Envelopes of Legendre Curves in the Unit Tangent Bundle over the Euclidean Plane

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May 9, 2016

Abstract

For singular plane curves, the classical definitions of envelopes are vague. In order to define envelopes for singular plane curves, we introduce a one-parameter family of Legendre curves in the unit tangent bundle over the Euclidean plane and the curvature. Then we give a definition of an envelope for the one-parameter family of Legendre curves. We investigate properties of the envelopes. For instance, the envelope is also a Legendre curve. Moreover, we consider bi-Legendre curves and give a relationship between envelopes.

1 Introduction

Envelopes are classical object in the differential geometry. There are many applications of envelopes to differential geometry, differential equations and physics, for instance [4, 5, 7, 9, 10, 15, 16, 18, 20]. An envelope of a family of curves in the plane is a curve that is "tangent" to each member of the family at some point. If the curves are regular, then the tangent is well-defined. However, the definitions of envelopes are vague for singular plane curves (smooth curves with singular points). In this paper, we would like to clarify the definition of the envelope for a family of singular curves. As singular curves, we consider Legendre curves in the unit tangent bundle over $\mathbb{R}^2$, see Appendix A (cf. [8]). The basic results on the singularity theory see [2, 4, 14, 17]. In §2, we quickly review on the definitions of envelopes which are given by implicit functions [3, 4, 12] and parametric curves [11, 19]. In §3, we consider one-parameter families of Legendre curves. We give a moving frame and the curvature of the one-parameter family of Legendre curves. Then we show that the existence and uniqueness theorem for one-parameter families of Legendre curves. In §4, we define an envelope of a one-parameter family of Legendre curves. Then the envelope is also a Legendre curve and hence we give a curvature of the envelope as a Legendre curve. Moreover, we give relationships between the envelopes given by implicit functions and one-parameter family of Legendre curves. In §5, we define a bi-Legendre curve as a special class of one-parameter family of Legendre curves and give a relationship between envelopes.

All maps and manifolds considered here are differential of class C^∞.

2010 Mathematics Subject classification: 58K05, 53A04

Key Words and Phrases. envelope, one-parameter family of Legendre curves, Legendre curve.

Acknowledgement. The author would like to thank Professor Goo Ishikawa and Professor Shyuichi Izumiya for valuable comments and constant encouragements. The author also would like to thank Professor Kazuyuki Fujii for informing references of applications of envelopes. This is a partially supported by JSPS KAKENHI Grant Number JP 26400078.

2 Previous results

Let $\mathbb{R}^2$ be the Euclidean plane equipped with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$, where $\boldsymbol{a} = (a_1, a_2), \boldsymbol{b} = (b_1, b_2) \in \mathbb{R}^2$.

We review two definitions of envelopes for one-parameter family of plane curves. These are given by implicit functions and parametrized curves. Here we denote these envelopes by E_I and E_P respectively. Other related definitions of envelopes see [3, 19, 21].

Let $F : V \times \Lambda \to \mathbb{R}, (x, y, \lambda) \mapsto F(x, y, \lambda)$ be a smooth function, where V is a domain in $\mathbb{R}^2$, and Λ is an interval or $\mathbb{R}$. A family of curves in the plane is given by $\Gamma_\lambda = \{(x, y) \in V \mid F(x, y, \lambda) = 0\}$ for each $\lambda \in \Lambda$. Then one of the classical definition of the envelope is as follows, see for instance [3, 4]:

Definition 2.1 The *envelope* of the family F is the set E_I of points given by

$$E_I = \left\{ (x, y) \in V \mid \text{for some } \lambda \in \Lambda, F(x, y, \lambda) = \frac{\partial F}{\partial \lambda}(x, y, \lambda) = 0 \right\}.$$

If $F(x, y, \lambda) = (\partial F/\partial \lambda)(x, y, \lambda) = 0$, we say that $(x, y) \in E_I$ *with respect to* λ.

On the other hand, let $\gamma : I \times \Lambda \to \mathbb{R}^2$ be a one-parameter family of smooth parametrized curves, and let $e_p : U \to I \times \Lambda, e_p(u) = (t(u), \lambda(u))$ be a regular curve, where I, Λ and U are intervals or $\mathbb{R}$. We denote $\Gamma_\lambda(t) = \gamma(t, \lambda)$ and $E_P(u) = \gamma \circ e_p(u)$.

Definition 2.2 ([11, Page 138]) We call E_P an *envelope* (and e_p a *pre-envelope*) for the family γ, when the following conditions are satisfied.

(i) The function λ is non-constant on any non-trivial subinterval of U. (The Variability Condition.)

(ii) For all u, the curve E_P is tangent at u to the curve $\Gamma_{\lambda(u)}$ at the parameter $t(u)$, meaning that the tangent vectors $E_P'(u) = (dE_P/du)(u)$ and $\dot{\Gamma}_{\lambda(u)}(t(u)) = (d\Gamma_{\lambda(u)}/dt)(t(u))$ are linearly dependent. (The Tangency Condition.)

We say that the *singular set of* $\gamma : I \times \Lambda \to \mathbb{R}^2, \gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda))$ is the subset of the domain $I \times \Lambda$ defined by

$$\det(\gamma_t(t, \lambda), \gamma_\lambda(t, \lambda)) = \det \begin{pmatrix} x_t(t, \lambda) & y_t(t, \lambda) \\ x_\lambda(t, \lambda) & y_\lambda(t, \lambda) \end{pmatrix} = 0. \tag{1}$$

Here we denote $\gamma_t(t, \lambda) = (\partial \gamma/\partial t)(t, \lambda) = (x_t(t, \lambda), y_t(t, \lambda))$ and $\gamma_\lambda(t, \lambda) = (\partial \gamma/\partial \lambda)(t, \lambda) = (x_\lambda(t, \lambda), y_\lambda(t, \lambda))$. Then the envelope theorem is as follows:

Theorem 2.3 ([11, Page 140]) *Let $\gamma : I \times \Lambda \to \mathbb{R}^2$ be a family of parametrized curves, and let $e_p : U \to I \times \Lambda$ be a regular curve satisfying the variability condition. Then e_p is a pre-envelope of γ (and E_P is an envelope) if and only if the trace of e_p lies in the singular set of γ.*

We consider one-parameter families of 3/2-cusps as examples. Other examples see [3, 11].

Example 2.4 Let $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, F(x, y, \lambda) = (x - \lambda)^3 - y^2$. Since $(\partial F/\partial \lambda)(x, y, \lambda) = -3(x - \lambda)^2$, the envelope is given by $E_I = \{(\lambda, 0) | \lambda \in \mathbb{R}\}$.

Let $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2, \gamma(t, \lambda) = (t^2 + \lambda, t^3)$. Since (1), we have $-3t^2 = 0$. By Theorem 2.3, the pre-envelope and the envelope are given by $e_p : \mathbb{R} \to \mathbb{R} \times \mathbb{R}, e_p(u) = (0, u)$ and $E_P : \mathbb{R} \to \mathbb{R}^2, E(u) = (u, 0)$.

Both cases, the envelopes are given by the x-axis, see Figure 1.

Example 2.5 Let $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, F(x, y, \lambda) = x^3 - (y - \lambda)^2$. Since $(\partial F/\partial \lambda)(x, y, \lambda) = -2(y - \lambda)$, the envelope is given by $E_I = \{(0, \lambda) | \lambda \in \mathbb{R}\}$.

Let $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2, \gamma(t, \lambda) = (t^2, t^3 + \lambda)$. Since (1), we have $2t = 0$. By Theorem 2.3, the pre-envelope and the envelope are given by $e_p : \mathbb{R} \to \mathbb{R} \times \mathbb{R}, e_p(u) = (0, u)$ and $E_P : \mathbb{R} \to \mathbb{R}^2, E_P(u) = (0, u)$.

Both cases, the envelopes are given by the y-axis, see Figure 2. However, in the sense of limit tangent of the 3/2-cusp, y-axis is not tangent to the 3/2-cusps. Moreover, as a solution of differential equations, the x-axis in Figure 1 is a singular solution of the ODE $-y+((2/3)y')^3 = 0$ and y-axis in Figure 2 is not a singular solution of the ODE $-x+((2/3)y')^2 = 0$ (cf. [15, 16, 20]). We would like to distinguish as envelopes, see Examples 4.2 and 4.3 below.

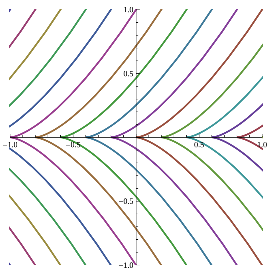


Figure 1.

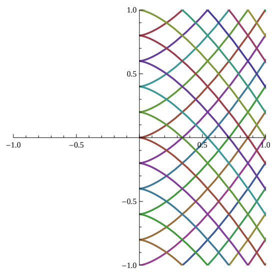


Figure 2.

3 One parameter families of Legendre curves

In this section, we consider one-parameter families of Legendre curves in the unit tangent bundle $T_1 S^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$. The fundamental results for Legendre curves in the unit tangent bundle over $\mathbb{R}^2$ see the Appendix or [8].

Definition 3.1 Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a smooth mapping. We say that (γ, ν) is a *one-parameter family of Legendre curves* if $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$ for all $(t, \lambda) \in I \times \Lambda$.

Then $(\gamma(\cdot, \lambda), \nu(\cdot, \lambda)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve for each fixed parameter $\lambda \in \Lambda$, that is, $(\gamma(\cdot, \lambda), \nu(\cdot, \lambda))$ is an integrable curve with respect to the canonical contact 1-form on $\mathbb{R}^2 \times S^1$. Therefore, $\gamma : I \times \Lambda \to \mathbb{R}^2$ is a one-parameter family of frontals.

We denote $J(\boldsymbol{a}) = (-a_2, a_1)$ the anticlockwise rotation by $\pi/2$ of a vector $\boldsymbol{a} = (a_1, a_2)$. We define $\boldsymbol{\mu}(t, \lambda) = J(\nu(t, \lambda))$. Since $\{\nu(t, \lambda), \boldsymbol{\mu}(t, \lambda)\}$ is a moving frame along $\gamma(t, \lambda)$ on $\mathbb{R}^2$, we have the Frenet type formula.

$$\begin{pmatrix} \nu_t(t, \lambda) \\ \boldsymbol{\mu}_t(t, \lambda) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t, \lambda) \\ -\ell(t, \lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t, \lambda) \\ \boldsymbol{\mu}(t, \lambda) \end{pmatrix},$$
$$\begin{pmatrix} \nu_\lambda(t, \lambda) \\ \boldsymbol{\mu}_\lambda(t, \lambda) \end{pmatrix} = \begin{pmatrix} 0 & m(t, \lambda) \\ -m(t, \lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t, \lambda) \\ \boldsymbol{\mu}(t, \lambda) \end{pmatrix}$$

and

$$\gamma_t(t,\lambda) = \beta(t,\lambda)\boldsymbol{\mu}(t,\lambda),$$

where $\ell(t,\lambda) = \nu_t(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda), m(t,\lambda) = \nu_\lambda(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda)$ and $\beta(t,\lambda) = \gamma_t(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda)$. By the integrability condition $\nu_{t\lambda}(t,\lambda) = \nu_{\lambda t}(t,\lambda)$, ℓ and m satisfies the condition

$$\ell_\lambda(t,\lambda) = m_t(t,\lambda) \tag{2}$$

for all $(t,\lambda)\in I\times\Lambda$. We call the pair (ℓ,m,β) with the integrability condition (2) a *curvature of the one-parameter family of Legendre curves* (γ,ν).

Remark 3.2 *Let $(\gamma,\nu): I\times\Lambda\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves with the curvature (ℓ,m,β). Then $(\gamma,-\nu)$ is also a one-parameter family of Legendre curves with the curvature $(\ell,m,-\beta)$.*

Example 3.3 (Example 2.4) Let $(\gamma,\nu):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2\times S^1, \gamma(t,\lambda) = (t^2+\lambda, t^3), \nu(t,\lambda) = (4+9t^2)^{-1/2}(-3t,2)$. Since $\gamma_t(t,\lambda) = (2t,3t^2), \nu_t(t,\lambda) = 6(4+9t^2)^{-3/2}(-2,-3t), \nu_\lambda(t,\lambda) = 0$ and $\boldsymbol{\mu}(t,\lambda) = (4+9t^2)^{-1/2}(-2,-3t)$, (γ,ν) is a one-parameter family of Legendre curves with the curvature $(\ell(t,\lambda), m(t,\lambda), \beta(t,\lambda)) = (6(4+9t^2)^{-1}, 0, -t(4+9t^2)^{1/2})$.

Example 3.4 (Example 2.5) Let $(\gamma,\nu):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2\times S^1, \gamma(t,\lambda) = (t^2, t^3+\lambda), \nu(t,\lambda) = (4+9t^2)^{-1/2}(-3t,2)$. Since $\gamma_t(t,\lambda) = (2t,3t^2), \nu_t(t,\lambda) = 6(4+9t^2)^{-3/2}(-2,-3t), \nu_\lambda(t,\lambda) = 0$ and $\boldsymbol{\mu}(t,\lambda) = (4+9t^2)^{-1/2}(-2,-3t)$, (γ,ν) is a one-parameter family of Legendre curves with the curvature $(\ell(t,\lambda), m(t,\lambda), \beta(t,\lambda)) = (6(4+9t^2)^{-1}, 0, -t(4+9t^2)^{1/2})$.

Definition 3.5 Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}): I\times\Lambda\to\mathbb{R}^2\times S^1$ be one-parameter families of Legendre curves. We say that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are *congruent as one-parameter family of Legendre curves* if there exist a constant rotation $A\in SO(2)$ and a smooth translation mapping $\boldsymbol{a}:\Lambda\to\mathbb{R}^2$ such that $\widetilde{\gamma}(t,\lambda) = A(\gamma(t,\lambda)) + \boldsymbol{a}(\lambda)$ and $\widetilde{\nu}(t,\lambda) = A(\nu(t,\lambda))$ for all $(t,\lambda)\in I\times\Lambda$.

We give the existence and uniqueness theorems for one-parameter families of Legendre curves.

Theorem 3.6 (The Existence Theorem for one-parameter families of Legendre curves.) *Let $(\ell,m,\beta): I\times\Lambda\to\mathbb{R}^3$ be a smooth mapping with the integrability condition. There exists a one-parameter family of Legendre curves $(\gamma,\nu): I\times\Lambda\to\mathbb{R}^2\times S^1$ whose associated curvature is (ℓ,m,β).*

Proof. Let $(t_0,\lambda_0)\in I\times\Lambda$ be fixed. We define a smooth mapping $\theta: I\times\Lambda\to\mathbb{R}$ by

$$\theta(t,\lambda) = \int_{t_0}^{t}\ell(t,\lambda)dt + \int_{\lambda_0}^{\lambda} m(t_0,\lambda)d\lambda.$$

Then θ satisfy the conditions $\theta_t(t,\lambda) = \ell(t,\lambda)$ and $\theta_\lambda(t,\lambda) = m(t,\lambda)$. We define a smooth mapping $(\gamma,\nu): I\times\Lambda\to\mathbb{R}^2\times S^1$ by

$$\begin{aligned}\gamma(t,\lambda) &= \left(-\int\beta(t,\lambda)\sin\theta(t,\lambda)dt, \int\beta(t,\lambda)\cos\theta(t,\lambda)dt\right),\\ \nu(t,\lambda) &= (\cos\theta(t,\lambda), \sin\theta(t,\lambda)).\end{aligned}$$

By a direct calculation, (γ,ν) is a one-parameter family of Legendre curves with the curvature (ℓ,m,β). $\square$

Theorem 3.7 (The Uniqueness Theorem for one-parameter families of Legendre curves.) *Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}) : I\times\Lambda \to \mathbb{R}^2\times S^1$ be one-parameter families of Legendre curves with the curvatures (ℓ,m,β) and $(\widetilde{\ell},\widetilde{m},\widetilde{\beta})$ respectively. Then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as one-parameter family of Legendre curves if and only if (ℓ,m,β) and $(\widetilde{\ell},\widetilde{m},\widetilde{\beta})$ coincides.*

Proof. Suppose that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as one-parameter families of Legendre curves. By a direct calculation, we have

$$\begin{aligned}
\widetilde{\gamma}_t(t,\lambda) &= \frac{\partial}{\partial t}(A(\gamma(t,\lambda))+\boldsymbol{a}(\lambda)) = A(\gamma_t(t,\lambda)) = \beta(t,\lambda)A(\boldsymbol{\mu}(t,\lambda)) = \beta(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda),\\
\widetilde{\nu}_t(t,\lambda) &= \frac{\partial}{\partial t}(A(\nu(t,\lambda))) = A(\nu_t(t,\lambda)) = \ell(t,\lambda)A(\boldsymbol{\mu}(t,\lambda)) = \ell(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda),\\
\widetilde{\nu}_\lambda(t,\lambda) &= \frac{\partial}{\partial \lambda}(A(\nu(t,\lambda))) = A(\nu_\lambda(t,\lambda)) = m(t,\lambda)A(\boldsymbol{\mu}(t,\lambda)) = m(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda).
\end{aligned}$$

Therefore the curvatures (ℓ,m,β) and $(\widetilde{\ell},\widetilde{m},\widetilde{\beta})$ coincides.

Conversely, suppose that (ℓ,m,β) and $(\widetilde{\ell},\widetilde{m},\widetilde{\beta})$ coincides. Let $(t_0,\lambda_0)\in I\times\Lambda$ be fixed. By using a congruence as one-parameter family of Legendre curves, we may assume $\gamma(t_0,\lambda_0) = \widetilde{\gamma}(t_0,\lambda_0)$ and $\nu(t_0,\lambda_0)=\widetilde{\nu}(t_0,\lambda_0)$. Moreover, we have $\theta(t,\lambda)=\widetilde{\theta}(t,\lambda)$ for all $(t,\lambda)\in I\times\Lambda$ in the proof of Theorem 3.6. It follows from the construction that we have $\nu(t,\lambda)=\widetilde{\nu}(t,\lambda)$, and $\gamma(t,\lambda)=\widetilde{\gamma}(t,\lambda)$ up to a smooth translation mapping $\boldsymbol{a}(\lambda)$ for all $(t,\lambda)\in I\times\Lambda$. □

4 Envelopes of Legendre curves

Let $(\gamma,\nu) : I\times\Lambda\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves with the curvature (ℓ,m,β), and let $e_L : U\to I\times\Lambda, e_L(u)=(t(u),\lambda(u))$ be a smooth curve. We denote $\Gamma_\lambda(t) = \gamma(t,\lambda)$ and $E_L=\gamma\circ e_L(u)$. Note that we don't assume e_L is a regular curve, see section 2.

Definition 4.1 We call E_L an *envelope* (and e_L a *pre-envelope*) for the family of Legendre curves (γ,ν), when the following conditions are satisfied.

(i) The function λ is non-constant on any non-trivial subinterval of U.(The Variability Condition.)

(ii) For all u the curve E_L is tangent at u to the curve $\Gamma_{\lambda(u)}$ at the parameter $t(u)$, meaning that $E_L'(u)$ and $\boldsymbol{\mu}(t(u),\lambda(u))$ are linearly dependent. (The Tangency Condition.)

Note that the tangency condition is equivalent to the condition $E_L'(u)\cdot\nu(e_L(u))=0$ for all $u\in U$.

Example 4.2 (Example 3.3) Let $(\gamma,\nu):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2\times S^1, \gamma(t,\lambda)=(t^2+\lambda,t^3), \nu(t,\lambda) = (4+9t^2)^{-1/2}(-3t,2)$. Let $e_L:\mathbb{R}\to\mathbb{R}\times\mathbb{R}, e_L(u)=(t(u),\lambda(u))=(0,u)$. Then $E_L(u) = \gamma\circ e_L(u)=(u,0)$. Since $\lambda'(u)=1$ and $E_L'(u)\cdot\nu(0,u)=0$, E_L is an envelope of (γ,ν).

Example 4.3 (Example 3.4) Let $(\gamma,\nu):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2\times S^1, \gamma(t,\lambda)=(t^2,t^3+\lambda), \nu(t,\lambda) = (4+9t^2)^{-1/2}(-3t,2)$. Let $e_L:\mathbb{R}\to\mathbb{R}\times\mathbb{R}, e_L(u)=(t(u),\lambda(u))=(0,u)$. Then $E_L(u) = \gamma\circ e_L(u)=(0,u)$ and $\lambda'(u)=1$. Since $E_L'(u)\cdot\nu(0,u)=1\neq 0$, E_L is not an envelope of (γ,ν).

Proposition 4.4 *Let $(\gamma,\nu):I\times\Lambda\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves with the curvature (ℓ,m,β). Suppose that $e_L:U\to I\times\Lambda$ is a pre-envelope and $E_L=\gamma\circ e_L:U\to\mathbb{R}^2$ is an envelope of (γ,ν). Then E_L is a frontal. More preciously, $(E_L,\nu\circ e_L):U\to\mathbb{R}^2\times S^1$ is a Legendre curve with the curvature*

$$\begin{aligned}\ell_E(u) &= t'(u)\ell(e_L(u))+\lambda'(u)m(e_L(u)),\\ \beta_E(u) &= t'(u)\beta(e_L(u))+\lambda'(u)\gamma_\lambda(e_L(u))\cdot\boldsymbol{\mu}(e_L(u)).\end{aligned}$$

Proof. We denote $e_L(u)=(t(u),\lambda(u))$. Since E_L is an envelope, $E_L'(u)\cdot\nu(e_L(u))=0$ for all $u\in U$. It follows that $(E_L,\nu\circ e_L):U\to\mathbb{R}^2\times S^1$ is a Legendre curve. Then $\ell_E(u)=(d/du)(\nu(e_L(u)))\cdot\boldsymbol{\mu}(e_L(u))=(t'(u)\nu_t(e_L(u))+\lambda'(u)\nu_\lambda(e_L(u)))\cdot\boldsymbol{\mu}(e_L(u))=t'(u)\ell(e_L(u))+\lambda'(u)m(e_L(u))$ and $\beta_E(u)=(d/du)(\gamma(e_L(u)))\cdot\boldsymbol{\mu}(e_L(u))=(t'(u)\gamma_t(e_L(u))+\lambda'(u)\gamma_\lambda(e_L(u)))\cdot\boldsymbol{\mu}(e_L(u))=t'(u)\beta(e_L(u))+\lambda'(u)\gamma_\lambda(e_L(u))\cdot\boldsymbol{\mu}(e_L(u))$. $\square$

We give the envelope theorem for one-parameter family of Legendre curves.

Theorem 4.5 *Let $(\gamma,\nu):I\times\Lambda\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves, and let $e_L:U\to I\times\Lambda$ be a smooth curve satisfying the variability condition. Then e_L is a pre-envelope of (γ,ν) (and E_L is an envelope) if and only if $\gamma_\lambda(e_L(u))\cdot\nu(e_L(u))=0$ for all $u\in U$.*

Proof. Suppose that e_L is a pre-envelope of (γ,ν). By the tangency condition, there exists a function $c(u)\in\mathbb{R}$ such that $E_L'(u)=c(u)\boldsymbol{\mu}(e_L(u))$. By differentiate $E_L(u)=\gamma\circ e_L(u)$, we have $E_L'(u)=t'(u)\gamma_t(e_L(u))+\lambda'(u)\gamma_\lambda(e_L(u))$. It follows from $\gamma_t(t,\lambda)=\beta(t,\lambda)\boldsymbol{\mu}(t,\lambda)$ that $(t'(u)\beta(e_L(u))-c(u))\boldsymbol{\mu}(e_L(u))+\lambda'(u)\gamma_\lambda(e_L(u))=0$. Then we have $\lambda'(u)\gamma_\lambda(e_L(u))\cdot\nu(e_L(u))=0$. By the variability condition, we have $\gamma_\lambda(e_L(u))\cdot\nu(e_L(u))=0$ for all $u\in U$.

Conversely, suppose that $\gamma_\lambda(e_L(u))\cdot\nu(e_L(u))=0$ for all $u\in U$. Since $E_L'(u)\cdot\nu(e_L(u))=(t'(u)\gamma_t(e_L(u))+\lambda'(u)\gamma_\lambda(e_L(u)))\cdot\nu(e_L(u))=0$, e_L is a pre-envelope of (γ,ν). $\square$

Example 4.6 Let i,j,m,n,j,k be natural numbers with $j=i+h, n=m+k$. Moreover, we take $h=1$ or $k=1$, or h,k are relatively prime numbers. Let $(\gamma,\nu):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2\times S^1$,

$$\gamma(t,\lambda)=\left(\frac{t^m}{m}+\frac{\lambda^i}{i},\frac{t^n}{n}+\frac{\lambda^j}{j}\right),\ \nu(t,\lambda)=\frac{1}{\sqrt{t^{2k}+1}}(-t^k,1).$$

Since $\gamma_t(t,\lambda)=(t^{m-1},t^{n-1})$, we have $\gamma_t(t,\lambda)\cdot\nu(t,\lambda)=0$ for all $(t,\lambda)\in\mathbb{R}\times\mathbb{R}$. Moreover, since $\gamma_\lambda(t,\lambda)=(\lambda^{i-1},\lambda^{j-1})$, we have $\gamma_\lambda(t,\lambda)\cdot\nu(t,\lambda)=(\lambda^{i-1}/\sqrt{t^{2k}+1})(-t^k+\lambda^h)$. If we take $e_L:\mathbb{R}\to\mathbb{R}\times\mathbb{R}, e_L(u)=(u^h,u^k)$, then the variability condition holds. Furthermore, since

$$\gamma_\lambda(e_L(u))\cdot\nu(e_L(u))=\frac{u^{k(i-1)}}{\sqrt{u^{2kh}+1}}(-u^{hk}+u^{hk})=0,$$

e_L is a pre-envelope of (γ,ν) by Theorem 4.5. Hence, the envelope $(E_L,\nu_L):\mathbb{R}\to\mathbb{R}^2\times S^1$ is given by

$$E_L(u)=\left(\frac{u^{mh}}{m}+\frac{u^{ik}}{i},\frac{u^{nh}}{n}+\frac{u^{jk}}{j}\right),\nu_L(u)=\frac{1}{\sqrt{u^{2kh}+1}}(-u^{kh},1).$$

Proposition 4.7 *Let $(\gamma,\nu):I\times\Lambda\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves. Suppose that $e_L:U\to I\times\Lambda$ is a pre-envelope and $E_L=\gamma\circ e_L$ is an envelope of (γ,ν). Then $e_L:U\to I\times\Lambda$ is also a pre-envelope and $E_L=\gamma\circ e_L$ is also an envelope of $(\gamma,-\nu)$.*

Proof. By Remark 3.2, $(\gamma, -\nu)$ is also a one-parameter family of Legendre curves. It follows from Theorem 4.5 that we have the same pre-envelopes and the envelopes of (γ, ν) and $(\gamma, -\nu)$. $\Box$

Definition 4.8 We say that a map $\Phi : \widetilde{I} \times \widetilde{\Lambda} \to I \times \Lambda$ is a one-parameter family of parameter change if Φ is a diffeomorphism and given by the form $\Phi(s, k) = (\phi(s, k), \varphi(k))$.

Proposition 4.9 *Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves. Suppose that $e_L : U \to I \times \Lambda$ is a pre-envelope, $E_L = \gamma \circ e_L$ is an envelope and $\Phi : \widetilde{I} \times \widetilde{\Lambda} \to I \times \Lambda$ is a one-parameter family of parameter change. Then $(\widetilde{\gamma}, \widetilde{\nu}) = (\gamma \circ \Phi, \nu \circ \Phi) : \widetilde{I} \times \widetilde{\Lambda} \to \mathbb{R}^2 \times S^1$ is also a one-parameter family of Legendre curves. Moreover, $\Phi^{-1} \circ e_L : U \to \widetilde{I} \times \widetilde{\lambda}$ is a pre-envelope and E_L is also an envelope of $(\widetilde{\gamma}, \widetilde{\nu})$.*

Proof. Since $\widetilde{\gamma}_s(s, k) = \phi_s(s, k)\gamma_t(\Phi(s, k))$ and $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$ for all $(t, \lambda) \in I \times \Lambda$, we have $\widetilde{\gamma}_s(s, k) \cdot \widetilde{\nu}(s, k) = 0$ for all $(s, k) \in \widetilde{I} \times \widetilde{\Lambda}$. Therefore, $(\widetilde{\gamma}, \widetilde{\nu})$ is a one-parameter family of Legendre curves. By the form of the diffeomorphism $\Phi(s, k) = (\phi(s, k), \varphi(k))$, $\Phi^{-1} : I \times \Lambda \to \widetilde{I} \times \widetilde{\Lambda}$ is given by the form $\Phi^{-1}(t, \lambda) = (\psi(t, \lambda), \varphi^{-1}(\lambda))$. It follows that $\Phi^{-1} \circ e_L(u) = (\phi(t(u), \lambda(u)), \varphi^{-1}(\lambda(u)))$. Since $(d/du)\varphi^{-1}(\lambda(u)) = \varphi^{-1}_\lambda(\lambda(u))\lambda'(u)$, the variability condition holds. Moreover, we have $\widetilde{\gamma}_k(s, k) \cdot \widetilde{\nu}(s, k) = (\gamma_t(\Phi(s, k))\phi_k(s, k) + \gamma_\lambda(\Phi(s, k))\varphi'(k)) \cdot \nu(\Phi(s, k)) = \varphi'(k)\gamma_\lambda(\Phi(s, k)) \cdot \nu(\Phi(s, k))$. It follows that $\widetilde{\gamma}_k(\Phi^{-1} \circ e_L(u)) \cdot \widetilde{\nu}(\Phi^{-1} \circ e_L(u)) = \varphi'(\varphi^{-1}(\lambda(u)))\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$. By Theorem 4.5, $\Phi^{-1} \circ e_L$ is a pre-envelope of $(\widetilde{\gamma}, \widetilde{\nu})$. Therefore, $\widetilde{\gamma} \circ \Phi^{-1} \circ e_L = \gamma \circ \Phi \circ \Phi^{-1} \circ e_L = \gamma \circ e_L = E_L$ is also an envelope of $(\widetilde{\gamma}, \widetilde{\nu})$. $\Box$

We give a relationship between envelopes which are given by implicit functions (Definition 2.1) and one-parameter families of Legendre curves.

Proposition 4.10 *Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves, and let $F(x, y, \lambda) = 0$ be an implicit function of the one-parameter family of frontals, that is, $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$, where $\gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda))$. If $e_L : U \to I \times \Lambda$ is a pre-envelope and $E_L : U \to \mathbb{R}^2$ is an envelope of (γ, ν), then $E_L(U) \subset E_I$.*

Proof. By differentiate $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$, we have

$$x_t(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_t(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) = 0$$

and

$$x_\lambda(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_\lambda(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) + F_\lambda(x(t, \lambda), y(t, \lambda), \lambda) = 0.$$

Equivalently, $\gamma_t(t, \lambda) \cdot (F_x, F_y)(x(t, \lambda), y(t, \lambda), \lambda) = 0$ and $\gamma_\lambda(t, \lambda) \cdot (F_x, F_y)(x(t, \lambda), y(t, \lambda), \lambda) + F_\lambda(x(t, \lambda), y(t, \lambda), \lambda) = 0$. Since (γ, ν) is a one-parameter family of Legendre curves, there exists a function $c(t, \lambda)$ such that $(F_x, F_y)(x(t, \lambda), y(t, \lambda), \lambda) = c(t, \lambda)\nu(t, \lambda)$. Moreover, $e_L(u) = (t(u), \lambda(u))$ is a pre-envelope of (γ, ν), we have $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ for all $u \in U$. It follows that $F_\lambda(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) = 0$. Therefore, we have $E(u) = \gamma \circ e_L(u) \in E_I$ with respect to $\lambda(u)$ for all $u \in U$. $\Box$

In order to consider the converse result, we need the following lemma and proposition.

Lemma 4.11 *Let $\boldsymbol{a}, \boldsymbol{b} : U \to \mathbb{R}^2$ be smooth maps. Suppose that the set of non-zero points of smooth function $k : U \to \mathbb{R}$ is dense in U. If $k(u)\boldsymbol{a}(u)$ and $\boldsymbol{b}(u)$ are linearly dependent, then $\boldsymbol{a}(u)$ and $\boldsymbol{b}(u)$ are linearly dependent for all $u \in U$.*

Proof. Since $\det(k(u)\boldsymbol{a}(u), \boldsymbol{b}(u)) = 0$, we have $k(u)\det(\boldsymbol{a}(u), \boldsymbol{b}(u)) = 0$. By the condition and continuous property, we have $\det(\boldsymbol{a}(u), \boldsymbol{b}(u)) = 0$ for all $u \in U$. $\square$

Proposition 4.12 *Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2$ be a one-parameter family of Legendre curves, and let $e_L : U \to I \times \Lambda$ be a smooth curve satisfying the variability condition. If the set of regular points of γ on $e_L(U)$ is dense in U and the trace of e_L lies in the singular set of γ, then e_L is a pre-envelope of (γ, ν) (and E_L is an envelope).*

Proof. Since $e_L(u)$ belong to the singular set of γ, we have $\det(\gamma_t(e_L(u)), \gamma_\lambda(e_L(u))) = 0$ for all $u \in U$. Therefore $\gamma_t(e_L(u)) = \beta(e_L(u))\boldsymbol{\mu}(e_L(u))$ and $\gamma_\lambda(e_L(u))$ are linearly dependent. By the assumption, the set of non-zero points of $\beta \circ e_L$ is dense in U. It follows from Lemma 4.11 that $\boldsymbol{\mu}(e_L(u))$ and $\gamma_\lambda(e_L(u))$ are linearly dependent. Therefore $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ for all $u \in U$. By Theorem 4.5, e_L is a pre-envelope of (γ, ν). $\square$

Proposition 4.13 *Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves, and let $F(x, y, \lambda) = 0$ be an implicit function of the one-parameter family of frontals, that is, $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$, where $\gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda))$. Let $e_L : U \to I \times \Lambda, e(u) = (t(u), \lambda(u))$ be a smooth curve satisfying the variability condition. If the set of regular points of γ on $e_L(U)$ is dense in U, $E_L(u) = \gamma \circ e_L(u) \in E_I$ with respect to $\lambda(u)$ and*

$$(F_x, F_y)(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) \neq (0, 0)$$

for all $u \in U$, then e_L is a pre-envelope of (γ, ν) (and E_L is an envelope).

Proof. By differentiate $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$, we have

$$x_t(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_t(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) = 0$$

and

$$x_\lambda(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_\lambda(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) + F_\lambda(x(t, \lambda), y(t, \lambda), \lambda) = 0.$$

Since $E_L(u) = \gamma \circ e_L(u) \in E_I$ with respect to $\lambda(u)$, we have $F_\lambda(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) = 0$. It follows that

$$\begin{pmatrix} x_t(t(u), \lambda(u)) & y_t(t(u), \lambda(u)) \\ x_\lambda(t(u), \lambda(u)) & y_\lambda(t(u), \lambda(u)) \end{pmatrix} \begin{pmatrix} F_x(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) \\ F_y(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then the trace of e_L lies in the singular set of γ. By Proposition 4.12, e_L is a pre-envelope of (γ, ν). $\square$

We give interesting examples of envelopes of one-parameter families of Legendre curves by using two Legendre curves. Also see [6, 9, 10].

Let $(\boldsymbol{p}, \nu_p) : I \to \mathbb{R}^2 \times S^1$ and $(\boldsymbol{q}, \nu_q) : \Lambda \to \mathbb{R}^2 \times S^1$ be Legendre curves with the curvature (ℓ_p, β_p) and (ℓ_q, β_q) respectively, see Appendix A. We denote

$$\boldsymbol{p}(t) = (p_1(t), p_2(t)), \nu_p(t) = (\nu_{p1}(t), \nu_{p2}(t)), \boldsymbol{\mu}_p(t) = (-\nu_{p2}(t), \nu_{p1}(t)),$$

$$\boldsymbol{q}(\lambda) = (q_1(\lambda), q_2(\lambda)), \nu_q(\lambda) = (\nu_{q1}(\lambda), \nu_{q2}(\lambda)), \boldsymbol{\mu}_q(\lambda) = (-\nu_{q2}(\lambda), \nu_{q1}(\lambda)),$$

respectively. Suppose that $\boldsymbol{p}(0) = (0,0)$ and $\nu_p(0) = (0,1)$.

We define $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ by

$$\gamma(t,\lambda) = \boldsymbol{q}(\lambda) + A(\theta(\lambda))\boldsymbol{p}(t), \nu(t,\lambda) = A(\theta(\lambda))\nu_p(t), \tag{3}$$

where $\theta : \Lambda \to \mathbb{R}$ and

$$A(\theta(\lambda)) = \begin{pmatrix} \cos\theta(\lambda) & -\sin\theta(\lambda) \\ \sin\theta(\lambda) & \cos\theta(\lambda) \end{pmatrix}.$$

By a direct calculation, (γ, ν) is a one-parameter family of Legendre curves.

First, we consider a Legendre curve $\boldsymbol{p}$ along a Legendre curve $\boldsymbol{q}$ which satisfying the both unit normal vectors coincide. Suppose that $\nu(0,\lambda) = \nu_q(\lambda)$. This means that the unit normal vector of γ at $(0,\lambda)$ coincide with the unit normal vector of $\boldsymbol{q}$ at λ. It follows that $\cos\theta(\lambda) = \nu_{q2}(\lambda)$ and $\sin\theta(\lambda) = -\nu_{q1}(\lambda)$. By a direct calculation, we have $\gamma_\lambda(t,\lambda) \cdot \nu(t,\lambda) = -\beta_q(\lambda)\nu_{p1}(t) - \ell_q(\lambda)\boldsymbol{p}(t) \cdot \boldsymbol{\mu}_p(t)$. By a corollary of Theorem 4.5, we have the following.

Corollary 4.14 *Under the above notations, let (γ, ν) be given by* (3) *with the conditions* $\boldsymbol{p}(0) = (0,0), \nu_p(0) = (0,1)$ *and* $\nu(0,\lambda) = \nu_q(\lambda)$. *Let* $e_L : U \to I \times \Lambda$ *be a smooth curve satisfying the variability condition. If* $\beta_q(\lambda(u))\nu_{p1}(t(u)) + \ell_q(\lambda(u))\boldsymbol{p}(t(u)) \cdot \boldsymbol{\mu}_p(t(u)) = 0$, *then* e_L *is a pre-envelope of* (γ, ν) *(and* E_L *is an envelope).*

Note that $e_L(u) = (0,u)$ is a pre-envelope of (γ, ν). Thus, $E_L(u) = \boldsymbol{q}(\lambda)$ is always an envelope of (γ, ν).

Second, we consider a Legendre curve $\boldsymbol{p}$ along a Legendre curve $\boldsymbol{q}$ which satisfying the unit normal vector of $\boldsymbol{p}$ coincide with the tangent vector of $\boldsymbol{q}$. Suppose that $\nu(0,\lambda) = \boldsymbol{\mu}_q(\lambda)$. This means that the unit normal vector of γ at $(0,\lambda)$ coincide with the unit tangent vector of $\boldsymbol{q}$ at λ. It follows that $\cos\theta(\lambda) = \nu_{q1}(\lambda)$ and $\sin\theta(\lambda) = \nu_{q2}(\lambda)$. By a direct calculation, we have $\gamma_\lambda(t,\lambda) \cdot \nu(t,\lambda) = \beta_q(\lambda)\nu_{p2}(t) - \ell_q(\lambda)\boldsymbol{p}(t) \cdot \boldsymbol{\mu}_p(t)$. By a corollary of Theorem 4.5, we have the following.

Corollary 4.15 *Under the above notations, let (γ, ν) be given by* (3) *with the conditions* $\boldsymbol{p}(0) = (0,0), \nu_p(0) = (0,1)$ *and* $\nu(0,\lambda) = \boldsymbol{\mu}_q(\lambda)$. *Let* $e_L : U \to I \times \Lambda$ *be a smooth curve satisfying the variability condition. If* $\beta_q(\lambda(u))\nu_{p2}(t(u)) - \ell_q(\lambda(u))\boldsymbol{p}(t(u)) \cdot \boldsymbol{\mu}_p(t(u)) = 0$, *then* e_L *is a pre-envelope of* (γ, ν) *(and* E_L *is an envelope).*

Example 4.16 Let $(\boldsymbol{p}, \nu_p) : [0,2\pi) \to \mathbb{R}^2 \times S^1$ be an astroid $\boldsymbol{p}(t) = (\cos^3 t - 1, \sin^3 t), \nu_p(t) = (\sin t, \cos t)$ and $(\boldsymbol{q}, \nu_q) : [0,2\pi) \to \mathbb{R}^2 \times S^1$ be the unit circle $\boldsymbol{q}(\lambda) = (\cos\lambda, \sin\lambda), \nu_q(\lambda) = (\cos\lambda, \sin\lambda)$, see Figure 3. Then we have $\beta_p(t) = 3\cos t \sin t, \ell_p(t) = -1, \beta_q(\lambda) = 1$ and $\ell_q(\lambda) = 1$. Moreover, the conditions $\boldsymbol{p}(0) = (0,0)$ and $\nu_p(0) = (0,1)$ are satisfied.

First, we consider a Legendre curve $\boldsymbol{p}$ along a Legendre curve $\boldsymbol{q}$ which satisfying the both unit normal vectors coincide. By (3) and the condition $\nu(0,\lambda) = \nu_q(\lambda)$, the one-parameter family of Legendre curves $(\gamma, \nu) : [0,2\pi) \times [0,2\pi) \to \mathbb{R}^2 \times S^1$ is given by

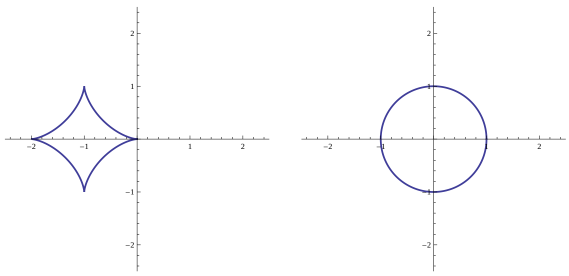
$$\begin{aligned} \gamma(t,\lambda) &= \begin{pmatrix} \cos\lambda \\ \sin\lambda \end{pmatrix} + \begin{pmatrix} \sin\lambda & \cos\lambda \\ -\cos\lambda & \sin\lambda \end{pmatrix} \begin{pmatrix} \cos^3 t - 1 \\ \sin^3 t \end{pmatrix}, \\ \nu(t,\lambda) &= \begin{pmatrix} \sin\lambda & \cos\lambda \\ -\cos\lambda & \sin\lambda \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}. \end{aligned}$$

By a direct calculation, we have $\gamma_\lambda(t,\lambda)\cdot\nu(t,\lambda) = -4\cos(t-(\pi/4))\cos((t/2)-(\pi/4))\sin t/2$. It follows that $e_L : [0,2\pi) \to [0,2\pi)\times[0,2\pi)$, $e_L(u) = (0,u), (3\pi/4,u), (3\pi/2,u), (7\pi/4,u)$ are pre-envelopes of (γ,ν) respectively, by Corollary 4.14. Therefore, the envelopes $E_L : [0,2\pi) \to \mathbb{R}^2$ of (γ,ν) are given by $E_L(u) = (\cos u, \sin u), (\sqrt{2}+(1/2))(\cos(u+\pi/4), \sin(u+\pi/4)), (\cos(u+\pi/2), \sin(u+\pi/2)), (\sqrt{2}-(1/2))(\cos(u+\pi/4), \sin(u+\pi/4))$, respectively see Figure 4 left.

Second, we consider a Legendre curve $\boldsymbol{p}$ along a Legendre curve $\boldsymbol{q}$ which satisfying the unit normal vector of $\boldsymbol{p}$ coincide with the tangent vector of $\boldsymbol{q}$. By (3) and the condition $\nu(0,\lambda) = \boldsymbol{\mu}_q(\lambda) = (-\sin\lambda, \cos\lambda)$, the one-parameter family of Legendre curves $(\gamma,\nu) : [0,2\pi)\times[0,2\pi) \to \mathbb{R}^2\times S^1$ is given by

$$\begin{aligned}
\gamma(t,\lambda) &= \begin{pmatrix} \cos\lambda \\ \sin\lambda \end{pmatrix} + \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} \cos^3 t - 1 \\ \sin^3 t \end{pmatrix}, \\
\nu(t,\lambda) &= \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.
\end{aligned}$$

By a direct calculation, we have $\gamma_\lambda(t,\lambda)\cdot\nu(t,\lambda) = \cos 2t$. It follows that $e_L : [0,2\pi) \to [0,2\pi)\times[0,2\pi)$, $e_L(u) = (\pi/4,u), (3\pi/4,u), (5\pi/4,u), (7\pi/4,u)$ are pre-envelopes of (γ,ν) respectively, by Corollary 4.15. Therefore the envelopes $E_L : [0,2\pi) \to \mathbb{R}^2$ of (γ,ν) are given by $E_L(u) = (1/2)(\cos(u+\pi/4), \sin(u+\pi/4)), (1/2)(\cos(u+3\pi/4), \sin(u+3\pi/4)), (1/2)(\cos(u+5\pi/4), \sin(u+5\pi/4)), (1/2)(\cos(u+7\pi/4), \sin(u+7\pi/4))$, respectively see Figure 4 right.



an astroid the unit circle

Figure 3.

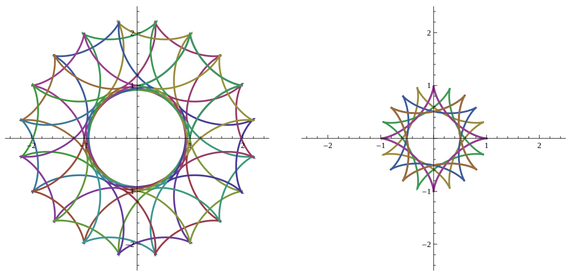


Figure 4.

5 Bi-Legendre curves and envelopes

We consider a special class of one-parameter families of Legendre curves. Let $(\gamma,\nu) : I\times\Lambda \to \mathbb{R}^2\times S^1$ be a smooth mapping.

Definition 5.1 We say that $(\gamma,\nu) : I\times\Lambda \to \mathbb{R}^2\times S^1$ is a *bi-Legendre curve* if $\gamma_t(t,\lambda)\cdot\nu(t,\lambda) = 0$ and $\gamma_\lambda(t,\lambda)\cdot\nu(t,\lambda) = 0$ for all $(t,\lambda) \in I\times\Lambda$.

Then (γ,ν) is a one-parameter family of Legendre curves with respect to both parameters t and λ. We define $\boldsymbol{\mu}(t,\lambda)=J(\nu(t,\lambda))$. Since $\{\nu(t,\lambda),\boldsymbol{\mu}(t,\lambda)\}$ is a moving frame along $\gamma(t,\lambda)$, we have the Frenet type formula.

$$
\begin{aligned}
\begin{pmatrix} \nu_t(t,\lambda) \\ \boldsymbol{\mu}_t(t,\lambda) \end{pmatrix} &= \begin{pmatrix} 0 & \ell(t,\lambda) \\ -\ell(t,\lambda) & 0 \end{pmatrix}\begin{pmatrix} \nu(t,\lambda) \\ \boldsymbol{\mu}(t,\lambda) \end{pmatrix}, \\
\begin{pmatrix} \nu_\lambda(t,\lambda) \\ \boldsymbol{\mu}_\lambda(t,\lambda) \end{pmatrix} &= \begin{pmatrix} 0 & m(t,\lambda) \\ -m(t,\lambda) & 0 \end{pmatrix}\begin{pmatrix} \nu(t,\lambda) \\ \boldsymbol{\mu}(t,\lambda) \end{pmatrix}, \\
\gamma_t(t,\lambda) &= \beta(t,\lambda)\boldsymbol{\mu}(t,\lambda), \\
\gamma_\lambda(t,\lambda) &= \alpha(t,\lambda)\boldsymbol{\mu}(t,\lambda),
\end{aligned}
$$

where

$$
\begin{aligned}
&\ell(t,\lambda)=\nu_t(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda),\ m(t,\lambda)=\nu_\lambda(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda), \\
&\beta(t,\lambda)=\gamma_t(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda),\ \alpha(t,\lambda)=\gamma_\lambda(t,\lambda)\cdot\boldsymbol{\mu}(t,\lambda).
\end{aligned}
$$

By the integrability conditions $\nu_{t\lambda}(t,\lambda)=\nu_{\lambda t}(t,\lambda),\gamma_{t\lambda}(t,\lambda)=\gamma_{\lambda t}(t,\lambda)$, ℓ,m,β,α satisfies the conditions

$$\ell_\lambda(t,\lambda)=m_t(t,\lambda),\beta_\lambda(t,\lambda)=\alpha_t(t,\lambda),\ell(t,\lambda)\alpha(t,\lambda)=m(t,\lambda)\beta(t,\lambda) \tag{4}$$

for all $(t,\lambda)\in I\times\Lambda$. We call the pair (ℓ,m,β,α) with the integrability conditions (4) a *curvature of the bi-Legendre curve* (γ,ν).

Definition 5.2 Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}):I\times\Lambda\to\mathbb{R}^2\times S^1$ be bi-Legendre curves. We say that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are *congruent as bi-Legendre curves* if there exist a constant rotation $A\in SO(2)$ and a translation $\boldsymbol{a}$ on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t,\lambda)=A(\gamma(t,\lambda))+\boldsymbol{a}$ and $\widetilde{\nu}(t,\lambda)=A(\nu(t,\lambda))$ for all $(t,\lambda)\in I\times\Lambda$.

Theorem 5.3 (The Existence Theorem for bi-Legendre curves.) *Let $(\ell,m,\beta,\alpha):I\times\Lambda\to\mathbb{R}^4$ be a smooth mapping with the integrability conditions. There exists a bi-Legendre curve (γ,ν) : $I\times\Lambda\to\mathbb{R}^2\times S^1$ whose associated curvature is (ℓ,m,β,α).*

Proof. Let $(t_0,\lambda_0)\in I\times\Lambda$ be fixed. We define a smooth mapping $\theta:I\times\Lambda\to\mathbb{R}$ by

$$\theta(t,\lambda)=\int_{t_0}^{t}\ell(t,\lambda)dt+\int_{\lambda_0}^{\lambda}m(t_0,\lambda)d\lambda.$$

Then θ satisfy the conditions $\theta_t(t,\lambda)=\ell(t,\lambda)$ and $\theta_\lambda(t,\lambda)=m(t,\lambda)$. We also define a smooth mapping $(x,y):I\times\Lambda\to\mathbb{R}^2$ by

$$
\begin{aligned}
x(t,\lambda) &= -\int_{t_0}^{t}\beta(t,\lambda)\sin\theta(t,\lambda)dt-\int_{\lambda_0}^{\lambda}\alpha(t_0,\lambda)\sin\theta(t_0,\lambda)d\lambda \\
y(t,\lambda) &= \int_{t_0}^{t}\beta(t,\lambda)\cos\theta(t,\lambda)dt+\int_{\lambda_0}^{\lambda}\alpha(t_0,\lambda)\cos\theta(t_0,\lambda)d\lambda.
\end{aligned}
$$

By the integrability condition (4), we have

$$x_t(t,\lambda)=-\beta(t,\lambda)\sin\theta(t,\lambda),x_\lambda(t,\lambda)=-\alpha(t,\lambda)\sin\theta(t,\lambda),$$

$$y_t(t,\lambda)=\beta(t,\lambda)\cos\theta(t,\lambda), y_\lambda(t,\lambda)=\alpha(t,\lambda)\cos\theta(t,\lambda).$$

We define a smooth mapping $(\gamma,\nu): I\times\Lambda\to\mathbb{R}^2\times S^1$ by

$$\gamma(t,\lambda)=(x(t,\lambda),y(t,\lambda)), \nu(t,\lambda)=(\cos\theta(t,\lambda),\sin\theta(t,\lambda)).$$

By a direct calculation, (γ,ν) is a bi-Legendre curve with the curvature (ℓ,m,β,α). $\Box$

Theorem 5.4 (The Uniqueness Theorem for bi-Legendre curves.) *Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}): I\times\Lambda\to\mathbb{R}^2\times S^1$ be bi-Legendre curves with the curvatures (ℓ,m,β,α) and $(\widetilde{\ell},\widetilde{m},\widetilde{\beta},\widetilde{\alpha})$ respectively. Then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as bi-Legendre curves if and only if (ℓ,m,β,α) and $(\widetilde{\ell},\widetilde{m},\widetilde{\beta},\widetilde{\alpha})$ coincides.*

Proof. Suppose that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as bi-Legendre curves. By a direct calculation, we have

$$\begin{aligned}
\widetilde{\gamma}_t(t,\lambda) &= \frac{\partial}{\partial t}(A(\gamma(t,\lambda))+\boldsymbol{a})=A(\gamma_t(t,\lambda))=\beta(t,\lambda)A(\boldsymbol{\mu}(t,\lambda))=\beta(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda),\\
\widetilde{\gamma}_\lambda(t,\lambda) &= \frac{\partial}{\partial \lambda}(A(\gamma(t,\lambda))+\boldsymbol{a})=A(\gamma_\lambda(t,\lambda))=\alpha(t,\lambda)A(\boldsymbol{\mu}(t,\lambda))=\alpha(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda),\\
\widetilde{\nu}_t(t,\lambda) &= \frac{\partial}{\partial t}A(\nu(t,\lambda))=A(\nu_t(t,\lambda))=\ell(t,\lambda)A(\boldsymbol{\mu}(t,\lambda))=\ell(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda),\\
\widetilde{\nu}_\lambda(t,\lambda) &= \frac{\partial}{\partial \lambda}A(\nu(t,\lambda))=A(\nu_\lambda(t,\lambda))=m(t,\lambda)A(\boldsymbol{\mu}(t,\lambda))=m(t,\lambda)\widetilde{\boldsymbol{\mu}}(t,\lambda).
\end{aligned}$$

Therefore the curvatures (ℓ,m,β,α) and $(\widetilde{\ell},\widetilde{m},\widetilde{\beta},\widetilde{\alpha})$ coincides.

Conversely, suppose that (ℓ,m,β,α) and $(\widetilde{\ell},\widetilde{m},\widetilde{\beta},\widetilde{\alpha})$ coincides. Let $(t_0,\lambda_0)\in I\times\Lambda$ be fixed. By using a congruence as bi-Legendre curves, $\gamma(t_0,\lambda_0)=\widetilde{\gamma}(t_0,\lambda_0)$ and $\nu(t_0,\lambda_0)=\widetilde{\nu}(t_0,\lambda_0)$. Moreover, we have $\theta(t,\lambda)=\widetilde{\theta}(t,\lambda)$ in the proof of Theorem 5.3. It follows from the construction that $\nu(t,\lambda)=\widetilde{\nu}(t,\lambda)$ and $\gamma(t,\lambda)=\widetilde{\gamma}(t,\lambda)$ for all $(t,\lambda)\in I\times\Lambda$. $\Box$

Let $(\gamma,\nu): I\times\Lambda\to\mathbb{R}^2\times S^1$ be a bi-Legendre curve. Then (γ,ν) is a one-parameter family of Legendre curves with respect to the parameter λ. We denote a smooth map $e_L: U\to I\times\Lambda, e_L(u)=(t(u),\lambda(u))$. Since $\gamma_\lambda(t,\lambda)\cdot\nu(t,\lambda)=0$ for all $(t,\lambda)\in I\times\Lambda$, we have $\gamma_\lambda(e_L(u))\cdot\nu(e_L(u))=0$ for all $u\in U$. If the function λ is non-constant on any non-trivial subinterval of U, then $E_L=\gamma\circ e_L$ is an envelope of (γ,ν) with respect to the parameter λ by Theorem 4.5. Moreover, (γ,ν) is also a one-parameter family of Legendre curves with respect to the parameter t. Since $\gamma_t(t,\lambda)\cdot\nu(t,\lambda)=0$ for all $(t,\lambda)\in I\times\Lambda$, we have $\gamma_t(e_L(u))\cdot\nu(e_L(u))=0$ for all $u\in U$. If the function t is non-constant on any non-trivial subinterval of U, then $E_L=\gamma\circ e_L$ is an envelope of (γ,ν) with respect to the parameter t by Theorem 4.5. Summary we have the following result.

Proposition 5.5 *Let $(\gamma,\nu): I\times\Lambda\to\mathbb{R}^2\times S^1$ be a bi-Legendre curve. If $e_L: U\to I\times\Lambda, e_L(u)=(t(u),\lambda(u))$ satisfy the conditions that the functions t and λ are non-constant on any non-trivial subinterval of U, then $E_L=\gamma\circ e_L$ is an envelope of (γ,ν) with respect to the both parameter t and λ respectively.*

Let $(\gamma,\nu): I\times\Lambda\to\mathbb{R}^2\times S^1$ be a bi-Legendre curve. Since $\gamma_t(t,\lambda)=\beta(t,\lambda)\boldsymbol{\mu}(t,\lambda)$ and $\gamma_\lambda(t,\lambda)=\alpha(t,\lambda)\boldsymbol{\mu}(t,\lambda)$, we have $\det(\gamma_t(t,\lambda),\gamma_\lambda(t,\lambda))=0$ for all $(t,\lambda)\in I\times\Lambda$. It follows that for any $(t,\lambda)\in I\times\Lambda$ are singular points of $\gamma: I\times\Lambda\to\mathbb{R}^2$. Hence, at a rank 1 point, the image of γ is a curve at least locally. We give a concrete example of bi-Legendre curves.

Example 5.6 Let k, n be natural numbers. We define $(\ell, m, \beta, \alpha) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^4$ by $\ell(t,\lambda) = \lambda^{k+1}t^k, m(t,\lambda) = \lambda^k t^{k+1}, \beta(t,\lambda) = \lambda^{n+1}t^n, \alpha(t,\lambda) = \lambda^n t^{n+1}$. Then the integrability conditions $\ell_\lambda(t,\lambda) = m_t(t,\lambda), \beta_\lambda(t,\lambda) = \alpha_t(t,\lambda), \alpha(t,\lambda)\ell(t,\lambda) = \beta(t,\lambda)m(t,\lambda)$ hold for all $(t,\lambda) \in \mathbb{R} \times \mathbb{R}$. It follows that $\theta(t,\lambda) = \lambda^{k+1}t^{k+1}/(k+1)$. By the construction in the proof of Theorem 5.3, we give a bi-Legendre curve $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$,

$$\begin{aligned}
\gamma(t,\lambda) &= (x(t,\lambda), y(t,\lambda)) = \left(-\int_0^t \lambda^{n+1}t^n \sin\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right) dt, \int_0^\lambda \lambda^{n+1}t^n \cos\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right) dt \right), \\
\nu(t,\lambda) &= (\cos\theta(t,\lambda), \sin\theta(t,\lambda)) = \left(\cos\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right), \sin\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right) \right).
\end{aligned}$$

A Legendre curves in the unit tangent bundle

We quickly review on the theory of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$, see detail [8]. We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma, \nu)^*\theta = 0$ for all $t \in I$, where θ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$ (cf. [1, 2]). This condition is equivalent to $\dot\gamma(t) \cdot \nu(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^2$ is a frontal if there exists $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve. Examples of Legendre curves see [13, 14]. We have the Frenet formula of a frontal γ as follows. We put on $\boldsymbol{\mu}(t) = J(\nu(t))$. Then we call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame of a frontal* $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet formula of a frontal (or, Legendre curve),

$$\begin{pmatrix} \dot\nu(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \ \dot\gamma(t) = \beta(t)\boldsymbol{\mu}(t),$$

where $\ell(t) = \dot\nu(t) \cdot \boldsymbol{\mu}(t)$ and $\beta(t) = \dot\gamma(t) \cdot \boldsymbol{\mu}(t)$. We call the pair (ℓ, β) *the curvature of the Legendre curve.*

Definition A.1 Let (γ, ν) and $(\widetilde\gamma, \widetilde\nu) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ, ν) and $(\widetilde\gamma, \widetilde\nu)$ are *congruent as Legendre curves* if there exist a constant rotation $A \in SO(2)$ and a translation $\boldsymbol{a}$ on $\mathbb{R}^2$ such that $\widetilde\gamma(t) = A(\gamma(t)) + \boldsymbol{a}$ and $\widetilde\nu(t) = A(\nu(t))$ for all $t \in I$.

Theorem A.2 (The Existence Theorem for Legendre curves.) *Let $(\ell, \beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is (ℓ, β).*

Theorem A.3 (The Uniqueness Theorem for Legendre curves.) *Let (γ, ν) and $(\widetilde\gamma, \widetilde\nu) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves with the curvatures of Legendre curves (ℓ, β) and $(\widetilde\ell, \widetilde\beta)$. Then (γ, ν) and $(\widetilde\gamma, \widetilde\nu)$ are congruent as Legendre curves if and only if (ℓ, β) and $(\widetilde\ell, \widetilde\beta)$ coincides.*

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