Period of the adelic Ikeda lift for U(m, m)

ON THE PERIOD OF THE IKEDA LIFT FOR $U(m,m)$

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Abstract. Let $K=\mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$, and χ the Dirichlet character corresponding to the extension $K/\mathbf{Q}$. Let $m=2n$ or $2n+1$ with n a positive integer. Let f be a primitive form of weight $2k+1$ and character χ for $\Gamma_0(D)$, or a primitive form of weight $2k$ for $SL_2(\mathbf{Z})$ according as $m=2n$, or $m=2n+1$. For such an f let $I_m(f)$ be the lift of f to the space of modular forms of weight $2k+2n$ and character $\det^{-k-n}$ for the Hermitian modular group $\Gamma_K^{(m)}$ constructed by Ikeda. We then express the period $\langle I_m(f), I_m(f)\rangle$ of $I_m(f)$ in terms of special values of the adjoint L-function of f and its twist by the character χ. This proves the conjecture concerning the period of the Hermitian Ikeda lift proposed by Ikeda. Period, Hermitian Ikeda lift

1. Introduction

It is an important and interesting problem to consider the relation between the period of an elliptic modular form and that of its lift. Here, we say that F is a lift of an elliptic modular form f if F or the adelization of F is a Hecke eigenform in the space of Siegel cusp forms or Hermitian cusp forms whose certain L-function is expressed in terms of L-functions related to f. There are several results concerning this problem in the Siegel modular form case (cf. [2], [19]). This type of period relation sometimes gives rise to congruence between the lift and non-lift, and are important also from the view point of arithmetic geometry (cf. [2], [4], [12]). In [16], we proved a conjecture on the period of the Duke-Imamoglu-Ikeda lift (DII lift) proposed by Ikeda [9]. As a result, in [13], we characterized prime ideals giving congruence between the DII lift and non-DII lift. (See also [5].) Klosin [17] gave the congruence between the Hermitian Maass lift and non-Hermitian Maass lift using the period relation in [10]. In this paper we prove a result similar to [16] for the period of the lift of an elliptic modular form to the space of Hermitian modular forms constructed by Ikeda. This also proves Ikeda's conjecture in [10] with some modification.

Let $K=\mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$, and χ the Kronecker character corresponding to the extension $K/\mathbf{Q}$. Let k be a non-negative integer. Then for a primitive form $f\in\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$ Ikeda [10] constructed a lift $I_{2n}(f)$ of f to the space of modular forms of weight $2k+2n$ and a character $\det^{-k-n}$ for the Hermitian group $\Gamma_K^{(2n)}$ of degree m. This is a generalization of the Maass lift considered by Kojima [18], Gritsenko [6], Krieg [20], Oda [21], and Sugano [27]. Similarly for a primitive form $f\in\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ he constructed a lift $I_{2n+1}(f)$ of f to the space of modular forms of weight $2k+2n$ and a character $\det^{-k-n}$ for $\Gamma_K^{(2n+1)}$. For the rest of this section, let $m=2n$ or $m=2n+1$. We then call $I_m(f)$ the Ikeda lift of f for $U(m,m)$ or the Hermitian Ikeda lift of degree m. Then our main result (Theorem 2.1) can be stated as follows:

The period $\langle I_m(f), I_m(f)\rangle$ of $I_m(f)$ is expressed as

$$L(1,f,\mathrm{Ad})\prod_{i=2}^{m} L(i,f,\mathrm{Ad},\chi^{i-1})L(i,\chi^i)$$

up to elementary factor, where $L(s,f,\mathrm{Ad},\chi^{i-1})$ is the "modified twist" of the adjoint L-function of f by χ^{i-1}, and $L(i,\chi^i)$ is the Dirichlet L-function for χ^i.

This result was already obtained in the case $m=2$, and was conjectured in general case by Ikeda [10].

We note that $I_m(f)$ is not likely to be a theta lift except in the case $m=2$, and therefore the method in [22] cannot be applied to prove our main result. The method we use is similar to that in the proof of the main result of [16] and to give an explicit formula of the Dirichlet series of Rankin-Selberg type associated to $I_m(f)$, and to compare its residue with $\langle I_m(f), I_m(f)\rangle$. We explain it more precisely. In Section 3, we consider the Dirichlet series $R(s, I_m(f))$ of Rankin Selberg type associated with $I_m(f)$. For the precise definition, see Section 3. This type of Dirichlet series was studied by Shimura [25] for a classical Hermitian modular form F of weight $2k+2n$. In particular we can express its residue at $2k+2n$ in terms of the period of F (cf. Proposition 3.1). Thus to prove Theorem 2.1, we have to get an explicit formula of $R(s, I_m(f))$ in terms of $L(s,f,\mathrm{Ad},\chi^i)$. To get it, in Section 4, we reduce our computation to a computation of certain formal power series $\hat{H}_{m,p}(d;X,Y,t)$ in t associated with local Siegel series similarly to [16] (cf. Theorem 4.1).

Section 5 is devoted to the computation of them. This computation is similar to that in [16], but we should be careful in dealing with the case where p is ramified in K. After such an elaborate computation, we can get explicit formulas of $\hat{H}_{m,p}(d;X,Y,t)$ for all prime numbers p (cf. Theorem 5.5.4). In Section 6, by using explicit formulas for $\hat{H}_{m,p}(d;X,Y,t)$, we immediately get an explicit formula of $R(s, I_m(f))$ (cf. Theorems 6 .1 and 6.2) and by taking the residue of it at $2k+2n$ we prove the Theorem 2.1.

We note that we can give a similar period relation for the adelic Ikeda lift, and we can apply it to a problem concerning congruence between the adelic Ikeda lifts and Hecke eigenforms not coming from the adelic Ikeda lifts. These will be discussed in subsequent papers.

Notation. Let R be a commutative ring. We denote by $R^\times$ and R^* the semigroup of non-zero elements of R and the unit group of R, respectively. For a subset S of R we denote by $M_{mn}(S)$ the set of (m,n)-matrices with entries in S. In particular put $M_n(S)=M_{nn}(S)$. Put $GL_m(R)=\{A\in M_m(R)\mid \det A\in R^*\}$, where $\det A$ denotes the determinant of a square matrix A. Let K_0 be a field, and K a quadratic extension of K_0, or $K=K_0\oplus K_0$. In the latter case, we regard K_0 as a subring of K via the diagonal embedding. We also identify $M_{mn}(K)$ with $M_{mn}(K_0)\oplus M_{mn}(K_0)$ in this case. If K is a quadratic extension of K_0, let ρ be the non-trivial automorphism of K over K_0, and if $K=K_0\oplus K_0$, let ρ be the automorphism of K defined by $\rho(a,b)=(b,a)$ for $(a,b)\in K_0$. We sometimes write $\overline{x}$ instead of $\rho(x)$ for $x\in K$ in both cases. Let R be a subring of K. For an (m,n)-matrix $X=(x_{ij})_{m\times n}$ write $\overline{X}=(\overline{x_{ij}})_{m\times n}$ and $X^*={}^t\overline{X}$, and for an (m,m)-matrix

A, we write $A[X]=X^*AX$. Let $\mathrm{Her}_n(R)$ denote the set of Hermitian matrices of degree n with entries in R, that is the subset of $M_n(R)$ consisting of matrices X such that $X^*=X$. Then a Hermitian matrix A of degree n with entries in K is said to be semi-integral over R if $\mathrm{tr}(AB)\in K_0\cap R$ for any $B\in \mathrm{Her}_n(R)$, where tr denotes the trace of a matrix. We denote by $\widehat{\mathrm{Her}}_n(R)$ the set of semi-integral matrices of degree n over R.

For a subset S of $M_n(R)$ we denote by $S^\times$ the subset of S consisting of non-degenerate matrices. If S is a subset of $\mathrm{Her}_n(\mathbf{C})$ with $\mathbf{C}$ the field of complex numbers, we denote by S^+ the subset of S consisting of positive definite matrices. The group $GL_n(R)$ acts on the set $\mathrm{Her}_n(R)$ from the right in the following way:

$$GL_n(R)\times \mathrm{Her}_n(R)\ni (g,A)\longrightarrow g^*Ag\in \mathrm{Her}_n(R).$$

Let G be a subgroup of $GL_n(R)$. For a G-stable subset $\mathcal{B}$ of $\mathrm{Her}_n(R)$ we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ under the action of G. We sometimes identify $\mathcal{B}/G$ with a complete set of representatives of $\mathcal{B}/G$. We abbreviate $\mathcal{B}/GL_n(R)$ as $\mathcal{B}/\sim$ if there is no fear of confusion. Two Hermitian matrices A and A' with entries in R are said to be G-equivalent and write $A\sim_G A'$ if there is an element X of G such that $A'=A[X]$. For square matrices X and Y we write $X\bot Y=\begin{pmatrix} X & O\\ O & Y\end{pmatrix}$.

We put $\mathbf{e}(x)=\exp(2\pi\sqrt{-1}x)$ for $x\in\mathbf{C}$, and for a prime number p we denote by $\mathbf{e}_p(*)$ the continuous additive character of $\mathbf{Q}_p$ such that $\mathbf{e}_p(x)=\mathbf{e}(x)$ for $x\in\mathbf{Z}[p^{-1}]$.

For a prime number p we denote by $\mathrm{ord}_p(*)$ the additive valuation of $\mathbf{Q}_p$ normalized so that $\mathrm{ord}_p(p)=1$, and put $|x|_p=p^{-\mathrm{ord}_p(x)}$. Moreover we denote by $|x|_\infty$ the absolute value of $x\in\mathbf{C}$.

2. Period of the Ikeda lift for $U(m,m)$

For a positive integer N let $\Gamma_0(N)=\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})\mid c\equiv 0 \bmod N\}$, and for a Dirichlet character ψ mod N, we denote by $\mathfrak{M}_l(\Gamma_0(N),\psi)$ the space of modular forms of weight l for $\Gamma_0(N)$ and nebentype ψ, and by $\mathfrak{S}_l(\Gamma_0(N),\psi)$ its subspace consisting of cusp forms. We simply write $\mathfrak{M}_l(\Gamma_0(N),\psi)$ (resp. $\mathfrak{S}_l(\Gamma_0(N),\psi)$) as $\mathfrak{M}_l(\Gamma_0(N))$ (resp. as $\mathfrak{S}_l(\Gamma_0(N))$) if ψ is the trivial character.

Throughout the paper, we fix an imaginary quadratic extension K of $\mathbf{Q}$ with the discriminant $-D$, and denote by $\mathcal{O}$ the ring of integers in K. For a prime number p put $K_p=K\otimes\mathbf{Q}_p$, and $\mathcal{O}_p=\mathcal{O}\otimes\mathbf{Z}_p$. Then K_p is a quadratic extension of $\mathbf{Q}_p$ or $K_p\cong\mathbf{Q}_p\oplus\mathbf{Q}_p$. In the former case, for $x\in K_p$, we denote by $\overline{x}$ the conjugate of x over $\mathbf{Q}_p$. In the latter case, we identify K_p with $\mathbf{Q}_p\oplus\mathbf{Q}_p$, and for $x=(x_1,x_2)\in\mathbf{Q}_p\oplus\mathbf{Q}_p$, we put $\overline{x}=(x_2,x_1)$. For $x\in K_p$ we define the norm $N_{K_p/\mathbf{Q}_p}(x)$ by $N_{K_p/\mathbf{Q}_p}(x)=x\overline{x}$, and put $\nu_{K_p}(x)=\mathrm{ord}_p(N_{K_p/\mathbf{Q}_p}(x))$, and $|x|_{K_p}=|N_{K_p/\mathbf{Q}_p}(x)|_p$. Moreover put $|x|_{K_\infty}=|x\overline{x}|_\infty$ for $x\in\mathbf{C}$.

For a non-degenerate Hermitian matrix or alternating matrix T with entries in K, let $\mathcal{U}_T$ be the unitary group defined over $\mathbf{Q}$, whose group $\mathcal{U}_T(R)$ of R-valued points is given by

$$\mathcal{U}_T(R)=\{g\in GL_m(R\otimes K)\mid {}^t\overline{g}Tg=T\}$$

for any $\mathbf{Q}$-algebra R, where $g\mapsto\overline{g}$ denotes the automorphism of $M_n(R\otimes K)$ induced by the non-trivial automorphism of K over $\mathbf{Q}$. We also define the special unitary group $\mathcal{SU}_T$ over $\mathbf{Q}_p$ by $\mathcal{SU}_T=\mathcal{U}_T\cap R_{K/\mathbf{Q}}(SL_m)$, where $R_{K/\mathbf{Q}}$ is the Weil restriction. In particular we write $\mathcal{U}_{J_m}$ as $\mathcal{U}^{(m)}$ or $U(m,m)$, where $J_m = \left(\begin{smallmatrix} O & -1_m \\ 1_m & O \end{smallmatrix}\right)$. Then

$$\mathcal{U}^{(m)}(\mathbf{Q}) = \{M \in GL_{2m}(K) \mid J_m[M] = J_m\}.$$

Put

$$\Gamma^{(m)} = \Gamma_K^{(m)} = \mathcal{U}^{(m)}(\mathbf{Q}) \cap GL_{2m}(\mathcal{O}).$$

Let $\mathfrak{H}_m$ be the Hermitian upper half-space defined by

$$\mathfrak{H}_m = \{Z \in M_m(\mathbf{C}) \mid \frac{1}{2\sqrt{-1}}(Z - Z^*) \text{ is positive definite}\}.$$

The group $\mathcal{U}^{(m)}(\mathbf{R})$ acts on $\mathfrak{H}_m$ by

$$g\langle Z\rangle = (AZ+B)(CZ+D)^{-1} \text{ for } g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{U}^{(m)}(\mathbf{R}), Z \in \mathfrak{H}_m.$$

We also put $j(g,Z) = \det(CZ+D)$ for such Z and g. Let l be an integer. For a subgroup Γ of $\mathcal{U}^{(m)}(\mathbf{Q})$ which is commensurable with $\Gamma^{(m)}$ and a character ψ of Γ, we denote by $\mathfrak{M}_l(\Gamma,\psi)$ the space of holomorphic modular forms of weight l with character ψ for Γ. We denote by $\mathfrak{S}_l(\Gamma,\psi)$ the subspace of $\mathfrak{M}_l(\Gamma,\psi)$ consisting of cusp forms. In particular, if ψ is the character of Γ defined by $\psi(\gamma) = (\det\gamma)^{-l}$ for $\gamma \in \Gamma$, we write $\mathfrak{M}_{2l}(\Gamma,\psi)$ as $\mathfrak{M}_{2l}(\Gamma,\det^{-l})$, and so on. Write the variable Z on $\mathfrak{H}_m$ as $Z = X + \sqrt{-1}Y$ with $X, Y \in \mathrm{Her}_m(\mathbf{C})$. We can identify $\mathrm{Her}_m(\mathbf{C})$ with $\mathbf{R}^{m^2}$ through the map $X = (x_{ij}) \longrightarrow (x_{ii}, \mathrm{Re}(x_{ij}), \mathrm{Im}(x_{ij})\ (i<j))$, and define a measure dX on $\mathrm{Her}_m(\mathbf{C})$ by pulling back the standard measure on $\mathbf{R}^{m^2}$. Similarly we define a measure dY on $\mathrm{Her}_m(\mathbf{C})$ in the same way as above. For two cusp forms F and G of weight l with respect to $\Gamma^{(m)}$ with character χ we define the Petersson scalar product $\langle F, G\rangle$ by

$$\langle F, G\rangle = \int_{\Gamma^{(m)}\backslash\mathfrak{H}_m} F(Z)\overline{G(Z)}(\det Y)^{l-2m} dX dY,$$

where $X = \frac{Z + {}^t\overline{Z}}{2}$, and $Y = \frac{Z - {}^t\overline{Z}}{2\sqrt{-1}}$. We call $\langle F, F\rangle$ the period of F. Similarly for two elements $f, g \in \mathfrak{S}_l(\Gamma_0(N),\psi)$, we define the Petersson scalar product $\langle f, g\rangle$ by

$$\langle f, g\rangle = [SL_2(\mathbf{Z}) : \Gamma_0(N)]^{-1} \int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)} y^{l-2} dx dy,$$

where $\mathfrak{H}$ is the complex upper half space.

Now we consider adelic modular forms. Let $\mathbf{A}$ be the adele ring of $\mathbf{Q}$, and $\mathbf{A}_f$ the non-archimedean factor of $\mathbf{A}$. Let $h = h_K$ be a class number of K. Let $G^{(m)} = \mathrm{Res}_{K/\mathbf{Q}}(GL_m)$, and $G^{(m)}(\mathbf{A})$ be the adelization of $G^{(m)}$. Moreover put $\mathcal{C}^{(m)} = \prod_p GL_m(\mathcal{O}_p)$. Let $\mathcal{U}^{(m)}(\mathbf{A})$ be the adelization of $\mathcal{U}^{(m)}$. We define the compact subgroup $\mathcal{K}_0^{(m)}$ of $\mathcal{U}^{(m)}(\mathbf{A}_f)$ by $\mathcal{U}^{(m)}(\mathbf{A}) \cap \prod_p GL_{2m}(\mathcal{O}_p)$, where p runs over all rational primes. Then we have

$$\mathcal{U}^{(m)}(\mathbf{A}) = \bigsqcup_{i=1}^{h} \mathcal{U}^{(m)}(\mathbf{Q})\gamma_i \mathcal{K}_0^{(m)} \mathcal{U}^{(m)}(\mathbf{R})$$

with some subset $\{\gamma_1, ..., \gamma_h\}$ of $\mathcal{U}^{(m)}(\mathbf{A}_f)$. We can take γ_i as

$$\gamma_i = \begin{pmatrix} t_i & 0 \\ 0 & t_i^{*-1} \end{pmatrix},$$

where $\{t_i\}_{i=1}^h=\{(t_{i,p})\}_{i=1}^h$ is a certain subset of $G^{(m)}(\mathbf{A}_f)$ such that $t_1=1$, and

$$G^{(m)}(\mathbf{A})=\bigsqcup_{i=1}^h G^{(m)}(\mathbf{Q})t_iG^{(m)}(\mathbf{R})\mathcal{C}^{(m)}.$$

Put $\mathit{\Gamma}_i=\mathcal{U}^{(m)}(\mathbf{Q})\cap\gamma_i\mathcal{K}_0\gamma_i^{-1}\mathcal{U}^{(m)}(\mathbf{R})$. Then for an element $(F_1,...,F_h)\in\bigoplus_{i=1}^h\mathfrak{M}_{2l}(\mathit{\Gamma}_i,\det^{-l})$, we define $(F_1,...,F_h)^\sharp$ by

$$(F_1,...,F_h)^\sharp(g)=F_i(x\langle\mathbf{i}\rangle)j(x,\mathbf{i})^{-2l}(\det x)^l$$

for $g=u\gamma_ix\kappa$ with $u\in\mathcal{U}^{(m)}(\mathbf{Q}),x\in\mathcal{U}^{(m)}(\mathbf{R}),\kappa\in\mathcal{K}_0$. We denote by $\mathcal{M}_l(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}),\det^{-l})$ the space of automorphic forms obtained in this way. We also put

$$\mathcal{S}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}),\det{}^{-l})=\{(F_1,...,F_h)^\sharp\mid F_i\in\mathfrak{S}_{2l}(\mathit{\Gamma}_i,\det{}^{-l})\}.$$

We can define the Hecke operators which act on the space $\mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}),\det^{-l})$. For the precise definition of them, see [10].

Let $\widehat{\mathrm{Her}}_m(\mathcal{O})$ be the set of semi-integral Hermitian matrices over $\mathcal{O}$ of degree m as in the Notation. We note that $A\in\mathrm{Her}_m(K)$ belongs to $\widehat{\mathrm{Her}}_m(\mathcal{O})$ if and only if its diagonal components are rational integers and $\sqrt{-D}A\in M_m(\mathcal{O})$.

For a non-degenerate Hermitian matrix B with entries in K_p of degree m, put $\gamma(B)=(-D)^{[m/2]}\det B$. Let $\widehat{\mathrm{Her}}_m(\mathcal{O}_p)$ be the set of semi-integral matrices over $\mathcal{O}_p$ of degree m as in the Notation. We put $\xi_p=1,-1$, or 0 according as $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$, K_p is an unramified quadratic extension of $\mathbf{Q}_p$, or K_p is a ramified quadratic extension of $\mathbf{Q}_p$. For $T\in\widehat{\mathrm{Her}}_m(\mathcal{O}_p)^\times$ we define the local Siegel series $b_p(T,s)$ by

$$b_p(T,s)=\sum_{R\in\mathrm{Her}_n(K_p)/\mathrm{Her}_n(\mathcal{O}_p)}\mathbf{e}_p(\mathrm{tr}(TR))p^{-\mathrm{ord}_p(\mu_p(R))s},$$

where $\mu_p(R)=[R\mathcal{O}_p^m+\mathcal{O}_p^m:\mathcal{O}_p^m]^{1/2}$.

Remark. In [14], we defined $\mu_p(R)$ as $\mu_p(R)=[R\mathcal{O}_p^m+\mathcal{O}_p^m:\mathcal{O}_p^m]$. However, it should be defined as above.

We remark that there exists a unique polynomial $F_p(T,X)$ in X such that

$$b_p(T,s)=F_p(T,p^{-s})\prod_{i=0}^{[(m-1)/2]}(1-p^{2i-s})\prod_{i=1}^{[m/2]}(1-\xi_pp^{2i-1-s})$$

(cf. Shimura [24]). We then define a Laurent polynomial $\widetilde{F}_p(T,X)$ as

$$\widetilde{F}_p(T,X)=X^{-\mathrm{ord}_p(\gamma(T))}F_p(T,p^{-m}X^2).$$

We remark that we have

$$\widetilde{F}_p(T,X^{-1})=(-D,\gamma(T))_p\widetilde{F}_p(T,X)\qquad\text{if } m \text{ is even},$$

$$\widetilde{F}_p(T,\xi_pX^{-1})=\widetilde{F}_p(T,X)\qquad\text{if } m \text{ is even and } p\nmid D,$$

and

$$\widetilde{F}_p(T,X^{-1})=\widetilde{F}_p(T,X)\qquad\text{if } m \text{ is odd}$$

(cf. [10]). Here $(a,b)_p$ is the Hilbert symbol of $a,b\in\mathbf{Q}_p^\times$. Hence we have

$$\widetilde{F}_p(T,X)=(-D,\gamma(B))_p^{m-1}X^{\mathrm{ord}_p(\gamma(T))}F_p(T,p^{-m}X^{-2}).$$

Now we put

$$\widehat{\mathrm{Her}}_m(\mathcal{O})_i^+=\{T\in\mathrm{Her}_m(K)^+\mid t_{i,p}^*Tt_{i,p}\in\widehat{\mathrm{Her}}_m(\mathcal{O}_p)\text{ for any }p\}.$$

Let k be a non-negative integer. First let $m = 2n$ be a positive even integer and let

$$f(z) = \sum_{N=1}^{\infty} a(N)\mathbf{e}(Nz)$$

be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$. For a prime number p not dividing D let $\alpha_p \in \mathbf{C}$ such that $\alpha_p + \chi(p)\alpha_p^{-1} = p^{-k}a(p)$, and for $p \mid D$ put $\alpha_p = p^{-k}a(p)$. We note that $\alpha_p \neq 0$ even if $p|D$. Then for the Kronecker character χ we define Hecke's L-function $L(s, f, \chi^i)$ twisted by χ^i as

$$L(s, f, \chi^i) = \prod_{p \nmid D} \{(1 - \alpha_p p^{-s+k}\chi(p)^i)(1 - \alpha_p^{-1} p^{-s+k}\chi(p)^{i+1})\}^{-1}$$

$$\times \begin{cases} \prod_{p|D}(1 - \alpha_p p^{-s+k})^{-1} & \text{if } i \text{ is even} \\ \prod_{p|D}(1 - \alpha_p^{-1} p^{-s+k})^{-1} & \text{if } i \text{ is odd.} \end{cases}$$

In particular, if i is even, we sometimes write $L(s, f, \chi^i)$ as $L(s, f)$ as usual. Moreover we define a Fourier series

$$I_m(f)(Z) = \sum_{T \in \widehat{\mathrm{Her}}_m(\mathcal{O})^+} a_{I_m(f)}(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where

$$a_{I_{2n}(f)}(T) = |\gamma(T)|^k \prod_p \widetilde{F}_p(T, \alpha_p^{-1}).$$

Next let $m = 2n + 1$ be a positive odd integer and let

$$f(z) = \sum_{N=1}^{\infty} a(N)\mathbf{e}(Nz)$$

be a primitive form in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$. For a prime number p let $\alpha_p \in \mathbf{C}$ such that $\alpha_p + \alpha_p^{-1} = p^{-k+1/2}a(p)$. Then we define Hecke's L-function $L(s, f, \chi^i)$ twisted by χ^i as

$$L(s, f, \chi^i)$$
$$= \prod_p \{(1 - \alpha_p p^{-s+k-1/2}\chi(p)^i)(1 - \alpha_p^{-1} p^{-s+k-1/2}\chi(p)^i)\}^{-1}.$$

In particular, if i is even we write $L(s, f, \chi^i)$ as $L(s, f)$ as usual. We define a Fourier series

$$I_{2n+1}(f)(Z) = \sum_{T \in \widehat{\mathrm{Her}}_{2n+1}(\mathcal{O})^+} a_{I_{2n+1}(f)}(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where

$$a_{I_{2n+1}(f)}(T) = |\gamma(T)|^{k-1/2} \prod_p \widetilde{F}_p(T, \alpha_p^{-1}).$$

Remark. In [10], Ikeda defined $\widetilde{F}_p(T, X)$ as

$$\widetilde{F}_p(T, X) = X^{\mathrm{ord}_p(\gamma(T))} F_p(T, p^{-m}X^{-2}),$$

and we define it by replacing X with X^{-1} in this paper. This change does not affect the results.

Then Ikeda [10] showed the following:

Let $m = 2n$ or $2n+1$. Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$ or in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ according as $m = 2n$ or $m = 2n+1$. Then $I_m(f)(Z)$ is an element of $\mathfrak{S}_{2k+2n}(\Gamma^{(m)}, \det^{-k-n})$.

To state our main result, put
$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$$
and
$$\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1).$$
We note that
$$\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$
For an integer i let $L(s,\chi^i) = \zeta(s)$ or $L(s,\chi)$ according as i is even or odd, where $\zeta(s)$ and $L(s,\chi)$ are Riemann's zeta function, and Dirichlet L-function for χ, respectively, and put
$$\widetilde{\Lambda}(s,\chi^i) = \Gamma_{\mathbf{C}}(s)L(s,\chi^i).$$
For a primitive form f in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$, we define the adjoint L-function $L(s,\ f,\ \mathrm{Ad})$ and its twist $L(s,\ f,\ \mathrm{Ad},\chi)$ by χ as
$$L(s,f,\mathrm{Ad}) = \prod_{p\nmid D}\{(1-\alpha_p^2\chi(p)p^{-s})(1-\alpha_p^{-2}\chi(p)p^{-s})(1-p^{-s})\}^{-1}\prod_{p\mid D}(1-p^{-s})^{-1},$$
and
$$L(s,f,\mathrm{Ad},\chi) = \prod_{p\nmid D}\{(1-\alpha_p^2 p^{-s})(1-\alpha_p^{-2}p^{-s})(1-\chi(p)p^{-s})\}^{-1}$$
$$\times\prod_{p\mid D}\{(1-\alpha_p^2 p^{-s})(1-\alpha_p^{-2}p^{-s})\}^{-1}.$$
For a primitive form f in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$, we define the adjoint L-function $L(s,\ f,\ \mathrm{Ad})$ and its twist $L(s,\ f,\ \mathrm{Ad},\chi)$ by χ as
$$L(s,f,\mathrm{Ad}) = \prod_{p}\{(1-\alpha_p^2 p^{-s})(1-\alpha_p^{-2}p^{-s})(1-p^{-s})\}^{-1},$$
and
$$L(s,f,\mathrm{Ad},\chi) = \prod_{p}\{(1-\alpha_p^2\chi(p)p^{-s})(1-\alpha_p^{-2}\chi(p)p^{-s})(1-\chi(p)p^{-s})\}^{-1}.$$
Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$ or in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ according as $m = 2n$ or $m = 2n+1$. We then put
$$L(s,\ f,\ \mathrm{Ad},\chi^i) = \begin{cases} L(s,\ f,\mathrm{Ad}) & \text{if } i \text{ is even}\\ L(s,f,\mathrm{Ad},\chi) & \text{if } i \text{ is odd}\end{cases}$$
Moreover put
$$\widetilde{\Lambda}(s,\ f,\ \mathrm{Ad},\chi^i) = \Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s+l-1)L(s,\ f,\ \mathrm{Ad},\chi^i),$$
where $l = 2k+1$ or $l = 2k$ according as $f \in \mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$ or $f \in \mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$. Let Q_D be the set of prime divisors of D. For each prime $q \in Q_D$, put $D_q = q^{\mathrm{ord}_q(D)}$. We define a Dirichlet character χ_q by
$$\chi_q(a) = \begin{cases}\chi(a') & \text{if } (a,q)=1\\ 0 & \text{if } q\mid a\end{cases},$$

where a' is an integer such that

$$a' \equiv a \bmod D_q \quad \text{ and } a' \equiv 1 \bmod DD_q^{-1}.$$

For a subset Q of Q_D put $\chi_Q = \prod_{q \in Q} \chi_q$ and $\chi'_Q = \prod_{q \in Q_D, q \notin Q} \chi_q$. Here we make the convention that $\chi_Q = 1$ and $\chi'_Q = \chi$ if Q is the empty set. Let

$$f(z) = \sum_{N=1}^{\infty} c_f(N)\mathbf{e}(Nz)$$

be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$. Then there exists a primitive form

$$f_Q(z) = \sum_{N=1}^{\infty} c_{f_Q}(N)\mathbf{e}(Nz)$$

such that

$$c_{f_Q}(p) = \chi_Q(p)c_f(p) \text{ for } p \notin Q$$

and

$$c_{f_Q}(p) = \chi'_Q(p)\overline{c_f(p)} \text{ for } p \in Q.$$

Then our main result in this paper is:

Theorem 2.1. (1) *Let $m = 2n$ be a positive even integer. For a primitive form f in $\mathfrak{S}_{2k+1}(\Gamma_0(D), \chi)$, we have*

$$\langle I_{2n}(f),\, I_{2n}(f)\rangle$$
$$= 2^{-4nk-4n^2-4n+2} D^{2nk+5n^2-3n/2-1/2} \eta_n(f) \prod_{i=1}^{2n} \widetilde{\Lambda}(i,\, f,\, \mathrm{Ad}, \chi^{i-1}) \prod_{i=2}^{2n} \widetilde{\Lambda}(i, \chi^i),$$

where

$$\eta_n(f) = \sum_{\substack{Q \subset Q_D \\ f_Q = f}} \chi_Q((-1)^n).$$

(2) *Let $m = 2n+1$ be a positive odd integer. For a primitive form f in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$, we have*

$$\langle I_{2n+1}(f),\, I_{2n+1}(f)\rangle$$
$$= 2^{-2(2n+1)k-4n^2-6n} D^{2nk+5n^2+5n/2} \prod_{i=1}^{2n+1} \widetilde{\Lambda}(i,\, f,\, \mathrm{Ad}, \chi^{i-1}) \prod_{i=2}^{2n+1} \widetilde{\Lambda}(i, \chi^i).$$

Remark. In [10] Ikeda showed that $I_m(f)$ is identically zero if and only if $m = 2n$ and $\eta_n(f) = 0$. Therefore the above theorem remains valid even if $I_m(f)$ is identically zero.

This type of result was conjectured by Ikeda [10]. When $m = 2$, by using the result of Sugano [27], Ikeda [10] has been already proved that

$$\langle I_2(f),\, I_2(f)\rangle = \eta_1(f) 2^{-4k-6} D^{2k+3} \widetilde{\Lambda}(2)\widetilde{\Lambda}(1, f, \mathrm{Ad})\widetilde{\Lambda}(2, f, \mathrm{Ad}, \chi).$$

His conjecture holds true up to a power of D. In fact, he conjectured that integer powers of D should appear on the right-hand sides of the above formulas. However, half-integer powers of D appear in some cases as shown in the above theorem.

Now put

$$\mathbf{L}(i, f, \mathrm{Ad}, \chi^{i-1}) = \frac{\widetilde{\Lambda}(i,\, f,\, \mathrm{Ad}, \chi^{i-1})}{\langle f,\, f\rangle}$$

for $i=1,...,m$

$$\mathbf{L}(2i,\chi^{2i})=\widetilde{\Lambda}(2i,\chi^{2i}),$$

and

$$\mathbf{L}(2i+1,\chi^{2i+1})=\widetilde{\Lambda}(2i+1,\chi^{2i+1})D^{2i+1/2}$$

for an integer $i \geq 1$. We note that

$$\mathbf{L}(1,f,\mathrm{Ad})=\begin{cases} 2^{2k+1}\prod_{q|D}(1+q^{-1}) & \text{if } f \in \mathfrak{S}_{2k+1}(\mathit{\Gamma}_0(D),\chi) \\ 2^{2k} & \text{if } f \in \mathfrak{S}_{2k}(SL_2(\mathbf{Z})). \end{cases}$$

Hence we obtain the following:

Theorem 2.2. *Let the notation be as above. Then we have*

$$\frac{\langle I_m(f),\, I_m(f)\rangle}{\langle f,\, f\rangle^m}=2^{\beta_{n,k}}\prod_{i=2}^{m}\mathbf{L}(i,\, f,\, \mathrm{Ad},\chi^{i-1})\mathbf{L}(i,\chi^i)$$

$$\times\begin{cases} \eta_n(f)D^{2nk+4n^2-n}\prod_{q|D}(1+q^{-1}) & \text{if } m=2n \\ D^{2nk+4n^2+n} & \text{if } m=2n+1, \end{cases}$$

where $\beta_{n,k}$ is an integer depending on n and k.

It is well known that $\mathbf{L}(i,\chi^i)$ is a rational number for any positive integer i. Moreover $\mathbf{L}(i,f,\mathrm{Ad},\chi^{i-1})$ is an algebraic number and belongs to the Hecke field $\mathbf{Q}(f)$ for $i=2,....,k'$ where $k'=2k$ or $2k-1$ according as if m is even or odd (cf. Shimura [24], [25]). Thus we have

Theorem 2.3. *In addition to the above notation and the assumption, suppose that $m \leq 2k$ or $m \leq 2k-1$ according as m is even or odd. Then $\dfrac{\langle I_m(f),\, I_m(f)\rangle}{\langle f,\, f\rangle^m}$ is algebraic, and in particular it belongs to $\mathbf{Q}(f)$.*

3. Rankin-Selberg convolution product

To prove Theorem 2.1, we rewrite it in terms of the residue of the Rankin-Selberg convolution product of $I_m(f)$. Let

$$F(z)=\sum_{A\in\widehat{\mathrm{Her}}_m(\mathcal{O})^+}a_F(A)\mathbf{e}(\mathrm{tr}(Az))$$

be an element of $\mathfrak{S}_{2l}(\mathit{\Gamma}^{(m)},\det^{-l})$. We then define the Rankin-Selberg series $R(s,F)$ for F by

$$R(s,F)=\sum_{A\in\widehat{\mathrm{Her}}_m(\mathcal{O})^+/SL_m(\mathcal{O})}\frac{a_F(A)\overline{a_F(A)}}{(\det A)^s e^*(A)},$$

where $e^*(A)=\#(\{g \in SL_m(\mathcal{O}) \mid g^*Ag=A\})$.

Proposition 3.1. *Put*

$$R_m=\frac{2^{2lm+m-1}\prod_{i=2}^{m}L(i,\chi^{i+1})}{D^{m(m-1)/2}\prod_{i=0}^{m-1}L(2m-i,\chi^i)\prod_{i=1}^{m}\Gamma_{\mathbf{C}}(i)\Gamma_{\mathbf{C}}(2l-i+1)}.$$

Let $F \in \mathfrak{S}_{2l}(\Gamma^{(m)}, \det^{-l})$. Then $R(s,F)$ is holomorphic in s for $\mathrm{Re}(s) > 2l$. Moreover it can be continued to a meromorphic function on the whole s-plane, and has a simple pole at $s = 2l$ with the residue $R_m\langle F, F\rangle$.

Proof. The assertion can be proved by a careful analysis of the proof of [[25], Proposition 22.2]. However, for the convenience of the readers we here give an outline of the proof. We define another Rankin-Selberg series $\widetilde{R}(s,F)$ for F by

$$\widetilde{R}(s,F) = \sum_{A \in \widehat{\mathrm{Her}}_m(\mathcal{O})^+/GL_m(\mathcal{O})} \frac{a_F(A)\overline{a_F(A)}}{(\det A)^s e(A)},$$

where $e(A) = \#(\{g \in GL_m(\mathcal{O}) \mid g^*Ag = A\})$. Remark that

$$R(s,F) = \#(\mathcal{O}^*)\widetilde{R}(s,F).$$

We define the non-holomorphic Eisenstein series $E(Z,s)$ for $\Gamma^{(m)}$ by

$$E(Z,s) = (\det Y)^s \sum_{M \in \Gamma_\infty^{(m)} \backslash \Gamma^{(m)}} |j(M,Z)|^{-2s},$$

where $\Gamma_\infty^{(m)} = \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma^{(m)}\}$. Then by using the same argument as in Page 179 of [25], we obtain

$$\widetilde{R}(s,F) = \frac{1}{\#(\mathcal{O}^*)\mathrm{vol}(\mathrm{Her}_m(\mathbf{C})/\mathrm{Her}_m(\mathcal{O}))\widetilde{\Gamma}_m(s)(4\pi)^{-ms}}$$

$$\times \int_{\Gamma^{(m)} \backslash \mathfrak{H}_m} F(Z)\overline{F(Z)E(Z, \bar{s} - 2l + m)}(\det Y)^{2l-2m} dX dY,$$

where $\mathrm{vol}(\mathrm{Her}_m(\mathbf{C})/\mathrm{Her}_m(\mathcal{O}))$ is the volume of $\mathrm{Her}_m(\mathbf{C})/\mathrm{Her}_m(\mathcal{O})$ with respect to the measure dX, and

$$\widetilde{\Gamma}_m(s) = \pi^{m(m-1)/2} \prod_{i=0}^{m-1} \Gamma(s-i).$$

By [[24],Theorem 19.7], $E(Z, s-2l+m)$ is holomorphic in s for $\mathrm{Re}(s) > 2l$. Moreover it has a meromorphic continuation to the whole s-plane, and has a simple pole at $s = 2l$ with the residue of the following form:

$$\pi^{m^2}\widetilde{\Gamma}_m(m)^{-1} \frac{2^{m(1-m)-1}\prod_{i=2}^m L(i, \chi^{i+1})}{\mathrm{vol}(\mathrm{Her}_m(\mathbf{C})/\mathrm{Her}_m(\mathcal{O}))\prod_{i=0}^{m-1} L(2m-i, \chi^i)}.$$

We note that

$$\mathrm{vol}(\mathrm{Her}_m(\mathbf{C})/\mathrm{Her}_m(\mathcal{O})) = 2^{m(1-m)/2} D^{m(m-1)/4}.$$

Thus we prove the assertion. $\square$

4. Reduction to local computations

To prove our main result, we give an explicit formula for $R(s, I_m(f))$. To do this, we reduce the problem to local computations. Let K_p and $\mathcal{O}_p$ be as in Notation. Then K_p is a quadratic extension of $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. In the former case let $\mathcal{O}_p$ be the ring of integers in K_p, and f_p the exponent of the conductor of $K_p/\mathbf{Q}_p$. If K_p is ramified over $\mathbf{Q}_p$, put $e_p = f_p - \delta_{2,p}$, where $\delta_{2,p}$ is Kronecker's delta. If K_p is unramified over $\mathbf{Q}_p$, put $e_p = 0$. In the latter case, put $\mathcal{O}_p = \mathbf{Z}_p \oplus \mathbf{Z}_p$, and $e_p = f_p = 0$. Moreover put $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p) = p^{e_p}\widehat{\mathrm{Her}}_m(\mathcal{O}_p)$. We note that $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p) = \mathrm{Her}_m(\mathcal{O}_p)$ if K_p is not ramified over $\mathbf{Q}_p$. Let K be an imaginary quadratic extension of $\mathbf{Q}$ with the discriminant $-D$. We then put $\widetilde{D} = \prod_{p|D} p^{e_p}$, and $\widetilde{\mathrm{Her}}_m(\mathcal{O}) = \widetilde{D}\widehat{\mathrm{Her}}_m(\mathcal{O})$. Now let m and l be positive integers such that $m \geq l$. Then for an integer a and $A \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p), B \in \widetilde{\mathrm{Her}}_l(\mathcal{O}_p)$ put

$$\mathcal{A}_a(A,B) = \{X \in M_{ml}(\mathcal{O}_p)/p^a M_{ml}(\mathcal{O}_p) \mid A[X] - B \in p^a \widetilde{\mathrm{Her}}_l(\mathcal{O}_p)\},$$

and

$$\mathcal{B}_a(A,B) = \{X \in \mathcal{A}_a(A,B) \mid \mathrm{rank}_{\mathcal{O}_p/p\mathcal{O}_p} X = l\}.$$

Suppose that A and B are non-degenerate. Then the number $p^{a(-2ml+l^2)}\#\mathcal{A}_a(A,B)$ is independent of a if a is sufficiently large. Hence we define the local density $\alpha_p(A,B)$ representing B by A as

$$\alpha_p(A,B) = \lim_{a\to\infty} p^{a(-2ml+l^2)}\#\mathcal{A}_a(A,B).$$

Similarly we can define the primitive local density $\beta_p(A,B)$ as

$$\beta_p(A,B) = \lim_{a\to\infty} p^{a(-2ml+l^2)}\#\mathcal{B}_a(A,B)$$

if A is non-degenerate. We remark that the primitive local density $\beta_p(A,B)$ can be defined even if B is not non-degenerate. In particular we write $\alpha_p(A) = \alpha_p(A,A)$.

Let $\mathcal{U}_1$ be the unitary group defined in Section 1. Namely let

$$\mathcal{U}_1 = \{u \in R_{K/\mathbf{Q}}(GL_1) \mid \overline{u}u = 1\}.$$

For an element $T \in \mathrm{Her}_m(\mathcal{O}_p)$, let

$$\widetilde{U_{p,T}} = \{\det X \mid X \in \mathcal{U}_T(K_p) \cap GL_m(\mathcal{O}_p))\}.$$

Then $\widetilde{U_{p,T}}$ is a subgroup of $U_{1,p}$ of finite index. We then put
$l_{p,T} = [U_{1,p} : \widetilde{U_{p,T}})]$. We also put

$$u_p = \begin{cases} (1+p^{-1})^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is unramified} \\ (1-p^{-1})^{-1} & \text{if } K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p \\ 2^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is ramified.} \end{cases}$$

For a subset $\mathcal{T}$ of $\mathcal{O}_p$ put

$$\mathrm{Her}_m(\mathcal{T}) = \mathrm{Her}_m(\mathcal{O}_p) \cap M_m(\mathcal{T}),$$

and for a subset $\mathcal{S}$ of $\mathcal{O}_p$ put

$$\mathrm{Her}_m(\mathcal{S},\mathcal{T}) = \{A \in \mathrm{Her}_m(\mathcal{T}) \mid \det A \in \mathcal{S}\},$$

and $\widetilde{\mathrm{Her}}_m(\mathcal{S},\mathcal{T}) = \mathrm{Her}_m(\mathcal{S},\mathcal{T}) \cap \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. In particular if $\mathcal{S}$ consists of a single element d we write $\mathrm{Her}_m(\mathcal{S},\mathcal{T})$ as $\mathrm{Her}_m(d,\mathcal{T})$, and so on. For $d \in \mathbf{Z}_{>0}$ we also define the set $\mathrm{Her}_m(d,\mathcal{O})^+$ in a similar way. For each $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)^\times$ put

$$F_p^{(0)}(T,X) = F_p(p^{-e_p}T,X)$$

and

$$\widetilde{F}_p^{(0)}(T,X) = \widetilde{F}_p(p^{-e_p}T,X).$$

We remark that

$$\widetilde{F}_p^{(0)}(T,X) = X^{-\mathrm{ord}_p(\det T)}X^{e_p m - f_p[m/2]}F_p^{(0)}(T,p^{-m}X^2).$$

For $d \in \mathbf{Z}_p^\times$ put

$$\lambda_{m,p}(d,X,Y) = \sum_{A \in \widetilde{\mathrm{Her}}_m(d,\mathcal{O}_p)/SL_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(A,X^{-1})\widetilde{F}_p^{(0)}(A,Y^{-1})}{u_p l_{p,A}\alpha_p(A)}.$$

An explicit formula for $\lambda_{m,p}(p^i d_0,X,Y)$ will be given in the next section for $d_0 \in \mathbf{Z}_p^*$ and $i \geq 0$.

Theorem 4.1. *Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$ or in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ according as $m = 2n$ or $2n+1$. For such an f and a positive integer d_0 put*

$$a_m(f;d_0) = \prod_p \lambda_{m,p}(d_0,\alpha_p,\overline{\alpha}_p),$$

where α_p is the Satake p-parameter of f. Moreover put

$$\mu_{m,k,D} = D^{m(s-2k+l_0)+(2k-l_0)[m/2]-m(m+1)/4-1/2}$$
$$\times 2^{-c_D m(s-2k-2n)-m+1}\prod_{i=2}^m \Gamma_{\mathbf{C}}(i),$$

where $l_0 = 0$ or 1 according as m is even or odd, and $c_D = 1$ or 0 according as 2 divides D or not. Then for $\mathrm{Re}(s) >> 0$, we have

$$R(s,I_m(f)) = \mu_{m,k,D}\sum_{d_0=1}^\infty a_m(f;d_0)d_0^{-s+2k+2n}.$$

Proof. We note that $R(s,I_m(f))$ can be rewritten as

$$R(s,I_m(f)) = \widetilde{D}^{ms}\sum_{T \in \widetilde{\mathrm{Her}}_m(\mathcal{O})^+/SL_m(\mathcal{O})} \frac{a_{I_m(f)}(\widetilde{D}^{-1}T)\overline{a_{I_m(f)}(\widetilde{D}^{-1}T)}}{e^*(T)(\det T)^s}.$$

For $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O})^+$ the Fourier coefficient $a_{I_m(f)}(\widetilde{D}^{-1}T)$ of $I_m(f)$ is uniquely determined by the genus to which T belongs, and can be expressed as

$$|a_{I_m(f)}(\widetilde{D}^{-1}T)|^2 = (D^{[m/2]}\widetilde{D}^{-m}\det T)^{2k-l_0}\prod_p \widetilde{F}_p^{(0)}(T,\alpha_p)\widetilde{F}_p^{(0)}(T,\overline{\alpha_p}).$$

Thus the assertion follows from [[14], Corollary to Proposition 3.2 and Proposition 3.3]. (See also the proof of [[14], Theorem 3.4].) □

5. Formal power series associated with local Siegel series

Let K_p be a quadratic extension of $\mathbf{Q}_p$, and $\varpi = \varpi_p$ and $\pi = \pi_p$ be prime elements of K_p and $\mathbf{Q}_p$, respectively. If K_p is unramified over $\mathbf{Q}_p$, we take $\varpi = \pi = p$. If K_p is ramified over $\mathbf{Q}_p$, we take π so that $\pi = N_{K_p/\mathbf{Q}_p}(\varpi)$. Let $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then put $\varpi = \pi = p$. For $d_0 \in \mathbf{Z}_p^\times$ put

$$\hat{H}_{m,p}(d_0, X, Y, t) = \sum_{i=0}^{\infty} \lambda^*_{m,p}(p^i d_0, X, Y) t^i,$$

where for $d \in \mathbf{Z}_p^\times$ we define $\lambda^*_{m,p}(p^i d_0, X, Y)$ as

$$\lambda^*_{m,p}(d, X, Y) = \sum_{A \in \widetilde{\mathrm{Her}}_m(dN_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*), \mathcal{O}_p)/GL_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(A, X^{-1})\widetilde{F}_p^{(0)}(A, Y^{-1})}{\alpha_p(A)}.$$

We note that

$$\lambda^*_{m,p}(d, X, Y) = \sum_{A \in \widetilde{\mathrm{Her}}_m(dN_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*), \mathcal{O}_p)/GL_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(A, X)\widetilde{F}_p^{(0)}(A, Y)}{\alpha_p(A)}.$$

In Proposition 5.5.1 we will show that we have

$$\lambda^*_{m,p}(d, X, Y) = u_p \lambda_{m,p}(d, X, Y)$$

for $d \in \mathbf{Z}_p^\times$ and therefore

$$\hat{H}_{m,p}(d_0, X, Y, t) = u_p \sum_{i=0}^{\infty} \lambda_{m,p}(p^i d_0, X, Y) t^i.$$

We also define $H_{m,p}(d_0, X, Y, t)$ as

$$H_{m,p}(d_0, X, Y, t) = \sum_{i=0}^{\infty} \lambda^*_{m,p}(\pi^i d_0, X, Y) t^i.$$

We note that $H_{m,p}(d_0, X, Y, t) = \hat{H}_{m,p}(d_0, X, Y, t)$ if K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$, but it is not necessarily the case if K_p is ramified over $\mathbf{Q}_p$. In this section, we give explicit formulas of $H_{m,p}(d_0, X, Y, t)$ for all prime numbers p (cf. Theorems 5.5.2 and 5.5.3), and therefore explicit formulas for $\hat{H}_{m,p}(d_0, X, Y, t)$ (cf. Theorem 5.5.4).

From now on we fix a prime number p. Throughout this section we simply write ord_p as ord and so on if the prime number p is clear from the context. We also write ν_{K_p} as ν. We also simply write $\widetilde{\mathrm{Her}}_{m,p}$ instead of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$, and so on. For a $GL_m(\mathcal{O}_p)$-stable subset $\mathcal{B}$ of $\mathrm{Her}_m(K_p)$ we simply write $\sum_{T \in \mathcal{B}}$ instead of $\sum_{T \in \mathcal{B}/GL_m(\mathcal{O}_p)}$ if there is no fear of confusion.

5.1. **Preliminaries.**

Let m be a positive integer. For a non-negative integer $i \le m$ let

$$\mathcal{D}_{m,i} = GL_m(\mathcal{O}_p) \begin{pmatrix} 1_{m-i} & 0 \\ 0 & \varpi 1_i \end{pmatrix} GL_m(\mathcal{O}_p),$$

and for $W \in \mathcal{D}_{m,i}$, put $\Pi_p(W) = (-1)^i p^{i(i-1)a/2}$, where $a = 2$ or 1 according as K_p is unramified over $\mathbf{Q}_p$ or not. Let $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then for a pair $i = (i_1, i_2)$ of non-negative integers such that $i_1, i_2 \le m$, let

$$\mathcal{D}_{m,i} = GL_m(\mathcal{O}_p) \left(\begin{pmatrix} 1_{m-i_1} & 0 \\ 0 & p1_{i_1} \end{pmatrix}, \begin{pmatrix} 1_{m-i_2} & 0 \\ 0 & p1_{i_2} \end{pmatrix} \right) GL_m(\mathcal{O}_p),$$

and for $W \in \mathcal{D}_{m,i}$ put $\Pi_p(W) = (-1)^{i_1+i_2} p^{i_1(i_1-1)/2+i_2(i_2-1)/2}$. In either case K_p is a quadratic extension of $\mathbf{Q}_p$, or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$, we put $\Pi_p(W) = 0$ for $W \in M_n(\mathcal{O}_p^\times) \setminus \bigcup_{i=0}^m \mathcal{D}_{m,i}$.

For non-degenerate Hermitian matrices S and T of degree m, we put

$$\alpha_p(S,T;i) = \lim_{e \longrightarrow \infty} p^{-m^2 e} \mathcal{A}_e(S,T;i),$$

where

$$\mathcal{A}_e(S,T;i) = \{\bar{X} \in M_m(\mathcal{O}_p)/p^e M_m(\mathcal{O}_p) \in \mathcal{A}_e(S,T) \mid X \in \mathcal{D}_{m,i}\}.$$

For two elements $A, A' \in \mathrm{Her}_m(\mathcal{O}_p)$ we simply write $A \sim_{GL_m(\mathcal{O}_p)} A'$ as $A \sim A'$ if there is no fear of confusion. For a variables U and q put

$$(U,q)_m = \prod_{i-1}^m (1 - q^{i-1}U), \qquad \phi_m(q) = (q,q)_m.$$

We note that $\phi_m(q) = \prod_{i=1}^m (1-q^i)$. Moreover for a prime number p put

$$\phi_{m,p}(q) = \begin{cases} \phi_m(q^2) & \text{if } K_p/\mathbf{Q}_p \text{ is unramified} \\ \phi_m(q)^2 & \text{if } K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p \\ \phi_m(q) & \text{if } K_p/\mathbf{Q}_p \text{ is ramified} \end{cases}$$

Lemma 5.1.1. (1) *Let* $\Omega(S,T) = \{w \in M_m(\mathcal{O}_p) \mid S[w] \sim T\}$, *and* $\Omega(S,T;i) = \Omega(S,T) \cap \mathcal{D}_{m,i}$. *Then we have*

$$\frac{\alpha_p(S,T)}{\alpha_p(T)} = \#(\Omega(S,T)/GL_m(\mathcal{O}_p)) p^{-m(\mathrm{ord}(\det T) - \mathrm{ord}(\det S))},$$

and

$$\frac{\alpha_p(S,T;i)}{\alpha_p(T)} = \#(\Omega(S,T;i)/GL_m(\mathcal{O}_p)) p^{-m(\mathrm{ord}(\det T) - \mathrm{ord}(\det S))}.$$

(2) *Let* $\widetilde{\Omega}(S,T) = \{w \in M_m(\mathcal{O}_p) \mid S \sim T[w^{-1}]\}$, *and* $\widetilde{\Omega}(S,T;i) = \widetilde{\Omega}(S,T) \cap \mathcal{D}_{m,i}$. *Then we have*

$$\frac{\alpha_p(S,T)}{\alpha_p(S)} = \#(GL_m(\mathcal{O}_p) \backslash \widetilde{\Omega}(S,T)),$$

and

$$\frac{\alpha_p(S,T;i)}{\alpha_p(S)} = \#(GL_m(\mathcal{O}_p) \backslash \widetilde{\Omega}(S,T;i)).$$

Proof. The assertions for $\frac{\alpha_p(S,T)}{\alpha_p(T)}$ and $\frac{\alpha_p(S,T)}{\alpha_p(S)}$ have been proved in [[14], Lemma 4.1.3]. The assertions for $\frac{\alpha_p(S,T;i)}{\alpha_p(T)}$ and $\frac{\alpha_p(S,T;i)}{\alpha_p(S)}$ can also be proved in a similar way. □

We define a reduced matrix. A non-degenerate square matrix $W=(d_{ij})_{m\times m}$ with entries in $\mathbf{Z}_p$ is said to be reduced if $d_{ii}=p^{e_i}$ with e_i a non-negative integer, d_{ij} is a non-negative integer such that $d_{ij}\le p^{e_j}-1$ for $i<j$, and $d_{ij}=0$ for $i>j$. Let $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then an element $W=(W_1,W_2)$ of $M_m(\mathcal{O}_p)^\times$ with $W_1,W_2\in M_m(\mathbf{Z}_p)^\times$ is said to be reduced if W_1 and W_2 are reduced. Let K_p be an unramified quadratic extension of $\mathbf{Q}_p$, and θ be an element of $\mathcal{O}_p$ such that $\mathcal{O}_p=\mathbf{Z}_p+\mathbf{Z}_p\theta$. Then a non-degenerate square matrix $W=(d_{ij})_{m\times m}$ with entries in $\mathcal{O}_p$ is said to be reduced if $d_{ii}=p^{e_i}$ with e_i a non-negative integer, $d_{ij}=d_{ij}^{(1)}+d_{ij}^{(2)}\theta$ with $d_{ij}^{(1)},d_{ij}^{(2)}$ non-negative integers such that $d_{ij}^{(1)},d_{ij}^{(2)}\le p^{e_j}-1$ for $i<j$, and $d_{ij}=0$ for $i>j$. Let K_p be a ramified quadratic extension of $\mathbf{Q}_p$, and ϖ be a prime element of K_p. Then a non-degenerate square matrix $W=(d_{ij})_{m\times m}$ with entries in $\mathcal{O}_p$ is said to be reduced if $d_{ii}=\varpi^{e_i}$ with e_i a non-negative integer, $d_{ij}=d_{ij}^{(1)}+d_{ij}^{(2)}\varpi$ with $d_{ij}^{(1)},d_{ij}^{(2)}$ non-negative integers such that $d_{ij}^{(1)}\le p^{[(e_j+1)/2]}-1, 0\le d_{ij}^{(2)}\le p^{[e_j/2]}-1$ for $i<j$, and $d_{ij}=0$ for $i>j$. In any case, we can take the set of all reduced matrices as a complete set of representatives of $GL_m(\mathcal{O}_p)\backslash M_m(\mathcal{O}_p)^\times$. Let m be an integer. For $B\in\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$ put

$$\widetilde{\Omega}(B)=\{W\in GL_m(K_p)\cap M_m(\mathcal{O}_p)\mid B[W^{-1}]\in\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)\}.$$

Moreover put $\widetilde{\Omega}(B,i)=\widetilde{\Omega}(B)\cap\mathcal{D}_{m.i}$. Let $r\le m$, and $\psi_{r,m}$ be the mapping from $GL_r(K_p)$ into $GL_m(K_p)$ defined by $\psi_{r,m}(W)=1_{m-r}\perp W$.

For a subset $\mathcal{T}$ of $\mathcal{O}_p$, we put

$$\mathrm{Her}_m(\mathcal{T})_k=\{A=(a_{ij})\in\mathrm{Her}_m(\mathcal{T})\mid a_{ii}\in\pi^k\mathbf{Z}_p\}.$$

From now on put

$$\mathrm{Her}_{m,*}(\mathcal{O}_p)=\begin{cases}\mathrm{Her}_m(\mathcal{O}_p)_1 & \text{if } p=2 \text{ and } f_p=3,\\ \mathrm{Her}_m(\varpi\mathcal{O}_p)_1 & \text{if } p=2 \text{ and } f_p=2\\ \mathrm{Her}_m(\mathcal{O}_p) & \text{otherwise,}\end{cases}$$

where ϖ is a prime element of K_p. Moreover put $i_p=0$, or 1 according as $p=2$ and $f_2=2$, or not. Suppose that $K_p/\mathbf{Q}_p$ is unramified or $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then an element B of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$ can be expressed as $B\sim_{GL_m(\mathcal{O}_p)}1_r\perp pB_2$ with some integer r and $B_2\in\mathrm{Her}_{m-r,*}(\mathcal{O}_p)$. Suppose that $K_p/\mathbf{Q}_p$ is ramified. For an even positive integer r define Θ_r by

$$\Theta_r=\overbrace{\begin{pmatrix}0&\varpi^{i_p}\\ \overline{\varpi}^{i_p}&0\end{pmatrix}\perp\ldots\perp\begin{pmatrix}0&\varpi^{i_p}\\ \overline{\varpi}^{i_p}&0\end{pmatrix}}^{r/2},$$

where $\overline{\varpi}$ is the conjugate of ϖ over $\mathbf{Q}_p$. Then an element B of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$ is expressed as $B\sim_{GL_m(\mathcal{O}_p)}\Theta_r\perp\pi^{i_p}B_2$ with some even integer r and $B_2\in\mathrm{Her}_{m-r,*}(\mathcal{O}_p)$. For these results, see Jacobowitz [11].

Lemma 5.1.2.

(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Let $B_1\in\mathrm{Her}_{m-n_0}(\mathcal{O}_p)$. Then $\psi_{m-n_0,m}$ induces a bijection from $GL_{m-n_0}(\mathcal{O}_p)\backslash\widetilde{\Omega}(pB_1)$ to $GL_m(\mathcal{O}_p)\backslash\widetilde{\Omega}(1_{n_0}\perp pB_1)$, which will be also denoted by $\psi_{m-n_0,m}$.*
(2) *Suppose that K_p is ramified over $\mathbf{Q}_p$ and that n_0 is even. Let $B_1\in\mathrm{Her}_{m-n_0}(\mathcal{O}_p)$. Then $\psi_{m-n_0,m}$ induces a bijection from $GL_{m-n_0}(\mathcal{O}_p)\backslash\widetilde{\Omega}(\pi^{i_p}B_1)$ to $GL_m(\mathcal{O}_p)\backslash\widetilde{\Omega}(\Theta_{n_0}\perp\pi^{i_p}B_1)$,*

which will be also denoted by $\psi_{m-n_0,m}$. Here i_p is the integer defined above.
(3) *The assertions remain valid if we replace $\widetilde{\Omega}(B)$ with $\widetilde{\Omega}(B,i)$.*

Proof. The assertions (1) and (2) are due to [[14], Lemma 4.1.4]. We prove (3). Assume that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Clearly $\psi_{m-n_0,m}$ is injective. To prove the surjectivity, take a representative W of an element of $GL_m(\mathcal{O}_p)\backslash\widetilde{\Omega}(1_{n_0}\bot B_1)$. Without loss of generality we may assume that W is a reduced matrix with diagonal elements p^r $(0 \le r \le 1)$. Since we have $(1_{n_0}\bot B_1)[W^{-1}] \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$, we have $W = \begin{pmatrix} 1_{n_0} & 0 \\ 0 & W_1 \end{pmatrix}$ with $W_1 \in \widetilde{\Omega}(B_1,i)$. This proves the assertion. Similarly the assertion holds in the case K_p is ramified over $\mathbf{Q}_p$.

□

5.2. **Formal power series of Andrianov type.**

For an element $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$, we define a polynomial $\widetilde{G}_p(T,X,t)$ in X and t by

$$\widetilde{G}_p(T,X,t) = \sum_{i=0}^{m} \sum_{W \in GL_m(\mathcal{O}_p)\backslash \mathcal{D}_{m,i}} \Pi_p(W) t^{\nu(\det W)} \widetilde{F}_p^{(0)}(T[W^{-1}],X).$$

We also define a polynomial $G_p(T,X)$ in X by

$$G_p(T,X) = \sum_{i=0}^{m} \sum_{W \in GL_m(\mathcal{O}_p)\backslash \mathcal{D}_{m,i}} (Xp^m)^{\nu(\det W)} \Pi_p(W) F_p^{(0)}(T[W^{-1}],X).$$

Moreover for an element $T \in \widetilde{\mathrm{Her}}_{m,p}$ we define a polynomial $B_p(T,t)$ in t by

$$B_p(T,t) = \frac{\prod_{i=0}^{m-1}(1-\tau_p^{m+i}p^{m+i}t^2)}{G_p(T,t^2)},$$

where $\tau_p^j = 1$ or ξ_p according as j is even or odd. We note that

$$\widetilde{G}_p(T,X,1) = X^{-\mathrm{ord}(\det T)} X^{e_p m - f_p[m/2]} G_p(T, Xp^{-m}).$$

Now we recall several results in [[14]].

Lemma 5.2.1. [[14], Corollary to Lemma 4.2.2] (1) *Suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Let $T = 1_{m-r}\bot pB_1$ with $B_1 \in \mathrm{Her}_r(\mathcal{O}_p)$. Then we have*

$$G_p(T,Y) = \prod_{i=0}^{r-1}(1-(\xi_p p)^{m+i}Y).$$

(2) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Let $T = \Theta_{m-r}\bot\pi^{i_p}B_1$ with $B_1 \in \mathrm{Her}_{r,*}(\mathcal{O}_p)$. Suppose that $m-r$ is even. Then*

$$G_p(T,Y) = \prod_{i=0}^{[(r-2)/2]} (1-p^{2i+2[(m+1)/2]}Y).$$

Lemma 5.2.2. [[14], Lemma 4.2.3] *Let $B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. Then we have*

$$F_p^{(0)}(B,X) = \sum_{W \in GL_m(\mathcal{O}_p)\backslash \widetilde{\Omega}(B)} G_p(B[W^{-1}],X)(p^m X)^{\nu(\det W)}.$$

Corollary. [[14], Corollary to Lemma 4.2.3] *Let* $B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. *Then we have*

$$\widetilde{F}_p^{(0)}(B,X) = X^{e_p m - f_p[m/2]} \sum_{B' \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)/GL_m(\mathcal{O}_p)} X^{-\mathrm{ord}(\det B')} \frac{\alpha_p(B',B)}{\alpha_p(B')}$$

$$\times G_p(B', p^{-m}X^2) X^{\mathrm{ord}(\det B) - \mathrm{ord}(\det B')}.$$

By Lemma 5.2.1, we easily obtain:

Lemma 5.2.3. (1) *Suppose that* K_p *is unramified over* $\mathbf{Q}_p$ *or* $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. *Let* $T = 1_{m-r} \perp pB_1$ *with* $B_1 \in \mathrm{Her}_r(\mathcal{O}_p)$. *Then we have*

$$B_p(T,t) = \prod_{i=r}^{m-1} (1 - (\xi_p p)^{m+i} t^2).$$

(2) *Suppose that* K_p *is ramified over* $\mathbf{Q}_p$. *Let* $T = \Theta_{m-r} \perp p^{i_p} B_1$ *with* $B_1 \in \mathrm{Her}_{r,*}(\mathcal{O}_p)$. *Then*

$$B_p(T,t) = \prod_{i=[(r-1)/2]+1}^{[(m-2)/2]} (1 - p^{2i+2[(m+1)/2]} t^2).$$

For a non-degenerate semi-integral matrix T over $\mathcal{O}_p$ of degree n, put

$$S_p(T,X,t) = \sum_{W \in M_m(\mathcal{O}_p)^\times / GL_m(\mathcal{O}_p)} \widetilde{F}_p^{(0)}(T[W],X) t^{\nu(\det W)}.$$

This type of formal power series was first introduced by Andrianov [A] to study the standard L-functions of Siegel modular forms of integral weight. Thus we call it the formal power series of Andrianov type. (See also [3], [15]). The following proposition can easily be proved by (1) of Lemma 5.1.1.

Proposition 5.2.4. *Let* $T \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$. *Then we have*

$$\sum_{B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(B,X)\alpha_p(T,B)}{\alpha_p(B)} t^{\mathrm{ord}(\det B)} = t^{\mathrm{ord}(\det T)} S_p(T,X,p^{-m}t).$$

Put $\mathcal{K}^{(m)} = \mathcal{K}_0^{(m)} \mathcal{U}^{(m)}(\mathbf{R})$. Let $\mathcal{H}(\mathcal{U}^{(m)}(\mathbf{A}), \mathcal{K}^{(m)})$ be the Hecke ring associated with the Hecke pair $(\mathcal{U}^{(m)}(\mathbf{A}), \mathcal{K}^{(m)})$. Then $\mathcal{H}(\mathcal{U}^{(m)}(\mathbf{A}), \mathcal{K}^{(m)})$ acts on $\mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q}) \backslash \mathcal{U}^{(m)}(\mathbf{A}), \det^{-l})$ as in [10]. We call an element F of $\mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q}) \backslash \mathcal{U}^{(m)}(\mathbf{A}), \det^{-l})$ a Hecke eigenform if it is a common eigenfunction of all Hecke operators T in $\mathcal{H}(\mathcal{U}^{(m)}(\mathbf{A}), \mathcal{K}^{(m)})$. Then for each element $r \in GL_m(\mathbf{A}) \cap \prod_p M_m(\mathcal{O}_p)$, let $\lambda_F(r)$ be the eigenvalue of $\mathcal{K}^{(m)} \begin{pmatrix} r^{-1} & 0 \\ 0 & r^* \end{pmatrix} \mathcal{K}^{(m)}$ with respect to F, and define a Dirichlet series $\mathfrak{T}(s,F)$ by

$$\mathfrak{T}(s,F) = \sum_{r \in \mathcal{K}^{(m)} \backslash (GL_m(\mathbf{A}) \cap \prod_p M_m(\mathcal{O}_p))/\mathcal{K}^{(m)}} \lambda_F(r) |\det r|_{\mathbf{A}}^s,$$

where $|\det r|_{\mathbf{A}} = \prod_p |\det r_p|_{K_p}$ for $r=(r_p)\in GL_m(\mathbf{A})\cap\prod_p M_m(\mathcal{O}_p)$. Then there exists an Euler product $\mathcal{Z}(s,F)$ such that

$$\mathfrak{T}(s,F)=\prod_{i=1}^m L(2s-i+1,\chi^{i-1})\mathcal{Z}(s,F).$$

We then put

$$\mathcal{L}(s,F,\mathrm{st})=\mathcal{Z}(s+m-1/2,F),$$

and call it the standard L-function of F in the sense of Shimura. We note that our standard L-function coincides with that in [10] up to Euler factors at ramified primes.

Now we define the Eisenstein series on $\mathcal{U}^{(m)}(\mathbf{A})$ and consider its standard L-function in the sense of Shimura. Let $\mathcal{P}$ be the maximal parabolic subgroup of $\mathcal{U}^{(m,m)}$ defined by

$$\mathcal{P}(R)=\{\gamma=\left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right)\in\mathcal{U}^{(m,m)}(R)\}$$

for any $\mathbf{Q}$-algebra R. Write an element $g=(g_v)\in\mathcal{U}^{(m)}(\mathbf{A})$ as

$$(g_p)_{p<\infty}=\left(\left(\begin{smallmatrix} a_p & b_p\\ 0 & d_p\end{smallmatrix}\right)\right)_{p<\infty}(\kappa_p)_{p<\infty}$$

with $\left(\left(\begin{smallmatrix} a_p & b_p\\ 0 & d_p\end{smallmatrix}\right)\right)_{p<\infty}\in\prod_{p<\infty}\mathcal{P}(\mathbf{Q}_p)$ and $(\kappa_p)_{p<\infty}\in\mathcal{K}_0$, and define the function on $\mathcal{U}^{(m)}(\mathbf{A})$ by

$$\mathbf{f}_{2l}(g)=\prod_p|\det(d_p\overline{d}_p)|_p^{-l}j(g_\infty,\mathbf{i})^{-2l}(\det g_\infty)^l.$$

Let l be a integer such that $l>m$. We then define the normalized Eisenstein series as

$$\mathbf{E}_{2l}^{(m)}(g)=2^{-m}\prod_{i=1}^m L(i-2l,\chi^{i-1})\sum_{\gamma\in\mathcal{P}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{Q})}\mathbf{f}_{2l}(\gamma g).$$

Put

$$\mathcal{E}_{2l,m}^{(i)}(Z)=2^{-m}\prod_{j=1}^m L(j-2l,\chi^{j-1})$$
$$\times\prod_p|\det(t_{i,p})\det(\overline{t_{i,p}})|_p^l\sum_{g\in(\Gamma_i\cap\mathcal{P}(\mathbf{Q}))\backslash\Gamma_i}(\det g)^l j(g,Z)^{-2l}$$

for $i=1,\ldots,h$, where $(t_{i,p})$ be the element of $\mathbf{G}^{(m)}(\mathbf{A}_f)$ defined in Section 2. Then $\mathbf{E}_{2l}^{(m)}$ is written as

$$\mathbf{E}_{2l}^{(m)}=(\mathcal{E}_{2l,m}^{(1)},\mathcal{E}_{2l,m}^{(2)},\ldots,\mathcal{E}_{2l,m}^{(h)})^\sharp.$$

Now put

$$\mathcal{L}_{m,p}(X,t)$$
$$=\begin{cases}\prod_{i=1}^m\{(1-p^{-m+2i-1}X^2t^2)(1-p^{-m+2i-1}X^{-2}t^2)\}^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is unramified}\\ \prod_{i=1}^m\{(1-p^{-m/2+i-1/2}Xt)^2(1-p^{-m/2+i-1/2}X^{-1}t)^2\}^{-1} & \text{if } K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p\\ \prod_{i=1}^m\{(1-p^{-m/2+i-1/2}Xt)(1-p^{-m/2+i-1/2}X^{-1}t)\}^{-1} & \text{if } K_p/\mathbf{Q}_p \text{ is ramified.}\end{cases}$$

Proposition 5.2.5. $\mathbf{E}_{2l}^{(m)}$ *is a Hecke eigenform in* $\mathcal{M}_{2l}(\mathcal{U}^{(m)}(\mathbf{Q})\backslash\mathcal{U}^{(m)}(\mathbf{A}),\det^{-l})$, *and its standard L-function* $\mathcal{L}(s,\mathbf{E}_{2l}^{(m)},\mathrm{st})$ *in the sense of Shimura is given by*

$$\mathcal{L}(s,\mathbf{E}_{2l}^{(m)},\mathrm{st})=\prod_p \mathcal{L}_{m,p}(p^{-l+m/2},p^{-s}).$$

Proof. The assertion is more or less well known (cf. [[10], Proposition 13.5]). But for the sake of completeness, we here give an outline of the proof. For each prime number p let $\mathcal{K}_p^{(m)}=\mathcal{U}_m(\mathbf{Q}_p)\cap GL_{2m}(\mathcal{O}_p)$. Moreover, for each $\eta\in\mathcal{U}_m(\mathbf{Q}_p)$ we write $\eta=\begin{pmatrix} a_\eta & b_\eta \\ c_\eta & d_\eta\end{pmatrix}$ with a_η,b_η,c_η and $d_\eta\in M_m(K_p)$. First assume that K_p is a field. Then for any $u\in\mathcal{U}_m(\mathbf{Q}_p)$, we can write the coset $\mathcal{K}_p^{(m)}u\mathcal{K}_p^{(m)}$ as

$$\mathcal{K}_p^{(m)}u\mathcal{K}_p^{(m)}=\bigsqcup_\eta \mathcal{K}_p^{(m)}\begin{pmatrix} a_\eta & b_\eta \\ 0 & d_\eta\end{pmatrix},$$

where d_η is an upper triangular matrix whose diagonal components are $\varpi^{e_1(\eta)},\ldots,\varpi^{e_m(\eta)}$ with $e_1(\eta),\ldots,e_m(\eta)\in\mathbf{Z}$. Then, by a simple computation we have

$$\mathbf{E}_{2l}^{(m)}|\mathcal{K}_p^{(m)}u\mathcal{K}_p^{(m)}=\sum_\eta q^{-l(e_1(\eta)+\cdots+e_m(\eta))}\mathbf{E}_{2l}^{(m)},$$

where $q=p^2$ or p according as $K_p/\mathbf{Q}_p$ is unramified or ramified. We note that $q^{-l(e_1(\eta)+\cdots+e_m(\eta))}=\prod_{i=1}^m(q^{-i}q^{-l+i})^{e_i(\eta)}$. Thus, by [[24], (16.1.3)], [[25], Theorem 19.8] and [[25], 20.6], we can prove that the Euler factor of $\mathcal{L}(s,\mathbf{E}_{2l}^{(m)},\mathrm{st})$ at p is $\mathcal{L}_{m,p}(p^{-l+m/2},p^{-s})$. Next assume that $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then, by [[25], p. 163], for any $u\in\mathcal{U}_m(\mathbf{Q}_p)$, we can write the coset $\mathcal{K}_p^{(m)}u\mathcal{K}_p^{(m)}$ as

$$\mathcal{K}_p^{(m)}u\mathcal{K}_p^{(m)}=\bigsqcup_\eta \mathcal{K}_p^{(m)}\begin{pmatrix} a_\eta & b_\eta \\ 0 & d_\eta\end{pmatrix},$$

where d_η is a pair of upper triangular matrices whose diagonal components are $p^{e_1(\eta)},\ldots,p^{e_m(\eta)}$ with $e_1(\eta),\ldots,e_m(\eta)\in\mathbf{Z}$ and $p^{e_{m+1}(\eta)},\ldots,p^{e_{2m}(\eta)}$ with $e_{m+1}(\eta),\ldots,e_{2m}(\eta)\in\mathbf{Z}$, respectively. Then, by a simple computation we have

$$\mathbf{E}_{2l}^{(m)}|\mathcal{K}_p^{(m)}u\mathcal{K}_p^{(m)}=\sum_\eta p^{-l(e_1(\eta)+\cdots+e_{2m}(\eta))}\mathbf{E}_{2l}^{(m)}.$$

We note that $p^{-l(e_1(\eta)+\cdots+e_{2m}(\eta))}=\prod_{i=1}^m(p^{-i}p^{-l+i})^{e_i(\eta)}(p^{-i}p^{-l+i})^{e_{m+i}(\eta)}$. Thus, by [[25], p. 163], [[25], Theorem 19.8] and [[25], 20.6], we can also prove that the Euler factor of $\mathcal{L}(s,\mathbf{E}_{2l}^{(m)},\mathrm{st})$ at p is $\mathcal{L}_{m,p}(p^{-l+m/2},p^{-s})$. This completes the proof. $\square$

For an element $x=(x_v)\in\mathbf{A}$ put $\mathbf{e}_{\mathbf{A}}(x)=\mathbf{e}(x_\infty)\prod_{p<\infty}\mathbf{e}_p(-x_p)$. We also denote by $\mathcal{HER}_m$ the algebraic group defined over $\mathbf{Q}$ such that $\mathcal{HER}_m(S)=\mathrm{Her}_m(S\otimes K)$ for any $\mathbf{Q}$-algebra S. Then for any $u\in G_m(\mathbf{A})$ and $s\in\mathcal{HER}_m(\mathbf{A})$ we have the following Fourier expansion:

$$\mathbf{E}_{2l}^{(m)}\left(\begin{pmatrix} u & (u^*)^{-1}s \\ 0 & (u^*)^{-1}\end{pmatrix}\right)=(\det u\,\overline{\det u})^l\sum_{T\in\mathrm{Her}_m(K)}c_{2l}^{(m)}(T;u)\mathbf{e}(\sqrt{-1}\mathrm{tr}(u^*Tu))\mathbf{e}_{\mathbf{A}}(\mathrm{tr}(As)),$$

where $c_{2l}^{(m)}(T;u)$ is a complex number depending only on $\mathbf{E}_{2l}^{(m)}, T, (u_p)_{p<\infty}$ and $(uu^*)_\infty$ (cf. [[24], Proposition 18.3). Here we have $c_{2l}^{(m)}(T;u) \neq 0$ only if T is semi-positive definite.

Remark. For any $T \in \mathrm{Her}_m(K)^+$, the T-th Fourier coefficient $c_{2l,m}^{(i)}(T)$ of $\mathcal{E}_{2l,m}^{(i)}(Z)$ is equal to $c_{2l}^{(m)}(T,(t_{i,p}))$ (cf. [[25], (20.9f)]), and it is given by

$$A_m|\gamma(T)|^{l-m/2}\prod_p |\det(t_{i,p})|_{K_p}^{m/2}\widetilde{F}_p(t_{i,p}^*Tt_{i,p}, p^{-l+m/2}),$$

where $A_m = (-1)^m$ or 1 according as $m = 2n$ or $m = 2n+1$ (cf. [9], pages 1134-1135). We notice that A_m appears in the above formula because the definition of $\widetilde{F}_p(*, X)$ is a slightly different from that in [9] as remarked in Section 2. In general, for any $T \in \mathrm{Her}_m(K)^+$ and $u = (u_p) \in \mathbf{G}^{(m)}(\mathbf{A}_f)$ we have

$$c_{2l}^{(m)}(T;u) = A_m|\gamma(T)|^{l-m/2}\prod_p |\det u_p|_{K_p}^{m/2}\widetilde{F}_p(u_p^*Tu_p, p^{-l+m/2}).$$

This can be proved in the same way as above.

Theorem 5.2.6. *Let T be an element of $\widetilde{\mathrm{Her}}_m(\mathcal{O}_p)^\times$. Then we have*

$$S_p(T,X,t) = B_p(T,p^{-m/2}t)\widetilde{G}_p(T,X,t)\mathcal{L}_{m,p}(X,p^{m/2-1/2}t).$$

Proof. Take an element $\widetilde{T} \in \widetilde{\mathrm{Her}}_m(\mathcal{O})^+$ such that $\widetilde{T} \sim_{GL_m(\mathcal{O}_p)} T$. Then we have

$$S_p(\widetilde{T},X,t) = S_p(T,X,t)$$

and

$$B_p(\widetilde{T},p^{-m/2}t)\widetilde{G}_p(\widetilde{T},X,t) = B_p(T,p^{-m/2}t)\widetilde{G}_p(T,X,t).$$

Write $S_p(\widetilde{T},X,t)$ and $B_p(\widetilde{T},p^{-m/2}t)\widetilde{G}_p(\widetilde{T},X,t)\mathcal{L}_{m,p}(X,p^{m/2-1/2}t)$ as

$$S_p(\widetilde{T},X,t) = \sum_{i=0}^{\infty} r_i(X)t^i,$$

and

$$B_p(\widetilde{T},p^{-m/2}t)\widetilde{G}_p(\widetilde{T},X,t)\mathcal{L}_{m,p}(X,p^{m/2-1/2}t) = \sum_{i=0}^{\infty} s_i(X)t^i.$$

Then $r_i(X)$ and $s_i(X)$ are polynomials in X and X^{-1}. For a positive integer l and $A \in \widehat{\mathrm{Her}}_m(\mathcal{O})^+$, put

$$D_p(s,A,\mathbf{E}_{2l}^{(m)}) = \sum_{W \in M_m(\mathcal{O}_p)^\times/GL_m(\mathcal{O}_p)} |\det W|_{K_p}^{-m} c_{2l}^{(m)}(A,\widetilde{W})p^{-s\nu_{K_p}(\det W)},$$

and

$$\widetilde{G}_{2l,m}(A,s) = \sum_{W \in GL_m(\mathcal{O}_p)\backslash M_m(\mathcal{O}_p)^\times} \mathit{\Pi}_p(W)c_{2l}^{(m)}(A,\widetilde{W}^{-1})p^{-s\nu_{K_p}(\det W)},$$

where for $V \in M_m(\mathcal{K}_p)^\times$ we denote by $\widetilde{V} = (V_q)$ the element of $\mathbf{G}^{(m)}(\mathbf{A}_f)$ such that $V_p = V$ and $V_q = 1_m$ for any $q \neq p$. Then by Proposition 5.2.5 and by using the same argument as in the proof of [[25], Theorem 20.7], we obtain

$$D_p(s+m/2,\widetilde{D}^{-1}\widetilde{T},\mathbf{E}_{2l}^{(m)})$$
$$= \widetilde{G}_{2l,m}(\widetilde{D}^{-1}\widetilde{T},s+m/2)B_p(\widetilde{T},p^{-s-m/2})\mathcal{L}_{m,p}(p^{-l+m/2},p^{m/2-1/2-s})$$

for any positive integer $l > m$. By the above remark, for any $A \in \mathrm{Her}_m(K)^+$ and $V \in M_m(\mathcal{K}_p)^\times$ we have

$$c_{2l}^{(m)}(A, \widetilde{V}) = d(l,m;A)|\det V|_{K_p}^{m/2} \widetilde{F}_p(V^*AV, p^{-l+m/2}),$$

where $d(l,m;A) = A_m|\gamma(A)|^{l-m/2} \prod_{q\neq p} \widetilde{F}_q(A, q^{-l+m/2})$. Hence we have

$$D_p(s+m/2, \widetilde{D}^{-1}\widetilde{T}, \mathbf{E}_{2l}^{(m)}) = d(l,m;\widetilde{D}^{-1}\widetilde{T}) S_p(\widetilde{T}, p^{-l+m/2}, p^{-s}),$$

and

$$\widetilde{G}_{2l,m}(\widetilde{D}^{-1}\widetilde{T}, s+m/2) = d(l,m;\widetilde{D}^{-1}\widetilde{T})\widetilde{G}_p(\widetilde{T}, p^{-l+m/2}, p^{-s}),$$

and therefore

$$\begin{aligned} &d(l,m;\widetilde{D}^{-1}\widetilde{T}) S_p(\widetilde{T}, p^{-l+m/2}, p^{-s}) \\ &= d(l,m;\widetilde{D}^{-1}\widetilde{T}) B_p(\widetilde{T}, p^{-s-m/2})\widetilde{G}_p(\widetilde{T}, p^{-l+m/2}, p^{-s})\mathcal{L}_{m,p}(p^{-l+m/2}, p^{m/2-1/2-s}) \end{aligned}$$

for any positive integer $l > m$. We note that $d(l,m;\widetilde{D}^{-1}\widetilde{T}) \neq 0$ for $l > m$. Hence we have

$$S_p(\widetilde{T}, p^{-l+m/2}, t) = B_p(\widetilde{T}, p^{-m/2}t)\widetilde{G}_p(\widetilde{T}, p^{-l+m/2}, t)\mathcal{L}_{m,p}(p^{-l+m/2}, p^{m/2-1/2}t)$$

for any integer $l > m$. This implies that $r_i(p^{-l+m/2}) = s_i(p^{-l+m/2})$ for infinitely many positive integers l. Hence we have $r_i(X) = s_i(X)$. □

Now by Theorem 5.2.6, we can rewrite $H_{m,p}(d_0, X, Y, t)$ in terms of $G_p(B', Y)$, $B_p(T,t)$ and $\widetilde{G}_p(T, X, t)$ in the following way: For $d_0 \in \mathbf{Z}_p^\times$ put

$$\widetilde{\mathcal{F}}_{m,p}(d_0) = \bigcup_{i=0}^{\infty} \widetilde{\mathrm{Her}}_m(\pi^i d_0 N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*), \mathcal{O}_p),$$

and define a formal power series $R_m(d_0, X, Y, t)$ in t by

$$\begin{aligned} R_m(d_0, X, Y, t) = &\sum_{B' \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{\widetilde{G}_p(B', X, p^{-m}Yt)}{\alpha_p(B')} \\ &\times (tY^{-1})^{\mathrm{ord}(\det B')} B_p(B', p^{-3m/2}Yt) G_p(B', p^{-m}Y^2). \end{aligned}$$

Theorem 5.2.7. *We have*

$$H_{m,p}(d_0, X, Y, t) = Y^{e_p m - f_p[m/2]} R_{m,p}(d_0, X, Y, t)\mathcal{L}_{m,p}(X, tYp^{-m/2-1/2})$$

for $d_0 \in \mathbf{Z}_p^\times$.

Proof. We note that $H_{m,p}(d_0, X, Y, t)$ can be written as

$$H_{m,p}(d_0, X, Y, t) = \sum_{B \in \widetilde{\mathcal{F}}_{m,p}(d_0)} t^{\mathrm{ord}(\det B)} \frac{\widetilde{F}_p^{(0)}(B, X)\widetilde{F}_p^{(0)}(B, Y)}{\alpha_p(B)}.$$

Hence by Corollary to Lemma 5.2.2, we have

$$\begin{aligned} H_{m,p}(d_0, X, Y, t) = Y^{e_p m - f_p[m/2]} &\sum_{B \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{t^{\mathrm{ord}(\det B)} \widetilde{F}_p^{(0)}(B, X)}{\alpha_p(B)} \\ \times \sum_{B' \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)} &\frac{Y^{-\mathrm{ord}(\det B')} G_p(B', p^{-m}Y^2)\alpha_p(B', B)}{\alpha_p(B')} Y^{\mathrm{ord}(\det B) - \mathrm{ord}(\det B')}. \end{aligned}$$

Let $B, B' \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)$, and suppose that $\alpha_p(B', B) \neq 0$. Then we note that $B \in \widetilde{\mathcal{F}}_{m,p}(d_0)$ if and only if $B' \in \widetilde{\mathcal{F}}_{m,p}(d_0)$. Hence by Proposition 5.2.4 and Theorem 5.2.6 we have

$$Y^{-e_p m + f_p[m/2]} H_{m,p}(d_0, X, Y, t) = \sum_{B' \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{G_p(B', p^{-m}Y^2) Y^{-2\mathrm{ord}(\det B')}}{\alpha_p(B')}$$

$$\times \sum_{B \in \widetilde{\mathrm{Her}}_m(\mathcal{O}_p)} \frac{\widetilde{F}_p^{(0)}(B, X)\alpha_p(B', B)}{\alpha_p(B)} (tY)^{\mathrm{ord}(\det B)}$$

$$= \sum_{B' \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{G_p(B', p^{-m}Y^2) Y^{-2\mathrm{ord}(\det B')}}{\alpha_p(B')} (tY)^{\mathrm{ord}(\det B')} S_p(B', X, tYp^{-m})$$

$$= \sum_{B' \in \widetilde{\mathcal{F}}_{m,p}(d_0)} \frac{\widetilde{G}_p(B', X, p^{-m}Yt)}{\alpha_p(B')} (tY^{-1})^{\mathrm{ord}(\det B')}$$

$$\times B_p(B', p^{-3m/2}Yt) G_p(B', p^{-m}Y^2) \mathcal{L}_{m,p}(X, tYp^{-m/2-1/2}).$$

□

5.3. **Formal power series of modified Koecher-Maass type.**

Let r be a positive integer, and $d_0 \in \mathbf{Z}_p^*$. We then define a formal power series $P_r(d_0, X, t)$ in t by

$$P_r(d_0, X, t) = \sum_{B \in \widetilde{\mathcal{F}}_{r,p}(d_0)} \frac{\widetilde{F}_p^{(0)}(B, X)}{\alpha_p(B)} t^{\mathrm{ord}(\det B)}.$$

This type of formal power series appears in an explicit formula of the Koecher-Maass series associated with the Siegel Eisenstein series and the Ikeda lift (cf. [7], [8]). Thus we call this the formal power series of Koecher-Maass type. To prove Theorems 5.5.1 and 5.5.2, the main results of Section 5, we define a formal power series $\widetilde{P}_r(d_0, X, Y, t)$ in t by

$$\widetilde{P}_r(d_0, X, Y, t) = \sum_{B' \in \widetilde{\mathcal{F}}_{r,p}(d_0)} \frac{\widetilde{G}_p(B', X, tY)}{\alpha_p(B')} (tY^{-1})^{\mathrm{ord}(\det B')}.$$

The relation between $\widetilde{P}_r(d_0, X, Y, t)$ and $P_r(d_0, X, t)$ will be given in the following proposition:

Proposition 5.3.1.
(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Then*

$$\widetilde{P}_r(d_0, X, Y, t) = P_r(d_0, X, tY^{-1}) \prod_{i=1}^{r} (1 - t^4 p^{-2r-2+2i}).$$

(2) *Suppose that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$\widetilde{P}_r(d_0, X, Y, t) = P_r(d_0, X, tY^{-1}) \prod_{i=1}^{r} (1 - t^2 p^{-r-1+i})^2.$$

(3) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Then*

$$\widetilde{P}_r(d_0,X,Y,t)=P_r(d_0,X,tY^{-1})\prod_{i=1}^{r}(1-t^2p^{-r-1+i}).$$

Proof. First suppose that K_p is a quadratic extension of $\mathbf{Q}_p$. For each non-negative integer $i \le r$ put

$$P_{r,i}(d_0,X,t)=\sum_{B\in\widetilde{\mathcal{F}}_{r,p}(d_0)}\ \sum_{W\in GL_r(\mathcal{O}_p)\backslash\mathcal{D}_{r,i}}\frac{\widetilde{F}_p^{(0)}(B[W^{-1}],X)}{\alpha_p(B)}t^{\mathrm{ord}(\det B)}.$$

Then by (2) of Lemma 5.1.1 we have

$$P_{r,i}(d_0,X,t)=\sum_{B\in\widetilde{\mathcal{F}}_{r,p}(d_0)}\frac{1}{\alpha_p(B)}\sum_{B'\in\widetilde{\mathrm{Her}}_r(\mathcal{O}_p)}\frac{\widetilde{F}_p^{(0)}(B',X)\alpha_p(B',B;i)}{\alpha_p(B')}t^{\mathrm{ord}(\det B)}.$$

Let $B,B'\in\widetilde{\mathrm{Her}}_r(\mathcal{O}_p)$, and suppose that $\alpha_p(B',B;i)\neq 0$. Then we note that $B\in\widetilde{\mathcal{F}}_{r,p}(d_0)$ if and only if $B'\in\widetilde{\mathcal{F}}_{r,p}(d_0)$. Thus by (1) of Lemma 5.1.1 we have

$$\begin{aligned}
&P_{r,i}(d_0,X,t)\\
&=\sum_{B'\in\widetilde{\mathcal{F}}_{r,p}(d_0)}\frac{\widetilde{F}_p^{(0)}(B',X)}{\alpha_p(B')}\sum_{B\in\widetilde{\mathrm{Her}}_r(\mathcal{O}_p)}t^{\mathrm{ord}(\det B)}\frac{\alpha_p(B',B;i)}{\alpha_p(B)}\\
&=\sum_{B'\in\widetilde{\mathcal{F}}_{r,p}(d_0)}\frac{\widetilde{F}_p^{(0)}(B',X)}{\alpha_p(B')}t^{\mathrm{ord}(\det B')}\#(\mathcal{D}_{r,i}/GL_r(\mathcal{O}_p))(tp^{-r})^{ei},
\end{aligned}$$

where $e=2$ or 1 according as $K_p/\mathbf{Q}_p$ is unramified or ramified. By using the same argument as in the proof of Lemma 3.2.18 of Andrianov [1], we have

$$\#(\mathcal{D}_{r,i}/GL_r(\mathcal{O}_p))=\frac{\phi_r(p^e)}{\phi_i(p^e)\phi_{r-i}(p^e)}.$$

Hence we have

$$\begin{aligned}
&P_{r,i}(d_0,X,t)\\
&=\sum_{B'\in\widetilde{\mathcal{F}}_{r,p}(d_0)}\frac{\widetilde{F}_p^{(0)}(B',X)}{\alpha_p(B')}t^{\mathrm{ord}(\det B')}\frac{\phi_r(p^e)}{\phi_i(p^e)\phi_{r-i}(p^e)}(tp^{-r})^{ei}\\
&=\frac{\phi_r(p^e)}{\phi_i(p^e)\phi_{r-i}(p^e)}P_r(d_0,X,t)(tp^{-r})^{ei}.
\end{aligned}$$

Then we have

$$\widetilde{P}_r(d_0,X,Y,t)=\sum_{i=0}^{r}(-1)^ip^{i(i-1)e/2}(tY)^{ei}P_{r,i}(d_0,X,tY^{-1}).$$

Hence we have

$$\widetilde{P}_r(d_0,X,Y,t)=\sum_{i=0}^{r}(-1)^ip^{i(i+1)e/2}(p^{e(-r-1)}t^{2e})^i\frac{\phi_r(p^e)}{\phi_i(p^e)\phi_{r-i}(p^e)}P_r(d_0,X,tY^{-1})$$
$$=P_r(d_0,X,tY^{-1})\prod_{i=1}^{r}(1-t^{2e}p^{e(-r-1+i)}).$$

Next suppose that $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. For a pair $i=(i_1,i_2)$ of non-negative integers such that $i_1,i_2\le r$, put

$$P_{r,i}(d_0,X,t)=\sum_{B\in\widetilde{\mathcal{F}}_{r,p}(d_0)}\sum_{W\in GL_r(\mathcal{O}_p)\backslash\mathcal{D}_{r,i}}\frac{\widetilde{F}_p^{(0)}(B[W^{-1}],X)}{\alpha_p(B)}t^{\mathrm{ord}(\det B)}.$$

Then by using the same argument as above we can prove that

$$P_{r,i}(d_0,X,t)=\frac{\phi_r(p)}{\phi_{i_1}(p)\phi_{r-i_1}(p)}\frac{\phi_r(p)}{\phi_{i_2}(p)\phi_{r-i_2}(p)}P_r(d_0,X,t)(tp^{-r})^{i_1+i_2}.$$

Hence we have

$$\widetilde{P}_r(d_0,X,Y,t)$$
$$=\sum_{i_1=0}^{r}\sum_{i_2=0}^{r}(-1)^{i_1+i_2}p^{i_1(i_1+1)/2+i_2(i_2+1)/2}(p^{-r-1}t^2)^{i_1+i_2}$$
$$\times\frac{\phi_r(p)}{\phi_{i_1}(p)\phi_{r-i_1}(p)}\frac{\phi_r(p)}{\phi_{i_2}(p)\phi_{r-i_2}(p)}P_r(d_0,X,tY^{-1})$$
$$=P_r(d_0,X,tY^{-1})\prod_{i=1}^{r}(1-t^2p^{-r-1+i})^2.$$

This proves the assertion.

□

Now we consider a partial series of $\widetilde{P}_r(d_0,X,Y,t)$. For $d_0\in\mathbf{Z}_p^*$, we put

$$Q_r(d_0,X,Y,t)$$
$$=\sum_{B'\in\pi^{-i_p}\widetilde{\mathcal{F}}_{r,p}(d_0)\cap\mathrm{Her}_{r,*}(\mathcal{O}_p)}\frac{\widetilde{G}_p(\pi^{i_p}B',X,tY)}{\alpha_p(\pi^{i_p}B')}(tY^{-1})^{\mathrm{ord}(\det\pi^{i_p}B')}.$$

To consider the relation between $\widetilde{P}_r(d_0,X,Y,t)$ and $Q_r(d_0,X,Y,t)$, and to express $R_m(d_0,X,Y,t)$ in terms of $\widetilde{P}_r(d_0,X,Y,t)$, we provide some more preliminary results.

Let X be a variable. First suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Put $\hat{\xi}_p=\sqrt{-1}$ or 1 according as K_p is unramified over $\mathbf{Q}_p$ or not. Let $H_m=H_m(\cdot,X)$ be a function on $\mathrm{Her}_m(\mathcal{O}_p)^\times$ with values in $\mathbf{C}[X,X^{-1}]$ satisfying the following condition:

$$H_m(1_{m-r}\perp pB,X)=\hat{\xi}_p^{(m-r)\mathrm{ord}(\det(pB))}H_r(pB,\hat{\xi}_p^{m-r}X)\text{ for any }B\in\mathrm{Her}_r(\mathcal{O}_p).$$

Let $d_0\in\mathbf{Z}_p^*$. Then we put

$$Q(d_0,H_m,r,X,t)=\sum_{B\in p^{-1}\mathcal{F}_{r,p}(d_0)\cap\mathrm{Her}_r(\mathcal{O}_p)}\frac{H_m(1_{m-r}\perp pB,X)}{\alpha_p(1_{m-r}\perp pB)}t^{\mathrm{ord}(\det(pB))}.$$

Next suppose that K_p is ramified over $\mathbf{Q}_p$. Let $H_m = H_m(\cdot, X)$ be a function on $\mathrm{Her}_m(\mathcal{O}_p)^\times$ with values in $\mathbf{C}[X, X^{-1}]$ satisfying the following condition:

$$H_m(\Theta_{m-r}\perp\pi^{i_p}B, X) = H_r(\pi^{i_p}B, X) \text{ for any } B \in \mathrm{Her}_{r,*}(\mathcal{O}_p) \text{ if } m-r \text{ is even.}$$

Let $d_0 \in \mathbf{Z}_p^*$ and $m-r$ be even. Then we put

$$Q(d_0, H_m, r, X, t) = \sum_{B\in\pi^{-i_p}\widetilde{\mathcal{F}}_{r,p}(d_0)\cap\mathrm{Her}_{r,*}(\mathcal{O}_p)} \frac{H_m(\Theta_{m-r}\perp\pi^{i_p}B, X)}{\alpha_p(\Theta_{m-r}\perp\pi^{i_p}B)} t^{\mathrm{ord}(\det(\pi^{i_p}B))}.$$

Then we have the following (cf. [[14], Proposition 4.2.4]).

Proposition 5.3.2.
(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p\oplus\mathbf{Q}_p$. Then for any $d_0 \in \mathbf{Z}_p^*$ and a non-negative integer r we have*

$$Q(d_0, H_m, r, X, t) = \frac{Q(d_0, H_r, r, \hat{\xi}_p^{m-r}X, \hat{\xi}_p^{m-r}t)}{\phi_{m-r}(\xi_p p^{-1})}.$$

(2) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Then for any $d_0 \in \mathbf{Z}_p^*$ and a non-negative integer r such that $m-r$ is even, we have*

$$Q(d_0, H_m, r, X, t) = \frac{Q(d_0, H_r, r, X, t)}{\phi_{(m-r)/2}(p^{-2})}.$$

Now to apply Proposition 5.3.2 to the formal power series $R_m(d_0, X, Y, t)$ and $Q_r(d_0, X, Y, t)$ we give the following lemma.
Lemma 5.3.3. *Let m be an integer.*
(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p = \mathbf{Q}_p\oplus\mathbf{Q}_p$. Then for any integer such that $r \le m$, and $B' \in \mathrm{Her}_r(\mathcal{O}_p)$ we have*

$$\widetilde{G}_p(1_{m-r}\perp pB', X, t) = \widetilde{G}_p(pB', \hat{\xi}_p^{m-r}X, \hat{\xi}_p^{m-r}t).$$

(2) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Then for any non-negative integer r such that $m-r$ is even, and $B' \in \mathrm{Her}_{r,*}(\mathcal{O}_p)$, we have*

$$\widetilde{G}_p(\Theta_{m-r}\perp\pi^{i_p}B', X, t) = \widetilde{G}_p(\pi^{i_p}B', X, t).$$

Proof. By Lemma 5.2.1 (1), we have

$$G_p(1_{m-r}\perp pB', X) = G_p(pB', \xi_p^{m-r}p^{m-r}X)$$

for $B' \in \mathrm{Her}_r(\mathcal{O}_p)$. Hence by Corollary to Lemma 5.2.2 we have

$$\widetilde{F}_p^{(0)}(1_{m-r}\perp pB', X) = \hat{\xi}_p^{(m-r)\mathrm{ord}(\det(pB'))}\widetilde{F}_p^{(0)}(pB', \hat{\xi}_p^{m-r}X)$$

for $B' \in \mathrm{Her}_r(\mathcal{O}_p)$. Thus the assertion (1) follows from (3) of Lemma 5.1.2. The assertion (2) can be proved in a similar way. □

Let $R_m(d_0, X, Y, t)$ be the formal power series defined at the beginning of Section 5. We express $R_m(d_0, X, Y, t)$ in terms of $Q_r(d_0, X, Y, t)$.
Theorem 5.3.4. *Let $d_0 \in \mathbf{Z}_p^*$.*

(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Then*

$$R_m(d_0,X,Y,t)=\sum_{r=0}^{m}\frac{\prod_{i=0}^{r-1}(1-(-1)^m(-p)^iY^2)\prod_{i=r}^{m-1}(1-(-1)^m(-p)^{-2m+i}Y^2t^2)}{\phi_{m-r}(-p^{-1})}$$
$$\times Q_r(d_0,\hat{\xi}_p^{m-r}X,p^{-m/2}Y,\hat{\xi}_p^{m-r}p^{-m/2}t).$$

(2) *Suppose that $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then*

$$R_m(d_0,X,Y,t)=\sum_{r=0}^{m}\frac{\prod_{i=0}^{r-1}(1-p^iY^2)\prod_{i=r}^{m-1}(1-p^{-2m+i}Y^2t^2)}{\phi_{m-r}(p^{-1})}$$
$$\times Q_r(d_0,X,p^{-m/2}Y,p^{-m/2}t).$$

Throughout (1) *and* (2), *we understand that $Q_0(d_0,X,Y,t)=1$.*
(3) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Let $i_p=0$, or 1 according as $p=2$ and $f_2=2$, or not as defined in Section 5.1.*

(3.1) *Let m be odd. Then*

$$R_m(d_0,X,Y,t)=\sum_{r=0}^{(m-1)/2}\frac{\prod_{i=0}^{r-1}(1-p^{2i+1}Y^2)\prod_{i=r}^{(m-3)/2}(1-p^{-2m+2i+1}Y^2t^2)}{\phi_{(m-2r-1)/2}(p^{-2})}$$
$$\times (tY^{-1})^{(m-2r-1)i_p/2}Q_{2r+1}((-1)^{(m-2r-1)/2}d_0,X,p^{-m/2}Y,p^{-m/2}t).$$

(3.2) *Let m be even. Then*

$$R_m(d_0,X,Y,t)=\sum_{r=0}^{m/2}\frac{\prod_{i=0}^{r-1}(1-p^{2i}Y^2)\prod_{i=r}^{(m-2)/2}(1-p^{-2m+2i}Y^2t^2)}{\phi_{(m-2r)/2}(p^{-2})}$$
$$\times (tY^{-1})^{(m-2r)i_p/2}Q_{2r}((-1)^{(m-2r)/2}d_0,X,p^{-m/2}Y,p^{-m/2}t).$$

Here, for $u\in\mathbf{Z}_p^*$ we understand that $Q_0(u,X,Y,t)=1$ or 0 according as $u\in N_{K_p/\mathbf{q}_p}(\mathcal{O}_p^*)$ or not.

Proof. First suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Let B be an element of $\widetilde{\mathrm{Her}}_r(\mathcal{O}_p)$. Then we note that $1_{m-r}\perp pB$ belongs to $\widetilde{\mathcal{F}}_{m,p}(d_0)$ if and only if $B\in p^{-1}\widetilde{\mathcal{F}}_{r,p}(d_0)\cap\widetilde{\mathrm{Her}}_r(\mathcal{O}_p)$. Thus the assertions (1) and (2) follow from Lemmas 5.2.1, 5.2.3, and 5.3.3, and Proposition 5.3.2.

Next suppose that K_p is ramified over $\mathbf{Q}_p$. Let B be an element of $\widetilde{\mathrm{Her}}_r(\mathcal{O}_p)$. Let $m-r$ be even. Then we note that $\Theta_{m-r}\perp\pi^{i_p}B$ belongs to $\widetilde{\mathcal{F}}_{m,p}(d_0)$ if and only if $B\in\pi^{-i_p}\widetilde{\mathcal{F}}_{r,p}((-1)^{(m-r)/2}d_0)\cap\mathrm{Her}_{r,*}(\mathcal{O}_p)$. Moreover we note that $\mathrm{ord}(\det(\Theta_{m-r}\perp\pi^{i_p}B))=(m-r)i_p/2+\mathrm{ord}(\det(\pi^{i_p}B))$. Thus the assertion (3) can be proved similarly to above. $\square$

Now to rewrite the above theorem, first we express $\widetilde{P}_m(d_0,X,Y,t)$ in terms of $Q_r(d_0,X,Y,t)$.

Proposition 5.3.5. *Let $d_0\in\mathbf{Z}_p^*$.*
(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then*

$$\widetilde{P}_m(d_0,\hat{\xi}_p^mX,Y,\hat{\xi}_p^mt)=\sum_{r=0}^{m}\frac{1}{\phi_{m-r}(\xi_pp^{-1})}Q_r(d_0,\hat{\xi}_p^rX,Y,\hat{\xi}_p^rt).$$

(2) *Suppose that K_p is ramified over $\mathbf{Q}_p$.*

(2.1) *Let m be odd. Then*

$$(tY^{-1})^{(1-m)i_p/2}\widetilde{P}_m((-1)^{(m-1)/2}d_0,X,Y,t)=\sum_{r=0}^{(m-1)/2}\frac{1}{\phi_{(m-2r-1)/2}(p^{-2})} \times (tY^{-1})^{-ri_p}Q_{2r+1}((-1)^r d_0,X,Y,t).$$

(2.2) *Let m be even. Then*

$$(tY^{-1})^{-mi_p/2}\widetilde{P}_m((-1)^{m/2}d_0,X,Y,t)=\sum_{r=0}^{m/2}\frac{1}{\phi_{(m-2r)/2}(p^{-2})} \times (tY^{-1})^{-ri_p}Q_{2r}((-1)^r d_0,X,Y,t).$$

Proof. The assertion can be proved in the same argument as in the proof of Theorem 5.3.4. □

Corollary. *Let d_0 be an element of $\mathbf{Z}_p^*$.*
(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then*

$$Q_r(d_0,\hat{\xi}_p^r X,Y,\hat{\xi}_p^r t)=\sum_{m=0}^{r}\frac{(-1)^m(\xi_p p)^{(m-m^2)/2}}{\phi_m(\xi_p p^{-1})}\widetilde{P}_{r-m}(d_0,\hat{\xi}_p^{r-m}X,Y,\hat{\xi}_p^{r-m}t).$$

Here we understand that $\widetilde{P}_0(d_0,X,Y,t)=1$.
(2) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Then*

$$(tY^{-1})^{-ri_p}Q_{2r+1}((-1)^r d_0,X,Y,t)=\sum_{m=0}^{r}\frac{(-1)^m p^{m-m^2}}{\phi_m(p^{-2})}(tY^{-1})^{(m-r)i_p}\widetilde{P}_{2r+1-2m}((-1)^{r-m}d_0,X,Y,t),$$

and

$$(tY^{-1})^{-ri_p}Q_{2r}((-1)^r d_0,X,Y,t)=\sum_{m=0}^{r}\frac{(-1)^m p^{m-m^2}}{\phi_m(p^{-2})}(tY^{-1})^{(m-r)i_p}\widetilde{P}_{2r-2m}((-1)^{r-m}d_0,X,Y,t).$$

Here, for $u\in\mathbf{Z}_p^*$ we understand that $\widetilde{P}_0(u,X,Y,t)=1$ or 0 according as $u\in N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*)$ or not.

Proof. We can prove the assertions by induction on r (cf. [[16], Corollary 5.1.2]). □

The following lemma follows from [[8], Lemma 3.4].

Lemma 5.3.6. *Let l be a positive integer. Then we have the following identity on the three variables q,U and Q :*

$$\prod_{i=1}^{l}(1-U^{-1}Qq^{-i+1})U^l$$
$$=\sum_{m=0}^{l}\frac{\phi_l(q^{-1})}{\phi_{l-m}(q^{-1})\phi_m(q^{-1})}\prod_{i=1}^{l-m}(1-Qq^{-i+1})\prod_{i=1}^{m}(1-Uq^{i-1})(-1)^m q^{(m-m^2)/2}.$$

Theorem 5.3.7. *Let the notation be as in Theorem 5.3.5.*

(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p=\mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$R_m(d_0,X,Y,t)=\sum_{l=0}^{m}((p^l\xi_pY^2)^{m-l}\widetilde{P}_l(d_0,\hat{\xi}_p^{m-l}X,p^{-m/2}Y,\hat{\xi}_p^{m-l}p^{-m/2}t)$$
$$\times \frac{\prod_{i=1}^{m-l}(1-(\xi_pp)^{-l-m-i}t^2)\prod_{i=0}^{l-1}(1-\xi_p^m(\xi_pp)^iY^2)}{\phi_{m-l}(\xi_pp^{-1})}.$$

(2) *Suppose that K_p is ramified over $\mathbf{Q}_p$.*
(2.1) *Let m be odd. Then*

$$R_m(d_0,X,Y,t)=\sum_{l=0}^{(m-1)/2}(tY^{-1})^{(m-2l-1)i_p/2}\widetilde{P}_{2l+1}((-1)^{(m-2l-1)/2}d_0,X,p^{-m/2}Y,p^{-m/2}t)$$
$$\times \frac{(p^{2l+1}Y^2)^{(m-2l-1)/2}\prod_{i=0}^{l-1}(1-p^{2i+1}Y^2)\prod_{i=1}^{(m-2l-1)/2}(1-p^{-2l-m-2i-1}t^2)}{\phi_{(m-2l-1)/2}(p^{-2})}.$$

(2.2) *Let m be even. Then*

$$R_m(d_0,X,Y,t)=\sum_{l=0}^{m/2}(tY^{-1})^{(m-2l)i_p/2}\widetilde{P}_{2l}((-1)^{(m-2l)/2}d_0,X,p^{-m/2}Y,p^{-m/2}t)$$
$$\times \frac{(p^{2l}Y^2)^{(m-2l)/2}\prod_{i=0}^{l-1}(1-p^{2i}Y^2)\prod_{i=1}^{(m-2l)/2}(1-p^{-2l-m-2i}t^2)}{\phi_{(m-2l)/2}(p^{-2})}.$$

Proof. (1) By Theorem 5.3.4 and Corollary to Proposition 5.3.5, we have

$$R_m(d_0,X,Y,t)$$
$$=\sum_{r=0}^{m}\frac{\prod_{i=0}^{r-1}(1-\xi_p^m(\xi_pp)^iY^2)\prod_{i=0}^{m-r-1}(1-(\xi_pp)^{-m+i+r}p^{-m}Y^2t^2)}{\phi_{m-r}((\xi_pp)^{-1})}$$
$$\times\sum_{j=0}^{r}\frac{(-1)^j(\xi_pp)^{(j-j^2)/2}}{\phi_j((\xi_pp)^{-1})}\widetilde{P}_{r-j}(d_0,\hat{\xi}_p^{m-r+j}X,p^{-m/2}Y,\hat{\xi}_p^{m-r+j}p^{-m/2}t)$$
$$=\sum_{l=0}^{m}\widetilde{P}_l(d_0,\hat{\xi}_p^{m-l}X,p^{-m/2}Y,\hat{\xi}_p^{m-l}p^{-m/2}t)$$
$$\times\sum_{j=0}^{m-l}(-1)^j(\xi_pp)^{(j-j^2)/2}\frac{\prod_{i=0}^{l+j-1}(1-\xi_p^m(\xi_pp)^iY^2)\prod_{i=0}^{m-l-j-1}(1-(\xi_pp)^{-m+i+l+j}p^{-m}Y^2t^2)}{\phi_j(\xi_pp^{-1})\phi_{m-j-l}(\xi_pp^{-1})}.$$

Then the assertion (1) follows from Lemma 5.3.6.

(2) Let m be odd. Then, again by Theorem 5.3.4 and Corollary to Proposition 5.3.5,

$$\begin{aligned}
&R_m(d_0,X,Y,t)\\
&=\sum_{r=0}^{(m-1)/2}\frac{\prod_{i=0}^{r-1}(1-p^{2i+1}Y^2)\prod_{i=0}^{(m-1)/2-r-1}(1-p^{-2m+2i+2r+1}Y^2t^2)}{\phi_{(m-2r-1)/2}(p^{-2})}\\
&\times (tY^{-1})^{(m-1)i_p/2}\sum_{j=0}^{r}\frac{(-1)^jp^{j-j^2}}{\phi_j(p^{-2})}(tY^{-1})^{(j-r)i_p}\\
&\times \widetilde{P}_{2r+1-2j}((-1)^{(m-1-2r+2j)/2}d_0,X,p^{-m/2}Y,p^{-m/2}t)\\
&=(tY^{-1})^{(m-1)i_p/2}\sum_{l=0}^{(m-1)/2}(tY^{-1})^{-li_p}\widetilde{P}_{2l+1}((-1)^{(m-1-2l)/2}d_0,X,Y^{-m/2}Y,p^{-m/2}t)\\
&\times\sum_{j=0}^{(m-1)/2-l}(-1)^jp^{j-j^2}\frac{\prod_{i=0}^{l+j-1}(1-p^{2i+1}Y^2)\prod_{i=0}^{(m-1)/2-l-j-1}(1-p^{-2m+2i+2l+2j+1}Y^2t^2)}{\phi_j(p^{-2})\phi_{(m-1)/2-j-l}(p^{-2})}.
\end{aligned}$$

Hence the assertion (2.1) follows from Lemma 5.3.6. The assertion (2.2) can be proved in the same manner as above. □

By Proposition 5.3.1 we obtain:

Corollary. (1) *Suppose that K_p is unramified over $\mathbf{Q}_p$ or $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then*

$$\begin{aligned}
&R_m(d_0,X,Y,t)=\prod_{i=1}^{m}(1-p^{-2m}(\xi_pp)^{i-1}t^2)\\
&\times\sum_{l=0}^{m}(p^l\xi_pY^2)^{m-l}P_l(d_0,\hat{\xi}_p^{m-l}X,\hat{\xi}_p^{m-l}tY^{-1})\frac{\prod_{i=1}^{l}(1-\xi_p(\xi_pp)^{-l-m+i-1}t^2)\prod_{i=0}^{l-1}(1-\xi_p^m(\xi_pp)^iY^2)}{\phi_{m-l}(\xi_pp^{-1})}.
\end{aligned}$$

Here we understand that $P_0(d_0,X,t)=1$.
(2) *Suppose that K_p is ramified over $\mathbf{Q}_p$.*
(2.1) *Let m be odd. Then*

$$\begin{aligned}
R_m(d_0,X,Y,t)=&\prod_{i=1}^{(m+1)/2}(1-p^{-2m+2i-2}t^2)\\
\times&\sum_{l=0}^{(m-1)/2}(tY^{-1})^{(m-2l-1)i_p/2}P_{2l+1}((-1)^{(m-2l-1)/2}d_0,X,tY^{-1})\\
\times&\frac{(p^{2l+1}Y^2)^{(m-2l-1)/2}\prod_{i=0}^{l-1}(1-p^{2i+1}Y^2)\prod_{i=1}^{l}(1-p^{-2l-2+2i-m}t^2)}{\phi_{(m-2l-1)/2}(p^{-2})}.
\end{aligned}$$

(2.2) *Let m be even. Then*

$$R_m(d_0,X,Y,t)=\prod_{i=1}^{m/2}(1-p^{-2m+2i-2}t^2)$$
$$\times\sum_{l=0}^{m/2}(tY^{-1})^{(m-2l)i_p/2}P_{2l}((-1)^{(m-2l)/2}d_0,X,tY^{-1})$$
$$\times\frac{(p^{2l}Y^2)^{(m-2l)/2}\prod_{i=0}^{l-1}(1-p^{2i}Y^2)\prod_{i=1}^{l}(1-p^{-2l-1+2i-m}t^2)}{\phi_{(m-2l)/2}(p^{-2})}.$$

Here, for $u\in\mathbf{Z}_p^*$ we understand that $P_0(u,X,t)=1$ or 0 according as $u\in N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*)$ or not.

5.4. **Explicit formulas of formal power series of Koecher-Maass type.**

In this section we review explicit formulas for $P_m(d_0,X,t)$.

Theorem 5.4.1. [[14], Theorem 4.3.1] *Let m be even, and $d_0\in\mathbf{Z}_p^*$.*
(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t)=\frac{1}{\phi_m(-p^{-1})\prod_{i=1}^m(1-t(-p)^{-i}X)(1+t(-p)^{-i}X^{-1})}.$$

(2) *Suppose that $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t)=\frac{1}{\phi_m(p^{-1})\prod_{i=1}^m(1-tp^{-i}X)(1-tp^{-i}X^{-1})}.$$

(3) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Let χ_{K_p} be the character of $\mathbf{Q}_p^{\cdot}$ defined by $\chi_{K_p}(a)=(-D,a)$ for $a\in\mathbf{Q}_p^*$. Then*

$$P_m(d_0,X,t)=\frac{t^{mi_p/2}}{2\phi_{m/2}(p^{-2})}$$
$$\times\left\{\frac{1}{\prod_{i=1}^{m/2}(1-tp^{-2i+1}X)(1-tp^{-2i}X^{-1})}+\frac{\chi_{K_p}((-1)^{m/2}d_0)}{\prod_{i=1}^{m/2}(1-tp^{-2i}X)(1-tp^{-2i+1}X^{-1})}\right\}.$$

Theorem 5.4.2. [[14], Theorem 4.3.2] *Let m be odd, and $d_0\in\mathbf{Z}_p^*$.*
(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t)=\frac{1}{\phi_m(-p^{-1})\prod_{i=1}^m(1+t(-p)^{-i}X)(1+t(-p)^{-i}X^{-1})}.$$

(2) *Suppose that $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t)=\frac{1}{\phi_m(p^{-1})\prod_{i=1}^m(1-tp^{-i}X)(1-tp^{-i}X^{-1})}.$$

(3) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Then*

$$P_m(d_0,X,t)=\frac{t^{(m+1)i_p/2+\delta_{2p}}}{2\phi_{(m-1)/2}(p^{-2})\prod_{i=1}^{(m+1)/2}(1-tp^{-2i+1}X)(1-tp^{-2i+1}X^{-1})}.$$

5.5. **Explicit formulas of formal power series of Rankin-Selberg type.**

We give an explicit formula for $H_m(d,X,Y,t)$. First we remark the following.

Proposition 5.5.1. *Let $d \in \mathbf{Z}_p^{\times}$. Then we have*

$$\lambda^*_{m,p}(d,X,Y) = u_p \lambda_{m,p}(d,X,Y).$$

Proof. This can be proved in the same way as [[14], Proposition 4.3.7] □

It is well known that $\#(\mathbf{Z}_p^*/N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*)) = 2$ if $K_p/\mathbf{Q}_p$ is ramified. Hence we can take a complete set $\mathcal{N}_p$ of representatives of $\mathbf{Z}_p^*/N_{K_p/\mathbf{Q}_p}(\mathcal{O}_p^*)$ so that $\mathcal{N}_p = \{1, \xi_0\}$ with $\chi_{K_p}(\xi_0) = -1$.

Theorem 5.5.2. *Let $m = 2n$ be even, and $d_0 \in \mathbf{Z}_p^*$.*
(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Then*

$$\begin{aligned} H_{2n}(d_0,X,Y,t) &= \frac{\prod_{i=1}^{2n}(1-p^{-4n}(-p)^{i-1}t^2)}{\phi_{2n}(-p^{-1})} \\ &\times \frac{1}{\prod_{i=1}^{2n}(1+(-p)^{-2n+i-1}XYt)(1-(-p)^{-2n+i-1}XY^{-1}t)} \\ &\times \frac{1}{\prod_{i=1}^{2n}(1-(-p)^{-2n+i-1}X^{-1}Yt)(1+(-p)^{-2n+i-1}X^{-1}Y^{-1}t)}. \end{aligned}$$

(2) *Suppose that $K_p = \mathbf{Q}_p \oplus \mathbf{Q}_p$. Then*

$$\begin{aligned} H_{2n}(d_0,X,Y,t) &= \frac{\prod_{i=1}^{2n}(1-p^{-4n}p^{i-1}t^2)}{\phi_{2n}(p^{-1})} \\ &\times \frac{1}{\prod_{i=1}^{2n}(1-p^{-2n+i-1}XYt)(1-p^{-2n+i-1}XY^{-1}t)} \\ &\times \frac{1}{\prod_{i=1}^{2n}(1-p^{-2n+i-1}X^{-1}Yt)(1-p^{-2n+i-1}X^{-1}Y^{-1}t)}. \end{aligned}$$

(3) *Suppose that K_p is ramified over $\mathbf{Q}_p$. For $l = 0,1$ put*

$$H_{2n}^{(l)}(X,Y,t) = \sum_{d \in \mathcal{N}_p} \chi_{K_p}((-1)^n d)^l H_{2n}(d,X,Y,t).$$

Then we have

$$H_{2n}(d_0,X,Y,t) = \frac{1}{2}(H_{2n}^{(0)}(X,Y,t) + \chi_{K_p}((-1)^n d_0) H_{2n}^{(1)}(X,Y,t)),$$

and

$$\begin{aligned} H_{2n}^{(0)}(X,Y,t) &= t^{ni_p} \frac{\prod_{i=1}^{n}(1-p^{-4n}p^{2i-2}t^2)}{\phi_n(p^{-2})} \\ &\times \frac{1}{\prod_{i=1}^{n}(1-p^{-2n+2i-1}XYt)(1-p^{-2n+2i-1}X^{-1}Y^{-1}t)} \\ &\times \frac{1}{\prod_{i=1}^{n}(1-p^{-2n+2i-2}X^{-1}Yt)(1-p^{-2n+2i-2}XY^{-1}t)}, \end{aligned}$$

and

$$H_{2n}^{(1)}(X,Y,t) = t^{ni_p}\frac{\prod_{i=1}^{n}(1-p^{-4n}p^{2i-2}t^2)}{\phi_n(p^{-2})}$$
$$\times \frac{1}{\prod_{i=1}^{n}(1-p^{-2n+2i-1}X^{-1}Yt)(1-p^{-2n+2i-1}XY^{-1}t)}$$
$$\times \frac{1}{\prod_{i=1}^{n}(1-p^{-2n+2i-2}XYt)(1-p^{-2n+2i-2}X^{-1}Y^{-1}t)}\}.$$

Proof. First we prove (1). By Theorems 5.4.1and 5.4.2, we have

$$P_l(d_0,\hat{\xi}_p^{m-l}X,\hat{\xi}_p^{m-l}X) = P_l(d_0,X,t)$$

if l is even, and

$$P_l(d_0,\hat{\xi}_p^{m-l}X,\hat{\xi}_p^{m-l}X) = \frac{1}{\phi_m(-p^{-1})\prod_{i=1}^{l}(1-t(-p)^{-i}X)(1+t(-p)^{-i}X^{-1})}$$

if l is odd. Hence, by Corollary to Theorem 5.3.7, $R_{2n}(d_0,X,Y,t)$ can be expressed as

$$R_{2n}(d_0,X,Y,t)$$
$$= \frac{\prod_{i=1}^{2n}(1-p^{-4n}(-p)^{i-1}t^2)S(X,Y,t)}{\phi_{2n}(-p)\prod_{i=1}^{2n}(1-t(-p)^{-2n+i-1}XY^{-1})(1+t(-p)^{-2n+i-1}X^{-1}Y^{-1})},$$

where $S(X,Y,t)$ is a polynomial in t of degree at most $4n$. Then by Theorem 5.2.8, we have

$$H_{2n}(d_0,X,Y,t)$$
$$= \frac{\prod_{i=1}^{2n}(1-p^{-4n}(-p)^{i-1}t^2)S(X,Y,t)}{\phi_{2n}(-p)\prod_{i=1}^{2n}(1-t(-p)^{-2n+i-1}XY^{-1})(1+t(-p)^{-2n+i-1}X^{-1}Y^{-1})}$$
$$\times \frac{1}{\prod_{i=1}^{2n}(1-t^2p^{-4n+2i-2}X^2Y^2)(1-t^2p^{-4n+2i-2}X^{-2}Y^2)}.$$

Recall that we have the following functional equation

$$H_{2n}(d_0,X,Y^{-1},t) = H_{2n}(d_0,X,-Y,t).$$

Hence the reduced denominator of the rational function $H_{2n}(d_0,X,Y^{-1},t)$ in t is at most

$$\prod_{i=1}^{2n}\{(1-t(-p)^{-2n+i-1}XY^{-1})(1+t(-p)^{-2n+i-1}X^{-1}Y^{-1})$$
$$\times (1+t(-p)^{-2n+i-1}XY)(1-t(-p)^{-2n+i-1}X^{-1}Y)\},$$

and therefore we have

$$H_{2n}(d_0,X,Y,t) = \frac{c\prod_{i=1}^{2n}(1-(-p)^{-2n-i}t^2)}{\phi_{2n}(-p)}$$
$$\times \frac{1}{\prod_{i=1}^{2n}(1-t(-p)^{-2n+i}XY^{-1})(1+t(-p)^{-2n+i}X^{-1}Y^{-1})}$$
$$\times \frac{1}{\prod_{i=1}^{2n}(1+t(-p)^{-2n+i-1}XY)(1-t(-p)^{-2n+i-1}X^{-1}Y)}$$

with some constant c. We easily see that we have $c=1$. This proves the assertion (1). Similarly the assertions (2) and (3) can be proved. $\square$

Similarly to Theorem 5.5.2, we have

Theorem 5.5.3. *Let $m=2n+1$ be odd, and $d_0 \in \mathbf{Z}_p^*$.*
(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$. Then*

$$H_{2n+1}(d_0,X,Y,t)=\frac{\prod_{i=1}^{2n+1}(1-p^{-4n-2}(-p)^{i-1}t^2)}{\phi_{2n+1}(-p^{-1})}$$
$$\times \frac{1}{\prod_{i=1}^{2n+1}(1+(-p)^{-2n+i-2}XYt)(1+(-p)^{-2n+i-2}XY^{-1}t)}$$
$$\times \frac{1}{\prod_{i=1}^{2n}(1+(-p)^{-2n+i-2}X^{-1}Yt)(1+(-p)^{-2n+i-2}X^{-1}Y^{-1}t)}.$$

(2) *Suppose that $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then*

$$H_{2n+1}(d_0,X,Y,t)=\frac{\prod_{i=1}^{2n+1}(1-p^{-4n-2}p^{i-1}t^2)}{\phi_{2n+1}(p^{-1})}$$
$$\times \frac{1}{\prod_{i=1}^{2n+1}(1-p^{-2n+i-2}XYt)(1-p^{-2n+i-2}XY^{-1}t)}$$
$$\times \frac{1}{\prod_{i=1}^{2n+1}(1-p^{-2n+i-2}X^{-1}Yt)(1-p^{-2n+i-2}X^{-1}Y^{-1}t)}.$$

(3) *Suppose that K_p is ramified over $\mathbf{Q}_p$. Then*

$$H_{2n+1}(d_0,X,Y,t)=t^{(n+1)i_p+\delta_{2p}}\frac{\prod_{i=1}^{n+1}(1-p^{-4n-2}p^{2i-2}t^2)}{2\phi_n(p^{-2})}$$
$$\times \frac{1}{\prod_{i=1}^{n+1}(1-p^{-2n+2i-3}XYt)(1-p^{-2n+2i-3}X^{-1}Y^{-1}t)}$$
$$\times \frac{1}{(1-p^{-2n+2i-3}X^{-1}Yt)(1-p^{-2n+2i-3}XY^{-1}t)}.$$

By using the same argument as in the proof of [[14],Theorem 4.3.6 and its corollary] we obtain the following:

Theorem 5.5.4. *Let $d_0 \in \mathbf{Z}_p^*$.*
(1) *Suppose that K_p is unramified over $\mathbf{Q}_p$ or that $K_p=\mathbf{Q}_p\oplus\mathbf{Q}_p$. Then*

$$\hat{H}_m(d_0,X,Y,t)=H_m(d_0,X,Y,t)$$

for any $m>0$.
(2) *Suppose that K_p is ramified over $\mathbf{Q}_p$.*
(2.1) *For $l=0,1$ put*

$$\hat{H}_{2n}^{(l)}(X,Y,t)=\sum_{d\in\mathcal{N}_p}\chi_{K_p}((-1)^n d)^l\hat{H}_m(d,X,Y,t).$$

Then we have

$$\hat{H}_{2n}(d_0,X,Y,t)=\frac{1}{2}(\hat{H}_{2n}^{(0)}(X,Y,t)+\chi_{K_p}((-1)^n d_0)\hat{H}_{2n}^{(1)}(X,Y,t)),$$

and
$$\hat{H}_{2n}^{(0)}(X,Y,t) = H_{2n}^{(0)}(X,Y,t),$$
and
$$\hat{H}_{2n}^{(1)}(X,Y,t) = H_{2n}^{(1)}(X,Y,\chi_{K_p}(p)t).$$
(2.2) *We have*
$$\hat{H}_{2n+1}(d_0,X,Y,t) = H_{2n+1}(d_0,X,Y,t)$$

6. Proof of the main theorem

Theorem 6.1. *Let k and n be positive integers. Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$. For a subset Q of Q_D and a Dirichlet character $\eta = \chi^{i-1}$ with a positive integer i put*
$$M(s,f,\mathrm{Ad},\eta,\chi_Q)$$
$$= \{\prod_{p \notin Q}(1-\alpha_p^2\chi(p)^i\chi_Q(p)p^{-s})(1-\alpha_p^{-2}\chi(p)^i\chi_Q(p)p^{-s})(1-\chi^{i-1}(p)\chi_Q(p)p^{-s})^2$$
$$\times \prod_{p \in Q}(1-\alpha_p^2\chi'_Q(p)\chi^{i-1}(p)p^{-s})(1-\alpha_p^{-2}\chi'_Q(p)\chi^{i-1}(p)p^{-s})(1-\chi'_Q(p)\chi(p)^ip^{-s})^2\}^{-1},$$
where for $\psi = \chi_Q$ or $\psi = \chi'_Q$ we make the convention $\psi(p)\chi^j(p) = \psi(p)$ or 0 according as j is even or odd. Then, we have
$$R(s,I_{2n}(f)) = D^{ns+n^2-n/2-1/2}2^{-2n+1}$$
$$\times \prod_{i=2}^{2n}\widetilde{\Lambda}(i,\chi^i)\prod_{i=0}^{2n-1}L(2s-4k-i,\chi^i)^{-1}$$
$$\times \sum_{Q \subset Q_D}\chi_Q((-1)^n)\prod_{i=1}^{2n}M(s-2k-2n+i,f,\mathrm{Ad},\chi^{i-1},\chi_Q).$$

Proof. The assertion can be proved by using Theorems 4.1, 5.5.2 and 5.5.4 similarly to [[14], Theorem 2.3]. □

Theorem 6.2. *Let k and n be positive integers. Given a primitive form $f \in \mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$. Then, we have*
$$R(s,I_{2n+1}(f)) = D^{ns+n^2+3n/2+1/2}2^{-2n}$$
$$\times \prod_{i=2}^{2n+1}\widetilde{\Lambda}(i,\chi^i)\prod_{i=0}^{2n}L(2s-4k-i+2,\chi^i)^{-1}$$
$$\times \prod_{i=1}^{2n+1}L(s-2k-2n+i,f,\mathrm{Ad},\chi^{i-1})L(s-2k-2n+i,\chi^{i-1}).$$

Proof. The assertion follows directly from Theorems 4.1 and 5.5.3. □

Lemma 6.3. *Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$. Suppose that $f_Q = f$ for $Q \subset Q_D$. Then for a positive integer i we have*
$$M(s,f,\mathrm{Ad},\chi^{i-1},\chi_Q) = L(s,f,\mathrm{Ad},\chi^{i-1})L(s,\chi^{i-1}).$$

Proof. For a prime number p let $M_p(s)$ and $L_p(s)$ be the p-Euler factor of $M(s,f,\mathrm{Ad},\chi^{i-1},\chi_Q)$ and $L(s,f,\mathrm{Ad},\chi^{i-1})L(s,\chi^{i-1})$, respectively. We have $M_p(s)=L_p(s)$ if $p\notin Q$ and $\chi_Q(p)=1$. By the assumption we have

$$\chi_Q(p)c_f(p)=c_f(p).$$

Since f is a primitive form, we have $c_f(p)\neq 0$ for $p|D$. Hence we have $M_p(s)=L_p(s)$ if $p\notin Q$ and $p|D$. Suppose $p\nmid D$ and $\chi_Q(p)=-1$. Then $c_f(p)=0$ and hence $\alpha_p+\chi(p)\alpha_p^{-1}=0$. Then by a simple computation we have

$$M_p(s)=(1-p^{-2s})^{-2}.$$

Similarly we have

$$L_p(s)=(1-p^{-2s})^{-2}.$$

Suppose that $p\in Q$. Then $|\alpha_p|=|c_f(p)|=1$, and $\chi'_Q(p)\overline{c_f(p)}=c_f(p)$. Hence α_p is a real number or a purely imaginary number according as $\chi'_Q(p)=1$ or -1. Hence $\chi'_Q(p)\alpha_p^2=\chi'_Q(p)\alpha_p^{-2}=1$, and

$$M_p(s)=L_p(s).$$

This completes the assertion. □

Proposition 6.4. (1) *Let f be a primitive form in $\mathfrak{S}_{2k+1}(\Gamma_0(D),\chi)$, and Q be a subset of Q_D. Then for a positive integer $i\geq 2$ the Euler product $M(s+i-1,f,\mathrm{Ad},\chi^{i-1},\chi_Q)$ is holomorphic at $s=1$. Moreover $M(s,f,\mathrm{Ad},1,\chi_Q)$ has a non-zero residue at $s=1$ if and only if $f=f_Q$. In this case the residue of $M(s,f,\mathrm{Ad},1,\chi_Q)$ at $s=1$ is $L(1,f,\mathrm{Ad})$.*

(2) *Let f be a primitive form in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ and χ be a primitive quadratic odd character. Then for a positive integer $i\geq 2$ the Euler product $L(s+i-1,f,\mathrm{Ad},\chi^{i-1})L(s+i-1,\chi^{i-1})$ is holomorphic at $s=1$, and $L(s,f,\mathrm{Ad},1)L(s,1)$ has a simple pole at $s=1$ with the residue $L(1,f,\mathrm{Ad})$.*

Proof. (1) Clearly $M(s+i-1,f,\mathrm{Ad},\chi^{i-1},\chi_Q)$ is holomorphic at $s=1$ if $i\geq 2$. To prove the latter half of the assertion, let $R(s,f_Q\otimes f_\rho)$ be the tensor product L-function of f_Q and f_ρ, where

$$f_\rho(z)=\sum_{e=1}^{\infty}\overline{c_f(e)}\mathbf{e}(ez).$$

We note that $\overline{c_f(e)}=\chi(e)c_f(n)$ and $c_{f_Q}(e)=\chi_Q(e)c_f(n)$ if $(e,D)=1$. Hence we have

$$M(s,f,\mathrm{Ad},1,\chi_Q)=R(s,f_Q\otimes f_\rho)\times\prod_{p|D}\frac{M_p(s,f,\mathrm{Ad},1,\chi_Q)}{R_p(s,f_Q\otimes f_\rho)},$$

where $M_p(s,f,\mathrm{Ad},1,\chi_Q)$ and $R_p(s,f_Q\otimes f_\rho)$ are the p-Euler factors of $M(s,f,\mathrm{Ad},1,\chi_Q)$ and $R(s,f_Q\otimes f_\rho)$, respectively. We note $\prod_{p|D}\frac{M_p(s,f,\mathrm{Ad},1,\chi_Q)}{R_p(s,f_Q\otimes f_\rho)}$ is holomorphic and nonzero at $s=1$. Hence we have

$$\mathrm{Res}_{s=1}M(s,f,\mathrm{Ad},1,\chi_Q)=c(f_Q,f)$$

with c a nonzero complex numbers (cf. [[23], p. 788] and [[26], p. 831]). Hence $M(s,f,\mathrm{Ad},1,\chi_Q)$ has a non-zero residue at $s=1$ if and only if $(f,f_Q)\neq 0$. Since f and f_Q are primitive forms, this is equivalent to say that $f=f_Q$. In this case, we have

$$M(s,f,\mathrm{Ad},1,\chi_Q)=L(s,f,\mathrm{Ad})\zeta(s),$$

and hence the last assertion holds.

(2) The assertion can easily be proved.

□

Proof of Theorem 2.1.
(1) By Theorem 6.1 and Lemma 6.3, we have

$$R(s, I_m(f)) = D^{ns+n^2-n/2-1/2}2^{-2n+1}\prod_{i=1}^{2n}\widetilde{\Lambda}(i,\chi^i)\prod_{i=0}^{2n-1}L(2s-4k-i,\chi^i)^{-1}$$

$$\times\{\eta_m(f)\prod_{i=1}^{2n}L(s-2k-2n+i,f,\mathrm{Ad},\chi^{i-1})L(s-2k-2n+i,\chi^{i-1})$$

$$+\sum_{\substack{Q\in\mathcal{Q}_D\\ f_Q\neq f}}\chi_Q((-1)^n)\prod_{i=1}^{2n}M(s-2k-2n+i,f,\mathrm{Ad},\chi^{i-1},\chi_Q)\}.$$

By (1) of Lemma 6.4, the term

$$\prod_{i=0}^{2n-1}L(2s-4k-i,\chi^i)^{-1}\prod_{i=1}^{2n}M(2s-2k+i,f,\mathrm{Ad},\chi^{i-1},\chi_Q)$$

is holomorphic at $s=2k+2n$ if $f_Q\neq f$. On the other hand, the term

$$\prod_{i=0}^{2n-1}L(2s-4k-i,\chi^i)^{-1}\prod_{i=1}^{2n}L(s-2k-2n+i,f,\mathrm{Ad},\chi^{i-1})L(s-2k-2n+i,\chi^{i-1})$$

has a simple pole at $s=2k+2n$ with the residue

$$\prod_{i=0}^{2n-1}L(4n-i,\chi^i)^{-1}\prod_{i=1}^{2n}L(i,f,\mathrm{Ad},\chi^{i-1})\prod_{i=2}^{2n}L(i,\chi^{i-1}).$$

Hence $R(s,I_m(f))$ has a simple at $s=2k+2n$ with the residue

$$D^{n(2k+2n)+n^2-n/2-1/2}2^{-2n+1}$$

$$\times\,\eta_m(f)\prod_{i=2}^{2n}\widetilde{\Lambda}(i,\chi^i)\prod_{i=0}^{2n-1}L(4n-i,\chi^i)^{-1}\prod_{i=1}^{2n}L(i,f,\mathrm{Ad},\chi^{i-1})\prod_{i=2}^{2n}L(i,\chi^{i-1}).$$

Thus the assertion can be proved by comparing the above result with Proposition 3.1.

(2) The assertion holds if $m=1$. In the case $m\geq 3$, the assertion can be proved by Theorem 6.2, (2) of Lemma 6.4, and Proposition 3.1 in the same manner as above.

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