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PAPER

OBDD Representation of Intersection Graphs

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SUMMARY Ordered Binary Decision Diagrams (OBDDs for short) are popular dynamic data structures for Boolean functions. In some modern applications, we have to handle such huge graphs that the usual explicit representations by adjacency lists or adjacency matrices are infeasible. To deal with such huge graphs, OBDD-based graph representations and algorithms have been investigated. Although the size of OBDD representations may be large in general, it is known to be small for some special classes of graphs. In this paper, we show upper bounds and lower bounds of the size of OBDDs representing some intersection graphs such as bipartite permutation graphs, biconvex graphs, convex graphs, (2-directional) orthogonal ray graphs, and permutation graphs.
key words: *implicit representation of graphs, ordered binary decision diagrams, orthogonal ray graphs, permutation graphs*

1. Introduction

In some modern applications such as nano-circuit design and bioinformatics, we have to handle such huge graphs that the usual explicit representations by adjacency lists or adjacency matrices may exceed the memory limitations, and even polynomial time algorithms may be infeasible. To deal with such huge graphs, some *implicit* representations of graphs have been proposed [17], [28], [30]. *Ordered Binary Decision Diagrams* (*OBDDs* for short) [7], [31] are dynamic data structures used for representing and manipulating Boolean functions. Since the adjacency matrix of a graph can be considered as a Boolean function, a graph can be implicitly represented by an OBDD that represents the adjacency function. The OBDD representation of graphs has been considered as a promising implicit representation of graphs, since it realizes compact representations as well as efficient algorithms, based on existing OBDD operations (See [20] for survey). To realize OBDD-based graph algorithms that have good running times, the size of the OBDD representation of the input graph should be small, since the running time of one OBDD operation depends on the size of OBDD on which the operation is performed. For that reason, compact OBDD representations of graphs have been considered. Nunkesser and Woelfel [20] investigate the worst-case space complexities of the OBDD representations (OBDD sizes for short) of certain graphs as follows:

- The OBDD size of general graphs is $O(N^2/\log N)$, $O(M\log N)$, and $\Omega(N^2/\log N)$;
- The OBDD size of bipartite graphs is $O(N^2/\log N)$, $O(M\log N)$, and $\Omega(N^2/\log N)$;
- The OBDD sizes of cographs and its related graphs are $O(N\log N)$ and $\Omega(N/\log N)$;
- The OBDD size of interval graphs is $O(N^{3/2}/\log^{3/4} N)$ and $\Omega(N)$;
- The OBDD size of unit interval graphs is $O(N/\sqrt{\log N})$ and $\Omega(N/\log N)$,

where N and M are the number of vertices and edges of a graph, respectively. Recently, Gillé [10], [11] has been improved the results as follows:

- The OBDD size of interval graphs is $O(N\log N)$;
- The OBDD size of interval graphs is $\Omega(N\log N)$ if the variable ordering of OBDDs and the vertex labeling of graphs are fixed on some natural scheme;
- The OBDD size of unit interval graphs is $\Theta(N/\log N)$,

where variable ordering and vertex labeling are as defined in Sect. 3.

This paper considers the OBDD size of orthogonal ray graphs [24], which have been introduced in connection with the defect-tolerant design of nano-circuits [23], and other related graphs. We show the following:

- The OBDD sizes of (2-directional) orthogonal ray graphs, convex graphs, and permutation graphs are $O(N^{3/2}/\log^{3/4} N)$ and $\Omega(N)$;
- The OBDD sizes of biconvex graphs and bipartite permutation graphs are $\Theta(N/\log N)$.

The upper bounds are shown in Sect. 4, and the lower bounds are shown in Sect. 5.

It should be noted that the graphs above, except permutation graphs, are a special kind of bipartite graphs. The OBDD sizes of these graphs are substantially smaller than the size of general bipartite graphs.

2. Classes of Intersection Graphs

All graphs considered in this paper are finite, simple, and undirected. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively. Let $N = |V(G)|$. The *neighborhood* of a vertex v in G is the set $\Gamma_G(v) = \{u \in V(G) \mid (u,v) \in E(G)\}$.

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A bipartite graph G with a bipartition (U, W) is called a *grid intersection graph* [16] if there exist a set of disjoint horizontal line segments L_u, $u \in U$, in the xy-plane, and a set of disjoint vertical line segments L_w, $w \in W$, such that for any $u \in U$ and $w \in W$, $(u, w) \in E(G)$ if and only if L_u and L_w intersect. A grid intersection graph G is called a *unit grid intersection graph* [21] if every L_v, $v \in V(G)$, has the same (unit) length.

A bipartite graph G is called a *chordal bipartite graph* [13] if it contains no cycles of length at least 6 as an induced subgraph.

A bipartite graph G with a bipartition (U, W) is called an *orthogonal ray graph* [24] if there exist a set of disjoint horizontal rays (closed half-lines) R_u, $u \in U$, in the xy-plane, and a set of disjoint vertical rays R_w, $w \in W$, such that for any $u \in U$ and $w \in W$, $(u, w) \in E(G)$ if and only if R_u and R_w intersect. The set $\mathcal{R}(G) = \{R_v \mid v \in V(G)\}$ is called an *orthogonal ray representation* of G. An orthogonal ray graph G is called a *2-directional orthogonal ray graph* if every horizontal ray R_u, $u \in U$, has the same direction, and every vertical ray R_w, $w \in W$, has the same direction.

Let G be a bipartite graph with a bipartition (U, W). A *convex ordering* of U is a total ordering of the vertices in U such that for every vertex $w \in W$, the vertices in $\Gamma_G(w)$ occur consecutively in the ordering. A bipartite graph G is called a *convex graph* [12] if it has a convex ordering of U. A *biconvex ordering* of G is a pair of convex orderings of U and W. A bipartite graph G is called a *biconvex graph* [12] if it has a biconvex ordering.

A graph G with a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is called a *permutation graph* [8] if there exists a permutation π on $\{1, 2, \ldots, n\}$ such that for any $i, j \in \{1, 2, \ldots, n\}$, $(v_i, v_j) \in E(G)$ if and only if $(i - j)(\pi(i) - \pi(j)) < 0$. It is shown in [1] that a graph is a permutation graph if and only if there exists a set of points $p_v = (x_v, y_v)$, $v \in V(G)$, in the xy-plane such that for any $u, w \in V(G)$, $(u, w) \in E(G)$ if and only if $x_u < x_w$ and $y_u < y_w$. The set $\mathcal{P}(G) = \{p_v \mid v \in V(G)\}$ is called a *point representation* of G. To prove the upper bound of OBDD size of permutation graphs, we use point representations of graphs. A permutation graph is called a *bipartite permutation graph* [29] if it is bipartite.

The following relationships between bipartite graph classes have been known [24]: {Bipartite Permutation Graphs} ⊂ {Biconvex Graphs} ⊂ {Convex Graphs} ⊂ {2-Directional Orthogonal Ray Graphs} ⊂ {Chordal Bipartite Graphs}, and {2-Directional Orthogonal Ray Graphs} ⊂ {Orthogonal Ray Graphs} ⊂ {Unit Grid Intersection Graphs} ⊂ {Grid Intersection Graphs}, where $X \subset Y$ indicates a set X is a proper subset of Y.

Some comprehensive surveys with other results can be found in [6], [24], [28].

3. Representation of Graphs by OBDDs

3.1 OBDDs

Let $X_n = \{x_1, x_2, \ldots, x_n\}$ be a set of Boolean variables, and let B_n be the set of Boolean functions on X_n. A *variable ordering* π on X_n is a permutation of $\{1, 2, \ldots, n\}$, leading to the ordered list $(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ of the variables.

A *π-OBDD* on X_n is a single-rooted directed acyclic graph with two sinks such that each non-sink (inner) node has two outgoing edges. One of two sinks is labeled by Boolean constant 0, and the other sink is labeled by 1. One of two outgoing edges of each inner node is labeled by 0, and the other edge is labeled by 1. Each inner node is labeled by a Boolean variable from X_n such that the edges between inner nodes respect the variable ordering π, that is, if an edge leads from an inner node labeled by x_i (x_i-node for short) to an x_j-node, then $\pi^{-1}(i) < \pi^{-1}(j)$.

A π-OBDD is said to *represent* a Boolean function $f \in B_n$ if for any binary string $\boldsymbol{a} = (a_{n-1}, a_{n-2}, \ldots, a_0) \in \{0, 1\}^n$, the path starting at the root and leading from any x_i-node over the edge labeled by the value of a_{n-i}, ends at a sink with label $f(\boldsymbol{a})$.

The *size* of a π-OBDD is the number of its nodes. The *minimal π-OBDD* for a Boolean function $f \in B_n$ is the minimal-size π-OBDD representing f, and the *π-OBDD size* of f is the size of the minimal π-OBDD for f. It is known that the minimal π-OBDD for f is unique up to isomorphism, and it can be found in almost linear time [31]. Also, minimal π-OBDDs are known to be characterized by the following theorem. A function f is said to *essentially depends* on a variable x if $f_{|x=0} \neq f_{|x=1}$, where $f_{|x=0}$ and $f_{|x=1}$ are the subfunctions of f obtained by replacing x with constants 0 and 1, respectively.

Theorem A (Sieling and Wegener [25])**.** *The number of $x_{\pi(i)}$-nodes in the minimal π-OBDD for $f \in B_n$ is equal to the number of different subfunctions of f that are obtained by replacing each variable $x_{\pi(j)}$, $j < i$, with a Boolean constant, and that essentially depend on $x_{\pi(i)}$.* □

The *OBDD size* of a Boolean function f is the minimal π-OBDD size of f over all variable orderings π. It is known to be NP-hard to compute an optimal variable ordering that minimizes the OBDD size of f [5].

3.2 OBDD Representation of Graphs

For a binary string $\boldsymbol{a} = (a_{n-1}, a_{n-2}, \ldots, a_0) \in \{0, 1\}^n$, let $|\boldsymbol{a}|$ denote the natural number represented by $\boldsymbol{a}$, that is, $|\boldsymbol{a}| = \sum_{i=0}^{n-1} a_i \cdot 2^i$. Conversely, for a natural number $l \in \mathbb{N}$, let $[l]_n$, $n \geq \lceil \log(l + 1) \rceil$, denote the n-bit binary string representing l, that is, $|[l]_n| = l$.

Let $\mathcal{G}$ be a class of graphs, and

$$\mathcal{G}_N = \{G \in \mathcal{G} \mid |V(G)| = N\}.$$

For a graph $G \in \mathcal{G}_N$, a bijection

$$\lambda : V(G) \to \{[0]_n, [1]_n, \ldots, [N - 1]_n\}$$

is called a *vertex labeling* of G, where $n \geq \lceil \log N \rceil$. For a vertex v of G, a binary string $\lambda(v)$ is called the *label* of v. For a binary string $\boldsymbol{a} \in \{0, 1\}^n$, a vertex $\lambda^{-1}(\boldsymbol{a})$ is called the vertex *labeled* by $\boldsymbol{a}$. We refer to a pair (G, λ) as a *labeled graph*.

A labeled graph (G, λ) can be represented by its *characteristic function* $\chi_{G,\lambda} \in B_{2n}$, where for any $\boldsymbol{a}, \boldsymbol{b} \in \{0,1\}^n$, $\chi_{G,\lambda}(\boldsymbol{a}, \boldsymbol{b}) = 1$ if and only if $(\lambda^{-1}(\boldsymbol{a}), \lambda^{-1}(\boldsymbol{b})) \in E(G)$. If (G, λ) is clear from the context, we will omit the index. A vertex subset $S \subseteq V(G)$ of (G, λ) can be represented by its characteristic function $\chi_S \in B_n$, where for any $\boldsymbol{a} \in \{0,1\}^n$, $\chi_S(\boldsymbol{a}) = 1$ if and only if $\lambda^{-1}(\boldsymbol{a}) \in S$.

The *π-OBDD for a labeled graph* (G, λ) is the minimal π-OBDD for $\chi_{G,\lambda}$. Since the minimal π-OBDD for any Boolean function is unique up to isomorphism [31] as mentioned in the previous section, the minimal π-OBDD for (G, λ) is unique up to isomorphism, and we denote it by π-OBDD(G, λ). The *π-OBDD size of* (G, λ) is the size of π-OBDD(G, λ), and we denote it by $|\pi\text{-OBDD}(G, \lambda)|$. The *$\pi$-OBDD size* of G is the minimal of $|\pi\text{-OBDD}(G, \lambda)|$ over all vertex labelings λ of G. The *OBDD size* of G is the minimal of the π-OBDD size of G over all variable orderings π on X_{2n}. The *OBDD size of a graph class $\mathcal{G}_N$* is the maximal OBDD size of G over all graphs in $\mathcal{G}_N$, that is,

$$\max_{G \in \mathcal{G}_N} \min_{\pi \in \Pi, \lambda \in \Lambda} \{|\pi\text{-OBDD}(G, \lambda)|\}, \qquad (1)$$

where Π is the set of all variable orderings on X_{2n}, and Λ is the set of all vertex labelings of G.

3.3 Length of Labels

In the previous works, Nunkesser and Woelfel [20], and Gillé [10], [11] have used a $\lceil \log N \rceil$-bit label for each vertex, which is the minimal number of bits for encoding every vertices. As discussed in [4], [20], it is known to be important to keep the number of bits as low as possible by the following reason. Since the running time of one OBDD operation depends on the size of OBDD on which the operation is performed, the size of intermediate OBDDs during the computation of an OBDD-based algorithm should be small. Since the size of intermediate OBDDs may get larger, the worst-case OBDD size should also be small. Since the worst-case OBDD size is exponentially larger in the number of bits [18], it is desirable to keep the number of bits as low as possible. Thus, we use binary strings of length $\lceil \log N \rceil$ for vertex labels, and we assume that $n = \lceil \log N \rceil$ in the rest of this paper.

It should be noticed that increasing the length of labels may be valid for compact OBDD representations. Indeed, Meer and Rautenbach [19] investigate compact OBDD representations of graphs with bounded tree- and clique-width by using a label of length more than $\lceil \log N \rceil$ for each vertex. They show that the OBDD size of cographs can be improved from $O(N \log N)$ [20] to $O(N)$, if we use vertex labels of length $c \cdot \lceil \log N \rceil$ for some constant $c > 1$. However, the running time of an OBDD-based algorithm may be larger for longer vertex labels, since the worst-case OBDD size is exponentially larger in the number of bits.

3.4 Variable Orderings

To prove upper bounds mentioned in the introduction, Nunkesser and Woelfel [20], and Gillé [10], [11] have used the *interleaved variable ordering* σ on $\{a_{n-1}, a_{n-2}, \ldots, a_0, b_{n-1}, b_{n-2}, \ldots, b_0\}$, leading to the ordered list

$$(a_{n-1}, b_{n-1}, a_{n-2}, b_{n-2}, \ldots, a_0, b_0)$$

of variables. We also use the same variable ordering in this paper, since it can achieve compact OBDD representations compared with some other variable orderings.

Let ρ be the *naive ordering*, leading to the ordered list $(a_{n-1}, a_{n-2}, \ldots, a_0, b_{n-1}, b_{n-2}, \ldots, b_0)$ of variables. This ordering is natural, and has an advantage on an OBDD operation as follows. Computing the neighborhood $\Gamma_G(v)$ of a vertex v of a graph G is one of the basic operations. In terms of OBDD representations, it is to compute the minimal π-OBDD for the characteristic function of $\Gamma_G(v)$ from the π-OBDD for a labeled graph (G, λ). Notice that the characteristic function of $\Gamma_G(v)$ is obtained from $\chi_{G,\lambda}(\boldsymbol{a}, \boldsymbol{b})$ by replacing each variable a_i with a constant c_i for any i, $0 \leq i < n$, where $\boldsymbol{c} = \lambda(v)$. Since it is known that replacement by constants can be done in $O(|\pi\text{-OBDD}(G, \lambda)|)$ time [7], [31], we have the following.

Theorem 1. *For any labeled graph (G, λ) and variable ordering π, the OBDD for the characteristic function of the neighborhood of a vertex can be obtained in $O(|\pi\text{-OBDD}(G, \lambda)|)$ time from a π-OBDD(G, λ).* □

On the other hand, replacing a_i with c_i for any i, $0 \leq i < n$, can be done in $O(n)$ time for ρ-OBDD, since we only have to traverse from the root through at most n nodes labeled by a_i for some i, $0 \leq i < n$. Thus, we have the following.

Theorem 2. *For any labeled graph (G, λ), the OBDD for the characteristic function of the neighborhood of a vertex can be obtained in $O(\log N)$ time from a ρ-OBDD(G, λ).* □

We can see from Theorems 1 and 2 that the OBDD for the characteristic function of the neighborhood of a vertex can be obtained from a ρ-OBDD exponentially faster than from a σ-OBDD, if the size of the σ-OBDD is $\Omega(N)$.

Although the naive ordering has the advantage, we can see that the representations of most of graphs by ρ-OBDDs require $\Omega(N)$ size.

Theorem 3. *If G is a graph whose vertices have different neighborhoods, then the ρ-OBDD size of G is $\Omega(N)$.*

Proof. Let λ be a vertex labeling of G and let $\chi_{G,\lambda} \in B_{2n}$ be the characteristic function of labeled graph (G, λ). For any $\boldsymbol{c} = (c_{n-1}, c_{n-2}, \ldots, c_0) \in \{0,1\}^n$ with $|\boldsymbol{c}| < N$, all the subfunctions of $\chi_{G,\lambda}(\boldsymbol{a}, \boldsymbol{b})$ obtained by replacing each variable a_i with a constant c_i for any i, $1 \leq i < n$, is different and essentially depend on $a_0 = x_{\rho(n)}$, for otherwise there exists at least one pair of vertices having the same characteristic function of the neighborhood, contradicting the assumption that every vertices have different neighborhoods. It follows that the number of such subfunctions is at least $\lfloor N/2 \rfloor$. Since the number of a_0-nodes in the ρ-OBDD for $\chi_{G,\lambda}$ is equal to the number of such subfunctions by Theorem A, the size of the ρ-OBDD is at least $\lfloor N/2 \rfloor$, and we have the theorem. □

The theorem implies that the ρ-OBDD size of even a path or a complete graph is $\Omega(N)$, while for some graphs, the σ-OBDD size can be $o(N)$ as shown in [10], [11], [20] and this paper. This is the reason why we use the interleaved variable ordering.

4. Upper Bounds of OBDD Sizes

We show in this section upper bounds for the OBDD size of some special kinds of graphs. We give a vertex labeling $\lambda_{\mathcal{G}}$ for each graph class $\mathcal{G}$, and show the upper bound of $|\sigma\text{-OBDD}(G, \lambda_{\mathcal{G}})|$ for any graph $G \in \mathcal{G}_N$.

4.1 Preliminaries

Before showing the vertex labeling of graphs, we describe the scheme we use to estimate the size of σ-OBDD. We follow the arguments used in [20]. We assume in the rest of the paper that the indices of the bits of a binary string $\boldsymbol{c} \in \{0,1\}^k$ are $k-1, k-2, \ldots, 0$, that is, $\boldsymbol{c} = (c_{k-1}, c_{k-2}, \ldots, c_0)$.

Let λ be a vertex labeling of a graph G and let $\chi \in B_{2n}$ be the characteristic function of the labeled graph (G, λ). Recall that we use the interleaved variable ordering σ leading to the ordered list $(a_{n-1}, b_{n-1}, a_{n-2}, b_{n-2}, \ldots, a_0, b_0)$ of variables. Let $\chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}} \in B_{2(n-k)}$ be the subfunction of χ such that

$$\begin{aligned}&\chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}}(a_{n-k-1}, b_{n-k-1}, \ldots, a_0, b_0)\\&\quad = \chi(\alpha_{k-1}, \beta_{k-1}, \ldots, \alpha_0, \beta_0, a_{n-k-1}, b_{n-k-1}, \ldots, a_0, b_0),\end{aligned}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0,1\}^k$, $0 \le k \le n$. A Boolean function $f \in B_k$ is called 0 if $f(\boldsymbol{c}) = 0$ for every $\boldsymbol{c} \in \{0,1\}^k$, called 1 if $f(\boldsymbol{c}) = 1$ for every $\boldsymbol{c} \in \{0,1\}^k$, and called *non-constant*, otherwise.

We define for any k, $0 \le k < n$, that $s_{G,\lambda,k}$ is the number of non-constant different subfunctions $\chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ that do not necessarily depend on a_{n-k-1}. We can see from Theorem A that the number of a_{n-k-1}-nodes in σ-OBDD(G, λ) is equal to the number of different subfunctions $\chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ that essentially depend on a_{n-k-1}. It follows that the number of a_{n-k-1}-nodes is bounded above by $s_{G,\lambda,k}$.

Similarly, we define for any k, $0 \le k < n$, that $t_{G,\lambda,k}$ is the number of non-constant different subfunctions of χ obtained by replacing variables $a_{n-1}, b_{n-1}, a_{n-2}, b_{n-2}, \ldots, a_{n-k}, b_{n-k}, a_{n-k-1}$ with Boolean constants. We can see from Theorem A that the number of b_{n-k-1}-nodes in σ-OBDD(G, λ) is bounded above by $t_{G,\lambda,k}$.

Since $t_{G,\lambda,k} \le 2 s_{G,\lambda,k}$ for any G, λ, and k by definition, we have

$$\begin{aligned}|\sigma\text{-OBDD}(G, \lambda)| &\le \sum_{k=0}^{n-1} (s_{G,\lambda,k} + t_{G,\lambda,k}) + 2\\ &\le 3 \sum_{k=0}^{n-1} s_{G,\lambda,k} + 2. \qquad (2)\end{aligned}$$

We will estimate the size of σ-OBDD(G, λ) by using upper bounds of $s_{G,\lambda,k}$.

The following is the first upper bound of $s_{G,\lambda,k}$. The upper bound is derived from the fact that there exist 2^{2^m} Boolean functions on m variables.

Lemma B (Nunkesser and Woelfel [20])**.** *$s_{G,\lambda,k} \le 2^{2^{2(n-k)}}$ for any G, λ and k.* □

The upper bound in Lemma B will be used when k is large. We show in the following sections other upper bounds of $s_{G,\lambda,k}$, which will be used when k is small. The upper bounds are derived by structures of graphs. We need some definitions and notations.

We define for any k, $0 \le k \le n$, that

$$V(\boldsymbol{\alpha}) = \{v \in V(G) \mid \boldsymbol{\alpha} \in \{0,1\}^k \text{ is a prefix of } \lambda(v)\}.$$

We further define for any k, $0 \le k \le n$, that

$$S(G, \lambda, k) = \left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \;\middle|\; \begin{array}{l} \boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0,1\}^k, |\boldsymbol{\alpha}| < |\boldsymbol{\beta}|, \\ \chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}} \text{ is non-constant} \end{array} \right\}.$$

Notice that

$$s_{G,\lambda,k} \le 2|S(G, \lambda, k)| + 2^k, \qquad (3)$$

since there exist at most 2^k pairs of $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ such that $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$.

The *index of most significant different bit* of two binary strings $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0,1\}^k$ is the index i such that $\alpha_i \ne \beta_i$ and $\alpha_j = \beta_j$ for any $j > i$. For a binary string $\boldsymbol{\alpha} \in \{0,1\}^k$, let $\boldsymbol{\alpha}^e$ be the substring of $\boldsymbol{\alpha}$ consisting of bits with even indices, and let $\boldsymbol{\alpha}^o$ be the substring of $\boldsymbol{\alpha}$ consisting of bits with odd indices.

4.2 Biconvex Graphs and Bipartite Permutation Graphs

Let G be a biconvex graph with a bipartition (U, W) and a biconvex ordering $(u_0, u_1, \ldots, u_{p-1})$ and $(w_0, w_1, \ldots, w_{q-1})$. Let λ_1 be a vertex labeling of G such that $\lambda_1(u_i) = [i]_n$ for each $u_i \in U$, $0 \le i < p$, and $\lambda_1(w_j) = [p+j]_n$ for each $w_j \in W$, $0 \le j < q$. See Fig. 1 for example. We refer to λ_1 as a *biconvex labeling*.

We use some kind of adjacency matrix to show upper bounds of $s_{G,\lambda_1,k}$. Let M be a $2^n \times 2^n$ (0,1)-matrix such that for any $\boldsymbol{a}, \boldsymbol{b} \in \{0,1\}^n$, $M(|\boldsymbol{a}|, |\boldsymbol{b}|) = 1$ if and only if $(\lambda^{-1}(\boldsymbol{a}), \lambda^{-1}(\boldsymbol{b})) \in E(G)$. Figure 2 shows such matrix M_1 of the biconvex graph G_1 in Fig. 1. Notice that the submatrix induced by the rows $0, 1, \ldots, N-1$ and the columns $0, 1, \ldots, N-1$ is an adjacency matrix of G, and every other elements are 0. Since the rows and columns of M are sorted according to the biconvex ordering of G, 1-elements form two rectilinear polygons in M [32], which we call *1-polygons*. The polygons have two structural properties:

(P1) Each row and column of a 1-polygon contains consecutive 1-elements;

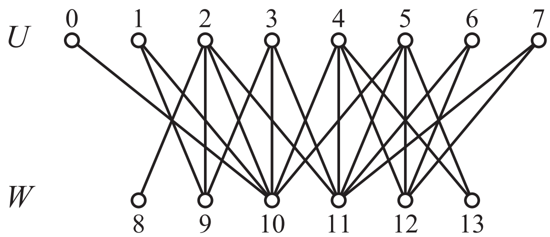

Fig. 1 A biconvex graph G_1 with a bipartition (U, W) such that $|U| = 8$ and $|W| = 6$. Each vertex v of the graph is labeled by $|\lambda_1(v)|$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13		
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
2	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
3	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0
4	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0
5	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0
6	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
8	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
10	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
11	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0
12	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0
13	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Fig. 2 The matrix M_1 of the biconvex graph G_1 in Fig. 1. The submatrix induced by the rows $0, 1, \ldots, 13$ and the columns $0, 1, \ldots, 13$ is an adjacency matrix of G_1. Gray parts denote the 1-polygons in M_1. The matrix is divided into 16 submatrices, each of which is $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ for some $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0, 1\}^2$.

(P2) The right and left boundary of a 1-polygon consists of two parts: one is non-decreasing from top to bottom and the other is non-increasing.

Gray parts in Fig. 2 denote the 1-polygons in M_1.

For any binary string $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0, 1\}^k$, $0 \leq k \leq n$, let $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ be the $2^{n-k} \times 2^{n-k}$ submatrix of M induced by rows $|\boldsymbol{\alpha}| \cdot 2^{n-k}, |\boldsymbol{\alpha}| \cdot 2^{n-k} + 1, \ldots, |\boldsymbol{\alpha}| \cdot 2^{n-k} + 2^{n-k} - 1$ and columns $|\boldsymbol{\beta}| \cdot 2^{n-k}, |\boldsymbol{\beta}| \cdot 2^{n-k} + 1, \ldots, |\boldsymbol{\beta}| \cdot 2^{n-k} + 2^{n-k} - 1$. See Fig. 2 for example. Each submatrix is $2^{n-k} \times 2^{n-k}$ matrix, and M has 2^{2k} such submatrices. Notice that the rows and columns of $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ correspond to $V(\boldsymbol{\alpha})$ and $V(\boldsymbol{\beta})$, respectively. Thus, each $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ represents the subfunction $\chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ such that for any i and j, $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}(i, j) = 1$ if and only if $\chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}}([i]_{n-k}, [j]_{n-k}) = 1$. A submatrix $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ is called *non-constant* if it contains both a 0-element and a 1-element. Notice that for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0, 1\}^k$, $\chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ is non-constant if and only if $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ is non-constant. Notice also that $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ is non-constant if and only if the boundary of 1-polygons, that is, the boundary between 0-elements and 1-elements, intersects $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$.

We have the following two upper bounds of $s_{G,\lambda_1,k}$. First upper bound will be used when k is small, and derived from the fact that $s_{G,\lambda_1,k}$ is bounded above by the number of non-constant subfunctions of χ.

Lemma 4. *$s_{G,\lambda_1,k} \leq 4 \cdot 2^k$ for any biconvex graph G and k, $0 \leq k < n$.*

Proof. Recall that $s_{G,\lambda_1,k}$ is bounded above by the number of non-constant subfunctions by definition. Recall also that for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0, 1\}^k$, the subfunction $\chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ is non-constant if and only if the boundary of 1-polygons intersects the submatrix $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$. By (P1) and (P2), the number of submatrices intersecting the boundary of 1-polygons is at most the number of submatrices bordering on the perimeter of M, that is, $4 \cdot 2^k$. Thus, we have the lemma. □

For biconvex graphs, we can obtain the following upper bound, which is substantially smaller than that in Lemma B. The second upper bound will be used when k is large, and derived from the fact that $s_{G,\lambda_1,k}$ is the number of non-constant different subfunctions of χ.

Lemma 5. *$s_{G,\lambda_1,k} = 2^{O(2^{n-k})}$ for any biconvex graph G and k, $0 \leq k < n$.*

Proof. Recall that $s_{G,\lambda_1,k}$ is the number of non-constant different subfunctions $\chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$, and thus, the number of non-constant different submatrices $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$. An *$n \times n$ grid $G_{n \times n}$* is the graph with

$$V(G_{n\times n}) = \{v_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq n\} \text{ and}$$
$$E(G_{n\times n}) = \{(v_{ij}, v_{i'j'}) \mid |i - i'| + |j - j'| = 1\}.$$

Recall that $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ is non-constant if and only if the boundary of 1-polygons intersects $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$. Notice that a part of boundary intersecting $M_{|\boldsymbol{\alpha},\boldsymbol{\beta}}$ corresponds to a walk in the $(2^{n-k}+1)\times(2^{n-k}+1)$ grid whose start point on the perimeter of the grid. Notice also that by (P1) and (P2), the length of such walk is less than the perimeter of the submatrix, that is, $4 \cdot 2^{n-k}$. Let l_k be the number of walks of length 2^{n-k+2} in the $(2^{n-k}+1)\times(2^{n-k}+1)$ grid whose start point on the perimeter of the grid. Since the $2^{n-k} \times 2^{n-k}$ matrix is uniquely determined by the boundary between 0-elements and 1-elements, and the side in which 1-elements lie, the number of different $2^{n-k} \times 2^{n-k}$ matrix is bounded above by $2l_k$. Since the maximum degree of a grid is 4, we have $l_k \leq 2^{n-k+2} \cdot 4^{2^{n-k+2}}$ (Consider random walks of length 2^{n-k+2} starting at a point on the perimeter). Thus, $s_{G,\lambda_1,k} \leq 2l_k \leq 2 \cdot 2^{n-k+2} \cdot 4^{2^{n-k+2}}$, and we have the lemma. □

Now, we have the following.

Theorem 6. *The OBDD size of biconvex graphs with N vertices is $O(N/\log N)$.*

Proof. We have from Lemmas 4 and 5 that $s_{G,\lambda_1,k} \leq 4 \cdot 2^k$ and $s_{G,\lambda_1,k} = 2^{O(2^{n-k})}$ for any biconvex graph G and k, $0 \leq k < n$. We have from (2) that

$$\begin{aligned}
|\sigma\text{-OBDD}(G, \lambda)| &\leq 3 \sum_{k=0}^{n-1} s_{G,\lambda_1,k} + 2 \\
&\leq 3 \sum_{k=0}^{n-\lfloor \log n \rfloor} 4 \cdot 2^k + 3 \sum_{k=n-\lfloor \log n \rfloor+1}^{n-1} 2^{O(2^{n-k})} + 2 \\
&\leq 12 \sum_{k=0}^{n-\lfloor \log n \rfloor} 2^k + 3 \sum_{k=1}^{\lfloor \log n \rfloor - 1} 2^{O(2^k)} + 2
\end{aligned}$$

$= O(N/\log N).$

Thus, we have the theorem. □

Since the class of biconvex graphs contains the class of bipartite permutation graphs, we have the following.

Corollary 7. *The OBDD size of bipartite permutation graphs with N vertices is $O(N/\log N)$.* □

4.3 Permutation Graphs

Let G be a permutation graph with a point representation $\mathcal{P}(G) = \{(x_v, y_v) \mid v \in V(G)\}$. We can assume without loss of generality that the x-coordinates are distinct and the y-coordinates are distinct [1].

Let λ_2 be a vertex labeling of G such that for any $\boldsymbol{a}$, $\boldsymbol{b} \in \{0,1\}^n$,

- $a_i < b_i$ implies $x_{\lambda_2^{-1}(\boldsymbol{a})} < x_{\lambda_2^{-1}(\boldsymbol{b})}$ if i is even, and
- $a_i < b_i$ implies $y_{\lambda_2^{-1}(\boldsymbol{a})} < y_{\lambda_2^{-1}(\boldsymbol{b})}$ if i is odd,

where i is the index of most significant different bit of $\boldsymbol{a}$ and $\boldsymbol{b}$, that is, $a_i \neq b_i$ and $a_j = b_j$ for any $j > i$. We refer to λ_2 as a *point partitioning labeling* of permutation graphs.

Lemma 8. *For any permutation graph G, there exists a point partitioning labeling λ_2 of G.*

Proof. The point partitioning labeling of a permutation graph G can be obtained by the following *point partitioning algorithm*, which assigns labels to the points in $\mathcal{P}(G)$ and corresponding vertices of G. We assume that $n-1$ is even. When $n-1$ is odd, the vertex labeling can be obtained by a similar way.

In the first step, we sort the points in $\mathcal{P}(G)$ in ascending order of the x-coordinates. Let P_0 be the set of first 2^{n-1} points in such ordering of $\mathcal{P}(G)$, and P_1 be the set of other points in $\mathcal{P}(G)$. We define that $(n-1)$-th bits of the labels of the points in P_0 are 0, and those in P_1 are 1.

In the second step, we sort the points in P_0 (resp., P_1) in ascending order of the y-coordinates. Let P_{00} (resp., P_{10}) be the set of first 2^{n-2} points in such ordering of P_0 (resp., P_1), and P_{01} (resp., P_{11}) be the set of other points in P_0 (resp., P_1). We define that $(n-2)$-th bits of the labels of the points in P_{00} and P_{10} are 0, and those in P_{01} and P_{11} are 1.

Similarly in the i-th step ($i \leq n$), for each $\boldsymbol{\alpha} \in \{0,1\}^{i-1}$, let $P_{\boldsymbol{\alpha}}$ be the set of points such that $\boldsymbol{\alpha}$ is the prefix of the labels of the points. We sort the points in $P_{\boldsymbol{\alpha}}$ in ascending order of the x-coordinates if $n-i$ is even, and in ascending order of the y-coordinates if $n-i$ is odd. We define that $(n-i)$-th bits of the labels of the first 2^{n-i} points in such ordering of $P_{\boldsymbol{\alpha}}$ are 0, and those of other points are 1.

We can see that the vertex labeling obtained from the point partitioning algorithm is indeed a point partitioning labeling of G, and we have the theorem. □

Recall that

$V(\boldsymbol{\alpha}) = \{v \in V(G) \mid \boldsymbol{\alpha} \in \{0,1\}^k$ is a prefix of $\lambda_2(v)\}$ and

$$S(G,\lambda_2,k) = \left\{ (\boldsymbol{\alpha},\boldsymbol{\beta}) \;\middle|\; \begin{array}{l} \boldsymbol{\alpha},\boldsymbol{\beta} \in \{0,1\}^k, |\boldsymbol{\alpha}| < |\boldsymbol{\beta}|, \\ \chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}} \text{ is non-constant} \end{array} \right\}.$$

where χ is the characteristic function of labeled graph (G,λ_2). To prove the upper bound of $s_{G,\lambda_2,k}$, we define as follows:

$$S_k^e = \left\{ (\boldsymbol{\alpha},\boldsymbol{\beta}) \in S(G,\lambda_2,k) \;\middle|\; \begin{array}{l} \text{The index of most sig-} \\ \text{-nificant different bit} \\ \text{of } \boldsymbol{\alpha} \text{ and } \boldsymbol{\beta} \text{ is even} \end{array} \right\};$$

$$S_k^o = \left\{ (\boldsymbol{\alpha},\boldsymbol{\beta}) \in S(G,\lambda_2,k) \;\middle|\; \begin{array}{l} \text{The index of most sig-} \\ \text{-nificant different bit} \\ \text{of } \boldsymbol{\alpha} \text{ and } \boldsymbol{\beta} \text{ is odd} \end{array} \right\}.$$

Notice that $S(G,\lambda_2,k) = S_k^e \cup S_k^o$. Recall that for a binary string $\boldsymbol{\alpha} \in \{0,1\}^k$, $\boldsymbol{\alpha}^e$ and $\boldsymbol{\alpha}^o$ are the substrings of $\boldsymbol{\alpha}$ consisting of bits with even indices and odd indices, respectively. We have the following for S_k^e.

Claim 9. *For every $(\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i)$ and $(\boldsymbol{\alpha}_j,\boldsymbol{\beta}_j)$ in S_k^e,*

$$(|\boldsymbol{\alpha}_i^e| = |\boldsymbol{\alpha}_j^e|) \wedge (|\boldsymbol{\beta}_i^e| = |\boldsymbol{\beta}_j^e|) \wedge (|\boldsymbol{\alpha}_i^o| < |\boldsymbol{\alpha}_j^o|) \Rightarrow |\boldsymbol{\beta}_i^o| \leq |\boldsymbol{\beta}_j^o|.$$

Proof. Recall that $\mathcal{P}(G) = \{(x_v, y_v) \mid v \in V(G)\}$ is the point representation of the permutation graph G, and for any $u, w \in V(G)$, $(u,w) \in E(G)$ if and only if $x_u < x_w$ and $y_u < y_w$. $(\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i) \in S_k^e$ together with $|\boldsymbol{\alpha}_i| < |\boldsymbol{\beta}_i|$ implies that $x_u < x_w$ for any vertex $u \in V(\boldsymbol{\alpha}_i)$ and $w \in V(\boldsymbol{\beta}_i)$, since the index of most significant different bit of $\lambda(u)$ and $\lambda(w)$ is even. There exists a pair of vertices $u' \in V(\boldsymbol{\alpha}_i)$ and $w' \in V(\boldsymbol{\beta}_i)$ such that $y_{u'} > y_{w'}$, for otherwise $x_u < x_w$ and $y_u < y_w$ for any $u \in V(\boldsymbol{\alpha}_i)$ and $w \in V(\boldsymbol{\beta}_i)$, which implies $\chi_{|\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i} = 1$, contradicting $(\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i) \in S(G,\lambda_2,k)$. $|\boldsymbol{\alpha}_i^o| < |\boldsymbol{\alpha}_j^o|$ together with $|\boldsymbol{\alpha}_i^e| = |\boldsymbol{\alpha}_j^e|$ implies that $y_{u'} < y_{u''}$ for any vertex $u'' \in V(\boldsymbol{\alpha}_j)$, since the index of most significant different bit of $\lambda(u')$ and $\lambda(u'')$ is odd.

Suppose contrary that $|\boldsymbol{\beta}_i^o| > |\boldsymbol{\beta}_j^o|$. Then $|\boldsymbol{\beta}_i^o| > |\boldsymbol{\beta}_j^o|$ together with $|\boldsymbol{\beta}_i^e| = |\boldsymbol{\beta}_j^e|$ implies that $y_{w'} > y_{w''}$ for any vertex $w'' \in V(\boldsymbol{\beta}_j)$. Since $y_{u''} > y_{u'} > y_{w'} > y_{w''}$ and $x_{u''} < x_{w''}$ for any $u'' \in V(\boldsymbol{\alpha}_j)$ and $w'' \in V(\boldsymbol{\beta}_j)$, we conclude that $\chi_{|\boldsymbol{\alpha}_j,\boldsymbol{\beta}_j} = 0$, contradicting $(\boldsymbol{\alpha}_j,\boldsymbol{\beta}_j) \in S(G,\lambda_2,k)$. Thus, we have $|\boldsymbol{\beta}_i^o| \leq |\boldsymbol{\beta}_j^o|$. □

Similarly, we have the following for S_k^o.

Claim 10. *For every $(\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i)$ and $(\boldsymbol{\alpha}_j,\boldsymbol{\beta}_j)$ in S_k^o,*

$$(|\boldsymbol{\alpha}_i^o| = |\boldsymbol{\alpha}_j^o|) \wedge (|\boldsymbol{\beta}_i^o| = |\boldsymbol{\beta}_j^o|) \wedge (|\boldsymbol{\alpha}_i^e| < |\boldsymbol{\alpha}_j^e|) \Rightarrow |\boldsymbol{\beta}_i^e| \leq |\boldsymbol{\beta}_j^e|.$$

Proof. The proof is similar to that of Claim 9, and is omitted. □

The following shows an upper bound for the number of pairs of binary strings satisfying the conditions in Claims 9 and 10.

Claim 11. *Let $(\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1), (\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2), \ldots, (\boldsymbol{\alpha}_p,\boldsymbol{\beta}_p)$ be a sequence of pairs of k-bit binary strings such that $|\boldsymbol{\alpha}_i| < |\boldsymbol{\alpha}_j|$ implies $|\boldsymbol{\beta}_i| \leq |\boldsymbol{\beta}_j|$ for every i and j, $1 \leq i, j \leq p$. Then, $p \leq 2 \cdot 2^k$.*

Proof. We assume without loss of generality that $(\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$, $(\boldsymbol{\alpha}_2, \boldsymbol{\beta}_2), \ldots, (\boldsymbol{\alpha}_p, \boldsymbol{\beta}_p)$ are ordered lexicographically. Then, $|\boldsymbol{\alpha}_i| \leq |\boldsymbol{\alpha}_j|$ and $|\boldsymbol{\beta}_i| \leq |\boldsymbol{\beta}_j|$ for every i and j such that $1 \leq i < j \leq p$. Thus, we have $p \leq (|\boldsymbol{\alpha}_p| + 1) + (|\boldsymbol{\beta}_p| + 1) \leq 2 \cdot 2^k$. □

The following is obtained from Claims 9, 10, and 11.

Lemma 12. *$s_{G,\lambda_2,k} \leq 65 \cdot 2^{3k/2}$ for any permutation graph G and k, $0 \leq k < n$.*

Proof. We have from Claims 9 and 11 that there exist at most $2 \cdot 2^{\lceil k/2 \rceil}$ 4-tuples in set $\{(\boldsymbol{\alpha}^e, \boldsymbol{\beta}^e, \boldsymbol{\alpha}^o, \boldsymbol{\beta}^o) \mid (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S^e_k\}$ for each pair of $\boldsymbol{\alpha}^e$ and $\boldsymbol{\beta}^e$, since the length of $\boldsymbol{\alpha}^e, \boldsymbol{\beta}^e, \boldsymbol{\alpha}^o$, and $\boldsymbol{\beta}^o$ is at most $\lceil k/2 \rceil$, respectively. Thus,

$$|S^e_k| \leq 2^{\lceil k/2 \rceil} \cdot 2^{\lceil k/2 \rceil} \cdot 2 \cdot 2^{\lceil k/2 \rceil} \leq 16 \cdot 2^{3k/2}.$$

Similarly, we have from Claims 10 and 11 that

$$|S^o_k| \leq 2^{\lceil k/2 \rceil} \cdot 2^{\lceil k/2 \rceil} \cdot 2 \cdot 2^{\lceil k/2 \rceil} \leq 16 \cdot 2^{3k/2}.$$

Since $S(G, \lambda_2, k) = S^e_k \cup S^o_k$, we have that

$$|S(G, \lambda_2, k)| = |S^e_k| + |S^o_k| \leq 32 \cdot 2^{3k/2}.$$

We have from (3) that

$$s_{G,\lambda_2,k} \leq 2|S(G, \lambda_2, k)| + 2^k \leq 65 \cdot 2^{3k/2}.$$

Thus, we have the lemma. □

Now, we have the following from Lemmas B and 12.

Theorem 13. *The OBDD size of permutation graphs with N vertices is $O(N^{3/2}/\log^{3/4} N)$.*

Proof. We have from Lemmas B and 12 that $s_{G,\lambda_2,k} \leq 2^{2^{2(n-k)}}$ and there exists a constant c such that $s_{G,\lambda_2,k} \leq c \cdot 2^{3k/2}$ for any permutation graph G and k, $0 \leq k < n$. Thus, we have from (2) that

$$\begin{aligned}
|\sigma\text{-OBDD}(G, \lambda)| &\leq 3 \sum_{k=0}^{n-1} s_{G,\lambda_2,k} + 2 \\
&\leq 3 \sum_{k=0}^{n-\lfloor \frac{2\log n - 1}{4} \rfloor} c \cdot 2^{3k/2} + 3 \sum_{k=n-\lfloor \frac{2\log n-1}{4} \rfloor + 1}^{n-1} 2^{2^{2(n-k)}} + 2 \\
&\leq 3c \sum_{k=0}^{n-\lfloor \frac{2\log n - 1}{4} \rfloor} 2^{3k/2} + 3 \sum_{k=1}^{\lfloor \frac{2\log n-1}{4} \rfloor - 1} 2^{2^{2k}} + 2 \\
&= O(N^{3/2}/\log^{3/4} N).
\end{aligned}$$

Thus, we have the theorem. □

4.4 Two-Directional Orthogonal Ray Graphs

Let G be a 2-directional orthogonal ray graph with a bipartition (U, W) and an orthogonal ray representation $\mathcal{R}(G) = \{R_v \mid v \in V(G)\}$. We assume without loss of generality that every R_u, $u \in U$, is a rightward ray, and every R_w, $w \in W$, is a downward ray. Let (x_v, y_v) be the endpoint of $R_v \in \mathcal{R}(G)$, and we assume without loss of generality that the x-coordinates are distinct and the y-coordinates are distinct [26]. Notice that for any $u \in U$ and $w \in W$, $(u, w) \in E(G)$ if and only if $x_u < x_w$ and $y_u < y_w$.

Let λ_3 be a vertex labeling of G satisfying the following conditions:

- For any $u \in U$, $0 \leq |\lambda_3(u)| < |U|$;
- For any $w \in W$, $|U| \leq |\lambda_3(w)| < |V(G)|$;
- For any $\boldsymbol{a}, \boldsymbol{b} \in \{0, 1\}^n$ such that both $\lambda_3^{-1}(\boldsymbol{a})$ and $\lambda_3^{-1}(\boldsymbol{b})$ are in U (respectively, W),
  - $a_i < b_i$ implies $x_{\lambda_3^{-1}(\boldsymbol{a})} < x_{\lambda_3^{-1}(\boldsymbol{b})}$ if i is even, and
  - $a_i < b_i$ implies $y_{\lambda_3^{-1}(\boldsymbol{a})} < y_{\lambda_3^{-1}(\boldsymbol{b})}$ if i is odd,

  where i is the index of most significant different bit of $\boldsymbol{a}$ and $\boldsymbol{b}$, that is, $a_i \neq b_i$ and $a_j = b_j$ for any $j > i$.

We refer to λ_3 as a *point partitioning labeling* of 2-directional orthogonal ray graphs.

Lemma 14. *For any 2-directional orthogonal ray graph G, there exists a point partitioning labeling λ_3 of G.*

Proof. The point partitioning labeling of a 2-directional orthogonal ray graph can be obtained by an algorithm similar to the point partitioning algorithm shown in Sect. 4.3. □

Recall that

$$V(\boldsymbol{\alpha}) = \{v \in V(G) \mid \boldsymbol{\alpha} \in \{0, 1\}^k \text{ is a prefix of } \lambda_3(v)\} \text{ and}$$

$$S(G, \lambda_3, k) = \left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \;\middle|\; \begin{array}{l} \boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0, 1\}^k, |\boldsymbol{\alpha}| < |\boldsymbol{\beta}|, \\ \chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}} \text{ is non-constant} \end{array} \right\}.$$

where χ is the characteristic function of labeled graph (G, λ_3). Notice that for any $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S(G, \lambda_3, k)$, $V(\boldsymbol{\alpha})$ contains a vertex in U and $V(\boldsymbol{\beta})$ contains a vertex in W since $|\boldsymbol{\alpha}| < |\boldsymbol{\beta}|$. We define that

$$S'_k = \{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S(G, \lambda_3, k) \mid V(\boldsymbol{\alpha}) \subseteq U \text{ and } V(\boldsymbol{\beta}) \subseteq W\}.$$

Notice that there exists at most one binary string $\boldsymbol{\gamma} \in \{0, 1\}^k$ for each k, $0 \leq k < n$, such that $V(\boldsymbol{\gamma})$ contains both a vertex in U and a vertex in W, since $|\lambda_3(u)| < |\lambda_3(w)|$ for any $u \in U$ and $w \in W$. Since the number of pairs in $S(G, \lambda_3, k)$ containing such a binary string $\boldsymbol{\gamma}$ is at most 2^k, we have that $|S(G, \lambda_3, k)| \leq |S'_k| + 2^k$.

For any $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S'_k$, there exists a pair of vertices $u \in V(\boldsymbol{\alpha})$ and $w \in V(\boldsymbol{\beta})$ such that $(u, w) \notin E(G)$, for otherwise $\chi_{|\boldsymbol{\alpha},\boldsymbol{\beta}} = 1$, contradicting $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S(G, \lambda_3, k)$. Since $(u, w) \notin E(G)$ implies that $x_u > x_w$ or $y_u > y_w$, we define that

$$S^x_k = \left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S'_k \;\middle|\; \begin{array}{l} \text{There exists a pair of verti-} \\ \text{ces } u \in V(\boldsymbol{\alpha}) \text{ and } w \in V(\boldsymbol{\beta}) \\ \text{such that } x_u > x_w \end{array} \right\};$$

$$S^y_k = \left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S'_k \;\middle|\; \begin{array}{l} \text{There exists a pair of verti-} \\ \text{ces } u \in V(\boldsymbol{\alpha}) \text{ and } w \in V(\boldsymbol{\beta}) \\ \text{such that } y_u > y_w \end{array} \right\}.$$

Notice that $S'_k = S^x_k \cup S^y_k$. Recall that for a binary string

$\alpha \in \{0,1\}^k$, α^e and α^o are the substrings of α consisting of bits with even indices and odd indices, respectively. We have the following for S_k^x.

Claim 15. *For every (α_i, β_i) and (α_j, β_j) in S_k^x,*

$$(|\alpha_i^o| = |\alpha_j^o|) \wedge (|\beta_i^o| = |\beta_j^o|) \wedge (|\alpha_i^e| < |\alpha_j^e|) \Rightarrow |\beta_i^e| \leq |\beta_j^e|.$$

Proof. Since $(\alpha_i, \beta_i) \in S_k^x$, there exists a pair of vertices $u \in V(\alpha_i)$ and $w \in V(\beta_i)$ such that $x_u > x_w$. $|\alpha_i^e| < |\alpha_j^e|$ together with $|\alpha_i^o| = |\alpha_j^o|$ implies that $x_u < x_{u'}$ for any vertex $u' \in V(\alpha_j)$, since the index of most significant different bit of $\lambda(u)$ and $\lambda(u')$ is even.

Suppose contrary that $|\beta_i^e| > |\beta_j^e|$. Then $|\beta_i^e| > |\beta_j^e|$ together with $|\beta_i^o| = |\beta_j^o|$ implies that $x_w > x_{w'}$ for any vertex $w' \in V(\beta_j)$. Since $x_{u'} > x_u > x_w > x_{w'}$ for any $u' \in V(\alpha_j)$ and $w' \in V(\beta_j)$, we conclude that $\chi_{|\alpha_j, \beta_j} = 0$, contradicting $(\alpha_j, \beta_j) \in S(G, \lambda_3, k)$. Thus, we have $|\beta_i^e| \leq |\beta_j^e|$. □

Similarly, we have the following for S_k^y.

Claim 16. *For every (α_i, β_i) and (α_j, β_j) in S_k^y,*

$$(|\alpha_i^e| = |\alpha_j^e|) \wedge (|\beta_i^e| = |\beta_j^e|) \wedge (|\alpha_i^o| < |\alpha_j^o|) \Rightarrow |\beta_i^o| \leq |\beta_j^o|.$$

Proof. The proof is similar to that of Claim 15, and is omitted. □

The following is obtained from Claims 11, 15, and 16.

Lemma 17. $s_{G,\lambda_3,k} \leq 67 \cdot 2^{3k/2}$ *for any 2-directional orthogonal ray graph G and k, $0 \leq k < n$.*

Proof. We have from Claims 11 and 15 that there exist at most $2 \cdot 2^{\lceil k/2 \rceil}$ 4-tuples in set $\{(\alpha^o, \beta^o, \alpha^e, \beta^e) \mid (\alpha, \beta) \in S_k^x\}$ for each pair of α^o and β^o, since the length of α^o, β^o, α^e, and β^e is at most $\lceil k/2 \rceil$, respectively. Thus,

$$|S_k^x| \leq 2^{\lceil k/2 \rceil} \cdot 2^{\lceil k/2 \rceil} \cdot 2 \cdot 2^{\lceil k/2 \rceil} \leq 16 \cdot 2^{3k/2}.$$

Similarly, we have from Claims 11 and 16 that

$$|S_k^y| \leq 2^{\lceil k/2 \rceil} \cdot 2^{\lceil k/2 \rceil} \cdot 2 \cdot 2^{\lceil k/2 \rceil} \leq 16 \cdot 2^{3k/2}.$$

Since $|S(G, \lambda_3, k)| \leq |S'_k| + 2^k$ and $S'_k = S_k^x \cup S_k^y$, we have that

$$|S(G, \lambda_3, k)| \leq |S'_k| + 2^k \leq |S_k^x| + |S_k^y| + 2^k \leq 33 \cdot 2^{3k/2}.$$

We have from (3) that

$$s_{G,\lambda_3,k} \leq 2|S(G, \lambda_3, k)| + 2^k \leq 67 \cdot 2^{3k/2}.$$

Thus, we have the lemma. □

Now, we have the following from Lemmas B and 17.

Theorem 18. *The OBDD size of 2-directional orthogonal ray graph with N vertices is $O(N^{3/2} / \log^{3/4} N)$.*

Proof. The proof is similar to that of Theorem 13, and is omitted. □

Since the class of 2-directional orthogonal ray graphs contains the class of convex graphs, we have the following.

Corollary 19. *The OBDD size of convex graphs with N vertices is $O(N^{3/2} / \log^{3/4} N)$.* □

4.5 Orthogonal Ray Graphs

Let G be an orthogonal ray graph with a bipartition (U, W) and an orthogonal ray representation $\mathcal{R}(G) = \{R_v \mid v \in V(G)\}$. Let (x_v, y_v) be the endpoint of $R_v \in \mathcal{R}(G)$, and we assume without loss of generality that the x-coordinates are distinct and the y-coordinates are distinct. We define that

$$\begin{aligned} U_l &= \{u \in U \mid R_u \text{ is a leftward ray}\}, \\ U_r &= \{u \in U \mid R_u \text{ is a rightward ray}\}, \\ W_u &= \{w \in W \mid R_w \text{ is a upward ray}\}, \text{ and} \\ W_d &= \{w \in W \mid R_w \text{ is a downward ray}\}. \end{aligned}$$

Let λ_4 be a vertex labeling of G satisfying the following conditions:

- For any $u \in U_l$, $0 \leq |\lambda_4(u)| < |U_l|$;
- For any $u \in U_r$, $|U_l| \leq |\lambda_4(u)| < |U|$;
- For any $w \in W_u$, $|U| \leq |\lambda_4(w)| < |U| + |W_u|$;
- For any $w \in W_d$, $|U| + |W_u| \leq |\lambda_4(w)| < |V(G)|$;
- For any $\boldsymbol{a}, \boldsymbol{b} \in \{0,1\}^n$ such that both $\lambda_4^{-1}(\boldsymbol{a})$ and $\lambda_4^{-1}(\boldsymbol{b})$ are in U_l (respectively, U_r, W_u, W_d),
  - $a_i < b_i$ implies $x_{\lambda_4^{-1}(\boldsymbol{a})} < x_{\lambda_4^{-1}(\boldsymbol{b})}$ if i is even, and
  - $a_i < b_i$ implies $y_{\lambda_4^{-1}(\boldsymbol{a})} < y_{\lambda_4^{-1}(\boldsymbol{b})}$ if i is odd,

  where i is the index of most significant different bit of $\boldsymbol{a}$ and $\boldsymbol{b}$, that is, $a_i \neq b_i$ and $a_j = b_j$ for any $j > i$.

We refer to λ_4 as a *point partitioning labeling* of orthogonal ray graphs.

Lemma 20. *For any orthogonal ray graph G, there exists a point partitioning labeling λ_4 of G.*

Proof. The point partitioning labeling of an orthogonal ray graph can be obtained by an algorithm similar to the point partitioning algorithm shown in Sect. 4.3. □

Lemma 21. $s_{G,\lambda_4,k} \leq 271 \cdot 2^{3k/2}$ *for any orthogonal ray graph G and k, $0 \leq k < n$.*

Proof. Recall that

$$V(\alpha) = \{v \in V(G) \mid \alpha \in \{0,1\}^k \text{ is a prefix of } \lambda_4(v)\} \text{ and}$$
$$S(G, \lambda_4, k) = \left\{ (\alpha, \beta) \;\middle|\; \begin{array}{l} \alpha, \beta \in \{0,1\}^k, |\alpha| < |\beta|, \\ \chi_{|\alpha,\beta} \text{ is non-constant} \end{array} \right\}.$$

where χ is the characteristic function of labeled graph (G, λ_4). We define as follows:

$$\begin{aligned} S_k^{lu} &= \{(\alpha, \beta) \in S(G, \lambda_4, k) \mid V(\alpha) \subseteq U_l \text{ and } V(\beta) \subseteq W_u\}; \\ S_k^{ld} &= \{(\alpha, \beta) \in S(G, \lambda_4, k) \mid V(\alpha) \subseteq U_l \text{ and } V(\beta) \subseteq W_d\}; \\ S_k^{ru} &= \{(\alpha, \beta) \in S(G, \lambda_4, k) \mid V(\alpha) \subseteq U_r \text{ and } V(\beta) \subseteq W_u\}; \end{aligned}$$

$$S_k^{rd} = \{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S(G, \lambda_4, k) \mid V(\boldsymbol{\alpha}) \subseteq U_r \text{ and } V(\boldsymbol{\beta}) \subseteq W_d\}.$$

Notice that there exist at most three binary strings $\boldsymbol{\gamma} \in \{0,1\}^k$ for each k, $0 \le k < n$, such that $V(\boldsymbol{\gamma})$ has vertices whose corresponding rays have different directions, since $|\lambda_4(u_l)| < |\lambda_4(u_r)| < |\lambda_4(w_u)| < |\lambda_4(w_d)|$ for any $u_l \in U_l$, $u_r \in U_r$, $w_u \in W_u$, and $w_d \in W_d$. Since the number of pairs in $S(G, \lambda_4, k)$ containing such binary string $\boldsymbol{\gamma}$ is at most $3 \cdot 2^k$, we have that $|S(G, \lambda_4, k)| \le |S_k^{lu}| + |S_k^{ld}| + |S_k^{ru}| + |S_k^{rd}| + 3 \cdot 2^k$.

Since the subgraphs of G induced by $U_l \cup W_u$, $U_l \cup W_d$, $U_r \cup W_u$, and $U_r \cup W_d$ are 2-directional orthogonal ray graphs, we have from Lemma 17 that $|S_k^{lu}|, |S_k^{ld}|, |S_k^{ru}|, |S_k^{rd}| \le 33 \cdot 2^{3k/2}$. Thus, we have that

$$|S(G, \lambda_4, k)| \le |S_k^{lu}| + |S_k^{ld}| + |S_k^{ru}| + |S_k^{rd}| + 3 \cdot 2^k \le 135 \cdot 2^{3k/2}.$$

We have from (3) that

$$s_{G,\lambda_4,k} \le 2|S(G, \lambda_4, k)| + 2^k \le 271 \cdot 2^{3k/2}.$$

Thus, we have the lemma. □

Now, we have the following from Lemmas B and 21.

Theorem 22. *The OBDD size of orthogonal ray graph with N vertices is $O(N^{3/2}/\log^{3/4} N)$.*

Proof. The proof is similar to that of Theorem 13, and is omitted. □

5. Lower Bounds of OBDD Sizes

The following theorem is implicit in [20]. We give here an explicit statement and proof for the following arguments. Recall that $\mathcal{G}_N$ is the set of all N-vertex graphs in a graph class $\mathcal{G}$.

Theorem C (Nunkesser and Woelfel [20])**.** *When we use binary strings $[0]_n, [1]_n, \ldots, [N-1]_n$ for vertex labels, the OBDD size of a graph class $\mathcal{G}_N$ is*

$$\begin{array}{ll} \Omega(N/\log N) & \text{if } |\mathcal{G}_N| = 2^{\Omega(N)}, \\ \Omega(N) & \text{if } |\mathcal{G}_N| = 2^{\Omega(N \log N)}, \text{ and} \\ \Omega(N \log N) & \text{if } |\mathcal{G}_N| = 2^{\Omega(N \log^2 N)}, \end{array}$$

where $n = \lceil \log N \rceil$ and $|\mathcal{G}_N|$ is the number of graphs in $\mathcal{G}_N$.

Proof. Since we use n-bit labels $[0]_n, [1]_n, \ldots, [N-1]_n$ for vertex labeling as we discussed in Sect. 3, each Boolean function in B_{2n} can represent one N-vertex graph up to isomorphism. Hence, the number of characteristic functions needed to represent all graphs in $\mathcal{G}_N$ is at least the number of graphs in $\mathcal{G}_N$. Since OBDDs on X_k of size s can represent at most $sk^s(s+1)^{2s}/s! = 2^{s \log s + s \log k + \Theta(s)}$ different functions $f \in B_k$ [31], if

$$\lim_{N \to \infty} \frac{2^{s(N) \log s(N) + s(N) \log \log N + \Theta(s(N))}}{|\mathcal{G}_N|} < 1,$$

where $s : \mathbb{N} \to \mathbb{N}$ is a function, then there exists a natural number $N \in \mathbb{N}$ such that $\mathcal{G}_N$ has a graph that cannot be represented by OBDD of size at most $s(N)$. Thus, $s(N) = \Omega(N/\log N)$ if $|\mathcal{G}_N| = 2^{\Omega(N)}$, $s(N) = \Omega(N)$ if $|\mathcal{G}_N| = 2^{\Omega(N \log N)}$, and $s(N) = \Omega(N \log N)$ if $|\mathcal{G}_N| = 2^{\Omega(N \log^2 N)}$. □

It should be noted that the lower bounds are valid only when we use $[0]_n, [1]_n, \ldots, [N-1]_n$ for vertex labels, since if we increase the range of vertex labeling, one Boolean function in B_{2n} might be able to represent more than one N-vertex graphs.

5.1 Biconvex Graphs and Bipartite Permutation Graphs

The following is shown in [22].

Theorem D (Saitoh, Otachi, Yamanaka, and Uehara [22])**.** *The number of unlabeled connected bipartite permutation graphs with $N \ge 2$ vertices is*

$$\begin{array}{ll} \frac{1}{4}\left(C(N-1) + C(N/2-1) + \binom{N}{N/2}\right) & \text{if } N \text{ is even, and} \\ \frac{1}{4}\left(C(N-1) + \binom{N-1}{(N-1)/2}\right) & \text{if } N \text{ is odd,} \end{array}$$

where $C(N) = \frac{1}{N+1}\binom{2N}{N}$ is the N-th Catalan number. □

The following is immediate from Theorem D, since $\binom{2N}{N} = 2^{2N + \Theta(\log N)}$ [15].

Corollary 23. *The number of unlabeled bipartite permutation graphs with N vertices is $2^{\Omega(N)}$.* □

Now, we have the following from Theorem C and Corollary 23.

Theorem 24. *The OBDD size of bipartite permutation graphs with N vertices is $\Omega(N/\log N)$.* □

Since the class of bipartite permutation graphs is contained in the class of biconvex graphs, we have the following from Theorem 24.

Corollary 25. *The OBDD size of biconvex graphs with N vertices is $\Omega(N/\log N)$.* □

5.2 Permutation Graphs

The following is shown in [2].

Theorem E (Bazzaro and Gavoille [2])**.** *The number of unlabeled permutation graphs with N vertices is $2^{\Omega(N \log N)}$.* □

We have the following from Theorems C and E.

Theorem 26. *The OBDD size of permutation graphs with N vertices is $\Omega(N)$.* □

5.3 Orthogonal Ray Graphs and Convex Graphs

We have the following.

Theorem 27. *The number of unlabeled convex graphs with N vertices is $2^{\Omega(N \log N)}$.*

Proof. A graph G is called an *interval graph* [3] if there exists a set of closed intervals I_v, $v \in V(G)$, on the real line such that for any $u, w \in V(G)$, $(u, w) \in E(G)$ if and only if I_u and I_w intersect. The set $\mathcal{I}(G) = \{I_v \mid v \in V(G)\}$ is called an *interval representation* of G. It is shown in [9] that the number of interval graphs with N vertices is $2^{\Theta(N \log N)}$.

We can assume without loss of generality that N can be divided by 3. Let $\mathcal{CG}_N$ be a class of convex graphs with N vertices. We assume that each graph in $\mathcal{CG}_N$ has a bipartition (U, W) and a convex ordering of U. Let

$$\mathcal{CG}_{2N/3,N/3} = \{G \in \mathcal{CB}_N \mid |U| = 2N/3 \text{ and } |W| = N/3\}.$$

Let $\mathcal{IG}_{N/3}$ be a class of interval graphs with $N/3$ vertices. Notice that $\mathcal{IG}_{N/3} = 2^{\Omega(N \log N)}$. Then, it suffices to show that there exists a surjection $\phi : \mathcal{CG}_{2N/3,N/3} \to \mathcal{IG}_{N/3}$, since it implies that $|\mathcal{IG}_{N/3}| \leq |\mathcal{CG}_{2N/3,N/3}| \leq |\mathcal{CB}_N|$.

For any convex graph G in $\mathcal{CG}_{2N/3,N/3}$, we define that $\phi(G)$ is a graph such that

$$V(\phi(G)) = W \text{ and}$$
$$E(\phi(G)) = \{(w, w') \mid \Gamma_G(w) \cap \Gamma_G(w') \neq \emptyset\}.$$

Since for any $w \in W$, the vertices in $\Gamma_G(w)$ occur consecutively in the convex ordering of U, we have $\phi(G) \in \mathcal{IG}_{N/3}$.

Now, we show that the mapping ϕ is a surjection. Let H be an interval graph in $\mathcal{IG}_{N/3}$ with an interval representation $\mathcal{I}(H)$. For each $I_v \in \mathcal{I}(H)$, there exist the left and right endpoints of I_v, and let P be the set of all such endpoints of $\mathcal{I}(H)$. We can assume without loss of generality that endpoints in P are distinct [14]. For each $p \in P$, we define a corresponding vertex u_p. Let G_H be a bipartite graph with a bipartition (U_H, W_H) such that

$$U_H = \{u_p \mid p \in P\},$$
$$W_H = V(H), \text{ and}$$
$$E(G_H) = \{(u_p, v) \mid v \in W_H \text{ and } p \in I_v\}.$$

Since it is immediate that $G_H \in \mathcal{CG}_{2N/3,N/3}$ and $\phi(G_H) = H$, we conclude that ϕ is a surjection. □

Now, we have the following from Theorems C and 27.

Theorem 28. *The OBDD size of convex graphs with N vertices is* $\Omega(N)$. □

Since {Convex Graphs} ⊂ {2-Directional Orthogonal Ray Graphs} ⊂ {Orthogonal Ray Graphs} ⊂ {Unit Grid Intersection Graphs} ⊂ {Grid Intersection Graphs}, where $X \subset Y$ indicates a set X is a proper subset of Y, we have the following from Theorem 28.

Corollary 29. *The OBDD sizes of* (2*-directional*) *orthogonal ray graphs and* (*unit*) *grid intersection graphs with N vertices are* $\Omega(N)$. □

6. Concluding Remarks

It should be noted that for any permutation graph G, labeling λ_2 of G, shown in Sect. 4.3, can be obtained in polynomial time by the point partitioning algorithm, since it requires sorting the points in the xy-plane and setting the bits of the labels of the points. See the proof of Lemma 8. Similarly, labeling λ_3 can be obtained in polynomial time for any 2-directional orthogonal ray graph, and labeling λ_4 can be obtained in polynomial time for any orthogonal ray graph.

Since the number of chordal bipartite graphs with N vertices is $2^{\Theta(N \log^2 N)}$ [27], the OBDD size of chordal bipartite graphs is $\Omega(N \log N)$ by Theorem C.

It should be noticed that Gillé mentions in [10], [11] that the OBDD size of convex graphs is $O(N \log N)$.

Upper bounds of the OBDD sizes of chordal bipartite graphs and (unit) grid intersection graphs remain open. Also, the upper and lower bounds of OBDD sizes of convex graphs, (2-directional) orthogonal ray graphs, and permutation graphs are not tight, and closing the gaps between the bounds are another open problems.

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