PAPER

A Note on Harmonious Coloring of Caterpillars

Asahi TAKAOKA[†a)], ***Member***, **Shingo OKUMA**[†], ***Nonmember***, **Satoshi TAYU**[†b)], ***Member***, ***and*** **Shuichi UENO**[†c)], ***Fellow***

SUMMARY The harmonious coloring of an undirected simple graph is a vertex coloring such that adjacent vertices are assigned different colors and each pair of colors appears together on at most one edge. The harmonious chromatic number of a graph is the least number of colors used in such a coloring. The harmonious chromatic number of a path is known, whereas the problem to find the harmonious chromatic number is NP-hard even for trees with pathwidth at most 2. Hence, we consider the harmonious coloring of trees with pathwidth 1, which are also known as caterpillars. This paper shows the harmonious chromatic number of a caterpillar with at most one vertex of degree more than 2. We also show the upper bound of the harmonious chromatic number of a 3-regular caterpillar.

key words: *caterpillars, Eulerian trail, harmonious coloring, harmonious chromatic number, pathwidth*

1. Introduction

A *proper coloring* of an undirected simple graph G is an assignment of colors (or numbers) to the vertices of G such that adjacent vertices are assigned different colors. A *harmonious coloring* of a graph is a proper vertex coloring such that each pair of colors appears together on at most one edge. The *harmonious chromatic number* $h(G)$ of a graph G is the least number of colors used in such a coloring of G. The *harmonious coloring problem* is to find $h(G)$ of a graph G.

The harmonious coloring [13], [17], [18] was developed from the closely related concept of *line-distinguishing coloring* [12], [14], and has been studied in the literature (e.g. [10], [16] for surveys and [1], [2], [8] for recent results). The harmonious coloring has potential applications to minimal perfect hash functions [9] and aviation guidance systems [16]. The harmonious coloring problem is very difficult in general, and it is known to be NP-hard for several restricted classes of graphs [3]–[5], [7], [11], [14], [15].

The harmonious chromatic number of a path is known [14], [17], whereas the problem is NP-hard even for trees with pathwidth at most 2 [11]. Hence, we consider the harmonious coloring of trees with pathwidth 1, which are also known as *caterpillars*. A caterpillar is a tree that has a central path such that every vertex of the tree is on the path or adjacent to a vertex on the path.

This paper shows the harmonious chromatic number of a caterpillar with at most one vertex of degree more than 2. The class of such caterpillars can be partitioned into four classes, the class of paths, stars, shooting stars, and comets. For each class, we show the harmonious chromatic number of a caterpillar in the class. In addition, we show the upper bound of the harmonious chromatic number of a 3-regular caterpillar.

The paper is organized as follows. Section 2 describes our main results. Sections 3 to 6 are devoted to the proofs. We conclude with some discussions and remarks in Sect. 7.

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[†]The authors are with the Department of Communications and Computer Engineering, Tokyo Institute of Technology, Tokyo, 152–8550 Japan.
a) E-mail: asahi@eda.ce.titech.ac.jp
b) E-mail: tayu@eda.ce.titech.ac.jp
c) E-mail: ueno@eda.ce.titech.ac.jp

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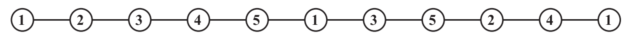


Fig. 1 A harmonious coloring of path P_{11} with 5 colors.

2. Harmonious Coloring of Caterpillars

In the rest of the paper, let $V(G)$ and $E(G)$ denote the set of vertices and edges of a graph G, respectively. Also, let $n = |V(G)|$ and $m = |E(G)|$. If a graph G can be harmoniously colored with k colors, then $m \le \binom{k}{2}$. Let $k(G)$ be the smallest integer k fulfilling this inequality. We can express $k(G)$ as a function of m, namely $k(G) = \left\lceil (1 + \sqrt{8m+1})/2 \right\rceil$. Since every graph can be colored harmoniously with n colors, we observe that for any graph G, $k(G) \le h(G) \le n$.

2.1 Paths and Stars

Let $P_t = (v_1, v_2, \dots, v_t)$ be a path on t vertices. The harmonious coloring of a path with k colors is obtained from a trail traversing edges of a complete graph K_k [14], [17]. The vertices on the path can be colored harmoniously according to the trail.

Theorem A.

$$h(P_t) = \begin{cases} k & \textit{if } k \textit{ is odd,} \\ k & \textit{if } k \textit{ is even and } m \le \binom{k}{2} - \frac{k}{2} + 1, \\ k+1 & \textit{otherwise,} \end{cases}$$

where $k = k(P_t)$. □

For example, path P_{11} in Fig. 1 can be colored harmoniously with 5 colors according to the Eulerian trail of K_5.

A complete bipartite graph $K_{1,\Delta}$ ($\Delta \ge 1$) is called a *star*. The following theorem shows the harmonious chromatic number of a star [10], [16]. The theorem is trivial since the harmonious coloring of stars has to assign a unique color to each vertex.

Theorem B. $h(K_{1,\Delta}) = \Delta + 1$. □

2.2 Shooting Stars and Comets

Let $P_t = (v_1, v_2, \ldots, v_t)$ be a path on t vertices. A *shooting star* is a caterpillar obtained from a path P_t ($t \geq 4$) and a star $K_{1,\Delta}$ ($\Delta \geq 1$) by identifying a vertex in $\{v_2, v_{t-1}\}$ of P_t with the degree-Δ vertex of $K_{1,\Delta}$. A *comet* is a caterpillar obtained from P_t ($t \geq 5$) and $K_{1,\Delta}$ ($\Delta \geq 1$) by identifying a vertex in $\{v_3, v_4, \ldots, v_{t-2}\}$ of P_t with the degree-Δ vertex of $K_{1,\Delta}$. In other words, a shooting star is a caterpillar obtained from $K_{1,\Delta}$ ($\Delta \geq 3$) by replacing an edge with a path of length at least 2, and a comet is a caterpillar obtained from $K_{1,\Delta}$ ($\Delta \geq 3$) by replacing two edges with paths of length at least 2. We denote by $S_{t,\Delta}$ ($\Delta \geq 3$) the shooting star with the longest path on t vertices and the maximum degree Δ, and we denote by $C_{t,\Delta}$ ($\Delta \geq 3$) a comet with the longest path on t vertices and the maximum degree Δ. Notice that a shooting star is uniquely determined by t and Δ, whereas a comet is not. Examples of shooting star and comet are shown in Figs. 2 (a) and 2 (b).

We can see that the class of caterpillars with at most one vertex of degree more than 2 can be partitioned into four classes, the class of paths, stars, shooting stars, and comets. The harmonious chromatic number of a path and a star is shown in Theorems A and B, respectively. In the following, we show the harmonious chromatic number of a shooting star and a comet, which we prove in Sects. 4 and 5, respectively.

The harmonious chromatic number of a shooting star is shown by Marszakowska (as cited in [16, Theorem 7.10]). We give here a simpler version of the theorem.

Theorem 1.

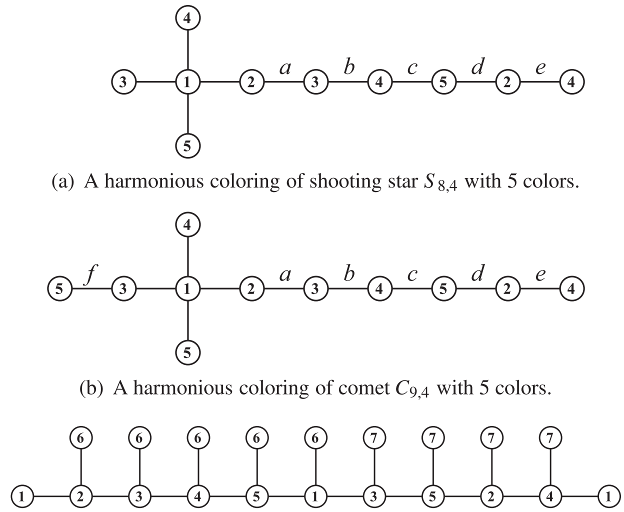
$$h(S_{t,\Delta}) = \begin{cases} \Delta + 1 & \text{if } \Delta \geq h, \\ h & \text{if } \Delta < h, \text{ and} \\ & \quad h \text{ is odd and } m \leq \binom{h}{2} - \lfloor \frac{\Delta-1}{2} \rfloor, \text{ or} \\ & \quad h \text{ is even and } m \leq \binom{h}{2} - \lfloor \frac{h-\Delta}{2} \rfloor, \\ h + 1 & \text{otherwise.} \end{cases}$$

where $h = h(P_t)$. □

We also show the harmonious chromatic number of a comet. Although the comet $C_{t,\Delta}$ is not uniquely determined by t and Δ, we can express $h(C_{t,\Delta})$ only with t and Δ.

Theorem 2.

$$h(C_{t,\Delta}) = \begin{cases} \Delta + 1 & \text{if } \Delta \geq h, \\ h & \text{if } \Delta < h, \text{ and} \\ & \quad h \text{ is odd and } m \leq \binom{h}{2} - \lfloor \frac{\Delta-3}{2} \rfloor, \text{ or} \\ & \quad h \text{ is even and } m \leq \binom{h}{2} - \lfloor \frac{h-\Delta}{2} \rfloor, \\ h + 1 & \text{otherwise.} \end{cases}$$



(a) A harmonious coloring of shooting star $S_{8,4}$ with 5 colors.

(b) A harmonious coloring of comet $C_{9,4}$ with 5 colors.

(c) A harmonious coloring of 3-regular caterpillar T_{11} with 7 colors.

Fig. 2 Examples of shooting star, comet, and 3-regular caterpillar.

where $h = h(P_t)$. □

The difference between the harmonious chromatic number of shooting star $S_{t,\Delta}$ and comet $C_{t,\Delta}$ arises only when $\Delta < h(P_t)$ and $h(P_t)$ is odd. For example, comet $C_{9,4}$ can be colored harmoniously with $h(P_9) = 5$ colors as shown in Fig. 2 (b), since $|E(C_{9,4})| \leq \binom{5}{2} - \lfloor \frac{4-3}{2} \rfloor = 10$. On the other hand, shooting star $S_{9,4}$ cannot be colored harmoniously with 5 colors, since $|E(S_{9,4})| > \binom{5}{2} - \lfloor \frac{4-1}{2} \rfloor = 9$. We note that $S_{8,4}$ can be colored harmoniously with 5 colors as shown in Fig. 2 (a).

2.3 Three-Regular Caterpillars

A caterpillar is said to be *3-regular* if every inner vertex (vertex with degree more than 1) has degree 3. In other words, a 3-regular caterpillar is obtained from a path by adding a degree-1 vertex to each inner vertex. An example of 3-regular caterpillar is shown in Fig. 2 (c). Let T_t be a 3-regular caterpillar with the longest path on t vertices, and let P_t be the longest path of T_t. We have the upper bound of the harmonious chromatic number of a 3-regular caterpillar, which we prove in Sect. 6.

Theorem 3. $h(T_t) \leq h + \lfloor (h-1)/2 \rfloor$, *where* $h = h(P_t)$. □

3. Preliminaries

Let K_h be a complete graph with $V(K_h) = \{v_1, v_2, \ldots, v_h\}$. We assume that each v_i has color i. We consider the graph K_h^Δ ($3 \leq \Delta < h$) obtained from K_h by deleting edges $(v_1, v_2), (v_1, v_3), \ldots, (v_1, v_{\Delta+1})$. See Fig. 3 for example. Trivially, we have $|E(K_h^\Delta)| = \binom{h}{2} - \Delta$. We define two subsets of $V(K_h^\Delta)$ as follows:

$$A(K_h^\Delta) = \{v_2, v_3, \ldots, v_{\Delta+1}\};$$

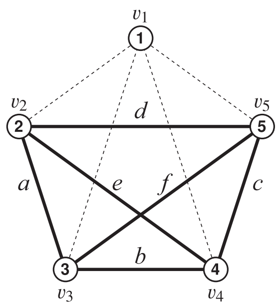


Fig. 3 Graph K_5^4. Dotted lines denote the edges deleted from K_5.

$$B(K_h^\Delta) = \{v_{\Delta+2}, v_{\Delta+3}, \ldots, v_h\}.$$

If K_h^Δ is clear from the context, we denote them by A and B instead of $A(K_h^\Delta)$ and $B(K_h^\Delta)$, respectively. Notice that sometimes $B = \emptyset$ as K_5^4 in Fig. 3.

A vertex with odd degree is called an *odd vertex*, and a vertex with even degree is called an *even vertex*. We have that $|A| = \Delta$ and $|B| = h-\Delta-1$. The degree of v_1 is $h-\Delta-1$, the degree of vertices in A is $h-2$, and the degree of vertices in B is $h-1$. Hence, we have the following:

- If h is odd and Δ is odd, then the degree of v_1 is odd, the degree of vertices in A is odd, and the degree of vertices in B is even, and hence, K_h^Δ has $\Delta+1$ odd vertices;
- If h is odd and Δ is even, then the degree of v_1 is even, the degree of vertices in A is odd, and the degree of vertices in B is even, and hence, K_h^Δ has Δ odd vertices;
- If h is even and Δ is odd, then the degree of v_1 is even, the degree of vertices in A is even, and the degree of vertices in B is odd, and hence, K_h^Δ has $h-\Delta-1$ odd vertices;
- If h is even and Δ is even, then the degree of v_1 is odd, the degree of vertices in A is even, and the degree of vertices in B is odd, and hence, K_h^Δ has $h-\Delta$ odd vertices.

A trail of a graph is called *Eulerian* if it traverses every edge of the graph. The following is a well-known fact. The proof can be found in standard textbooks on graph theory [6].

Theorem C. *A connected graph G has an Eulerian trail if and only if G has zero or two odd vertices. Moreover, if G has two odd vertices, all Eulerian trails start at one of them and end at the other. If G has no odd vertices, all Eulerian trails are closed.* □

4. Proof of Theorem 1

We prove Theorem 1 by a series of lemmas. Let P_t be the longest path of $S_{t,\Delta}$, and let $h = h(P_t)$. The following indicates the first case of Theorem 1.

Lemma 4. *If $\Delta \geq h$, then $h(S_{t,\Delta}) = \Delta + 1$.*

Proof. Since we have from Theorem B that $h(S_{t,\Delta}) \geq \Delta+1$, it suffices to show that $S_{t,\Delta}$ can be colored harmoniously with $\Delta+1$ colors if $\Delta \geq h$. First, we color the vertices on P_t harmoniously with colors $1, 2, \ldots, h$. Next, we recolor the degree-Δ vertex with color $\Delta+1$ (Recall that $\Delta+1 > h$). Finally, we color the remaining vertices, that is, the vertices adjacent to the degree-Δ vertex, with colors $1, 2, \ldots, \Delta$. It is straightforward to see that $S_{t,\Delta}$ is colored harmoniously. □

Similarly, we have the following.

Lemma 5. *If $\Delta < h$, then $h(S_{t,\Delta}) \leq h+1$.*

Proof. We show that $S_{t,\Delta}$ can be colored harmoniously with $h+1$ colors if $\Delta < h$. First, we color the vertices on P_t harmoniously with colors $1, 2, \ldots, h$. Next, we recolor the degree-Δ vertex with color $h+1$. Finally, we color the remaining vertices with colors $1, 2, \ldots, \Delta$ (Recall that $\Delta < h+1$). It is straightforward to see that $S_{t,\Delta}$ is colored harmoniously. □

Since P_t is a subgraph of $S_{t,\Delta}$, we have the following.

Lemma 6. $h(S_{t,\Delta}) \geq h$. □

The following lemma completes the proof of Theorem 1. The rest of the section is devoted to the proof of the lemma.

Lemma 7. *Suppose that $\Delta < h$. $S_{t,\Delta}$ can be colored harmoniously with h colors if and only if*

- $m \leq \binom{h}{2} - \frac{\Delta-1}{2}$ *when h is odd and Δ is odd,*
- $m \leq \binom{h}{2} - \frac{\Delta-2}{2}$ *when h is odd and Δ is even,*
- $m \leq \binom{h}{2} - \frac{h-\Delta-1}{2}$ *when h is even and Δ is odd, and*
- $m \leq \binom{h}{2} - \frac{h-\Delta}{2}$ *when h is even and Δ is even.*

Proof. We assume w.l.o.g. that color 1 is assigned to the degree-Δ vertex of $S_{t,\Delta}$, and colors $2, 3, \ldots, \Delta+1$ are assigned to the vertices adjacent to it (Recall that $\Delta+1 \leq h$). See Fig. 2 (a) for example. Then, $t-3$ vertices remains uncolored, and they induce the path P_{t-3} of $S_{t,\Delta}$. To prove Lemma 7, it suffices to show that the remaining path can be colored harmoniously with h colors if and only if the number of edges of $S_{t,\Delta}$ satisfies the inequality in the lemma. We assume without loss of generality that the end-vertex of the path is adjacent to the vertex with color 2 (See Fig. 2 (a)).

We use the graph K_h^Δ defined in Sect. 3. Recall that $V(K_h^\Delta) = \{v_1, v_2, \ldots, v_h\}$ and that each v_i has color i. Notice that the edges of K_h^Δ denote the pair of colors which we can use for the harmonious coloring of the remaining path, and the non-edges of K_h^Δ denote the pair of colors used for the harmonious coloring of the star $K_{1,\Delta}$ in $S_{t,\Delta}$. Coloring the remaining path of $S_{t,\Delta}$ harmoniously is equivalent to obtaining a trail of K_h^Δ with length $t-3$ starting at v_2. For example, K_5^4 in Fig. 3 has trail $(v_2, a, v_3, b, v_4, c, v_5, d, v_2, e, v_4)$ of length 5. As shown in Fig. 2 (a), the remaining path of $S_{8,4}$ can be colored harmoniously according to the trail. Then, the following claim proves Lemma 7 (Recall that $|E(K_h^\Delta)| = \binom{h}{2} - \Delta$).

Claim 8. *K_h^Δ has a trail starting at v_2 if and only if the length of the trail is at most*

- $\binom{h}{2} - \Delta - \frac{\Delta-1}{2}$ *when h is odd and Δ is odd,*
- $\binom{h}{2} - \Delta - \frac{\Delta-2}{2}$ *when h is odd and Δ is even,*
- $\binom{h}{2} - \Delta - \frac{h-\Delta-1}{2}$ *when h is even and Δ is odd, and*
- $\binom{h}{2} - \Delta - \frac{h-\Delta}{2}$ *when h is even and Δ is even.*

Proof of the "if" part of Claim 8. We distinguish four cases with respect to the parity of h and Δ.

Case 1 h is odd and Δ is odd: Let G_h^Δ be the subgraph of K_h^Δ obtained by deleting edges $(v_3, v_4), (v_5, v_6), \ldots, (v_\Delta, v_{\Delta+1})$. Since every vertex in $V(G_h^\Delta) \setminus \{v_1, v_2\}$ has even degree, G_h^Δ has an Eulerian trail from v_2 to v_1. Hence, K_h^Δ has the trail of length $\binom{h}{2} - \Delta - \frac{\Delta-1}{2}$.

Case 2 h is odd and Δ is even: Let G_h^Δ be the subgraph of K_h^Δ obtained by deleting edges $(v_4, v_5), (v_6, v_7), \ldots, (v_\Delta, v_{\Delta+1})$. Since every vertex in $V(G_h^\Delta) \setminus \{v_2, v_3\}$ has even degree, G_h^Δ has an Eulerian trail from v_2 to v_3. Hence, K_h^Δ has the trail of length $\binom{h}{2} - \Delta - \frac{\Delta-2}{2}$.

Case 3 h is even and Δ is odd: Let G_h^Δ be the subgraph of K_h^Δ obtained by deleting edges $(v_{\Delta+2}, v_{\Delta+3}), (v_{\Delta+4}, v_{\Delta+5}), \ldots, (v_{h-1}, v_h)$. Since every vertex of G_h^Δ has even degree, G_h^Δ has an Eulerian closed trail. Hence, K_h^Δ has the trail of length $\binom{h}{2} - \Delta - \frac{h-\Delta-1}{2}$.

Case 4 h is even and Δ is even: Let G_h^Δ be the subgraph of K_h^Δ obtained by deleting edges $(v_{\Delta+2}, v_{\Delta+3}), (v_{\Delta+4}, v_{\Delta+5}), \ldots, (v_{h-2}, v_{h-1})$, and (v_h, v_1) (Recall that the degree of v_1 is odd, and hence, at least 1). Since every vertex of G_h^Δ has even degree, G_h^Δ has an Eulerian closed trail. Hence, K_h^Δ has the trail of length $\binom{h}{2} - \Delta - \frac{h-\Delta}{2}$.

This completes the proof of the "if" part of Claim 8. □

Proof of the "only-if" part of Claim 8. We distinguish two cases.

Case 1 h is odd: If Δ is odd [resp., even], K_h^Δ has $\Delta + 1$ [resp., Δ] odd vertices. Since v_2 is an odd vertex, we have from Theorem C that v_2 and one more vertex can be remained odd vertices in the subgraph of K_h^Δ that has an Eulerian trail starting at v_2, and all the other odd vertices must be made into even vertices. Hence, at least $\frac{(\Delta+1)-2}{2}$ [resp., $\frac{\Delta-2}{2}$] edges must be deleted from K_h^Δ to obtain the subgraph, and it has at most $\binom{h}{2} - \Delta - \frac{\Delta-1}{2}$ [resp., $\binom{h}{2} - \Delta - \frac{\Delta-2}{2}$] edges.

Case 2 h is even: If Δ is odd [resp., even], K_h^Δ has $h - \Delta - 1$ [resp., $h - \Delta$] odd vertices. Since v_2 is even vertex, we have from Theorem C that all the odd vertices must be made into even vertices in the subgraph of K_h^Δ that has an Eulerian trail starting at v_2. Hence, at least $\frac{h-\Delta-1}{2}$ [resp., $\frac{h-\Delta}{2}$] edges must be deleted from K_h^Δ to obtain the subgraph, and it has at most $\binom{h}{2} - \Delta - \frac{h-\Delta-1}{2}$ [resp., $\binom{h}{2} - \Delta - \frac{h-\Delta}{2}$] edges.

This completes the proof of the "only-if" part of Claim 8. □

Now, we complete the proof of Lemma 7. □

5. Proof of Theorem 2

Let P_t be the longest path of $C_{t,\Delta}$, and let $h = h(P_t)$. We can prove the following by arguments similar to Lemmas 4, 5, and 6.

Lemma 9.

- *If $\Delta \geq h$, then $h(C_{t,\Delta}) = \Delta + 1$.*
- *If $\Delta < h$, then $h(C_{t,\Delta}) \leq h + 1$.*
- *$h(C_{t,\Delta}) \geq h$.* □

The following lemma completes the proof of Theorem 2. The rest of the section is devoted to the proof of the lemma.

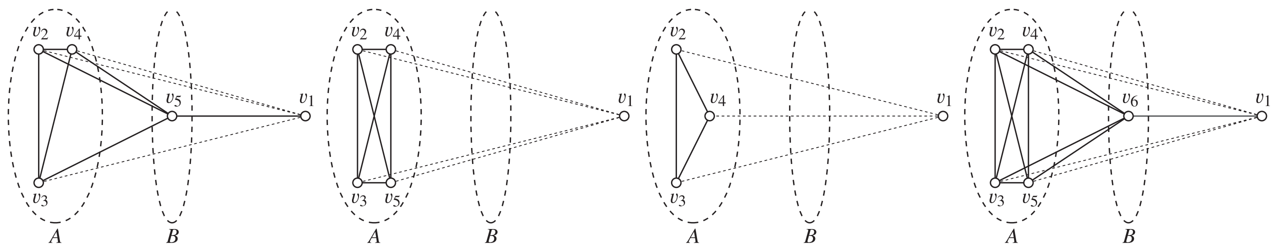
Lemma 10. *Suppose that $\Delta < h$. $C_{t,\Delta}$ can be colored harmoniously with h colors if and only if*

- $m \leq \binom{h}{2} - \frac{\Delta-3}{2}$ *when h is odd and Δ is odd,*
- $m \leq \binom{h}{2} - \frac{\Delta-4}{2}$ *when h is odd and Δ is even,*
- $m \leq \binom{h}{2} - \frac{h-\Delta-1}{2}$ *when h is even and Δ is odd, and*
- $m \leq \binom{h}{2} - \frac{h-\Delta}{2}$ *when h is even and Δ is even.*

Proof. As in the proof of Lemma 7, we assume w.l.o.g. that color 1 is assigned to the degree-Δ vertex, and colors $2, 3, \ldots, \Delta + 1$ are assigned to the vertices adjacent to it. See Fig. 2 (b) for example. Then, $t - 3$ vertices remains uncolored, and they induce two paths of $C_{t,\Delta}$ (Note that one vertex can be regarded as a path). To prove Lemma 10, it suffices to show that the remaining paths can be colored harmoniously with h colors if and only if the number of edges of $C_{t,\Delta}$ satisfies the inequality in the lemma. We assume w.l.o.g. that the end-vertex of the path is adjacent to the vertex with color 2, and the end-vertex of the other path is adjacent to the vertex with color 3 (See Fig. 2 (b)).

We use the graph K_h^Δ defined in Sect. 3. Recall that $V(K_h^\Delta) = \{v_1, v_2, \ldots, v_h\}$ and that each v_i has color i. Notice that the edges of K_h^Δ denote the pair of colors which we can use for the harmonious coloring of the remaining paths, and the non-edges of K_h^Δ denote the pair of colors used for the harmonious coloring of the star $K_{1,\Delta}$ in $C_{t,\Delta}$. Coloring the remaining paths of $C_{t,\Delta}$ harmoniously is equivalent to obtaining a pair of edge-disjoint trails W_1 and W_2 in K_h^Δ such that W_1 starts at v_2, W_2 starts at v_3, and the sum of the length of W_1 and W_2 is $t - 3$ (Notice that W_1 and W_2 must have length at least 1; otherwise the graph is not a comet but a shooting star). For example, K_5^4 in Fig. 3 has a pair of edge-disjoint trails $(v_2, a, v_3, b, v_4, c, v_5, d, v_2, e, v_4)$ and (v_3, f, v_5), the sum of the length of which is 6. As shown in Fig. 2 (b), the remaining paths of $C_{9,4}$ can be colored harmoniously according to the pair of trails. Then, the following claim proves Lemma 10 (Recall that $|E(K_h^\Delta)| = \binom{h}{2} - \Delta$).

Claim 11. *K_h^Δ has a pair of edge-disjoint trails W_1 and W_2*



(a) Graph K_5^3. (b) Graph K_5^4. (c) Graph K_4^3. (d) Graph K_6^4.

Fig. 4 Graphs K_5^3, K_5^4, K_4^3, and K_6^4 with tripartition (A, B, v_1). Dotted lines denote the edges deleted from the complete graphs. Edge (v_6, v_1) of K_6^4 is not used in the trails.

such that

- *W_1 starts at v_2 and has length l_1, and*
- *W_2 starts at v_3 and has length l_2,*

if and only if l_1 and l_2 are any positive integers such that $l_1 + l_2$ is at most

- $\binom{h}{2} - \Delta - \frac{\Delta-3}{2}$ *when h is odd and Δ is odd,*
- $\binom{h}{2} - \Delta - \frac{\Delta-4}{2}$ *when h is odd and Δ is even,*
- $\binom{h}{2} - \Delta - \frac{h-\Delta-1}{2}$ *when h is even and Δ is odd, and*
- $\binom{h}{2} - \Delta - \frac{h-\Delta}{2}$ *when h is even and Δ is even.*

Proof of the "if" part of Claim 11. We employ the arguments similar to those used in [13]. Let $K_{2,r}$ ($r \geq 2$) be the complete bipartite graph with bipartition (U, V) such that $|U| = 2$ and $|V| = r$. Let G_r be the graph obtained from $K_{2,r}$ by adding an edge between the vertices in U. A graph G is said to be *partitioned* into a pair of trails W_1 and W_2 if $E(G) = E(W_1) \cup E(W_2)$ and $E(W_1) \cap E(W_2) = \emptyset$.

(i). *Let r be a positive even integer, and let r_1 and r_2 be non-negative even integers with $r_1 + r_2 = r$. $K_{2,r}$ can be partitioned into two closed trails W_1 and W_2 such that $|E(W_1)| = 2r_1$ and $|E(W_2)| = 2r_2$ (Notice that $|E(W_1)|$ or $|E(W_2)|$ can be 0).*

Proof of (i). Let V_1 be a set of r_1 vertices in V, and let V_2 be the set of other r_2 vertices in V. We have from Theorem C that the subgraph K_{2,r_1} of $K_{2,r}$ induced by $U \cup V_1$ has an Eulerian closed trail, since every vertex of K_{2,r_1} has even degree. Similarly, the subgraph K_{2,r_2} of $K_{2,r}$ induced by $U \cup V_2$ has an Eulerian closed trail. Since any edge of $K_{2,r}$ is in either K_{2,r_1} or K_{2,r_2}, we have (i). □

(ii). *Let r be a positive odd integer, and let r_1 and r_2 be non-negative integers with $r_1 + r_2 = r$. We assume that r_1 is odd and r_2 is even. G_r can be partitioned into two closed trails W_1 and W_2 such that $|E(W_1)| = 2r_1 + 1$ and $|E(W_2)| = 2r_2$ (Notice that $|E(W_1)| \geq 3$ while $|E(W_2)|$ can be 0).*

Proof of (ii). Let V_1 be a set of r_1 vertices in V, and let V_2 be the set of other r_2 vertices in V. We have from Theorem C that the subgraph G_{r_1} of G_r induced by $U \cup V_1$ has an Eulerian closed trail, since every vertex of G_{r_1} has even degree. Similarly, the subgraph K_{2,r_2} of G_r induced by the edges between U and V_2 has an Eulerian closed trail. Since any edge of G_r is in either G_{r_1} or K_{2,r_2}, we have (ii). □

Now, we are ready to prove the "if" part of Claim 11. Recall that $A(K_h^\Delta) = \{v_2, v_3, \ldots, v_{\Delta+1}\}$ and $B(K_h^\Delta) = \{v_{\Delta+2}, v_{\Delta+3}, \ldots, v_h\}$. We prove by induction on $|A(K_h^\Delta)|$ and $|B(K_h^\Delta)|$. Since $|A(K_h^\Delta)| = \Delta$ and $|B(K_h^\Delta)| = h - \Delta - 1$, we have the following:

$$|A(K_{h+2}^{\Delta+2})| = |A(K_h^\Delta)| + 2; \quad |B(K_{h+2}^{\Delta+2})| = |B(K_h^\Delta)|;$$
$$|A(K_{h+2}^{\Delta})| = |A(K_h^\Delta)|; \quad |B(K_{h+2}^{\Delta})| = |B(K_h^\Delta)| + 2.$$

We take the pair of trails from $K_{h+2}^{\Delta+2}$ and from K_{h+2}^{Δ} with the pair of trails in K_h^Δ. We distinguish two cases.

Case 1 h is odd: Suppose that Δ is odd [resp., even]. Recall that $3 \leq \Delta < h$. In the base case, it can be verified that K_5^3 [resp., K_5^4] has the pair of trails, one of which starts at v_2 and the other starts at v_3, such that the length of the trails are any positive integers whose sum is $\binom{5}{2} - 3 - \frac{3-3}{2} = 7$ [resp., $\binom{5}{2} - 4 - \frac{4-4}{2} = 6$]. See Fig. 4 (a) [resp., Fig. 4 (b)].

Assume by induction that for any positive integers l_1 and l_2 such that $l_1 + l_2 = \binom{h}{2} - \Delta - \frac{\Delta-3}{2}$ [resp., $\binom{h}{2} - \Delta - \frac{\Delta-4}{2}$], K_h^Δ has a pair of edge-disjoint trails W_1 and W_2 such that W_1 starts at v_2 and has length l_1, and W_2 starts at v_3 and has length l_2.

We first show that for any positive integers l_1 and l_2 such that $l_1 + l_2 = \binom{h+2}{2} - (\Delta+2) - \frac{(\Delta+2)-3}{2}$ [resp., $\binom{h+2}{2} - (\Delta + 2) - \frac{(\Delta+2)-4}{2}$], $K_{h+2}^{\Delta+2}$ has a pair of edge-disjoint trails W_1 and W_2 such that W_1 starts at v_2 and has length l_1, and W_2 starts at v_3 and has length l_2. The subgraph I of $K_{h+2}^{\Delta+2}$ induced by $\{v_1, v_2, \ldots, v_{\Delta+1}, v_{\Delta+4}, v_{\Delta+5}, \ldots, v_{h+2}\}$ is isomorphic to K_h^Δ. Since $v_{\Delta+2}$ and $v_{\Delta+3}$ are adjacent to every vertex in $V(I) \backslash \{v_1\}$, the subgraph $J = K_{h+2}^{\Delta+2} - E(I) - (v_{\Delta+2}, v_{\Delta+3}) - v_1$ is isomorphic to $K_{2,h-1}$. Since $h - 1$ is even, we have from (i) that J can be partitioned into two closed trails whose length are $2r_1$ and $2r_2$, respectively. Here, r_1 and r_2 are non-negative even integers such that $r_1 + r_2 = h - 1$, $2r_1 < l_1$, and $2r_2 < l_2$. We can assume w.l.o.g. that the closed trail of length $2r_1$ has v_2 and the closed trail of length $2r_2$ has v_3. Let $l_1' = l_1 - 2r_1$ and $l_2' = l_2 - 2r_2$. It is straightforward to see that $l_1' + l_2' = \binom{h}{2} - \Delta - \frac{\Delta-3}{2}$ [resp., $\binom{h}{2} - \Delta - \frac{\Delta-4}{2}$]. By assumption, I has a pair of edge-disjoint trails W_1' and W_2' such that W_1' starts at v_2 and has length l'_1, and W'_2 starts at v_3 and has length l'_2. We can obtain the two trails W_1 and W_2 of $K^{\Delta+2}_{h+2}$ as follows: W_1 starts at v_2, traverses $2r_1$ edges of the closed trail in J, and traverses l'_1 edges in W'_1; W_2 starts at v_3, traverses $2r_2$ edges of the closed trail in J, and traverses l'_2 edges in W'_2.

We next show that for any positive integers l_1 and l_2 such that $l_1 + l_2 = \binom{h+2}{2} - \Delta - \frac{\Delta-3}{2}$ [resp., $\binom{h+2}{2} - \Delta - \frac{\Delta-4}{2}$], K^{Δ}_{h+2} has a pair of edge-disjoint trails W_1 and W_2 such that W_1 starts at v_2 and has length l_1, and W_2 starts at v_3 and has length l_2. We assume w.l.o.g. that $l_1 \geq l_2$. Since $l_1 + l_2 \geq 8$ for any h and Δ, we have $l_1 \geq 4$. The subgraph I of K^{Δ}_{h+2} induced by $\{v_1, v_2, \dots, v_h\}$ is isomorphic to K^{Δ}_h. Since v_{h+1} and v_{h+2} are adjacent to every vertex in I, the subgraph $J = K^{\Delta}_{h+2} - E(I)$ is isomorphic to G_h. Since h is odd, we have from (ii) that J can be partitioned into two closed trails whose length are $2r_1 + 1$ and $2r_2$, respectively. Here, r_1 is a non-negative odd integer and r_2 is a non-negative even integer such that $r_1 + r_2 = h$, $2r_1 + 1 < l_1$, and $2r_2 < l_2$. We can assume w.l.o.g. that the closed trail of length $2r_1 + 1$ has v_2 and the closed trail of length $2r_2$ has v_3. Let $l'_1 = l_1 - 2r_1 - 1$ (recall that $2r_1 + 1 \geq 3$ but $l_1 \geq 4$) and $l'_2 = l_2 - 2r_2$. It is straightforward to see that $l'_1 + l'_2 = \binom{h}{2} - \Delta - \frac{\Delta-3}{2}$ [resp., $\binom{h}{2} - \Delta - \frac{\Delta-4}{2}$]. By assumption, I has a pair of edge-disjoint trails W'_1 and W'_2 such that W'_1 starts at v_2 and has length l'_1, and W'_2 starts at v_3 and has length l'_2. We can obtain the two trails W_1 and W_2 of K^{Δ}_{h+2} as follows: W_1 starts at v_2, traverses $2r_1 + 1$ edges of the closed trail in J, and traverses l'_1 edges in W'_1; W_2 starts at v_3, traverses $2r_2$ edges of the closed trail in J, and traverses l'_2 edges in W'_2.

Case 2 h is even: Suppose that Δ is odd [resp., even]. Recall that $3 \leq \Delta < h$. In the base case, it can be verified that K^3_4 [resp., K^4_6] has the pair of trails, one of which starts at v_2 and the other starts at v_3, such that the length of the trails are any positive integers whose sum is $\binom{4}{2} - 3 - \frac{4-3-1}{2} = 3$ [resp., $\binom{6}{2} - 4 - \frac{6-4}{2} = 10$]. See Fig. 4 (c) [resp., Fig. 4 (d)].

Assume by induction that for any positive integers l_1 and l_2 such that $l_1 + l_2 = \binom{h}{2} - \Delta - \frac{h-\Delta-1}{2}$ [resp., $\binom{h}{2} - \Delta - \frac{h-\Delta}{2}$], K^{Δ}_h has a pair of edge-disjoint trails W_1 and W_2 such that W_1 starts at v_2 and has length l_1, and W_2 starts at v_3 and has length l_2.

We first show that for any positive integers l_1 and l_2 such that $l_1 + l_2 = \binom{h+2}{2} - (\Delta + 2) - \frac{(h+2)-(\Delta+2)-1}{2}$ [resp., $\binom{h+2}{2} - (\Delta + 2) - \frac{(h+2)-(\Delta+2)}{2}$], $K^{\Delta+2}_{h+2}$ has a pair of edge-disjoint trails W_1 and W_2 such that W_1 starts at v_2 and has length l_1, and W_2 starts at v_3 and has length l_2. We assume w.l.o.g. that $l_1 \geq l_2$. Since $l_1 + l_2 \geq 8$ for any h and Δ, we have $l_1 \geq 4$. The subgraph I of $K^{\Delta+2}_{h+2}$ induced by $\{v_1, v_2, \dots, v_{\Delta+1}, v_{\Delta+4}, v_{\Delta+5}, \dots, v_{h+2}\}$ is isomorphic to K^{Δ}_h. Since $v_{\Delta+2}$ and $v_{\Delta+3}$ are adjacent to every vertex in $V(I)\backslash\{v_1\}$, the subgraph $J = K^{\Delta+2}_{h+2} - E(I) - v_1$ is isomorphic to G_{h-1}. Since $h - 1$ is odd, we have from (ii) that J can be partitioned into two closed trails whose length are $2r_1 + 1$ and $2r_2$, respectively. Here, r_1 is a non-negative odd integer and r_2 is a non-negative even integer such that $r_1 + r_2 = h - 1$, $2r_1 + 1 < l_1$, and $2r_2 < l_2$. We can assume w.l.o.g. that the closed trail of length $2r_1 + 1$ has v_2 and the closed trail of length $2r_2$ has v_3. Let $l'_1 = l_1 - 2r_1 - 1$ (recall that $2r_1 + 1 \geq 3$ but $l_1 \geq 4$) and $l'_2 = l_2 - 2r_2$. It is straightforward to see that $l'_1 + l'_2 = \binom{h}{2} - \Delta - \frac{h-\Delta-1}{2}$ [resp., $\binom{h}{2} - \Delta - \frac{h-\Delta}{2}$]. By assumption, I has a pair of edge-disjoint trails W'_1 and W'_2 such that W'_1 starts at v_2 and has length l'_1, and W'_2 starts at v_3 and has length l'_2. We can obtain the two trails W_1 and W_2 of $K^{\Delta+2}_{h+2}$ as follows: W_1 starts at v_2, traverses $2r_1 + 1$ edges of the closed trail in J, and traverses l'_1 edges in W'_1; W_2 starts at v_3, traverses $2r_2$ edges of the closed trail in J, and traverses l'_2 edges in W'_2.

We next show that for any positive integers l_1 and l_2 such that $l_1 + l_2 = \binom{h+2}{2} - \Delta - \frac{(h+2)-\Delta-1}{2}$ [resp., $\binom{h+2}{2} - \Delta - \frac{(h+2)-\Delta}{2}$], K^{Δ}_{h+2} has a pair of edge-disjoint trails W_1 and W_2 such that W_1 starts at v_2 and has length l_1, and W_2 starts at v_3 and has length l_2. The subgraph I of K^{Δ}_{h+2} induced by $\{v_1, v_2, \dots, v_h\}$ is isomorphic to K^{Δ}_h. Since v_{h+1} and v_{h+2} are adjacent to every vertex in I, the subgraph $J = K^{\Delta}_{h+2} - E(I) - (v_{h+1}, v_{h+2})$ is isomorphic to $K_{2,h}$. Since h is even, we have from (i) that J can be partitioned into two closed trails whose length are $2r_1$ and $2r_2$, respectively. Here, r_1 and r_2 are non-negative even integers such that $r_1 + r_2 = h$, $2r_1 < l_1$, and $2r_2 < l_2$. We can assume w.l.o.g. that the closed trail of length $2r_1$ has v_2 and the closed trail of length $2r_2$ has v_3. Let $l'_1 = l_1 - 2r_1$ and $l'_2 = l_2 - 2r_2$. It is straightforward to see that $l'_1 + l'_2 = \binom{h}{2} - \Delta - \frac{h-\Delta-1}{2}$ [resp., $\binom{h}{2} - \Delta - \frac{h-\Delta}{2}$]. By assumption, I has a pair of edge-disjoint trails W'_1 and W'_2 such that W'_1 starts at v_2 and has length l'_1, and W'_2 starts at v_3 and has length l'_2. We can obtain the two trails W_1 and W_2 of K^{Δ}_{h+2} as follows: W_1 starts at v_2, traverses $2r_1$ edges of the closed trail in J, and traverses l'_1 edges in W'_1; W_2 starts at v_3, traverses $2r_2$ edges of the closed trail in J, and traverses l'_2 edges in W'_2.

This completes the proof of the "if" part of Claim 11. □

Proof of the "only-if" part of Claim 11. Recall that a graph G is said to be *partitioned* into a pair of trails W_1 and W_2 if $E(G) = E(W_1) \cup E(W_2)$ and $E(W_1) \cap E(W_2) = \emptyset$. It is easy to see the following.

Claim 12. *If a graph G can be partitioned into a pair of trails, G has at most four odd vertices. Moreover, if the trails start at distinct even vertices, G has no odd vertices.* □

Let G^{Δ}_h be a subgraph of K^{Δ}_h that can be partitioned into the pair of trails, one of which starts at v_2 and the other starts at v_3. We distinguish two cases.

Case 1 h is odd: If Δ is odd [resp., even], K^{Δ}_h has $\Delta + 1$ [resp., Δ] odd vertices. Since v_2 and v_3 have odd degree, we have from Claim 12 that four odd vertices including v_2 and v_3 can be remained odd vertices in G^{Δ}_h, and all the other odd vertices must be made into even vertices. Hence, at least $\frac{(\Delta+1)-4}{2}$ [resp., $\frac{\Delta-4}{2}$] edges must be deleted from K^{Δ}_h to obtain G^{Δ}_h, and G^{Δ}_h has at most $\binom{h}{2} - \Delta - \frac{\Delta-3}{2}$ [resp., $\binom{h}{2} - \Delta - \frac{\Delta-4}{2}$]

edges.

Case 2 h is even: If Δ is odd [resp., even], K_h^Δ has $h-\Delta-1$ [resp., $h-\Delta$] odd vertices. Notice that $h-\Delta-1$ [resp., $h-\Delta$] is even. Since v_2 and v_3 have even degree, we have from Claim 12 that all the odd vertices must be made into even vertices in G_h^Δ. Hence, at least $\frac{h-\Delta-1}{2}$ [resp., $\frac{h-\Delta}{2}$] edges must be deleted from K_h^Δ to obtain G_h^Δ, and G_h^Δ has at most $\binom{h}{2}-\Delta-\frac{h-\Delta-1}{2}$ [resp., $\binom{h}{2}-\Delta-\frac{h-\Delta}{2}$] edges.

This completes the proof of the "only-if" part of Claim 11. □

Now, we complete the proof of Lemma 10. □

6. Proof of Theorem 3

Let T_t be a 3-regular caterpillar with the longest path on t vertices, let P_t be the longest path of T_t, and let $h = h(P_t)$. We show in this section that T_t can be colored harmoniously with $h + \lfloor (h-1)/2 \rfloor$ colors. We first color the vertices on P_t harmoniously with h colors, then color the remaining vertices with $\lfloor (h-1)/2 \rfloor$ colors. See Fig. 2 (c) for example of such a coloring of a 3-regular caterpillar.

Now, it suffices to show that in any harmonious coloring of P_t, at most $\lfloor (h-1)/2 \rfloor$ inner vertices have the same color. Suppose contrary that at least $\lfloor (h-1)/2 \rfloor + 1$ inner vertices have, w.l.o.g., color h. Let V_h be the set of such inner vertices. Let $N(V_h)$ be the set of vertices on P_t adjacent to a vertex in V_h. Since the distance between any pair of vertices in V_h is at least 3, we have $|N(V_h)| = 2|V_h|$. Since the vertices on P_t are colored harmoniously, the vertices in $N(V_h)$ must have different colors from colors $1, 2, \ldots, h-1$. However, since $(h-1)/2 - 1 < \lfloor (h-1)/2 \rfloor$, we have that $h-1 < 2(\lfloor (h-1)/2 \rfloor + 1) \le |N(V_h)|$, a contradiction. Hence, at most $\lfloor (h-1)/2 \rfloor$ inner vertices on P_t have the same color, and we have Theorem 3.

7. Concluding Remarks

This paper showed in Theorems 1 and 2 the harmonious chromatic number of a shooting star and a comet, respectively. We also showed in Theorem 3 the upper bound of the harmonious chromatic number of a 3-regular caterpillar.

Akbari et al. [2] show that for any forest T, $h(T) = \Delta(T)+1$ if $\Delta(T) \ge \frac{n+2}{3}$ and T has no pair of non-adjacent vertices of degree $\Delta(T)$. Here, $n = |V(T)|$ and $\Delta(T)$ is the maximum degree of T. They also show that the bound $\Delta(T) \ge \frac{n+2}{3}$ is sharp, that is, for any integer $d \ge 3$ they present a caterpillar T_d such that (i) $\Delta(T_d) = d$, (ii) T_d has no pair of non-adjacent vertices of degree d, (iii) $|V(T_d)| = 3d-1$, and (iv) $h(T_d) \ge d+2$. In the case of shooting stars and comets (that is, caterpillars with at most one vertex of degree more than 2), Theorems 1 and 2 state that $h(T') = \Delta(T') + 1$ if $\Delta(T') \ge h(P')$ for any such caterpillar T', where P' is the longest path of T'. The theorems also imply that the bound $\Delta(T') \ge h(P')$ is sharp, that is, for any integer $d \ge 3$ we have a caterpillar T'_d with at most one vertex of degree more than 2 such that (i) $\Delta(T'_d) = d$, (ii) $\Delta(T'_d) < h(P'_d)$, and (iii) $h(T'_d) \ge d+2$. The bound $\Delta(T') \ge h(P')$ is significantly smaller than the general bound $\Delta(T') \ge \frac{n+2}{3}$, since $h(P') = O(\sqrt{n})$ (Recall that $k(G) = \left\lceil (1+\sqrt{8m+1})/2 \right\rceil$ for any graph G with m edges).

We also note that Edwards [10] conjectures that for any tree T, $h(T) \le k(T) + \Delta(T)$. Theorems 1 and 2 imply that this conjecture holds for caterpillars with at most one vertex of degree more than 2. On the other hand, Theorem 3 does not imply the conjecture since 3-regular caterpillar T_{315} is a counter example, that is, $h(P_{315}) + \lfloor (h(P_{315}) - 1)/2 \rfloor > k(T_{315}) + 3$, where T_{315} is the 3-regular caterpillar with the longest path on 315 vertices, and P_{315} is the longest path of T_{315}. (Note that $k(T_{315}) = 36$ and $h(P_{315}) = 27$). Hence, it remains an open problem to ask whether $h(T) \le k(T)+3$ for any 3-regular caterpillar T.

We finally note that the complexity of the harmonious coloring problem for general caterpillars also remains open.

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Asahi Takaoka received his B.E. degree in computer science and M.E. and D.E. degree in communications and integrated systems from Tokyo Institute of Technology, Tokyo, Japan, in 2010, 2012, and 2015, respectively. He has been a research fellow of Japan Society for the Promotion of Science (JSPS) from April 2014. He is now a postdoctoral researcher at Tokyo Institute of Technology. He is interested in algorithmic graph theory with applications. He is a member of IEEE and IEICE.

Shingo Okuma received his B.E. degree in computer science and M.E. degree in communications and integrated systems from Tokyo Institute of Technology, Tokyo, Japan, in 2012 and 2014, respectively.

Satoshi Tayu received the B.E., M.E., and D.E. degrees in electrical and electronic engineering from Tokyo Institute of Technology, Tokyo, Japan, in 1992, 1994, and 1997, respectively. From 1997 to 2003, he was a research associate in the School of Information Science, Japan Advanced Institute of Science and Technology, Ishikawa, Japan. He is currently an assistant professor in the Department of Communications and Computer Engineering, Graduate School of Science and Engineering, Tokyo Institute of Technology. His research interests are in parallel computation. He is a member of IPSJ.

Shuichi Ueno received the B.E. degree in electronic engineering from Yamanashi University, Yamanashi, Japan, in 1976, and M.E. and D.E. degrees in electronic engineering from Tokyo Institute of Technology, Tokyo, Japan, in 1978 and 1982, respectively. Since 1982 he has been with Tokyo Institute of Technology, where he is now a professor in the Department of Communications and Computer Engineering, Graduate School of Science and Engineering. His research interests are in the theory of parallel and VLSI computation. He received the best paper award from the Institute of Electronics and Communication Engineers of Japan in 1986, the 30th anniversary best paper award from the Information Processing Society of Japan in 1990, and the best paper award of APCCAS 2000 from IEEE in 2000. Dr Ueno is a Fellow of IEICE, and a member of IEEE, SIAM, and IPSJ.