, f, \mathrm{Sym}^m)}{(2\pi)^{(r+1)l-r(r+1)(k-1)/2}\langle f, f\rangle^{r(r+1)/2}}.$$

Then the following conjecture is a special case of the conjecture propose by Deligne [9].

Conjecture 3.1. *(Deligne's Conjecture)* $\mathbf{L}(l, f, \mathrm{Sym}^m) \in \mathbb{Q}(f)$.

We note that $\frac{\langle f,f\rangle}{\Omega_+\Omega_-} \in \mathbb{Q}(f)$, and that Deligne originally formulated the above conjecture in terms of Ω_+ and Ω_-. Deligne's conjecture holds true for $m = 1, 2$. (cf. Shimura [38], Sturm [41]) In the case of $m = 3$, more precisely we have the following:

Proposition 3.2. *(Orloff [34], Satoh [36], Böecherer and Panchishkin [5]) Let f be a primitive form in $S_k(SL_2(\mathbb{Z}))$. For an integer l such that $(k-1)/2 < l \le k-1$, and for a primitive Dirichlet character χ mod M such that $\chi^2 = 1$ put*

$$\mathbf{L}(l, f, \mathrm{Sym}^3, \chi) = \frac{L(l, f, \mathrm{Sym}^3, \chi)}{(2\pi)^{2l-(k-1)}\Omega_*^2\langle f, f\rangle\sqrt{M}},$$

where $\Omega_* = \Omega_+$ or Ω_- according as $\chi(-1) = (-1)^l$ or $\chi(-1) = (-1)^{l+1}$. Then $\mathbf{L}(l, f, \mathrm{Sym}^3, \chi) \in \mathbb{Q}(f)$.

4 Special values of the standard L-functions

In this section we define three L-functions, the spinor L-function, the standard L-function, and the adjoint L-function for a cuspidal Hecke eigenform F in $S_{k,m}(\Gamma^{(2)})$. For a prime number p let $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}$ be the Satake p-parameters of F. We then first define the spinor L-function $L(s, F, \mathrm{Sp}, \chi)$ of F twisted by a Dirichlet character χ as

$$L(s, F, \mathrm{Sp}, \chi)$$
$$= \prod_p \{(1 - \alpha_{0,p} p^{-s} \chi(p))(1 - \alpha_{0,p}\alpha_{1,p} p^{-s}\chi(p))(1 - \alpha_{0,p}\alpha_{2,p} p^{-s}\chi(p))$$
$$\times (1 - \alpha_{0,p}\alpha_{1,p}\alpha_{2,p} p^{-s}\chi(p))\}^{-1}.$$

Next we define the standard L-function $L(s, F, \mathrm{St})$ of F as

$$L(s, F, \mathrm{St}) = \prod_p \{(1-p^{-s})(1-\alpha_{1,p}p^{-s})(1-\alpha_{1,p}^{-1}p^{-s})(1-\alpha_{2,p}p^{-s})(1-\alpha_{2,p}^{-1}p^{-s})\}^{-1}.$$

Here we normalize the Satake p-parameters $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}$ so that $L(s, F, \mathrm{St})$ and $L(s, F, \mathrm{Sp})$ satisfy the functional equations for $s \longrightarrow 1 - s$ and for $s \longrightarrow 2k + m - 2 - s$, respectively. Finally we define the adjoint L-function $L(s, F, \mathrm{Ad})$ as

$$L(s, F, \mathrm{Ad})$$
$$= \prod_p \{(1 - p^{-s})^2 (1 - \alpha_{1,p}p^{-s})(1 - \alpha_{1,p}^{-1}p^{-s})(1 - \alpha_{2,p}p^{-s})(1 - \alpha_{2,p}^{-1}p^{-s})$$
$$\times (1-\alpha_{1,p}\alpha_{2,p}p^{-s})(1-\alpha_{1,p}^{-1}\alpha_{2,p}p^{-s})(1-\alpha_{1,p}\alpha_{2,p}^{-1}p^{-s})(1-\alpha_{1,p}^{-1}\alpha_{2,p}^{-1}p^{-s})\}^{-1}.$$

Now we review on the algebraicity of the standard L-function. For an even positive integer l, we define the Siegel Eisenstein series $E_{4,l}(Z, s)$ of degree 4 by

$$\begin{aligned} E_{4,l}(Z, s) &= \zeta(1 - l - 2s)\zeta(3 - 2l - 4s)\zeta(5 - 2l - 4s) \\ &\times \sum_{g=\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{(4)}{}_\infty \backslash \Gamma^{(4)}} \det(CZ + D)^{-l} (\det(\mathrm{Im}(g(Z))))^s \end{aligned}$$

$(Z \in \mathbb{H}_4, s \in \mathbb{C})$, where $\zeta(*)$ is Riemann's zeta function, and $\Gamma^{(4)}{}_\infty = \left\{ \left(\begin{smallmatrix} * & * \\ 0_4 & * \end{smallmatrix}\right) \in \Gamma^{(4)} \right\}$. This series converges for $2Re(s) + l > 5$ and is continued meromorphically to the whole plane as a function of s. Furthermore, for $l \geq 4$, $E_{4,l}(Z, 0)$ is a holomorphic Siegel modular form of weight l as a function of Z (cf. [39]). From now on we assume that $E_{4,l}(Z,0)$ is holomorphic as a function of Z, and write $E_{4,l}(Z) = E_{4,l}(Z,0)$. Put $V_1^{(m)} = \mathbb{C}[u_1,u_2]_m$ and $V_2^{(m)} = \mathbb{C}[u_3,u_4]_m$ and denote by $\mathrm{Hol}(\mathbb{H}_2, V_i^{(m)})$ the space of $V_i^{(m)}$-valued holomorphic functions on $\mathbb{H}_2$. We naturally identify elements in $V_1^{(m)} \otimes V_2^{(m)}$ with polynomials in $u_1,\ u_2,\ u_3,\ u_4$. We also denote by $\mathrm{Hol}(\mathbb{H}_4, V_1^{(m)} \otimes V_2^{(m)})$ the space of $V_1^{(m)} \otimes V_2^{(m)}$-valued functions on $\mathbb{H}_4$. We note that $\mathrm{Hol}(\mathbb{H}_2, V_1^{(0)}) = \mathrm{Hol}(\mathbb{H}_2, V_2^{(0)})$ is the space $\mathrm{Hol}(\mathbb{H}_2,\mathbb{C})$ of scalar valued holomorphic functions on $\mathbb{H}_2$ and $\mathrm{Hol}(\mathbb{H}_4, V_1^{(0)} \otimes V_2^{(0)})$ is the space $\mathrm{Hol}(\mathbb{H}_4,\mathbb{C})$ of scalar valued holomorphic functions on $\mathbb{H}_4$. Let $\widetilde{\mathcal{D}}_l^{k-l} = \tilde{\mathcal{D}}_{k-1} \circ \cdots \tilde{\mathcal{D}}_{l+1} \circ \tilde{\mathcal{D}}_l$ be the holomorphic differential operator acting on $\mathrm{Hol}(\mathbb{H}_4,\mathbb{C})$ defined in Böcherer [3]. Moreover let $L^{(m)}$ be the holomorphic vector valued differential operator defined in Böecherer, Satoh and Yamazaki [6]. It maps $\mathrm{Hol}(\mathbb{H}_4,\mathbb{C})$ to $\mathrm{Hol}(\mathbb{H}_2 \times \mathbb{H}_2, V_1^{(m)} \otimes V_2^{(m)})$. For any scalar valued holomorphic function f on $\mathbb{H}_4$, we write

$$\mathcal{D}_{l,(k,m)}(f) = \frac{1}{(2\pi\sqrt{-1})^{2(k-l)}}(L^{(m)}\tilde{\mathcal{D}}_l^{k-l}(f)).$$

Then $\mathcal{D}_{l,(k,m)}$ maps holomorphic functions on $\mathbb{H}_4$ to $\mathrm{Hol}(\mathbb{H}_2, V_1^{(m)}) \otimes \mathrm{Hol}(\mathbb{H}_2, V_2^{(m)})$. In particular, it preserves automorphy after restriction and maps $M_l(\Gamma^{(4)})$ to $M_{k,m}(\Gamma^{(2)}) \otimes M_{k,m}(\Gamma^{(2)})$. The image is contained in $S_{k,m}(\Gamma^{(2)}) \otimes S_{k,m}(\Gamma^{(2)})$ if $k-l>0$. For $Z=(z_{ij}) \in \mathbb{H}_4$ we write $\partial_{ij} = \frac{\delta_{ij}+1}{2}\frac{\partial}{\partial z_{ij}}$, and for an integer a and a non-negative integer μ put $(a)_\mu = a(a+1)\cdots(a+\mu-1)$. Then we note that $\widetilde{\mathcal{D}}_l^{k-l}$ and $\tilde{L}^{k,m}$ can be expressed as

$$\widetilde{\mathcal{D}}_l^{k-l} = \frac{1}{\prod_{\alpha=l}^{k}(\alpha-3/2)(\alpha-1)}$$

$$\times P(\partial_{ij}\ (1 \le i \le j \le 4),\ z_{ij}\ (1 \le i \le 2, 3 \le j \le 4)),$$

and

$$L^{(m)} = \frac{1}{(2\pi\sqrt{-1})^m (k)_m \prod_{\mu=0}^{[m/2]} \mu!(m-2\mu)!(1-k-m)_\mu}$$

$$\times Q(\partial_{ij}\ (1 \le i \le j \le 4),\ u_i\ (1 \le i \le 4))\big|_{\mathbb{H}_2\times\mathbb{H}_2},$$

where $P(X_{ij}\ (1 \le i \le j \le 4),\ z_{ij}\ (1 \le i \le 2, 3 \le j \le 4))$ is a polynomial in $X_{ij}\ (1 \le i \le j \le 4),\ z_{ij}\ (1 \le i \le 2, 3 \le j \le 4)$ with coefficients in $\mathbb{Z}$, and $Q(X_{ij}\ (1 \le i \le j \le 4),\ u_i\ (1 \le i \le 4))$ is a polynomial in $X_{ij}\ (1 \le i \le j \le 4),\ u_i\ (1 \le i \le 4)$ with coefficients in $\mathbb{Z}$, and $\big|_{\mathbb{H}_2 \times \mathbb{H}_2}$ means the restriction of functions of $Z = \begin{pmatrix} Z_1 & Z_{12} \\ Z_{12} & Z_2 \end{pmatrix} \in \mathbb{H}_4$ to the set $\mathbb{H}_2 \times \mathbb{H}_2 \cong \left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}; Z_i \in \mathbb{H}_2 \right\}$ (see, [12], pages 1312-1322). Hence, $\mathcal{D}_{l,(k,m)}$ can be expressed as

$$\mathcal{D}_{l,(k,m)} = \frac{1}{a_{l,(k,m)}(2\pi\sqrt{-1})^{2(k-l)+m}}$$
$$\times R(\partial_{ij}\ (1 \le i \le j \le 4),\ u_i\ (1 \le i \le 4)),$$

where

$$a_{l,(k,m)} = \prod_{\alpha=l}^{k} (\alpha - 3/2)(\alpha - 1) \prod_{\mu=0}^{[m/2]} \mu!(m-2\mu)!(1-k-m)_\mu,$$

and $R(X_{ij}\ (1 \le i \le j \le 4),\ u_i\ (1 \le i \le 4))$ is a polynomial in $X_{ij}\ (1 \le i \le j \le 4),\ u_i\ (1 \le i \le 4)$ with coefficients in $\mathbb{Z}$. We note that $a_{l,(k+1,k-2)}$ is an integer whose prime divisor is not greater than $2k-1$ for $4 \le l \le k$. For $R, S \in S_n(\mathbb{Q})$ and $T \in M_2(\mathbb{Q})$, put $(m_{ij})_{4\times 4} = \begin{pmatrix} R & T \\ {}^tT & S \end{pmatrix}$ and put

$$\phi_{l,(k,m)}(R, S, T; u_i\ (1 \le i \le 4)) = R(m_{ij}\ (1 \le i \le j \le 4), u_i(1 \le i \le 4)).$$

For $F \in S_{k,m}(\mathit{\Gamma}^{(4)})$, we define $\Lambda(r, F, \mathrm{St})$ by

$$\Lambda(r, F, \mathrm{St}) = 2^{14-6k-r-2m}(-1)^{r/2} \times \frac{\Gamma(r+1)\Gamma(2r)\Gamma(2k+m-3)}{(k-2)(k)_m m!}$$
$$\times \frac{L(r, F, \mathrm{St})}{\pi^{2k+m+3r-3}\langle F, F\rangle}.$$

(This is the same as the definition in [31] or [12] for $n = 2$, though apparently different looking.) Then the following result is a special case of the pullback formula for the Siegel Eisenstein series in [3], [6] and [31]. We can take a basis $F_1, \ldots, F_d$ of $S_{k,m}(\mathit{\Gamma}^{(2)})$ so that F_i belong to $S_{k,m}(\mathit{\Gamma}^{(2)})(\mathbb{Q}(F_i))$ for any $1 \le i \le d$. We define the function $\widetilde{F}_{l,(k,m)}(Z_1, Z_2)$ on $\mathrm{Hol}(\mathbb{H}_2, V_1^{(m)}) \otimes \mathrm{Hol}(\mathbb{H}_2, V_2^{(m)})$ by

$$\widetilde{F}_{l,(k,m)}(Z_1, Z_2) = \mathcal{D}_{l,(k,m)}(E_{4,l})(Z_1, Z_2).$$

We write the Fourier expansion of $\widetilde{F}_{l,(k,m)}(Z_1, Z_2)$ by

$$\widetilde{F}_{l,(k,m)}(Z_1, Z_2) = \sum_{R,S \in \mathcal{H}_2(\mathbb{Z})_{>0}} \epsilon_{l,(k,m)}(R, S; U) \exp(2\pi\sqrt{-1}\mathrm{tr}(RZ_1 + SZ_2)),$$

where $\epsilon_{l,(k,m)}(R, S; U)$ is a polynomial in $U = (u_1, u_2, u_3, u_4)$. Then for fixed R and S, we have

$$\epsilon_{l,(k,m)}(R, S; U) = \sum_{T \in M_2(\mathbb{Z})} c_{4,l}\left(\begin{pmatrix} R & T/2 \\ {}^tT/2 & S \end{pmatrix}\right) \phi_{l,(k,m)}(R, S, T/2; U),$$

where $c_{4,l}(\mathcal{T})$ denotes the Fourier coefficient of $E_{4,l}$ at $\mathcal{T} \in \mathcal{H}_4(\mathbb{Z})_{\geq 0}$ and is regarded as zero if $\mathcal{T}$ is not positive semi-definite. For a fixed $S \in \mathcal{H}_2(\mathbb{Z})_{>0}$, we write

$$G_{l,(k,m),S}(Z_1) = \sum_{R \in \mathcal{H}_2(\mathbb{Z})_{>0}} \epsilon_{l,(k,m)}(R, S; U) \exp(2\pi\sqrt{-1}\mathrm{tr}(RZ_1)).$$

Then we have

$$\widetilde{F}_{l,(k,m)}(Z_1, Z_2) = \sum_{S \in \mathcal{H}_2(\mathbb{Z})_{>0}} G_{l,(k,m),S}(Z_1) \exp(2\pi\sqrt{-1}\mathrm{tr}(SZ_2)).$$

For each $S \in \mathcal{H}_2(\mathbb{Z})_{>0}$, we denote by $c_i(S, v)$ the Fourier coefficient of $F_i(Z_2)$ at $S \in \mathcal{H}_2(\mathbb{Z})_{>0}$, which are polynomials in $v = (u_3, u_4)$. Then we have the following.

Theorem 4.1. *Let $F_1, \cdots, F_d$ be a basis of $S_{k,m}(\Gamma^{(2)})$ consisting of Hecke eigenforms. Assume that $F_i \in S_{k,m}(\Gamma^{(2)})(\mathbb{Q}(F_i))$ for any $i = 1, \cdots, d$. Then for any $S \in \mathcal{H}_2(\mathbb{Z})_{>0}$ and for any even integer l such that $4 \leq l \leq k-2$, we have*

$$G_{l,(k,m),S}(Z_1) = \sum_{i=1}^{d} c_{F_i}(S; v)\Lambda(l-2, F_i, \mathrm{St})F_i(Z_1), \tag{1}$$

where $v = (u_3, u_4)$.

Remark 4.2. We always take a basis $F_1, \cdots, F_d$ of $S_{k,m}(\Gamma^{(2)})$ satisfying the conditions in the above theorem.

Corollary 4.3. *Let F be a Hecke eigenform in $S_{k,m}(\Gamma^{(2)})(\mathbb{Q}(F))$. Then $\Lambda(l-2, F, \mathrm{St}) \in \mathbb{Q}(F)$ for any even integer l such that $4 \leq l \leq k-2$.*

5 Period and congruence of the K-R-S lift

For any $F \in S_{k,j}(\Gamma^{(2)})$, let $L(s, F, \mathrm{St})$ and $L(s, F, \mathrm{Sp})$ be the standard L function and the spinor L function, respectively, of F, normalized as in Section 4. Now we quote a part of the theorem in [35] we need.

Theorem 5.1 (Ramakrishnan-Shahidi [35])**.** *For any primitive Hecke eigenform $f \in S_k(SL_2(\mathbb{Z}))$, there exists a holomorphic Siegel modular form $F \in S_{k+1,k-2}(\Gamma^{(2)})$ which is a Hecke eigenform such that*

$$L(s, f, \mathrm{Sym}^3) = L(s, F, \mathrm{Sp}).$$

We call the above F the Kim-Ramakrishnan-Shahidi lift as in Section 1. We note that Ibukiyama [17] gave a precise conjecture on these liftings with numerical experiments on Δ and F as cited in [35]. The above theorem is a correspondence between automorphic representations, so even if multiplicity one theorem holds, F is defined only up to constants and there is no canonical way to choose normalization of F at moment. But from now on we write this F as $\mathrm{KS}(f)$. We note that $\mathbb{Q}(\mathrm{KS}(f)) = \mathbb{Q}(f)$, and therefore we can take $\mathrm{KS}(f)$ so that $\mathrm{KS}(f) \in S_{k+1,k-2}(\Gamma^{(2)})(\mathbb{Q}(f))$. We can easily see that for the same f and $\mathrm{KS}(f)$ above, we have also

$$L(s + 2k - 2, f, \mathrm{Sym}^4) = L(s, \mathrm{KS}(f), \mathrm{St})$$

by checking the relation between Satake parameters. Hence by Corollary 4.3, we have

Theorem 5.2. *For any primitive form $f \in S_k(SL_2(\mathbb{Z}))$, let $\mathrm{KS}(f)$ be the K-R-S lift of f such that $\mathrm{KS}(f) \in S_{k+1,k-2}(\mathbb{Q}(f))$. Then Deligne's conjecture holds true for the critical values of $L(s, f, \mathrm{Sym}^4)$ if and only if $\dfrac{\langle \mathrm{KS}(f), \mathrm{KS}(f)\rangle}{\langle f, f\rangle^3} \in \mathbb{Q}(f)$.*

Hence, taking Theorem 5.2 into account, we propose the following conjecture.

Conjecture 5.3. *Let f be a primitive form in $S_k(SL_2(\mathbb{Z}))$. Assume that $\mathrm{KS}(f) \in S_{k+1,k-2}(\Gamma^{(2)})(\mathbb{Q}(f))$. Then $\frac{\langle \mathrm{KS}(f), \mathrm{KS}(f)\rangle}{(f,f)^3} \in \mathbb{Q}(f)$.*

Assuming the above conjecture, we want to know more precise information about the $\langle \mathrm{KS}(f), \mathrm{KS}(f)\rangle/(f, f)^3$. Hence we would like to propose the following question.

Question. What L-value divides $\dfrac{\langle \mathrm{KS}(f), \mathrm{KS}(f)\rangle}{\langle f, f\rangle^3}$?
In the next section, we consider this question.

Now we consider the congruence of Siegel modular forms. We denote by $\mathbf{L}_n$ the $\mathbb{Z}$-free module whose genenerators over $\mathbb{Z}$ are the symbols $\Gamma\alpha\Gamma$ ($\alpha \in GSp_n(\mathbb{Q})^+ \cap M_{2n}(\mathbb{Z})$). Then $\mathbf{L}_n$ is a subring of $\widetilde{\mathbf{L}}_n$ and it acts on $M_{k,j}(\Gamma^{(n)})$. Moreover, if $k \geq n+1$, then for any $T \in \mathbf{L}_n$ and $F \in M_{k,j}(\Gamma^{(n)})(\mathbb{Z})$, we have $F|T \in M_{k,j}(\Gamma^{(n)})(\mathbb{Z})$. Let F be a Hecke eigenform in $S_{k,j}(\Gamma^{(n)})$. First we note that $\lambda_F(T) \in \mathfrak{O}_{\mathbb{Q}(F)}$ for any $T \in \mathbf{L}_n$. Let M be a Hecke stable subspace of $S_{k,j}(\Gamma^{(n)})$ such that $M \subset (\mathbb{C}F)^{\perp}$, where $(\mathbb{C}F)^{\perp}$ is the orthogonal complement of $\mathbb{C}F$ in $S_{k,j}(\Gamma^{(n)})$ with respect to the Petersson product. Let K be a algebraic number field containing $\mathbb{Q}(F)$. A prime ideal $\mathfrak{P}$ in K is called a congruence prime of F with respect to M if there exists a Hecke eigenform $G \in M$ such that

$$\lambda_G(T) \equiv \lambda_F(T) \bmod \widetilde{\mathfrak{P}}$$

for any $T \in \mathbf{L}_n$, where $\widetilde{\mathfrak{P}}$ is a prime ideal of $K \cdot \mathbb{Q}(G)$ lying above $\mathfrak{P}$. In this case, we also say $\mathfrak{P}$ gives a congruence between F and G. If $M = (\mathbb{C}F)^{\perp}$, we simply say that $\mathfrak{P}$ is a congruence prime of F. In this case, we simply write $G \equiv_{e.v} F \bmod \mathfrak{P}$. Let $\mathfrak{O}$ be the ring of integers in K, and $\mathfrak{P}$ a prime ideal of K. We denote by $\mathfrak{O}_{(\mathfrak{P})}$ the localization of $\mathfrak{O}$ at $\mathfrak{P}$. For a polynomial $P(u_1, \ldots, u_n) = \sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n} u_1^{i_1} \cdots u_n^{i_n} \in \mathfrak{O}_{(\mathfrak{P})}[u_1, \ldots, u_n]$, we write $P(u_1, \ldots, i_n) \equiv 0 \bmod \mathfrak{P}$ if $a_{i_1\ldots i_n} \equiv 0 \bmod \mathfrak{P}$ for any $i_1, \ldots, i_n$. First we give a lemma, which can be proved in the same way as [25], Lemma 5.1.

Lemma 5.4. *Let $F_1, \cdots, F_d$ be Hecke eigenforms in $S_{k,m}(\Gamma^{(n)})$ linearly independent over $\mathbb{C}$, and G an element of $S_{k,m}(\Gamma^{(n)})$. Write*

$$F_i(z) = \sum_A c_{F_i}(A; u) \exp(2\pi\sqrt{-1}\mathrm{tr}(Az))$$

for $i = 1, \cdots, d$, and

$$G(z) = \sum_A c_G(A; u) \exp(2\pi\sqrt{-1}\mathrm{tr}(Az)).$$

Let K be the composite field of $\mathbb{Q}(F_1), \mathbb{Q}(F_2), \cdots, \mathbb{Q}(F_d)$, and $\mathfrak{O} = \mathfrak{O}_K$. Let $\mathfrak{P}$ be a prime ideal of $\mathfrak{O}$. Assume that

(1) $c_G(A; u)$ belongs to $\mathfrak{O}_{(\mathfrak{P})}$ for any $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$, and $c_{F_1}(A_1; u)$ belongs to $\mathfrak{O}^*_{(\mathfrak{P})}$ for some $A_1 \in \mathcal{H}_2(\mathbb{Z})_{>0}$;

(2) *there exist* $c_1, ..., c_d \in K$ *such that* $\mathrm{ord}_{\mathfrak{P}}(c_1) < 0$ *and*

$$G(z) = \sum_{i=1}^{d} c_i F_i(z).$$

Then there exists $i \neq 1$ *such that we have*

$$\lambda_{F_i}(T) \equiv \lambda_{F_1}(T) \bmod \mathfrak{P}$$

for any $T \in \mathbf{L}_n$.

Theorem 5.5. *Assume that Deligne's conjecture holds true for the critical values of* $L(s, f, \mathrm{Sym}^4)$. *Assume that a prime ideal* $\mathfrak{P}$ *of* K *divides* $(\mathrm{KS}(f), \mathrm{KS}(f))/(f, f)^3$ *and does not divide*

$$(2k-1)!|c_{\mathrm{KS}(f)}(A; u)|^2 \mathbf{L}(l + 2k - 4, f, \mathrm{Sym}^4)$$

for some even integer l *such that* $4 \leq l \leq k-1$ *and an element* A *of* $\mathcal{H}_2(\mathbb{Z})_{>0}$. *Then there exists a Hecke eigenform* G *not constant multiple of* $\mathrm{KS}(f)$ *such that* $G \equiv_{e.v} \mathrm{KS}(f) \bmod \mathfrak{P}$.

Proof. Take a basis of $S_{k+1,k-2}(\Gamma^{(2)})$ in Theorem 4.1 so that $F_1 = \mathrm{KS}(f)$. Then

$$G_{l,(k+1,k-2),A}(Z) = \sum_{i=1}^{d} c_{F_i;u}(A, v)\Lambda(l-2, F_i)F_i.$$

By the assumption, $\mathfrak{P}$ divides the denominator of $\Lambda(l-2, \mathrm{KS}(f))$ for some $4 \leq l \leq k-1$ and $c_{\mathrm{KS}(f)}(A; u) \not\equiv 0 \bmod \mathfrak{P}$. By [2], $E_{4,l} \in M_l(\Gamma^{(4)})(\mathfrak{O}_{(\mathfrak{P})})$. Hence, by the property of $\mathcal{D}_{l,(k+1,k-2)}$ stated before, we have $G_{l,(k+1,k-2)}(Z) \in S_{k+1,k-2}(\Gamma^{(2)})(\mathfrak{O}_{(\mathfrak{P})})$. Then the assertion follows from Lemma 5.4. $\square$

6 Some observation on the period of the K-R-S lift

To consider the above question, we will make a "stupid" observation. First we review the result and the observation in [20]. Let G be a Hecke eigenform in $S_k(\Gamma^{(2)})$ with k even. First assume L-packet conjecture. This implies that there exists a generic modular form G_{gen} for $GSp_2(\mathbb{A}_{\mathbb{Q}})$ such that

$$L(s, G_{gen}, \mathrm{Ad}) = L(s, G, \mathrm{Ad}).$$

We define the Petersson norm $\langle G_{gen}, G_{gen}\rangle$ of G_{gen} by

$$\langle G_{gen}, G_{gen}\rangle = \int_{\mathbb{A}_{\mathbb{Q}}^{\times} GSp_2(\mathbb{Q})\backslash GSp_2(\mathbb{A}_{\mathbb{Q}})} |G_{gen}(g)|^2 dg,$$

where dg is the Tamagawa measure on $GSp_2(\mathbb{A}_{\mathbb{Q}})$. Then we have the following result. (See also [30].)

Proposition 6.1. *([20], Theorem 1.1) Let W be the Whittaker function of G_{gen}. Assume that G_{gen} is stable and that $W(1_4) = 1$. Then*

$$\langle G_{gen}, G_{gen}\rangle = dL(1, G_{gen}, \mathrm{Ad}),$$

where d is a constant depending only on k.

Moreover we assume the conjectural relative trace formula due to Furusawa and Shalika [14]. Then, for a fundamental discriminant $D < 0$

$$\frac{|B_G(|D|)|^2}{\langle G, G\rangle} = c\,\frac{L(3k/2-1, G, \mathrm{Sp})L(3k/2-1, G, \mathrm{Sp}, (\frac{*}{D}))}{L(1, G, \mathrm{Ad})},$$

where $B_G(|D|)$ is the D-th Bessel function of G defined in Section 2, and c is a constant depending only on k (cf. page 4 of [20].) We note that $3k/2-1$ is the central point of $L(s, G, \mathrm{Sp})$.

Now assume that for a primitive form g of odd weight k of level N of neben type, there is a lift $G \in S_{k+1,k-2}(\Gamma)$ with some arithmetic subgroup Γ of $Sp_2(\mathbb{Q})$ such that

$$L(s, G, \mathrm{Sp}) = L(s, g, \mathrm{Sym}^3).$$

Taking [17] into account, we might assume that Γ is the paramodular group of level N. Then we expect a formula similar to above for G. We remark that

$$L(s, G, \mathrm{Ad}) = L(k-1+s, g, \mathrm{Sym}^2)L(3k-3+s, g, \mathrm{Sym}^6),$$

$$L(s, G, \mathrm{Sp}, \chi) = L(s, g, \mathrm{Sym}^3, \chi)$$

for a Dirichlet character χ, and that $\dfrac{L(k,g,\mathrm{Sym}^2)}{\langle g,g\rangle\pi^{k+1}}$ is a rational number independent of g (cf. [16], Theorem 5.1). Hence we expect that

$$\frac{|B_G(|D|)|^2\langle g,g\rangle^3}{\langle G,G\rangle}=c\ \frac{\mathbf{L}(\frac{3k}{2}-1,g,\mathrm{Sym}^3)\mathbf{L}(\frac{3k}{2}-1,g,\mathrm{Sym}^3,(\frac{*}{D}))}{\mathbf{L}(3k-2,g,\mathrm{Sym}^6)}$$

with c a rational number depending only on k. Assume that Conjceture 3.1 holds for $L(s,g,\mathrm{Sym}^6)$. Consider a prime ideal $\mathfrak{P}$ dividing $\mathbf{L}(3k-2,g,\mathrm{Sym}^6)$. Then we normalize G so that

$$\min_{T\in\mathcal{H}_2(\mathbb{Z})_{>0}}\nu_{\mathfrak{P}}(c_G(T))=0.$$

Then $B_G(|D|)$ belongs to $\mathfrak{O}_{\mathfrak{P}}^*$ for some fundamental discriminant D (cf. Remark 8.11(2) of [4]). According to the first named author's experience (e.g. [18]), it is expected that $\mathfrak{P}$ does not divide

$$\mathbf{L}(\frac{3k}{2}-1,g,\mathrm{Sym}^3)\mathbf{L}(\frac{3k}{2}-1,g,\mathrm{Sym}^3,(\frac{*}{D}))$$

with some fundamental discriminant D if it is "big", for an example, if it does not divide $(2k-1)!$. Hence $\mathfrak{P}$ is expected to divide $\dfrac{\langle G,G\rangle}{\langle g,g\rangle^3}$. This observation makes no sense in the original K-R-S lift, because $k+1$ is odd in this case, and

$$B_{\mathrm{KS}(f)}(|D|)=L(3k/2-1,\mathrm{KS}(f),\mathrm{Sp})=L(3k/2-1,f,\mathrm{Sym}^3)=0.$$

Nevertheless we expect the above equality holds with some modification, for an example, replacing $L(3k/2-1,\mathrm{KS}(f),\mathrm{Sp})$ with the derivative of $L(s,\mathrm{KS}(f),\mathrm{Sp})$ at $s=3k/2-1$, and thus we propose the following two conjectures:

Conjecture 6.2. *Under a certain normalization of* $\mathrm{KS}(f)$*, the ratio* $\dfrac{\langle \mathrm{KS}(f),\mathrm{KS}(f)\rangle}{\langle f,f\rangle^3}$ *is algebraic. Moreover let* $\mathfrak{P}$ *be a prime ideal dividing* $\mathbf{L}(3k-2,f,\mathrm{Sym}^6)$ *not dividing* $(2k-1)!$*. Then* $\mathfrak{P}$ *divides* $\dfrac{\langle \mathrm{KS}(f),\mathrm{KS}(f)\rangle}{\langle f,f\rangle^3}$.

In view of Theorem 5.5, and taking Conjecture B in [18] into account, we can also expect:

Conjecture 6.3. *A prime ideal dividing* $\mathbf{L}(3k-2,f,\mathrm{Sym}^6)$ *but not dividing* $(2k-1)!$ *gives a congruence between* $\mathrm{KS}(f)$ *and non-K-R-S lift.*

7 Numerical Examples

In this section, we give numerical examples which support the Conjecture 6.3. Let $k = 16,\ 18$ or 20 and $f_k \in S_k(SL_2(\mathbb{Z}))$ be the primitive form of weight k. Note that $\dim_{\mathbb{C}} S_k(SL_2(\mathbb{Z})) = 1$ and $\mathbb{Q}(f_k) = \mathbb{Q}$ in this case. We compute the conjectural value $\mathbf{L}(3k-2, f_k, \mathrm{Sym}^6)$ of the symmetric 6-th L-function numerically. We also compute $\mathrm{KS}(f_k)$ and a basis of $(\mathbb{C}\mathrm{KS}(f_k))^{\perp}$.

7.1 Approximate value of $\mathbf{L}(3k-2, f_k, \mathrm{Sym}^6)$

First, we give numerical examples of $\mathbf{L}(3k-2, f_k, \mathrm{Sym}^6)$. We can compute a numerical value of a Dirichlet series which has a functional equation by Dokchitser's algorithm [11] and his script written in the GP language. For the computation of the Petersson norm $\langle f_k, f_k \rangle$, we compute the product of critical values of $L(s, f_k)$ at even and odd numbers. This method was used by Zagier [45]. We try to compute $\mathbf{L}(3k-2, f_k, \mathrm{Sym}^6)$ numerically to precision 150, though we lose some precision in the computation. In the following, we illustrate how we find a rational number which is close to an approximate value of $\mathbf{L}(52, f_{18}, \mathrm{Sym}^6)$. To compute $\mathbf{L}(52, f_{18}, \mathrm{Sym}^6)$ to precision 150, we have to compute the first 220619 coefficients of the Dirichlet series $L(s, f_{18}, \mathrm{Sym}^6)$. We use Sage [40] for the computation of Fourier coefficients of elliptic modular forms. A script for the computation of $\mathbf{L}(3k-2, f_k, \mathrm{Sym}^6)$, written in Sage, can be found at [27].

To reduce the denominator of the conjectural critical value, we normalize $\mathbf{L}(l, f_k, \mathrm{Sym}^6)$ as follows:

$$\mathbf{L}_k(l) = 2^{-6k}\Gamma(l)\Gamma(l-k)\mathbf{L}(l, f_k, \mathrm{Sym}^6).$$

Then $\mathbf{L}_{18}(52)$ is equal to $7.4539999121523788927335640138408304498269896193771860067423945961268805201173410067333355171380075089512666575514983221421026764836521568390659310675 7 \times 10^{17}$. The continued fraction expansion of this value is given as follows:

$$[a_0, 3, 1, 1, 1, 12, 2, 5121719897367775453576894048, 3, 3, 1, 2, 3, 4, 2, \dots],$$

where $a_0 = 745399991215237889$. If we consider the 8-th number as "big", then $\mathbf{L}_{18}(52)$ is "close" to the continued fraction with expansion $[a_0, 3, 1, 1, 1, 12, 2]$ which is equal to

$$2^4 \cdot 3^{10} \cdot 5^7 \cdot 7 \cdot 11 \cdot 17^{-2} \cdot 37903031.$$

In a similar way, we can guess the conjectural value of $\mathbf{L}_k(3k-2)$ for other weights. Table 1 shows those values.

Table 1: Conjectural value of $\mathbf{L}_k(3k-2)$

k	conjectural value of $\mathbf{L}_k(3k-2)$
12	$2^{21} \cdot 3^{13} \cdot 5^4 \cdot 7 \cdot 11^{-2} \cdot 13$
16	$2^{16} \cdot 3^9 \cdot 5^2 \cdot 7^5 \cdot 11 \cdot 13^{-3} \cdot 92467$
18	$2^4 \cdot 3^{10} \cdot 5^7 \cdot 7 \cdot 11 \cdot 17^{-2} \cdot 37903031$
20	$2^{14} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17^{-4} \cdot 19^{-2} \cdot 103 \cdot 5518029068479$

7.2 Rankin-Cohen-Ibukiyama type differential operators

In order to construct vector valued Siegel modular forms of odd weights, we use Rankin-Cohen-Ibukiyama type differential operators construed by Eholzer-Ibukiyama [13] and a differential operator constructed by van Dorp [44]. Theta series are also useful for the construction of vector valued Siegel modular forms. However we prefer differential operators since they are easy to compute. In this subsection, we review differential operators given in [13] and [44].

Let $k,\ l$ be positive integers and $\chi,\ \psi$ characters of $\Gamma^{(2)}$. For $F \in M_k(\Gamma^{(2)}, \chi)$, $G \in M_l(\Gamma^{(2)}, \psi)$, Eholzer-Ibukiyama [13] constructed vector valued Siegel modular forms

$$\{F, G\}_{\mathrm{Sym}(j)} \in M_{k+l,j}(\Gamma^{(2)}, \chi\psi),$$
$$\{F, G\}_{\det^2 \mathrm{Sym}(j)} \in M_{k+l+2,j}(\Gamma^{(2)}, \chi\psi),$$

by Rankin-Cohen-Ibukiyama type differential operators. For example, $\{F, G\}_{\mathrm{Sym}(2)}$ is given as follows:

$$\left(kF\frac{\partial G}{\partial z_1} - lG\frac{\partial F}{\partial z_1}\right)u_1^2 + \left(kF\frac{\partial G}{\partial z_2} - lG\frac{\partial F}{\partial z_2}\right)u_1u_2 + \left(kF\frac{\partial G}{\partial z_3} - lG\frac{\partial F}{\partial z_3}\right)u_2^2.$$

Here we write $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2$ and we consider $\mathrm{Sym}(2)$ as the space of homogeneous polynomials of u_1 and u_2 of degree 2. $\{F, G\}_{\mathrm{Sym}(2)}$ is the differential operator defined by Satoh [37].

Next we review the differential operator defined by van Dorp [44]. As before, let $k,\ l$ be positive integers and $\chi,\ \psi$ characters of $\Gamma^{(2)}$.

Let $F \in M_{k,j}(\Gamma^{(2)}, \chi)$ and $G \in M_l(\Gamma^{(2)}, \psi)$ be a vector valued Siegel modular form and a scalar valued Siegel modular form respectively. Then van Dorp [44, Proposition 3.6.1] constructed

$$\{F,\ G\}_{\det \mathrm{Sym}(j)} \in M_{k+l+1,j}(\Gamma^{(2)}, \chi\psi),$$

by a differential operator. Though he proved the proposition only when both χ and ψ are trivial characters, the same proof works for this case.

Next we define differential operators on three scalar valued Siegel modular forms. For $i = 1,\ 2,\ 3$, let k_i be a positive integer and χ_i be a character of $\Gamma^{(2)}$. For $F_i \in M_{k_i}(\Gamma^{(2)}, \chi_i)$ $(i = 1,\ 2,\ 3)$, we define differential operators as follows:

$$\begin{aligned}\{F_1,\ F_2,\ F_3\}_{\det \mathrm{Sym}(j)} &= \Big\{\{F_1,\ F_2\}_{\mathrm{Sym}(j)}, F_3\Big\}_{\det \mathrm{Sym}(j)} \\ &\in M_{k_1+k_2+k_3+1,\ j}(\Gamma^{(2)}, \chi_1\chi_2\chi_3), \\ \{F_1,\ F_2,\ F_3\}_{\det^3 \mathrm{Sym}(j)} &= \Big\{\{F_1,\ F_2\}_{\det^2 \mathrm{Sym}(j)}, F_3\Big\}_{\det \mathrm{Sym}(j)} \\ &\in M_{k_1+k_2+k_3+3,\ j}(\Gamma^{(2)}, \chi_1\chi_2\chi_3),\end{aligned}$$

Note that polynomials used when defining these differential operators have $\mathbb{Z}[1/2]$-integral coefficients. Therefore if $p \neq 2$ is a prime and $F_1,\ F_2$ and F_3 have p-integral Fourier coefficients then, $\{F_1,\ F_2,\ F_3\}_{\det \mathrm{Sym}(j)}$ and $\{F_1,\ F_2,\ F_3\}_{\det^3 \mathrm{Sym}(j)}$ also have p-integral Fourier coefficients.

7.3 Generators of the ring of scalar valued Siegel modular forms

For the construction of scalar valued Siegel modular forms, we recall generators of the ring of scalar valued Siegel modular forms of even weights. For a semi-positive definite matrix $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ and a Siegel modular form F of degree 2, we put

$$c((n, r, m); F) = c_F(T).$$

For an even integer k, we denote by ϕ_k the Siegel-Eisenstein series of degree 2, of level 1 and of weight k. We normalize ϕ_k so that the constant term $c((0, 0, 0); \phi_k)$ is equal to 1. We denote by $\chi_{10} \in S_{10,0}(\Gamma^{(2)})$ and $\chi_{12} \in S_{12,0}(\Gamma^{(2)})$ the nontrivial cusp form of weight 10 and 12 respectively. We normalize χ_{10} and χ_{12} so that $c((1,1,1);\chi_{10}) = c((1,1,1);\chi_{12}) = 1$. The following theorem is well-known.

Theorem 7.1 (Igusa [21], [22]). *Modular forms ϕ_4, ϕ_6, χ_{10} and χ_{12} are algebraically independent over $\mathbb{C}$. Moreover they have integral Fourier coefficients and generate the ring of scalar valued Siegel modular forms of degree 2, of even weights and of level* 1.

Let sgn be the unique nontrivial character of $\Gamma^{(2)}$ (see [32]). Then sgn is quadratic. There exists a square root $\chi_5 \in S_5(\Gamma^{(2)}, sgn)$ of χ_{10}. The cusp form χ_5 has the following Fourier expansion:

$$\chi_5(Z) = \sum_{\substack{n,\ m,\ 4nm-r^2>0 \\ n,\ r,\ m\in 1/2+\mathbb{Z}}} c((n,r,m);\chi_5)\mathbf{e}\left(nz_{11} + rz_{12} + mz_{22}\right),$$

where $Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}$. We normalize χ_5 so that $c((1/2,1/2,1/2);\chi_5) = 1$.

We note that Fourier coefficients of Siegel-Eisenstein ϕ_k are explicitly known, and χ_{10} and χ_{12} can be written as polynomials of Siegel-Eisenstein series. Since χ_5 is the Saito-Kurokawa lift of a Jacobi theta series (see [33], [15]), we can easily calculate Fourier coefficients of χ_5. By the same reason, it can be proved that χ_5 has integral Fourier coefficients.

7.4 Congruences between $\mathrm{KS}(f_k)$ and non-K-R-S lift

In this subsection, we prove a congruence between $\mathrm{KS}(f_k)$ and a non-K-R-S lift, modulo a prime ideal which divides the conjectural numerator of $\mathbf{L}(3k-2, f_k, \mathrm{Sym}^6)$, for $k = 16,\ 18,\ 20$. Since Fourier coefficients of $\mathrm{KS}(f_k)$ are not explicitly known, we construct a basis of $S_{k+1,k-2}(\Gamma^{(2)})$ and compute all eigenforms in this space. Source files for computing bases and eigenforms, written in Sage, can be found at [27].

By the dimension formula of Tsushima [43], dimensions of $S_{k+1,k-2}(\Gamma^{(2)})$ for $k = 16,\ 18,\ 20$ and $S_{21,14}(\Gamma^{(2)})$ are given in Table 2. To construct basis of $S_{19,16}(\Gamma^{(2)})$ and $S_{21,18}(\Gamma^{(2)})$, we use differential operators defined in Section 7.2 and the Hecke operator $T(2)$. In general, it is difficult to construct Siegel modular forms of small determinant

Table 2: Dimension of $S_{k,j}(\Gamma^{(2)})$

(k,j)	$(17,14)$	$(21,14)$	$(19,16)$	$(21,18)$
$\dim S_{k,j}(\Gamma^{(2)})$	13	24	23	39

weights with differential operators. Therefore, to construct a basis of $S_{17,14}(\Gamma^{(2)})$, we compute a basis of a space of larger determinant weight instead. We embed $S_{17,14}(\Gamma^{(2)})$ into $S_{21,14}(\Gamma^{(2)})$ by multiplying by ϕ_4, and construct a basis of $S_{21,14}(\Gamma^{(2)})$. Then we compute a basis of $S_{17,14}(\Gamma^{(2)})$ from the basis of $S_{21,14}(\Gamma^{(2)})$ and the Hecke operator $T(2)$. In the following, we explain this method. A similar method was used by van Dorp [44].

For a subring $R \subset \mathbb{C}$, we define the ring A_R of formal q-expansion as

$$A_R = R[q_{12},\ q_{12}^{-1}][\![q_{11},\ q_{22}]\!].$$

We put $A = A_{\mathbb{C}}$. For an half integral matrix $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$, we put

$$q^T = q_{11}^n q_{12}^r q_{22}^m.$$

We consider $M_{k,j}(\Gamma^{(2)})$ as a subspace of $A \otimes \mathrm{Sym}(j)$ by the embedding

$$\sum_{T \in \mathcal{H}_2(\mathbb{Z})_{\geq 0}} c_F(T)\mathbf{e}(\mathrm{tr}(TZ)) \mapsto \sum_{\nu=0}^{j} \left(\sum_{T \in \mathcal{H}_2(\mathbb{Z})_{\geq 0}} c_F(T)_\nu q^T \right) \otimes u_1^{j-\nu} u_2^{\nu}$$

for $F \in M_{k,j}(\Gamma^{(2)})$. Here for an element of $a \in \mathrm{Sym}(j)$, we write a_ν the coefficient of $u_1^{j-\nu}u_2^{\nu}$ in a. Next, we define a formal Hecke operator $T(p)_{k,j}$ on $A \otimes \mathrm{Sym}(j)$. Let $F = \sum_{T \in \mathcal{H}_2(\mathbb{Z})_{\geq 0}} c_F(T) q^T$ be an element of $A \otimes \mathrm{Sym}(j)$. Let p be a prime number and α, β and γ non negative integers. For $T \in \mathcal{H}_2(\mathbb{Z})_{\geq 0}$ and $U \in M_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Z})$, we put

$$b(\alpha, \beta, \gamma, T, U) = \rho_{k,j}\left({}^tU^{-1}\right) c_F(p^{\alpha-\beta-\gamma} T[U]),$$

if $p^{-\gamma}T \in \mathcal{H}_2(\mathbb{Z})$ and $p^{-\beta-\gamma}T[U] \in \mathcal{H}_2(\mathbb{Z})$. Otherwise we put

$$b(\alpha, \beta, \gamma, T, U) = 0.$$

Then we define $F|T(p)_{k,j}$ by

$$c_{F|T(p)_{k,j}}(T) = \sum_{\alpha+\beta+\gamma=1} \sum_{U} b(\alpha, \beta, \gamma, T, U),$$

where $\alpha,\ \beta,\ \gamma$ runs over all non-negative integers such that $\alpha+\beta+\gamma=1$ and U runs over the set $\mathrm{GL}_2(\mathbb{Z})\begin{pmatrix}1&0\\0&p\end{pmatrix}\mathrm{GL}_2(\mathbb{Z})/\mathrm{GL}_2(\mathbb{Z})$. By Arakawa [1, (2. 5)], $T(p)_{k,j}$ on $M_{k,j}(\Gamma^{(2)})$ coincides with the usual Hecke operator $T(p)$.

Since $\phi_4 S_{17,14}(\Gamma^{(2)})\subset S_{21,14}(\Gamma^{(2)})$ and ϕ_4 is a unit in A, we have

$$S_{17,14}(\Gamma^{(2)})\subset \widetilde{S}_{17,14}(\Gamma^{(2)})\subset A\otimes \mathrm{Sym}(j),$$

where $\widetilde{S}_{17,14}(\Gamma^{(2)})=\phi_4^{-1}S_{21,14}(\Gamma^{(2)})$. We can construct a basis of $S_{21,14}(\Gamma^{(2)})$ by differential operators and the Hecke operator $T(2)$. Therefore we can compute a basis of $\widetilde{S}_{17,14}(\Gamma^{(2)})$ explicitly.

Proposition 7.2. *Let $F_1,\ldots,F_{24}$ be Siegel modular forms of weight $\det^{21}\otimes\mathrm{Sym}(14)$ given as follows:*

$$\begin{aligned}
&F_1=\{\phi_4,\chi_5,\chi_5\phi_6\}_1, && F_2=\{\phi_4,\phi_6,\phi_4\phi_6\}_1, && F_3=\{\phi_4,\phi_6,\chi_{10}\}_1,\\
&F_4=\left\{\phi_4,\phi_4^2,\phi_4^2\right\}_1, && F_5=\{\phi_4,\chi_{10},\phi_6\}_1, && F_6=\{\phi_4,\phi_4\phi_6,\phi_6\}_1,\\
&F_7=\{\phi_4,\chi_5\phi_6,\chi_5\}_1, && F_8=\{\phi_4,\chi_{12},\phi_4\}_1, && F_9=\left\{\phi_4,\phi_4^3,\phi_4\right\}_1,\\
&F_{10}=\left\{\phi_4,\phi_6^2,\phi_4\right\}_1, && F_{11}=\{\chi_5,\phi_6,\phi_4\chi_5\}_1, && F_{12}=\{\chi_5,\phi_4\chi_5,\phi_6\}_1,\\
&F_{13}=\{\chi_5,\phi_4\phi_6,\chi_5\}_1, && F_{14}=\{\chi_5,\chi_{10},\chi_5\}_1, && F_{15}=\{\chi_5,\chi_5\phi_6,\phi_4\}_1,\\
&F_{16}=\left\{\phi_6,\phi_4^2,\phi_6\right\}_1, && F_{17}=\{\phi_6,\phi_4\chi_5,\chi_5\}_1, && F_{18}=\{\phi_6,\phi_4\phi_6,\phi_4\}_1,\\
&F_{19}=\{\phi_6,\chi_{10},\phi_4\}_1, && F_{20}=\{\phi_4,\chi_5,\phi_4\chi_5\}_3, && F_{21}=\left\{\phi_4,\phi_6,\phi_4^2\right\}_3,\\
&F_{22}=\left\{\phi_4,\phi_4^2,\phi_6\right\}_3, && F_{23}=F_1|T(2), && F_{24}=F_2|T(2).
\end{aligned}$$

Here $T(2)$ is the Hecke operator and we abbreviate $\{F,\ G,H\}_{\det\mathrm{Sym}(14)}$ as $\{F,\ G,H\}_1$ and $\{F,\ G,H\}_{\det^3\mathrm{Sym}(14)}$ as $\{F,\ G,H\}_3$. Then $F_1,\ldots,F_{24}$ forms a basis of $S_{21,14}(\Gamma^{(2)})$.

Remark 7.3.
1. To check the linear independence and compute the action of the Hecke operator, we computed all Fourier coefficients of $\{F_1,\ldots F_{24}\}$ and $\left\{\phi_4^{-1}F_1,\ldots\phi_4^{-1}F_{24}\right\}$ for the following finite set of half integral matrices.

$$\left\{\begin{pmatrix}n&r/2\\r/2&m\end{pmatrix}\in\mathcal{H}_2(\mathbb{Z})_{\geq 0}\ \middle|\ n,\ m\leq 6\right\}.$$

2. As explained in Section 7.2 and §7.3, modular forms $F_1,\ldots,F_{24}$ have $\mathbb{Z}[1/2]$-integral Fourier coefficients.

3. Since ϕ_4 is a unit in $A_{\mathbb{Z}}$, modular forms $\phi_4^{-1}F_1, \dots \phi_4^{-1}F_{24}$ also have $\mathbb{Z}[1/2]$-integral Fourier coefficients.

For $1 \le i \le 24$, we define $G_i = \phi_4^{-1}F_i$. Let $F \in A \otimes \mathrm{Sym}(j)$ be a formal q-expansion. For $T \in \mathcal{H}_2(\mathbb{Z})_{\ge 0}$ and an integer ν, we denote by $c_F(T)_\nu$ the coefficient of $u_1^{j-\nu}u_2^{\nu}$ in $c_F(T)$. Let $\{(T_i,\ \nu_i) \mid 1 \le i \le 24\}$ be a set of pairs of a half integral matrix and a positive integer such that

$$\det\left(c_{G_j}(T_i)_{\nu_i}\right)_{1\le i,j\le 24} \neq 0.$$

We can take this set as follows:

$$\begin{gathered}\{((1,0,1),1),\ ((1,0,1),3),\ ((1,0,1),5),\ ((1,1,1),1),\\ ((1,1,1),3),\ ((1,0,2),1),\ ((1,0,2),3),\ ((1,0,2),5),\\ ((1,0,2),7),\ ((1,0,2),9),\ ((1,0,2),11),\ ((1,0,2),13),\\ ((1,1,2),7),\ ((1,1,2),9),\ ((1,1,2),11),\ ((1,1,2),13),\\ ((2,0,2),1),\ ((2,0,2),3),\ ((2,0,2),5),\ ((2,1,2),2),\\ ((2,1,2),3),\ ((2,1,2),6),\ ((1,0,3),13),\ ((2,0,3),11)\}.\end{gathered}$$

Here we identify a triple of integers (n, r, m) as a half integral matrix $\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$. For $1 \le i \le 24$, we define (T_i, ν_i) by the i-th element in the sequence above. We define a $\mathbb{C}$-linear map $\psi : A \otimes \mathrm{Sym}(j) \to \mathbb{C}^{24}$ by

$$\psi(F) = (c_F(T_1)_{\nu_1}, \dots, c_F(T_{24})_{\nu_{24}}).$$

Then $\psi|_{\widetilde{S}_{17,14}(\Gamma^{(2)})}$ is an isomorphism by definition. We define an endomorphism $\widetilde{T}(2) \in \mathrm{End}_{\mathbb{C}}(\widetilde{S}_{17,14}(\Gamma^{(2)}))$ by

$$\widetilde{T}(2) = \psi|_{\widetilde{S}_{17,14}(\Gamma^{(2)})}^{-1} \circ \psi \circ T(2)_{17,14}.$$

Note that $S_{17,14}(\Gamma^{(2)})$ is stable under the action of $T(2)_{17,14}$ but $\widetilde{S}_{17,14}(\Gamma^{(2)})$ is not. By definition, $\widetilde{T}(2)|_{S_{17,14}(\Gamma^{(2)})}$ coincides with $T(2)$. The characteristic polynomial of $\widetilde{T}(2)$ is given as follows:

$$(x + 4078080)P(x)Q(x),$$

where $P(x)$ and $Q(x)$ are irreducible polynomials in $\mathbb{Q}[x]$ of degree 11 and 12 respectively. The polynomial $Q(x)$ is given by

$$Q(x) = x^{12} + \sum_{i=1}^{12} c_i x^{12-i},$$

where $c_1, \ldots, c_{12}$ is given as follows.

$c_1 = 2^6 \cdot 3 \cdot 134213,$
$c_2 = -2^{11} \cdot 3^2 \cdot 29 \cdot 1907 \cdot 14479,$
$c_3 = -2^{23} \cdot 3^4 \cdot 2081 \cdot 3378533243,$
$c_4 = -2^{24} \cdot 3^5 \cdot 887 \cdot 1171 \cdot 2767 \cdot 2284700807,$
$c_5 = 2^{36} \cdot 3^6 \cdot 5 \cdot 4211 \cdot 159508391107808527,$
$c_6 = 2^{44} \cdot 3^6 \cdot 5^2 \cdot 1279 \cdot 1730089 \cdot 1957827496395923,$
$c_7 = -2^{53} \cdot 3^7 \cdot 5^4 \cdot 7 \cdot 146201416370423477118231137,$
$c_8 = -2^{60} \cdot 3^8 \cdot 5^4 \cdot 4404227063914933919915630591809,$
$c_9 = -2^{70} \cdot 3^{10} \cdot 5^7 \cdot 59 \cdot 71 \cdot 45659 \cdot 1005269522640570835335 7,$
$c_{10} = 2^{79} \cdot 3^{12} \cdot 5^6 \cdot 7 \cdot 29 \cdot 1126202887 \cdot 110526993959 \cdot 561766099379,$
$c_{11} = 2^{94} \cdot 3^{14} \cdot 5^7 \cdot 7 \cdot 11 \cdot 17 \cdot 29^2 \cdot 47 \cdot 10093 \cdot 31469 \cdot 149371 \cdot 1618265278589,$
$c_{12} = -2^{99} \cdot 3^{17} \cdot 5^9 \cdot 7^2 \cdot 11^2 \cdot 17 \cdot 29^3 \cdot 17477 \cdot 903943377933063888 4999$

Therefore there exist 3 eigenvectors of $\widetilde{T}(2)$ in $\widetilde{S}_{17,14}(\Gamma^{(2)})$ up to $\mathrm{Aut}(\mathbb{C})$ conjugate and constant multiple. Let $\widetilde{\alpha}$ be a root of the polynomial $P(x)$. It is easy to check that an eigenvector of $\widetilde{T}(2)$ with eigenvalue $\widetilde{\alpha}$ is not an eigenvector of $T(2)_{17,14}$. Thus the following proposition holds.

Proposition 7.4. *The space $S_{17,14}(\Gamma^{(2)})$ is equal to the subspace of $\widetilde{S}_{17,14}(\Gamma^{(2)})$ annihilated by $(\widetilde{T}(2)+4078080)Q(\widetilde{T}(2))$. Thus there exist exactly two eigenforms in $S_{17,14}(\Gamma^{(2)})$ up to* $\mathrm{Aut}(\mathbb{C})$ *conjugate and constant multiple.*

By this proposition, we can calculate the lift $\mathrm{KS}(f_{16})$ and a non-lift eigenform $G_{17,14} \in S_{17,14}(\Gamma^{(2)})$ as linear combinations of $G_1, \ldots, G_{24}$. Let α be a root of $Q(x)$. The lift $\mathrm{KS}(f_{16})$ (resp. the non-lift eigenform $G_{17,14}$) is equal to the eigenform in $S_{17,14}(\Gamma^{(2)})$ whose eigenvalue of $T(2)$ is equal -4078080 (resp. α). Therefore the Hecke field of $\mathrm{KS}(f_{16})$ (resp. $G_{17,14}$) is the rational field (resp. the number field generated by α). We normalize $\mathrm{KS}(f_{16})$ so that

$$\begin{aligned} 9^{-1}c((1,\ 1,\ 1);\mathrm{KS}(f_{16})) = {}& 14u_1^{13}u_2+91u_1^{12}u_2^2+436u_1^{11}u_2^3+1397u_1^{10}u_2^4 \\ &+ 2466u_1^9u_2^5 + 2121u_1^8u_2^6 - 2121u_1^6u_2^8 - 2466u_1^5u_2^9 \\ &- 1397u_1^4u_2^{10} - 436u_1^3u_2^{11} - 91u_1^2u_2^{12} - 14u_1u_2^{13}. \quad (2) \end{aligned}$$

We normalize $G_{17,14}$ so that the coefficient of $u_1^{13}u_2$ in $c((1,\ 1,\ 1); G_{17,14})$ is equal to that of $\mathrm{KS}(f_{16})$. Let p be the prime 92467. For $F \in \widetilde{S}_{17,14}(\Gamma^{(2)})$, we define $\xi(F) = (v_1, \cdots, v_{24}) \in \mathbb{C}^{24}$ by the vector that satisfies the following equation

$$F = \sum_{i=1}^{24} v_i G_i.$$

Then every entry of $\xi\,(\mathrm{KS}(f_{16}))$ (resp. $\xi\,(G_{17,14})$) is p-integral (resp. $\mathfrak{P}'$-integral). Here $\mathfrak{P}'$ is any prime of the Hecke field $\mathbb{Q}(G_{17,14})$ above p. Therefore Fourier coefficients of $\mathrm{KS}(f_{16})$ are p-integral and Fourier coefficients of $G_{17,14}$ are $\mathfrak{P}'$-integral. The same statement also holds for Hecke eigenvalues. The factorization of $Q(x)$ in $\mathbb{F}_p[x]$ is as follows:

$$\begin{aligned}(x+9532)(x+62632)(x^{10}&+83373x^9+7236x^8+53688x^7+63576x^6\\ &+79102x^5+299x^4+77779x^3+56013x^2+33999x+83588).\end{aligned}$$

Thus any prime above p is unramified in $\mathbb{Q}(G_{17,14})/\mathbb{Q}$ and there exists a unique prime $\mathfrak{P}$ such that $\alpha \equiv -9532 \equiv -4078080 \bmod \mathfrak{P}$. It can be checked that every entry of $\xi\,(\mathrm{KS}(f_{16}) - G_{17,14})$ modulo $\mathfrak{P}$ is equal to 0. Therefore we have the following theorem.

Theorem 7.5. *Let* $\mathrm{KS}(f_{16})$ *be the K-R-S lift of* f_{16} *normalized so that the equation* (2) *holds and* $G_{17,14} \in S_{17,14}(\Gamma^{(2)})$ *the eigenform which is not K-R-S lift and normalized as above. Let* $p = 92467$ *be a prime and* $\mathfrak{P}$ *be the prime of the Hecke field* $\mathbb{Q}(G_{17,14})$ *defined as above. Then for every* $T \in \mathcal{H}_2(\mathbb{Z})_{\geq 0}$, $c_{\mathrm{KS}(f_{16})}(T)$ *and* $c_{G_{17,14}}(T)$ *are* $\mathfrak{P}$*-integral and the following congruence relation holds.*

$$c_{\mathrm{KS}(f_{16})}(T) \equiv c_{G_{17,14}}(T) \mod \mathfrak{P} \quad \textit{for } T \in \mathcal{H}_2(\mathbb{Z})_{\geq 0}.$$

In particular, the following congruence relation among Hecke eigenvalues holds.

$$\lambda_{\mathrm{KS}(f_{16})}(T(m)) \equiv \lambda_{G_{17,14}}(T(m)) \mod \mathfrak{P} \quad \textit{for } m \in \mathbb{Z}_{\geq 1}.$$

Remark 7.6. The prime 92467 appears in the conjectural numerator of $\mathbf{L}(46, f_{16}, \mathrm{Sym}^6)$ given in Section 7.1. Therefore this theorem gives an example that supports Conjecture 6.3.

Similar statements to the theorem above hold for $\mathrm{KS}(f_{18})$ and $\mathrm{KS}(f_{20})$. We introduce the statements briefly. The construction of a basis of $S_{k+1,k-2}(\Gamma^{(2)})$ is more straightforward. We can construct the basis by differential operators introduced in §7.2 and the Hecke operator $T(2)$.

Proposition 7.7. *Let $k = 18$ or 20. We define $\alpha_k = c_{f_k}(2)^3 - 2^k c_{f_k}(2)$, where $c_{f_k}(n)$ is the n-th Fourier coefficient of f_k. Explicitly, we have*

$$\alpha_k = \begin{cases} -8785920 & \text{if } k = 18, \\ -383331840 & \text{if } k = 20. \end{cases}$$

Then the characteristic polynomial of $T(2) \in S_{k+1,k-2}(\Gamma^{(2)})$ is equal to $P_1(x)P_2(x)$. Here $P_1(x) = x - \alpha_k$ and $P_2(x)$ is an irreducible polynomial of $\mathbb{Q}[x]$. Thus there exist exactly two eigenforms in $S_{k+1,k-2}(\Gamma^{(2)})$ up to $\mathrm{Aut}(\mathbb{C})$ conjugate and constant multiple.

We normalize $\mathrm{KS}(f_k)$ so that $c((1,1,1);\mathrm{KS}(f_{18}))$ is equal to

$$\begin{aligned} &260u_1^{15}u_2 + 1950u_1^{14}u_2^2 + 4844u_1^{13}u_2^3 + 1911u_1^{12}u_2^4 - 13818u_1^{11}u_2^5 \\ &\quad - 31955u_1^{10}u_2^6 - 29282u_1^9u_2^7 + 29282u_1^7u_2^9 + 31955u_1^6u_2^{10} \\ &+ 13818u_1^5u_2^{11} - 1911u_1^4u_2^{12} - 4844u_1^3u_2^{13} - 1950u_1^2u_2^{14} - 260u_1u_2^{15}, \end{aligned}$$

and $7^{-1}c((1,1,1);\mathrm{KS}(f_{20}))$ is equal to

$$\begin{aligned} &8926u_1^{17}u_2 + 75871u_1^{16}u_2^2 + 403888u_1^{15}u_2^3 + 1511740u_1^{14}u_2^4 \\ &+ 3842794u_1^{13}u_2^5 + 6652945u_1^{12}u_2^6 + 7722424u_1^{11}u_2^7 + 5266591u_1^{10}u_2^8 \\ &- 5266591u_1^8u_2^{10} - 7722424u_1^7u_2^{11} - 6652945u_1^6u_2^{12} - 3842794u_1^5u_2^{13} \\ &\quad - 1511740u_1^4u_2^{14} - 403888u_1^3u_2^{15} - 75871u_1^2u_2^{16} - 8926u_1u_2^{17} \end{aligned}$$

We denote by $G_{k+1,k-2}$ the eigenform in $S_{k+1,k-2}(\Gamma^{(2)})$ whose eigenvalue of $T(2)$ is equal to a root of $P_2(x)$. We normalize $G_{k+1,k-2}$ so that the coefficient of $u_1^{k-3}u_2$ in $c((1,1,1);G_{k+1,k-2})$ is equal to that of $c((1,1,1);\mathrm{KS}(f_k))$. We define a prime p by

$$p = \begin{cases} 37903031 & \text{if } k = 18, \\ 103 \text{ or } 5518029068479 & \text{if } k = 20. \end{cases}$$

By factoring of $P_2(x) \bmod p$, we see that there exists a unique prime $\mathfrak{P}$ of the Hecke field $\mathbb{Q}(G_{k+1,k-2})$ above p which is unramified in $\mathbb{Q}(G_{k+1,k-2})/\mathbb{Q}$ and $\lambda_{\mathrm{KS}(f_k)} \equiv \alpha_k \bmod \mathfrak{P}$.

Theorem 7.8. *Let $k = 18$ or 20 and $\mathrm{KS}(f_k)$, $G_{k+1,k-2}$ and $\mathfrak{P}$ as above. Then for every $T \in \mathcal{H}_2(\mathbb{Z})_{\geq 0}$, $c_{\mathrm{KS}(f_k)}(T)$ and $c_{G_{k+1,k-2}}(T)$ are $\mathfrak{P}$-integral and the following congruence relation holds.*

$$c_{\mathrm{KS}(f_k)}(T) \equiv c_{G_{k+1,k-2}}(T) \mod \mathfrak{P} \quad \text{for } T \in \mathcal{H}_2(\mathbb{Z})_{\geq 0}.$$

In particular, the following congruence relation among Hecke eigenvalues holds.

$$\lambda_{\mathrm{KS}(f_k)}(T(m)) \equiv \lambda_{G_{k+1,k-2}}(T(m)) \mod \mathfrak{P} \quad \textit{for } m \in \mathbb{Z}_{\geq 1}.$$

Remark 7.9. The primes 37903031, 103 and 5518029068479 appear in the conjectural numerators of $\mathbf{L}(3k-2, f_k, \mathrm{Sym}^6)$ given in Section 7.1. Therefore this theorem gives examples that support Conjecture 6.3.

Remark 7.10. Let $k = 12$. Then $\dim S_{12}(SL_2(\mathbb{Z})) = 1$ and $S_{12}(SL_2(\mathbb{Z}))$ is spanned by the Ramanujan delta function Δ. Then, approximately we have*

$$\mathbf{L}(34, \Delta, \mathrm{Sym}^6) = \frac{2^{44}}{3^{11} \cdot 5^7 \cdot 7^6 \cdot 11^6 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31}.$$

Therefore, this gives neither a numerical support nor a counter example of our conjecture.

7.5 Table of Fourier coefficients

Table 3 shows Fourier coefficients of $\mathrm{KS}(f_{16})$, $\mathrm{KS}(f_{18})$ and $\mathrm{KS}(f_{20})$. For integers (n, r, m, i) and $\mathrm{KS}(f_k)$, the corresponding number in the table shows the coefficient of $u_1^{j-i} u_2^i$ in $c((n, r, m); \mathrm{KS}(f_k))$, where $j = k-2$. If $i > j$, then the coefficient of $u_1^{j-i} u_2^i$ of the Fourier coefficient does not exist. Therefore we denote it by "None" if $i > j$. Fourier coefficients of non-lift eigenforms are too complicated to show here. See [27] for Fourier coefficients of non-lift eigenforms.

(n, r, m, i)	$\mathrm{KS}(f_{16})$	$\mathrm{KS}(f_{18})$	$\mathrm{KS}(f_{20})$
(1, 1, 1, 0)	0	0	0
(1, 1, 1, 1)	126	260	62482
(1, 1, 1, 2)	819	1950	531097
(1, 1, 1, 3)	3924	4844	2827216
(1, 1, 1, 4)	12573	1911	10582180
(1, 1, 1, 5)	22194	-13818	26899558
(1, 1, 1, 6)	19089	-31955	46570615
(1, 1, 1, 7)	0	-29282	54056968
(1, 1, 1, 8)	-19089	0	36866137
(1, 1, 1, 9)	-22194	29282	0

*This was first informed by A. Mellit

(1, 1, 1, 10)	-12573	31955	-36866137
(1, 1, 1, 11)	-3924	13818	-54056968
(1, 1, 1, 12)	-819	-1911	-46570615
(1, 1, 1, 13)	-126	-4844	-26899558
(1, 1, 1, 14)	0	-1950	-10582180
(1, 1, 1, 15)	None	-260	-2827216
(1, 1, 1, 16)	None	0	-531097
(1, 1, 1, 17)	None	None	-62482
(1, 1, 1, 18)	None	None	0
(1, 0, 1, 0)	0	0	0
(1, 0, 1, 1)	-1452	1040	-4780164
(1, 0, 1, 2)	0	0	0
(1, 0, 1, 3)	-28792	8096	-3464648
(1, 0, 1, 4)	0	0	0
(1, 0, 1, 5)	77812	192	-35156716
(1, 0, 1, 6)	0	0	0
(1, 0, 1, 7)	0	105776	-252085184
(1, 0, 1, 8)	0	0	0
(1, 0, 1, 9)	-77812	-105776	0
(1, 0, 1, 10)	0	0	0
(1, 0, 1, 11)	28792	-192	252085184
(1, 0, 1, 12)	0	0	0
(1, 0, 1, 13)	1452	-8096	35156716
(1, 0, 1, 14)	0	0	0
(1, 0, 1, 15)	None	-1040	3464648
(1, 0, 1, 16)	None	0	0
(1, 0, 1, 17)	None	None	4780164
(1, 0, 1, 18)	None	None	0
(1, 1, 2, 0)	0	0	0
(1, 1, 2, 1)	164064	14560	-760217472
(1, 1, 2, 2)	1066416	109200	-6461848512
(1, 1, 2, 3)	484000	3114496	-17710799680
(1, 1, 2, 4)	-9068576	18588024	-3594027360
(1, 1, 2, 5)	-22515744	21981456	115166666272
(1, 1, 2, 6)	-15710664	-79926616	411073936000
(1, 1, 2, 7)	415680	-183609888	494829268640
(1, 1, 2, 8)	-12169872	72099720	-688113915104
(1, 1, 2, 9)	4990336	729865136	-3721332519520
(1, 1, 2, 10)	69181048	1260570696	-7317582085968
(1, 1, 2, 11)	105468480	1091356224	-8483702524640

(1, 1, 2, 12)	74475568	443250976	-5844240474240
(1, 1, 2, 13)	27695104	22165248	-1399528146688
(1, 1, 2, 14)	4464000	-38099040	1556500998080
(1, 1, 2, 15)	None	-13922944	948294575200
(1, 1, 2, 16)	None	-2814336	-443455506256
(1, 1, 2, 17)	None	None	-402415446272
(1, 1, 2, 18)	None	None	-65643110400
(1, 0, 2, 0)	0	0	0
(1, 0, 2, 1)	308376	680160	2870692872
(1, 0, 2, 2)	0	0	0
(1, 0, 2, 3)	-10234368	-11585376	21170506128
(1, 0, 2, 4)	0	0	0
(1, 0, 2, 5)	14925744	-201397536	-207676697592
(1, 0, 2, 6)	0	0	0
(1, 0, 2, 7)	-90793440	342733248	515492430624
(1, 0, 2, 8)	0	0	0
(1, 0, 2, 9)	6773184	-244396416	5359795453320
(1, 0, 2, 10)	0	0	0
(1, 0, 2, 11)	-82904832	-1712994048	4673534432016
(1, 0, 2, 12)	0	0	0
(1, 0, 2, 13)	-13238784	1592051712	-7577714008032
(1, 0, 2, 14)	0	0	0
(1, 0, 2, 15)	None	-386488320	4021039796352
(1, 0, 2, 16)	None	0	0
(1, 0, 2, 17)	None	None	-92754851328
(1, 0, 2, 18)	None	None	0
(2, 2, 2, 0)	0	0	0
(2, 2, 2, 1)	-513838080	-2284339200	-23951340026880
(2, 2, 2, 2)	-3339947520	-17132544000	-203586390228480
(2, 2, 2, 3)	-16002385920	-42558996480	-1083761911357440
(2, 2, 2, 4)	-51273699840	-16789893120	-4056486530611200
(2, 2, 2, 5)	-90508907520	121403842560	-10311457063326720
(2, 2, 2, 6)	-77846469120	280754073600	-17851999537881600
(2, 2, 2, 7)	0	257269309440	-20721757008261120
(2, 2, 2, 8)	77846469120	0	-14131964129902080
(2, 2, 2, 9)	90508907520	-257269309440	0
(2, 2, 2, 10)	51273699840	-280754073600	14131964129902080
(2, 2, 2, 11)	16002385920	-121403842560	20721757008261120
(2, 2, 2, 12)	3339947520	16789893120	17851999537881600
(2, 2, 2, 13)	513838080	42558996480	10311457063326720

(2, 2, 2, 14)	0	17132544000	4056486530611200
(2, 2, 2, 15)	None	2284339200	1083761911357440
(2, 2, 2, 16)	None	0	203586390228480
(2, 2, 2, 17)	None	None	23951340026880
(2, 2, 2, 18)	None	None	0
(2, 1, 2, 0)	2343600000	1291780224	-18708286464000
(2, 1, 2, 1)	6599473920	60812237184	-283106276624640
(2, 1, 2, 2)	-4935304080	211298009760	-1532906227320720
(2, 1, 2, 3)	8449920000	367872146880	-400231772844960
(2, 1, 2, 4)	78047298360	1313340416640	8509377989028000
(2, 1, 2, 5)	142805970000	3235358847168	17839762657594080
(2, 1, 2, 6)	155413971000	3249774289608	-26204400458628000
(2, 1, 2, 7)	0	1319296867680	-116239043127809280
(2, 1, 2, 8)	-155413971000	0	-120617734193935440
(2, 1, 2, 9)	-142805970000	-1319296867680	0
(2, 1, 2, 10)	-78047298360	-3249774289608	120617734193935440
(2, 1, 2, 11)	-8449920000	-3235358847168	116239043127809280
(2, 1, 2, 12)	4935304080	-1313340416640	26204400458628000
(2, 1, 2, 13)	-6599473920	-367872146880	-17839762657594080
(2, 1, 2, 14)	-2343600000	-211298009760	-8509377989028000
(2, 1, 2, 15)	None	-60812237184	400231772844960
(2, 1, 2, 16)	None	-1291780224	1532906227320720
(2, 1, 2, 17)	None	None	283106276624640
(2, 1, 2, 18)	None	None	18708286464000
(2, 0, 2, 0)	0	0	0
(2, 0, 2, 1)	10082121728	71661715456	356358052100096
(2, 0, 2, 2)	0	0	0
(2, 0, 2, 3)	-44349837312	-173404717056	3687020553695232
(2, 0, 2, 4)	0	0	0
(2, 0, 2, 5)	-103716046848	1442227421184	-15714753453070336
(2, 0, 2, 6)	0	0	0
(2, 0, 2, 7)	0	-3081923395584	71719961448185856
(2, 0, 2, 8)	0	0	0
(2, 0, 2, 9)	103716046848	3081923395584	0
(2, 0, 2, 10)	0	0	0
(2, 0, 2, 11)	44349837312	-1442227421184	-71719961448185856
(2, 0, 2, 12)	0	0	0
(2, 0, 2, 13)	-10082121728	173404717056	15714753453070336
(2, 0, 2, 14)	0	0	0
(2, 0, 2, 15)	None	-71661715456	-3687020553695232

(2, 0, 2, 16)	None	0	0
(2, 0, 2, 17)	None	None	-356358052100096
(2, 0, 2, 18)	None	None	0

Table 3: Fourier coefficients of K-R-S lifts

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