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On Wohlfahrt series and wreath products

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Abstract. Suppose that a group A contains only a finite number of subgroups of index d for each positive integer d. Let $G \wr S_n$ be the wreath product of a finite group G with the symmetric group S_n on $\{1, \ldots, n\}$. For each positive integer n, let K_n be a subgroup of $G \wr S_n$ containing the commutator subgroup of $G \wr S_n$. If the sequence $\{K_n\}_0^\infty$ satisfies a certain compatible condition, then the exponential generating function $\sum_{n=0}^\infty |\mathrm{Hom}(A, K_n)| X^n / |G|^n n!$ of the sequence $\{|\mathrm{Hom}(A, K_n)|\}_0^\infty$ takes the form of a sum of exponential functions.

1. Introduction

Let A be a group and $\mathcal{F}_A$ the set of subgroups B of A of finite index $|A : B|$. Suppose that A contains only a finite number of subgroups of index d for each positive integer d. Then for any finite group K, the set $\mathrm{Hom}(A, K)$ of homomorphisms from A to K is a finite set. We denote by $|\mathrm{Hom}(A, K)|$ the number of homomorphisms from A to a finite group K. Let S_n be the symmetric group on $[n] = \{1, \ldots, n\}$ and S_0 the group consisting of only the identity. In [17] Wohlfahrt proves that

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(A, S_n)|}{n!} X^n = \exp\left(\sum_{B \in \mathcal{F}_A} \frac{1}{|A : B|} X^{|A:B|} \right). \qquad \text{(WF)}$$

This formula interests us in various exponential formulas.

Given a sequence $\{K_n\}_0^\infty$ of finite groups, the Wohlfahrt series $E_A(X : \{K_n\}_0^\infty)$ is the exponential generating function

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(A, K_n)|}{n!} X^n.$$

Previous studies of Wohlfahrt series have given some exponential formulas, each of which is a sum of exponential functions. In this paper we extend the approach to the exponential formulas. The approach is based on character theory of finite groups.

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Let G be a finite group and $G^{(n)}$ the direct product of n copies of G. If H is a subgroup of S_n, then the wreath product

$$G \wr H = \{(g_1, \ldots, g_n)h \mid (g_1, \ldots, g_n) \in G^{(n)},\, h \in H\}$$

is the semidirect product $G^{(n)} \rtimes H$, in which each $h \in H$ acts as an inner automorphism on $G^{(n)}$:

$$h(g_1, \ldots, g_n)h^{-1} = (g_{h^{-1}(1)}, \ldots, g_{h^{-1}(n)}).$$

We consider $G \wr S_0 = S_0$. In [10, 11, 15, 16] the Wohlfahrt formula (WF) is extended to formulas for $E_A(X : \{G \wr S_n\}_0^\infty)$ and $E_A(X/|G| : \{G \wr S_n\}_0^\infty)$ (cf. Corollary 2.7).

Let 1_{S_n} be the trivial $\mathbb{C}$-character of S_n and δ_n the linear $\mathbb{C}$-character of S_n such that $\delta_n(h)$ is the sign of h for all $h \in S_n$, where $\mathbb{C}$ is the complex numbers. We denote by $\mathbf{e}$ the sequence $\{1_{S_n}\}_0^\infty$ and denote by $\mathbf{sgn}$ the sequence $\{\delta_n\}_0^\infty$. Let χ be a linear $\mathbb{C}$-character of G, and let $\zeta(\chi, \mathbf{e}, n)$ and $\zeta(\chi, \mathbf{sgn}, n)$ be linear $\mathbb{C}$-characters of $G \wr S_n$ defined by

$$\zeta(\chi, \mathbf{e}, n)((g_1, \ldots, g_n)h) = \chi(g_1 \cdots g_n)1_{S_n}(h)$$

and

$$\zeta(\chi, \mathbf{sgn}, n)((g_1, \ldots, g_n)h) = \chi(g_1 \cdots g_n)\delta_n(h)$$

for all $(g_1, \ldots, g_n) \in G^{(n)}$ and $h \in S_n$. Given a linear $\mathbb{C}$-character ζ of $G \wr S_n$, there exists a linear $\mathbb{C}$-character χ_0 of G such that $\zeta = \zeta(\chi_0, \mathbf{e}, n)$ or $\zeta = \zeta(\chi_0, \mathbf{sgn}, n)$.

Let $\mathbf{z} \in \{\mathbf{e}, \mathbf{sgn}\}$. We define $K(\chi, \mathbf{z}, n)$ to be the kernel of $\zeta(\chi, \mathbf{z}, n)$, and consider $K(\chi, \mathbf{e}, 0) = K(\chi, \mathbf{sgn}, 0) = S_0$. Let 1_G be the trivial $\mathbb{C}$-character of G, and let A_n be the alternating group on $[n]$. Then $G \wr S_n = K(1_G, \mathbf{e}, n)$ and $G \wr A_n = K(1_G, \mathbf{sgn}, n)$. The Wohlfahrt series $E_A(X : \{K(1_G, \mathbf{z}, n) \cap K(\chi, \mathbf{e}, n)\}_0^\infty)$ with $|G/\mathrm{Ker}\,\chi| \leq 2$ is described as a sum of exponential functions by Müller and Shareshian [11]. The form of $E_A(X/|G| : \{G \wr A_n\}_0^\infty)$ is also studied in [16] (cf. Corollary 2.8). Moreover, $E_A(X/|G| : \{K(\chi, \mathbf{e}, n)\}_0^\infty)$ with $|G/\mathrm{Ker}\,\chi| = p$, where p is a prime, takes the form of a sum of exponential functions, and so does $E_A(X/|G| : \{K(\chi, \mathbf{sgn}, n)\}_0^\infty)$ with $|G/\mathrm{Ker}\,\chi| = 2$ [16, Theorem 1].

Given linear $\mathbb{C}$-characters $\chi_1, \ldots, \chi_s$ of G and an element $(\mathbf{z}_1, \ldots, \mathbf{z}_s)$ of the Cartesian product $\{\mathbf{e}, \mathbf{sgn}\}^{(s)}$ of s copies of $\{\mathbf{e}, \mathbf{sgn}\}$, we define

$$K(\chi_1, \ldots, \chi_s, \mathbf{z}_1, \ldots, \mathbf{z}_s, n) = \bigcap_{i \in \{1, \ldots, s\}} K(\chi_i, \mathbf{z}_i, n).$$

Every subgroup of $G \wr S_n$ containing the commutator subgroup of $G \wr S_n$ is considered as such a subgroup, because any subgroup of a finite abelian group is expressed as the intersection of kernels of linear $\mathbb{C}$-characters. In Section 2 we study the form of

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(A, K(\chi_1, \ldots, \chi_s, \mathbf{z}_1, \ldots, \mathbf{z}_s, n))|}{|G|^n n!} X^n,$$

which is described as a sum of exponential functions (cf. Theorem 2.1).

Let m be a positive integer, and let ω be a primitive mth root of unity in $\mathbb{C}$. If G is the cyclic group $\langle\omega\rangle$ generated by ω and if $\chi(\omega) = \omega^{m/r}$, where r is a divisor of m, then we identify $K(\chi, \mathbf{e}, n)$ with the imprimitive complex pseudo-reflection group $G(m, r, n)$ [8], and define

$$H(m, r, n) = K(\chi, \mathbf{e}, n) \cap (G \wr A_n)(= K(\chi, 1_G, \mathbf{e}, \mathbf{sgn}, n))$$

and

$$L(m, r, n) = K(\chi, \mathbf{sgn}, n).$$

The form of $E_A(X/p : \{G(p, p, n)\}_0^\infty)$ and the form of $E_A(X/2 : \{L(2, 2, n)\}_0^\infty)$ are studied in [16]. In Section 3 we study the form of $E_A(X/m : \{K_n\}_0^\infty)$ where K_n is $G(m, r, n)$, $H(m, r, n)$ or $L(m, r, n)$ (cf. Theorem 3.2).

The Weyl group $W(D_n)$ of type D_n is isomorphic to $G(2, 2, n)$. When A is a finite abelian group, the explicit forms of $E_A(X : \{G \wr A_n\}_0^\infty)$ and $E_A(X : \{W(D_n)\}_0^\infty)$ are given in [11]. In Section 4 we study the form of $E_P(X/p : \{G(p, p, n)\}_0^\infty)$ where P is a finite abelian p-group, together with that of $E_P(X/2 : \{L(2, 2, n)\}_0^\infty)$ and that of $E_P(X : \{A_n\}_0^\infty)$ where P is a finite abelian 2-group (cf. Theorems 4.8 and 4.12). The argument about the descriptions of these Wohlfahrt series is essentially due to Müller and Shareshian (see [11, Section 4]).

In Sections 5 and 6 we present some examples.

2. The form of Wohlfahrt series

Let $\chi_1, \ldots, \chi_s$ be linear $\mathbb{C}$-characters of G, and let $(\mathbf{z}_1, \ldots, \mathbf{z}_s) \in \{\mathbf{e}, \mathbf{sgn}\}^{(s)}$. In this section we study the form of $E_A(X/|G| : \{K(\chi_1, \ldots, \chi_s, \mathbf{z}_1, \ldots, \mathbf{z}_s, n)\}_0^\infty)$.

Let $i \in \{1, \ldots, s\}$. Suppose that the factor group $G/\mathrm{Ker}\,\chi_i$ is of order r'_i. Put $r_i = r'_i$ if r'_i is even or $\mathbf{z}_i = \mathbf{e}$, and $r_i = 2r'_i$ otherwise. Then the linear $\mathbb{C}$-character $\zeta(\chi_i, \mathbf{z}_i, n)$ is a homomorphism from $G \wr S_n$ to the cyclic group $\langle\omega_{r_i}\rangle$ generated by a primitive r_ith root ω_{r_i} of unity in $\mathbb{C}$. Define

$$\Phi_{r_i}(A) = \bigcap_{\alpha \in \mathrm{Hom}(A, \langle\omega_{r_i}\rangle)} \mathrm{Ker}\,\alpha.$$

Then $\Phi_{r_i}(A)$ is a normal subgroup of A and the factor group $A/\Phi_{r_i}(A)$ is a finite abelian group. Write $R_i = A/\Phi_{r_i}(A)$, and let $\overline{a}$ denote the coset $a\Phi_{r_i}(A)$ of $\Phi_{r_i}(A)$ in A containing $a \in A$. Given $\varphi \in \mathrm{Hom}(A, G \wr S_n)$ and $\overline{a} \in R_i$, it is clear that $\zeta(\chi_i, \mathbf{z}_i, n)(\varphi(c))$ with $c \in \overline{a}$ is independent of the choice of c in $\overline{a}$.

Let $B \in \mathcal{F}_A$. We define a homomorphism $\mathrm{sgn}_{A/B}$ from A to $\mathbb{C}$ by

$$\mathrm{sgn}_{A/B}(a) = \begin{cases} 1 & \text{if } a \in A \text{ is an even permutation on } A/B, \\ -1 & \text{if } a \in A \text{ is an odd permutation on } A/B, \end{cases}$$

where A/B is the left A-set consisting of all left cosets of B in A with the action given by $a.cB = acB$ for all $a, c \in A$.

Suppose that $|A:B| = d$ and $T_B^A = \{a_1, \ldots, a_d\}$ is a left transversal of B in A. For each normal subgroup N of B containing the commutator subgroup B', let $V_{A\to B/N}$ be the transfer from A to the factor group B/N defined by

$$V_{A\to B/N}(a) = \prod_{j=1}^{d} a_{j'}^{-1} a a_j N \quad \text{with} \quad aa_j \in a_{j'}B$$

for all $a \in A$, which is independent of the choice of T_B^A, and is a homomorphism.

Let $\alpha \in \mathrm{Hom}(B, \mathbb{C}^\times)$, $\mathbb{C}^\times$ the multiplicative group of $\mathbb{C}$. Then $B' \le \mathrm{Ker}\,\alpha$. Let α_0 be the homomorphism from B/B' to $\mathbb{C}^\times$ defined by $\alpha_0(bB') = \alpha(b)$ for all $b \in B$. Let $\alpha^{\otimes A}$ be the homomorphism from A to $\mathbb{C}^\times$ given by

$$\alpha^{\otimes A}(a) = \alpha_0(V_{A\to B/B'}(a))$$

for all $a \in A$, which is the representation afforded by a tensor induced $\mathbb{C}A$-module (see [4, (13.12) Proposition]). Let $\kappa \in \mathrm{Hom}(B, G)$. Given $\overline{a} \in R_i$, it is clear that $(\chi_i \circ \kappa)^{\otimes A}(c)$ with $c \in \overline{a}$ is independent of the choice of c in $\overline{a}$.

Set $I = \{i \mid \mathbf{z}_i = \mathbf{sgn}\}$. Given $\overline{a} \in R_i$ with $i \in I$, $\mathrm{sgn}_{A/B}(c)$ with $c \in \overline{a}$ is independent of the choice of c in $\overline{a}$.

Put $R = R_1 \times \cdots \times R_s$. Given $(\overline{c_1}, \ldots, \overline{c_s}) \in R$, we define

$$\rho_B(\overline{c_1}, \ldots, \overline{c_s}) = \mathrm{sgn}_{A/B}\left(\prod_{i\in I} c_i\right) \sum_{\kappa\in\mathrm{Hom}(B,G)} \prod_{i=1}^{s} (\chi_i \circ \kappa)^{\otimes A}(c_i).$$

We are successful in finding the following formula.

Theorem 2.1

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(A, K(\chi_1, \ldots, \chi_s, \mathbf{z}_1, \ldots, \mathbf{z}_s, n))|}{|G|^n n!} X^n = \frac{1}{|R|} \sum_{(\overline{c_1},\ldots,\overline{c_s})\in R} \exp\left(\sum_{B\in\mathcal{F}_A} \frac{\rho_B(\overline{c_1}, \ldots, \overline{c_s})}{|G|\,|A:B|} X^{|A:B|}\right).$$

Let us prove this theorem. We start with the following lemma, which plays a crucial role in this description of $E_A(X/|G| : \{K(\chi_1, \ldots, \chi_s, \mathbf{z}_1, \ldots, \mathbf{z}_s, n)\}_0^\infty)$.

Lemma 2.2 *Let $\varphi \in \mathrm{Hom}(A, G \wr S_n)$. Then for each integer i with $1 \le i \le s$,*

$$\frac{1}{|R_i|} \sum_{\overline{a}\in R_i} \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(a)) = \begin{cases} 1 & \text{if} \quad \mathrm{Im}\varphi \le K(\chi_i, \mathbf{z}_i, n), \\ 0 & \text{otherwise,} \end{cases}$$

where the sum $\sum_{\overline{a}\in R_i}$ is over all left cosets $\overline{a} \in R_i$ with $a \in A$.

Proof. Define a $\mathbb{C}$-character α_i of R_i by setting

$$\alpha_i(\overline{a}) = \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(a))$$

for all $\overline{a} \in R_i$ with $a \in A$. Then $\mathrm{Im}\varphi \leq K(\chi_i, \mathbf{z}_i, n)$ if and only if α_i is the trivial $\mathbb{C}$-character of R_i. Hence it follows from the first orthogonality relation [4, (9.21) Proposition] that

$$\frac{1}{|R_i|} \sum_{\overline{a} \in R_i} \alpha_i(\overline{a}) = \begin{cases} 1 & \text{if} \quad \mathrm{Im}\varphi \leq K(\chi_i, \mathbf{z}_i, n), \\ 0 & \text{otherwise}, \end{cases}$$

which proves the lemma. □

This lemma enables us to get the following proposition.

Proposition 2.3

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(A, K(\chi_1, \dots, \chi_s, \mathbf{z}_1, \dots, \mathbf{z}_s, n))|}{n!} X^n$$
$$= \frac{1}{|R|} \sum_{(\overline{c_1}, \dots, \overline{c_s}) \in R} \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \sum_{\varphi \in \mathrm{Hom}(A, G \wr S_n)} \prod_{i=1}^{s} \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(c_i)) \right\} X^n.$$

Proof. If $\varphi \in \mathrm{Hom}(A, G \wr S_n)$, then by Lemma 2.2, we have

$$\prod_{i=1}^{s} \left\{ \frac{1}{|R_i|} \sum_{\overline{c_i} \in R_i} \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(c_i)) \right\} = \begin{cases} 1 & \text{if} \quad \mathrm{Im}\varphi \leq \bigcap_{i \in \{1, \dots, s\}} K(\chi_i, \mathbf{z}_i, n), \\ 0 & \text{otherwise}. \end{cases}$$

Hence it turns out that

$$|\mathrm{Hom}(A, K(\chi_1, \dots, \chi_s, \mathbf{z}_1, \dots, \mathbf{z}_s, n))|$$
$$= \sum_{\varphi \in \mathrm{Hom}(A, G \wr S_n)} \prod_{i=1}^{s} \left\{ \frac{1}{|R_i|} \sum_{\overline{c_i} \in R_i} \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(c_i)) \right\}$$
$$= \frac{1}{|R|} \sum_{(\overline{c_1}, \dots, \overline{c_s}) \in R_1 \times \cdots \times R_s} \sum_{\varphi \in \mathrm{Hom}(A, G \wr S_n)} \prod_{i=1}^{s} \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(c_i)),$$

completing the proof of the proposition. □

We consider the Cartesian product $G \times [n]$ of G and $[n]$ to be the left $G \wr S_n$-set with the left action of $G \wr S_n$ given by

$$(g_1, \dots, g_n)h.(g, i) = (g_{h(i)}g, h(i))$$

for all $(g_1, \ldots, g_n) \in G^{(n)}$, $h \in S_n$, and $(g, i) \in G \times [n]$ [9, 2.11], so that $G \wr S_n$ is isomorphic to the automorphism group of the free right G-set $G \times [n]$ with the right action of G given by $(g, i).y = (gy, i)$ for all $(g, i) \in G \times [n]$ and $y \in G$ (see [1, Proposition 6.11], [16, Proposition 1]).

Let υ_n be the homomorphism from $G \wr S_n$ to S_n defined by

$$\upsilon_n((g_1, \ldots, g_n)h) = h$$

for all $(g_1, \ldots, g_n) \in G^{(n)}$ and $h \in S_n$.

Set $\mathcal{F}_A(n) = \{B \in \mathcal{F}_A \mid |A : B| \leq n\}$. We now show a recurrence formula like Dey's theorem [5, (6.10)], namely,

Proposition 2.4 *If n is a positive integer, then*

$$\sum_{\varphi \in \mathrm{Hom}(A, G \wr S_n)} \frac{\prod_{i=1}^{s} \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(c_i))}{|G|^n (n-1)!}$$

$$= \sum_{B \in \mathcal{F}_A(n)} \frac{\rho_B(\overline{c_1}, \ldots, \overline{c_s})}{|G|} \sum_{\psi \in \mathrm{Hom}(A, G \wr S_{n-|A:B|})} \frac{\prod_{i=1}^{s} \zeta(\chi_i, \mathbf{z}_i, n - |A : B|)(\psi(c_i))}{|G|^{n-|A:B|}(n - |A : B|)!}$$

with $c_1, \ldots, c_s \in A$.

The proof is analogous to that of [15, Theorem 3.1].

Proof of Proposition 2.4. If $B \in \mathcal{F}_A$, then we fix a left transversal T_B^A containing the identity ϵ_A of A. We denote by ϵ the identity of G.

Let $\varphi \in \mathrm{Hom}(A, G \wr S_n)$. Define a subgroup B of A by

$$B = \{a \in A \mid \upsilon_n(\varphi(a))(1) = 1\},$$

and define a homomorphism κ from B to G by

$$\varphi(b).(\epsilon, 1) = (\kappa(b), 1)$$

for all $b \in B$. We then have $|A : B| \leq n$. Suppose that $T_B^A = \{a_1, \ldots, a_d\}$ with $a_1 = \epsilon_A$ and $d = |A : B|$. Define an injection ι from $[d]$ into $[n]$ with $\iota(1) = 1$ by

$$\iota(j) = \upsilon_n(\varphi(a_j))(1)$$

for all $j \in [d]$, and define an element $(y_1, \ldots, y_d)$ of the Cartesian product $G^{(d)}$ of d copies of G with $y_1 = \epsilon$ by

$$\varphi(a_j).(\epsilon, 1) = (y_j, \iota(j))$$

for all $j \in [d]$. If $a \in A$ and if $j \in [d]$, then we have

$$\varphi(a).(\epsilon, \iota(j)) = (y_{j'}\kappa(a_{j'}^{-1}aa_j)y_j^{-1}, \iota(j')) \quad \text{with} \quad aa_j \in a_{j'}B. \tag{I}$$

Suppose that $\{\iota(1), \ldots, \iota(d)\} \cup \{k_1, \ldots, k_{n-d}\} = [n]$ and $k_1 < \cdots < k_{n-d}$. If $h \in \mathrm{Im}(\upsilon_n \circ \varphi)$, then we define a permutation $\hat{h}$ on $[n-d]$ by $h(k_t) = k_{\hat{h}(t)}$ for all $t \in [n-d]$. Let ψ be the mapping from A to $G \wr S_{n-d}$ defined by

$$\psi(a) = (g_{k_1}, \ldots, g_{k_{n-d}})\hat{h} \quad \text{with} \quad h = \upsilon_n(\varphi(a)),\ \varphi(a) = (g_1, \ldots, g_n)h \tag{II}$$

for all $a \in A$. Then it is easily checked that ψ is a homomorphism.

We have got a quintet $(B, \kappa, \iota, (y_1, \ldots, y_d), \psi)$ satisfying the condition

$$\begin{cases} B \in \mathcal{F}_A \text{ with } d = |A : B| \leq n, \\ \kappa \in \mathrm{Hom}(B, G), \\ \iota \text{ is an injection from } [d] \text{ to } [n] \text{ with } \iota(1) = 1 \\ (y_1, \ldots, y_d) \in G^{(d)} \text{ with } y_1 = \epsilon, \\ \psi \in \mathrm{Hom}(A, G \wr S_{n-d}), \end{cases} \tag{III}$$

and by (I) and (II), we obtain

$$\begin{aligned} &\prod_{i=1}^{s} \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(c_i)) \\ &\quad = \mathrm{sgn}_{A/B}\left(\prod_{i \in I} c_i\right) \cdot \prod_{i=1}^{s} (\chi_i \circ \kappa)^{\otimes A}(c_i) \cdot \zeta(\chi_i, \mathbf{z}_i, n-d)(\psi(c_i)). \end{aligned} \tag{IV}$$

The preceding map

$$\Gamma \,:\, \varphi \to (B, \kappa, \iota, (y_1, \ldots, y_d), \psi)$$

from $\mathrm{Hom}(A, G \wr S_n)$ to the set of quintets $(B, \kappa, \iota, (y_1, \ldots, y_d), \psi)$ satisfying (III) is clearly injective. Moreover, it is easily verified that Γ is surjective (see the proof of [15, Theorem 3.1]). Combining this fact with (IV), we have

$$\begin{aligned} &\sum_{\varphi \in \mathrm{Hom}(A, G \wr S_n)} \prod_{i=1}^{s} \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(c_i)) \\ &\quad = \sum_{B \in \mathcal{F}_A(n)} \left\{ \rho_B(\overline{c_1}, \ldots, \overline{c_s})) \, \frac{(n-1)!}{(n - |A : B|)!} \, |G|^{|A:B|-1} \right. \\ &\qquad \left. \times \sum_{\psi \in \mathrm{Hom}(A, G \wr S_{n-|A:B|})} \prod_{i=1}^{s} \zeta(\chi_i, \mathbf{z}_i, n - |A : B|)(\psi(c_i)) \right\}. \end{aligned}$$

This completes the proof of the proposition. □

If $\chi_1 = \cdots = \chi_s = 1_G$ and if $\mathbf{z}_1 = \cdots = \mathbf{z}_s = \mathbf{e}$, then this proposition is the recurrence formula [15, Theorem 3.1] of $|\mathrm{Hom}(A, G \wr S_n)|$, which is a generalization of the recurrence formula [17, Satz] of $|\mathrm{Hom}(A, S_n)|$.

As a result of Proposition 2.4, we obtain the following proposition.

Proposition 2.5 *Suppose that $c_1, \ldots, c_s \in A$. Then*

$$\sum_{n=0}^{\infty} \frac{1}{|G|^n\, n!} \left\{ \sum_{\varphi \in \mathrm{Hom}(A, G\wr S_n)} \prod_{i=1}^{s} \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(c_i)) \right\} X^n = \exp\left(\sum_{B \in \mathcal{F}_A} \frac{\rho_B(\overline{c_1}, \ldots, \overline{c_s})}{|G|\,|A:B|} X^{|A:B|} \right).$$

Proof. Put $\gamma_\varphi(n) = \prod_{i=1}^{s} \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(c_i))$ with $\varphi \in \mathrm{Hom}(A, G \wr S_n)$, and put $\beta(B) = \rho_B(\overline{c_1}, \ldots, \overline{c_s})$ with $B \in \mathcal{F}_A$ for convenience. We denote by $\Xi(n)$ the set of sequences $(n_B)_{B \in \mathcal{F}_A}$ of nonnegative integers n_B corresponding to $B \in \mathcal{F}_A$ such that $\sum_{B \in \mathcal{F}_A} n_B |A : B| = n$, and abbreviate $(n_B)_{B \in \mathcal{F}_A}$ to (n_B). It suffices to show that for each nonnegative integer n,

$$\sum_{\varphi \in \mathrm{Hom}(A, G\wr S_n)} \frac{\gamma_\varphi(n)}{|G|^n n!} = \sum_{(n_B) \in \Xi(n)} \prod_{B \in \mathcal{F}_A} \frac{\beta(B)^{n_B}}{|G|^{n_B} |A:B|^{n_B} n_B!}.$$

We use induction on n. Evidently, this formula is true if $n = 0$. Suppose that $n \geq 1$. Then Proposition 2.4 yields

$$\sum_{\varphi \in \mathrm{Hom}(A, G\wr S_n)} \frac{\gamma_\varphi(n)}{|G|^n (n-1)!} = \sum_{B \in \mathcal{F}_A(n)} \frac{\beta(B)}{|G|} \sum_{\psi \in \mathrm{Hom}(A, G\wr S_{n-|A:B|})} \frac{\gamma_\psi(n - |A:B|)}{|G|^{n-|A:B|}(n - |A:B|)!}.$$

Moreover, given $B \in \mathcal{F}_A(n)$, the inductive assumption means that

$$\sum_{\psi \in \mathrm{Hom}(A, G\wr S_{n-|A:B|})} \frac{\gamma_\psi(n - |A:B|)}{|G|^{n-|A:B|}(n - |A:B|)!} = \sum_{(n_K) \in \Xi(n-|A:B|)} \prod_{K \in \mathcal{F}_A} \frac{\beta(K)^{n_K}}{|G|^{n_K} |A:K|^{n_K} n_K!}.$$

Hence we obtain

$$\begin{aligned}
\sum_{\varphi\in\mathrm{Hom}(A,G\wr S_n)} \frac{\gamma_\varphi(n)}{|G|^n n!}
&= \frac{1}{n}\sum_{B\in\mathcal{F}_A(n)} \frac{\beta(B)}{|G|} \sum_{(n_K)\in\Xi(n-|A:B|)} \prod_{K\in\mathcal{F}_A} \frac{\beta(K)^{n_K}}{|G|^{n_K}|A:K|^{n_K} n_K!} \\
&= \frac{1}{n}\sum_{B\in\mathcal{F}_A(n)} \sum_{(n_K)\in\Xi(n)} n_B|A:B| \prod_{K\in\mathcal{F}_A} \frac{\beta(K)^{n_K}}{|G|^{n_K}|A:K|^{n_K} n_K!} \\
&= \frac{1}{n}\sum_{(n_K)\in\Xi(n)} \left(\sum_{B\in\mathcal{F}_A(n)} n_B|A:B|\right) \prod_{K\in\mathcal{F}_A} \frac{\beta(K)^{n_K}}{|G|^{n_K}|A:K|^{n_K} n_K!} \\
&= \sum_{(n_K)\in\Xi(n)} \prod_{K\in\mathcal{F}_A} \frac{\beta(K)^{n_K}}{|G|^{n_K}|A:K|^{n_K} n_K!},
\end{aligned}$$

as required. □

Remark 2.6 Proposition 2.5 is also a consequence of a categorical fact, namely, [16, Propsition 5] (see the second half of the proof of [16, Theroem 1]). It should be stated in this connection that the categorical proof of the Wohlfahrt formula (WF) was given by Yoshida (see [18, 6.4]).

By virtue of Propositions 2.3 and 2.5, we have established Theorem 2.1.

Recall that $G \wr S_n = K(1_G, \mathbf{e}, n)$ and $G \wr A_n = K(1_G, \mathbf{sgn}, n)$. The next results are corollaries to Theorem 2.1.

Corollary 2.7 ([10, 11, 15, 16]) *We have*

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(A, G\wr S_n)|}{|G|^n n!} X^n = \exp\left(\sum_{B\in\mathcal{F}_A} \frac{|\mathrm{Hom}(B,G)|}{|G|\,|A:B|} X^{|A:B|}\right).$$

Corollary 2.8 ([16]) *We have*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(A, G\wr A_n)|}{|G|^n n!} X^n \\
&\qquad = \frac{1}{|A:\Phi_2(A)|} \sum_{\overline{c}\in A/\Phi_2(A)} \exp\left(\sum_{B\in\mathcal{F}_A} \frac{\mathrm{sgn}_{\,A/B}(c)\cdot|\mathrm{Hom}(B,G)|}{|G|\,|A:B|} X^{|A:B|}\right).
\end{aligned}$$

Remark 2.9 When A is a finite cyclic group, Corollary 2.7 is shown in [3, 12] and Corollary 2.8 is shown in [3].

3. Imprimitive complex pseudo reflection groups and related groups

Keep the notation of Section 2, and suppose that $G = \langle \omega \rangle$ with ω a primitive mth root of unity in $\mathbb{C}$. Assume that for any integer i with $1 \leq i \leq s$, $\chi_i(\omega) = \omega^{q_i}$, where q_i is a positive integer. Let $B \in \mathcal{F}_A$, and define

$$\Phi_m(B) = \bigcap_{\alpha \in \mathrm{Hom}(B, \langle \omega \rangle)} \mathrm{Ker}\, \alpha.$$

Let $i \in \{1, \ldots, s\}$. Since the order of $\langle \omega \rangle / \mathrm{Ker}\, \chi_i$ divides r_i, it follows that $q_i r_i$ is a multiple of m. Then the order of $V_{A \to B/\Phi_m(B)}(c^{q_i})$ with $c \in A$ divides r_i. Hence, given $\overline{a} \in R_i$, $V_{A \to B/\Phi_m(B)}(c^{q_i})$ with $c \in \overline{a}$ is independent of the choice of c in $\overline{a}$.

Now define a homomorphism $F^{(q_1, \ldots, q_s)}_{A \to B/\Phi_m(B)}$ from R to $B/\Phi_m(B)$ by

$$F^{(q_1, \ldots, q_s)}_{A \to B/\Phi_m(B)}(\overline{c_1}, \ldots, \overline{c_s}) = V_{A \to B/\Phi_m(B)} \left(\prod_{i=1}^{s} c_i^{q_i} \right)$$

for all $(\overline{c_1}, \ldots, \overline{c_s}) \in R$. Let $c_1, \ldots, c_s \in A$. We can identify $\mathrm{Hom}(B, \langle \omega \rangle)$ with $\mathrm{Hom}(B/\Phi_m(B), \langle \omega \rangle)$. Hence it turns out that

$$\begin{aligned} \sum_{\kappa \in \mathrm{Hom}(B, \langle \omega \rangle)} \prod_{i=1}^{s} (\chi_i \circ \kappa)^{\otimes A}(c_i) &= \sum_{\kappa \in \mathrm{Hom}(B/\Phi_m(B), \langle \omega \rangle)} \prod_{i=1}^{s} \kappa \left(V_{A \to B/\Phi_m(B)}(c_i) \right)^{q_i} \\ &= \sum_{\kappa \in \mathrm{Hom}(B/\Phi_m(B), \langle \omega \rangle)} \kappa \left(F^{(q_1, \ldots, q_s)}_{A \to B/\Phi_m(B)}(\overline{c_1}, \ldots, \overline{c_s}) \right). \end{aligned}$$

Moreover, the $\mathbb{C}$-character

$$\sum_{\kappa \in \mathrm{Hom}(B/\Phi_m(B), \langle \omega \rangle)} \kappa$$

of $B/\Phi_m(B)$ is afforded by the left regular module $\mathbb{C}(B/\Phi_m(B))$. Thus

$$\sum_{\kappa \in \mathrm{Hom}(B, \langle \omega \rangle)} \prod_{i=1}^{s} (\chi_i \circ \kappa)^{\otimes A}(c_i) = \begin{cases} |B : \Phi_m(B)| & \text{if } (\overline{c_1}, \ldots, \overline{c_s}) \in \mathrm{Ker}\, F^{(q_1, \ldots, q_s)}_{A \to B/\Phi_m(B)}, \\ 0 & \text{otherwise.} \end{cases}$$

Combining the preceding fact with Theorem 2.1, we conclude that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(A, K(\chi_1, \ldots, \chi_s, \mathbf{z}_1, \ldots, \mathbf{z}_s, n))|}{m^n n!} X^n \\ &= \frac{1}{|R|} \sum_{(\overline{c_1}, \ldots, \overline{c_s}) \in R} \exp \left(\sum_{B \in \Omega_A(\overline{c_1}, \ldots, \overline{c_s})} \mathrm{sgn}_{A/B} \left(\prod_{i \in I} c_i \right) \frac{|B : \Phi_m(B)|}{m|A : B|} X^{|A:B|} \right), \end{aligned} \tag{V}$$

where

$$\Omega_A(\overline{c_1}, \ldots, \overline{c_s}) = \left\{ B \in \mathcal{F}_A \,\middle|\, (\overline{c_1}, \ldots, \overline{c_s}) \in \mathrm{Ker}\, F^{(q_1, \ldots, q_s)}_{A \to B/\Phi_m(B)} \right\}.$$

Remark 3.1 There exists a divisor r of m such that $K(\chi_1, \ldots, \chi_s, \mathbf{z}_1, \ldots, \mathbf{z}_s, n)$ is $G(m, r, n)$, $H(m, r, n)$, or $L(m, r, n)$.

The following theorem is an immediate consequence of the formula (V).

Theorem 3.2 *Let r be a divisor of m. Given $c \in A$, set*

$$\Omega_A(\overline{c}) = \{B \in \mathcal{F}_A \mid c^{m/r} \in \operatorname{Ker} V_{A \to B/\Phi_m(B)}\}.$$

Put $r_0 = r$ if r is even, and $r_0 = 2r$ if r is odd. Then

$$\sum_{n=0}^{\infty} \frac{|\operatorname{Hom}(A, G(m,r,n))|}{m^n n!} X^n = \frac{1}{|A : \Phi_r(A)|} \sum_{\overline{c} \in A/\Phi_r(A)} \exp\left(\sum_{B \in \Omega_A(\overline{c})} \frac{|B : \Phi_m(B)|}{m|A : B|} X^{|A:B|} \right),$$

$$\sum_{n=0}^{\infty} \frac{|\operatorname{Hom}(A, H(m,r,n))|}{m^n n!} X^n = \frac{1}{|A : \Phi_r(A)|\,|A : \Phi_2(A)|} \times \sum_{(\overline{c}_1, \overline{c}_2) \in (A/\Phi_r(A)) \times (A/\Phi_2(A))} \exp\left(\sum_{B \in \Omega_A(\overline{c}_1)} \operatorname{sgn}_{A/B}(c_2) \frac{|B : \Phi_m(B)|}{m|A : B|} X^{|A:B|} \right),$$

and

$$\sum_{n=0}^{\infty} \frac{|\operatorname{Hom}(A, L(m,r,n))|}{m^n n!} X^n = \frac{1}{|A : \Phi_{r_0}(A)|} \sum_{\overline{c} \in A/\Phi_{r_0}(A)} \exp\left(\sum_{B \in \Omega_A(\overline{c})} \operatorname{sgn}_{A/B}(c) \frac{|B : \Phi_m(B)|}{m|A : B|} X^{|A:B|} \right).$$

Corollary 3.3 ([16]) *Keep the notation of Theorem 3.2, and assume further that $m = r = 2$. Then*

$$\sum_{n=0}^{\infty} \frac{|\operatorname{Hom}(A, W(D_n))|}{2^n n!} X^n = \frac{1}{|A : \Phi_2(A)|} \sum_{\overline{c} \in A/\Phi_2(A)} \exp\left(\sum_{B \in \Omega_A(\overline{c})} \frac{|B : \Phi_2(B)|}{2|A : B|} X^{|A:B|} \right).$$

Example 3.4 Suppose that A is a finite cyclic group of order ℓ and is generated by an element c. Let p be a prime. For a subgroup B of A, we have

$$\operatorname{sgn}_{A/B}(c) = \begin{cases} 1 & \text{if } |A:B| \text{ is odd,} \\ -1 & \text{if } |A:B| \text{ is even,} \end{cases}$$

and $V_{A\to B/\Phi_p(B)}(c) = c^{|A:B|}\Phi_p(B)$. Considering A as $\mathbb{Z}/\ell\mathbb{Z}$, we obtain the following.

(1)

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}/\ell\mathbb{Z}, S_n)|}{n!} X^n = \exp\left(\sum_{d|\ell} \frac{1}{d} X^d\right).$$

(2)

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}/\ell\mathbb{Z}, A_n)|}{n!} X^n = \frac{1}{2}\exp\left(\sum_{d|\ell} \frac{1}{d} X^d\right) + \frac{1}{2}\exp\left(\sum_{d|\ell} \frac{(-1)^{d-1}}{d} X^d\right).$$

(3)

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}/\ell\mathbb{Z}, G(p,p,n))|}{p^n n!} X^n$$
$$= \frac{1}{p}\exp\left(\sum_{\substack{d|\ell \\ p\nmid(\ell/d)}} \frac{1}{pd} X^d\right)\left\{\exp\left(\sum_{\substack{d|\ell \\ p|(\ell/d)}} \frac{1}{d} X^d\right) + p - 1\right\}.$$

(4)

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}/\ell\mathbb{Z}, H(p,p,n))|}{p^n n!} X^n$$
$$= \frac{1}{2p}\exp\left(\sum_{\substack{d|\ell \\ p\nmid(\ell/d)}} \frac{1}{pd} X^d\right)\left\{\exp\left(\sum_{\substack{d|\ell \\ p|(\ell/d)}} \frac{1}{d} X^d\right) + p - 1\right\}$$
$$+ \frac{1}{2p}\exp\left(\sum_{\substack{d|\ell \\ p\nmid(\ell/d)}} \frac{(-1)^{d-1}}{pd} X^d\right)\left\{\exp\left(\sum_{\substack{d|\ell \\ p|(\ell/d)}} \frac{(-1)^{d-1}}{d} X^d\right) + p - 1\right\}.$$

(5)

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}/\ell\mathbb{Z}, L(2,2,n))|}{2^n n!} X^n$$
$$= \frac{1}{2}\exp\left(\sum_{\substack{d|\ell \\ 2\nmid(\ell/d)}} \frac{1}{2d} X^d\right)\left\{\exp\left(\sum_{\substack{d|\ell \\ 2|(\ell/d)}} \frac{1}{d} X^d\right) + \exp\left(-\sum_{\substack{d|\ell \\ 2\nmid(\ell/d),\, 2|d}} \frac{1}{d} X^d\right)\right\}.$$

Remark 3.5 The formula (1) is given in [2] and (2) is given in [13, Chapter 4, Problem 22] and [3]. When $p = 2$, the formula (3) is shown in [3].

4. Finite abelian p-groups

Suppose that A is a finite abelian group. Let $\widehat{A}$ be the set of irreducible $\mathbb{C}$-characters of A, and define a multiplication in $\widehat{A}$ by $\alpha_1\alpha_2(a) = \alpha_1(a)\alpha_2(a)$ for all α_1, $\alpha_2 \in \widehat{A}$ and $a \in A$. Then $\widehat{A}$ becomes a group, and the groups A and $\widehat{A}$ are isomorphic [7, 5.1]. If B is a subgroup of A, we put

$$B^\perp = \{\alpha \in \widehat{A} \mid \alpha(b) = 1 \text{ for all } b \in B\}.$$

If U is a subgroup of $\widehat{A}$, then we put

$$U^\perp = \{a \in A \mid \alpha(a) = 1 \text{ for all } \alpha \in U\}.$$

We use the following lemmas, which are parts of [7, 5.5, 5.6].

Lemma 4.1 ([7]) *Let B be a subgroup of A. Then*

$$\widehat{A/B} \cong B^\perp \qquad \text{and} \qquad \widehat{A}/B^\perp \cong \widehat{B}.$$

Lemma 4.2 ([7]) *Let B be a subgroup of A, and let U be a subgroup of $\widehat{A}$. Then*

$$B^{\perp\perp} = B \qquad \text{and} \qquad U^{\perp\perp} = U.$$

Lemma 4.3 ([7]) *Let B_1, B_2 be subgroups of A. Then*

$$(B_1 \cap B_2)^\perp = B_1^\perp B_2^\perp \qquad \text{and} \qquad (B_1B_2)^\perp = B_1^\perp \cap B_2^\perp.$$

Lemma 4.4 ([7]) *Let U_1, U_2 be subgroups of $\widehat{A}$. Then*

$$(U_1 \cap U_2)^\perp = U_1^\perp U_2^\perp \qquad \text{and} \qquad (U_1U_2)^\perp = U_1^\perp \cap U_2^\perp.$$

Let ϵ_A be the identity of A. For each positive integer k, we define

$$\Omega_k(A) = \{a \in A \mid a^k = \epsilon_A\} \qquad \text{and} \qquad \mho_k(A) = \{a^k \mid a \in A\}.$$

We provide a part of [7, 5.8], namely,

Lemma 4.5 ([7]) $\Omega_k(A)^\perp = \mho_k(\widehat{A})$, *and equivalently,* $\Omega_k(A) = \mho_k(\widehat{A})^\perp$.

A partition is a sequence $\lambda = (\lambda_1, \ldots, \lambda_t, \ldots)$ of nonnegative integers containing only finitely many non-zero terms where $\lambda_1 \geq \cdots \geq \lambda_t \geq \cdots$. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_t, \ldots)$, we define

$$m_i(\lambda) = \sharp\{t \mid \lambda_t = i\}$$

and

$$\lambda'_i = \sharp\{t \mid \lambda_t \geq i\}.$$

Then $\lambda' = (\lambda'_1, \ldots, \lambda'_i, \ldots)$ is a partition, and is called the conjugate of λ.

Let p be a prime. If P is a finite abelian p-group, then there is a unique partition $\lambda = (\lambda_1, \ldots, \lambda_\ell, 0, \ldots)$ such that P is isomorphic to the direct product

$$\mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_\ell}\mathbb{Z}$$

of cyclic p-groups $\mathbb{Z}/p^{\lambda_1}\mathbb{Z}, \ldots, \mathbb{Z}/p^{\lambda_\ell}\mathbb{Z}$, and we call λ the type of P.

Now let P be a finite abelian p-group, and let ϵ_P be the identity of P. We have

$$\Phi_p(P) = \mho_p(P) \qquad \text{and} \qquad P/\Phi_p(P) \cong \Omega_p(P).$$

In order to describe the Wohlfahrt series $E_P(X/p : \{G(p,p,n)\}_0^\infty)$, we must show the following.

Lemma 4.6 *Let P_0 be a subgroup of P. Suppose that $c \in P$ and $c \notin \Phi_p(P)$. Then $c \notin \operatorname{Ker} V_{P\to P_0/\Phi_p(P_0)}$ if and only if $P = \langle c\rangle P_0$ and P_0 contains $\Omega_p(P)$.*

Proof. We have $\Phi_p(P_0) = \mho_p(P_0)$ and $V_{P\to P_0/\Phi_p(P_0)}(c) = c^{|P:P_0|}\Phi_p(P_0)$. Assume that $c \notin \operatorname{Ker} V_{P\to P_0/\Phi_p(P_0)}$. Then $c^{|P:P_0|} \notin \mho_p(P_0)$, and thereby $P = \langle c\rangle P_0$. Moreover, if P_0 does not contain $\Omega_p(P)$, then $c^{|P:P_0|} = \epsilon_P$, contrary to the assumption. Hence P_0 contains $\Omega_p(P)$. Conversely, assume that $P = \langle c\rangle P_0$ and P_0 contains $\Omega_p(P)$. Since $c \notin \Phi_p(P)$, it follows that $c \notin \operatorname{Ker} V_{P\to P/\Phi_p(P)}$. Hence we assume that $P \neq P_0$. Clearly, $c^{|P:P_0|/p} \notin P_0$. Now suppose that $c^{|P:P_0|} \in \mho_p(P_0)$ and a is an element of P_0 such that $a^p = c^{|P:P_0|}$. Then $a^{-1}c^{|P:P_0|/p}$ is not contained in P_0 and is of order p. But every element of order p in P is contained in P_0. This is a contradiction. Thus $c^{|P:P_0|} \notin \mho_p(P_0)$, and hence $c \notin \operatorname{Ker} V_{P_0\to P_0/\Phi_p(P_0)}$, which proves the lemma. □

Suppose that P is of type $\lambda = (\lambda_1, \ldots, \lambda_\ell, 0, \ldots)$ and $P = \langle a_1\rangle \times \cdots \times \langle a_\ell\rangle$, where $\langle a_i\rangle$ is a cyclic group generated by a_i and is of order p^{λ_i}. We assume that $\lambda_\ell > 0$, and set

$$T(P) = \{a_1^{e_1} \cdots a_\ell^{e_\ell} \mid 0 \leq e_1, \ldots, e_\ell \leq p-1\},$$

which is a left transversal of $\Phi_p(P)$ in P. Given a positive integer j, we define $T_j(P)$ to be the set of all elements of order p^j in $T(P)$. Then

$$\sharp T_j(P) = p^{\lambda'_1 - \lambda'_{j+1}} - p^{\lambda'_1 - \lambda'_j}.$$

We have the following.

Lemma 4.7 *Suppose that $c \in T_j(P)$. Let k be a nonnegative integer, and let $\mathcal{M}(\langle c\rangle; k)$ be the set of all subgroups P_0 of P containing $\Omega_p(P)$ such that $P = \langle c\rangle P_0$ and $|P : P_0| = p^k$. Then*

$$\sharp\mathcal{M}(\langle c\rangle; k) = \begin{cases} 0 & \text{if} \quad k \geq j, \\ p^{w_\lambda(k)} & \text{if} \quad k < j, \end{cases}$$

where

$$w_\lambda(k) = \left\{ k \sum_{i=k+1}^{\lambda_1} m_i(\lambda) + \sum_{i=1}^{k} (i-1) m_i(\lambda) \right\} - k.$$

Proof. Suppose that $P_0 \in \mathcal{M}(\langle c\rangle; k)$. Then by Lemma 4.5, $P_0^\perp$ is contained in $\mho_p(\widehat{P})$. Since $P^\perp = \{1_P\}$, it follows from Lemma 4.3 that $\langle c\rangle^\perp \cap P_0^\perp = \{1_P\}$, where 1_P is the trivial character of P. Moreover, by Lemma 4.1 we have

$$P/P_0 \cong \widehat{P/P_0} \cong P_0^\perp.$$

Thus $P_0^\perp$ is a cyclic group of $\mho_p(\widehat{P})$ such that $\langle c\rangle^\perp \cap P_0^\perp = \{1_P\}$ and $|P_0^\perp| = p^k$.

Now let $\mathcal{N}(\langle c\rangle^\perp; k)$ be the set of all cyclic subgroups U of $\mho_p(\widehat{P})$ such that $\langle c\rangle^\perp \cap U = \{1_P\}$ and $|U| = p^k$. If $P_0 \in \mathcal{M}(\langle c\rangle; k)$, then by the preceding argument, $P_0^\perp \in \mathcal{N}(\langle c\rangle^\perp; k)$. Define a map f from $\mathcal{M}(\langle c\rangle; k)$ to $\mathcal{N}(\langle c\rangle^\perp; k)$ by $f(P_0) = P_0^\perp$ for all $P_0 \in \mathcal{M}(\langle c\rangle; k)$. Then Lemma 4.2 implies that f is injective.

Suppose that $U \in \mathcal{N}(\langle c\rangle^\perp; k)$. Then by Lemma 4.5, $U^\perp$ contains $\Omega_p(P)$. Since $\{1_P\}^\perp = P$, it follows from Lemmas 4.2 and 4.4 that $P = \langle c\rangle U^\perp$. Moreover, by Lemmas 4.1 and 4.2, we have

$$P/U^\perp \cong \widehat{P/U^\perp} \cong U,$$

whence $|P : U^\perp| = |U| = p^k$. Thus we obtain $U^\perp \in \mathcal{M}(\langle c\rangle; k)$. This fact, together with Lemma 4.2, means that f is surjective. Consequently, f is bijective.

In order to prove the statement, it suffices to verify that

$$\sharp\mathcal{N}(\langle c\rangle^\perp; k) = \begin{cases} 0 & \text{if} \quad k \geq j, \\ p^{w_\lambda(k)} & \text{if} \quad k < j. \end{cases}$$

Suppose that $c = a_1^{e_1} \cdots a_\ell^{e_\ell}$, where $e_1, \ldots, e_\ell$ are nonnegative integers less than p. Since $c \neq \epsilon_P$, we assume that $e_i = 0$ with $i < t_0$ and $e_{t_0} \neq 0$, where $1 \leq t_0 \leq \ell$. Put

$$D = \langle a_1\rangle \times \cdots \times \langle a_{t_0-1}\rangle \times \langle a_{t_0+1}\rangle \times \cdots \times \langle a_\ell\rangle.$$

Then $P = \langle c\rangle \times D$, and hence $\widehat{P} = \langle c\rangle^\perp \times D^\perp$ by Lemma 4.3. Moreover, it follows from Lemma 4.1 that

$$D^\perp \cong \widehat{P}/\langle c\rangle^\perp \cong \widehat{\langle c\rangle} \cong \langle c\rangle \quad \text{and} \quad \langle c\rangle^\perp \cong \widehat{P}/D^\perp \cong \widehat{D} \cong D.$$

Thus there exists a bijection from $\mathcal{N}(\langle c\rangle^{\perp};k)$ to the set $\mathcal{W}(D;k)$ of all cyclic subgroups Y of $\mho_p(P)$ such that $D\cap Y=\{\epsilon_P\}$ and $|Y|=p^k$. If $k\geq j$, then clearly $\mathcal{W}(D;k)=\emptyset$, and hence $\sharp\mathcal{N}(\langle c\rangle^{\perp};k)=\sharp\mathcal{W}(D;k)=0$. Suppose that $k<j$. We set $I_1=\{t\mid \lambda_t>k,\, t\neq t_0\}$ and $I_2=\{t\mid \lambda_t\leq k\}$. For each sequence $(n_1,\dots,n_{t_0-1},n_{t_0+1},\dots,n_\ell)$ of positive integers, put

$$y_{(n_1,\dots,n_{t_0-1},n_{t_0+1},\dots,n_\ell)}=c^{p^{j-k}}\left(\prod_{t\in I_1}a_t^{p^{\lambda_t-k}n_t}\right)\left(\prod_{t\in I_2}a_t^{pn_t}\right).$$

Then

$$\mathcal{W}(D;k)=\left\{\left\langle y_{(n_1,\dots,n_{t_0-1},n_{t_0+1},\dots,n_\ell)}\right\rangle \,\middle|\, \begin{array}{ll} 1\leq n_t\leq p^k & \text{if}\quad t\in I_1, \\ 1\leq n_t\leq p^{\lambda_t-1} & \text{if}\quad t\in I_2 \end{array}\right\},$$

and $\sharp\mathcal{W}(D;k)=p^{w_\lambda(k)}$. Thus we conclude that $\sharp\mathcal{N}(\langle c\rangle^{\perp};k)=p^{w_\lambda(k)}$, and the proof is completed. □

Theorem 3.2, together with Lemmas 4.6 and 4.7, enables us to get the following.

Theorem 4.8 *Keep the notation of Lemma* 4.7. *We have*

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{|\mathrm{Hom}(P,G(p,p,n))|}{p^n n!}X^n\\ &=\frac{1}{p^\ell}\left\{\sum_{n=0}^{\infty}\frac{|\mathrm{Hom}(P,(\mathbb{Z}/p\mathbb{Z})\wr S_n)|}{p^n n!}X^n\right\}\\ &\qquad\times\left\{1+\sum_{j\geq 1}(p^{\lambda_1'-\lambda_{j+1}'}-p^{\lambda_1'-\lambda_j'})\exp\left(-p^{\ell-1}\sum_{k=0}^{j-1}p^{w_\lambda(k)-k}X^{p^k}\right)\right\}.\end{aligned}$$

We now turn to the forms of $E_P(X/2:\{L(2,2,n)\}_0^\infty)$ and $E_P(X:\{A_n\}_0^\infty)$. First, we need a consequence of [15, Lemma 2.1], namely,

Lemma 4.9 *Let P_0 be a subgroup of P, and let $c\in P$. Then* $\operatorname{sgn}_{P/P_0}(c)=-1$ *if and only if $P\neq P_0$ and $P=\langle c\rangle P_0$.*

The proof of the next lemma is straightforward.

Lemma 4.10 *Let P_0 be a subgroup of P, and let $c\in P-\{\epsilon_P\}$. Then $P=\langle c\rangle\times P_0$ if and only if $P=\langle c\rangle P_0$ and P_0 does not contain $\Omega_p(P)$.*

By an argument similar to that in the proof of Lemma 4.7, we get the following.

Lemma 4.11 *Suppose that $c \in T_j(P)$. Let k be a nonnegative integer. Then the number of all subgroups P_0 of P such that $P = \langle c\rangle P_0$ and $|P : P_0| = p^k$ is 0 if $k > j$, and is $p^{s_\lambda(k)}$ if $k \leq j$, where*

$$s_\lambda(k) = \left\{ k \sum_{i=k+1}^{\lambda_1} m_i(\lambda) + \sum_{i=1}^{k} i m_i(\lambda) \right\} - k.$$

Combining Theorem 3.2 with Lemmas 4.6, 4.9, 4.10, and 4.11, we can now state the following.

Theorem 4.12 *Keep the notation of Lemma 4.11, and assume further that $p = 2$. Then*

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(P, L(2,2,n))|}{2^n n!} X^n \\ &\quad = \frac{1}{2^\ell} \left\{ \sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(P, (\mathbb{Z}/2\mathbb{Z}) \wr S_n)|}{2^n n!} X^n \right\} \\ &\qquad \times \left\{ 1 + \sum_{j \geq 1} (2^{\lambda'_1 - \lambda'_{j+1}} - 2^{\lambda'_1 - \lambda'_j}) \exp\left(-2^{\ell-1} \sum_{k=0}^{j} 2^{s_\lambda(k)-k} X^{2^k} \right) \right\}, \end{aligned}$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(P, A_n)|}{n!} X^n \\ &\quad = \frac{1}{2^\ell} \left\{ \sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(P, S_n)|}{n!} X^n \right\} \\ &\qquad \times \left\{ 1 + \sum_{j \geq 1} (2^{\lambda'_1 - \lambda'_{j+1}} - 2^{\lambda'_1 - \lambda'_j}) \exp\left(-2 \sum_{k=1}^{j} 2^{s_\lambda(k)-k} X^{2^k} \right) \right\}. \end{aligned}$$

Remark 4.13 The form of $E_P(X : \{A_n\}_0^\infty)$ in the theorem above is also a consequence of Lemma 4.11 and [15, Theorem 1.1].

5. Explicit formulas

Keep the notation of Section 4, and further assume that $\lambda_1 = \cdots = \lambda_{\ell-1} = u$ and $\lambda_\ell = v$, where $\ell \geq 1$ and $u \geq v > 0$. Then $P \simeq (\mathbb{Z}/p^u\mathbb{Z})^{(\ell-1)} \times \mathbb{Z}/p^v\mathbb{Z}$, whence $\sharp T_u(P) = p^\ell - p$ and $\sharp T_v(P) = p - 1$.

Example 5.1 By Theorem 4.8, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{|\mathrm{Hom}((\mathbb{Z}/p^u\mathbb{Z})^{(\ell-1)}\times\mathbb{Z}/p^v\mathbb{Z},G(p,p,n))|}{p^n n!}X^n\\&\quad=\frac{1}{p^\ell}\left\{\sum_{n=0}^{\infty}\frac{|\mathrm{Hom}((\mathbb{Z}/p^u\mathbb{Z})^{(\ell-1)}\times\mathbb{Z}/p^v\mathbb{Z},(\mathbb{Z}/p\mathbb{Z})\wr S_n)|}{p^n n!}X^n\right\}\\&\quad\times\left\{1+(p-1)\exp\left(-p^{\ell-1}\sum_{k=0}^{v-1}p^{(\ell-2)k}X^{p^k}\right)\right.\\&\qquad\left.+(p^\ell-p)\exp\left(-p^{\ell-1}\sum_{k=0}^{v-1}p^{(\ell-2)k}X^{p^k}-p^{\ell-1}\sum_{k=v}^{u-1}p^{(\ell-3)k+v-1}X^{p^k}\right)\right\}.\end{aligned}$$

By Theorem 4.12,

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{|\mathrm{Hom}((\mathbb{Z}/2^u\mathbb{Z})^{(\ell-1)}\times\mathbb{Z}/2^v\mathbb{Z},L(2,2,n))|}{2^n n!}X^n\\&\quad=\frac{1}{2^\ell}\left\{\sum_{n=0}^{\infty}\frac{|\mathrm{Hom}((\mathbb{Z}/2^u\mathbb{Z})^{(\ell-1)}\times\mathbb{Z}/2^v\mathbb{Z},(\mathbb{Z}/2\mathbb{Z})\wr S_n)|}{2^n n!}X^n\right\}\\&\quad\times\left\{1+\exp\left(-2^{\ell-1}\sum_{k=0}^{v}2^{(\ell-2)k}X^{2^k}\right)\right.\\&\qquad\left.+(2^\ell-2)\exp\left(-2^{\ell-1}\sum_{k=0}^{v}2^{(\ell-2)k}X^{2^k}-2^{\ell-1}\sum_{k=v+1}^{u}2^{(\ell-3)k+v}X^{2^k}\right)\right\},\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{|\mathrm{Hom}((\mathbb{Z}/2^u\mathbb{Z})^{(\ell-1)}\times\mathbb{Z}/2^v\mathbb{Z},A_n)|}{n!}X^n\\&\quad=\frac{1}{2^\ell}\left\{\sum_{n=0}^{\infty}\frac{|\mathrm{Hom}((\mathbb{Z}/2^u\mathbb{Z})^{(\ell-1)}\times\mathbb{Z}/2^v\mathbb{Z},S_n)|}{n!}X^n\right\}\\&\quad\times\left\{1+\exp\left(-2\sum_{k=1}^{v}2^{(\ell-2)k}X^{2^k}\right)\right.\\&\qquad\left.+(2^\ell-2)\exp\left(-2\sum_{k=1}^{v}2^{(\ell-2)k}X^{2^k}-2\sum_{k=v+1}^{u}2^{(\ell-3)k+v}X^{2^k}\right)\right\}.\end{aligned}$$

Remark 5.2 The formulas of $E_P(X:\{W(D_n)\}_0^\infty)$ and $E_P(X:\{A_n\}_0^\infty)$ where $P=(\mathbb{Z}/2^u\mathbb{Z})^{(\ell)}$ are due to Müller and Shareshian [11].

We next suppose that $P \cong \mathbb{Z}/p^u\mathbb{Z} \times \mathbb{Z}/p^v\mathbb{Z}$, where $u \geq v > 0$. Given a nonnegative integer k, let $N_P(k)$ be the number of subgroups of order p^k in P.

Proposition 5.3 *Let k be a nonnegative integer. Then*

$$N_P(k) = \begin{cases} 1+p+\cdots+p^k & \text{if } 0 \leq k < v, \\ 1+p+\cdots+p^v & \text{if } v \leq k \leq u, \\ 1+p+\cdots+p^{u+v-k} & \text{if } u < k \leq u+v. \end{cases}$$

Proof. We proceed by induction on $u+v$. Obviously, the assertion is true if $u+v=0$. Assume that $u+v>0$ and $P = \langle a \rangle \times \langle b \rangle$, where a has order p^u and b order p^v. Put $M = \langle a^p \rangle \times \langle b \rangle$. If $k < v$, then $N_P(k) = N_M(k)$ because every subgroup of order less than p^v is contained in M, and hence by the inductive assumption,

$$N_P(k) = 1+p+\cdots+p^k.$$

Case (1) Assume that $u = v$. Then by [14, Corollary], we obtain

$$N_P(v) = N_M(v-1) + p^v.$$

Hence by the inductive assumption,

$$N_P(v) = 1+p+\cdots+p^v.$$

Case (2) Assume that $u > v$. If $v \leq k < u$, then clearly $N_P(k) = N_M(k)$. Moreover, it follows from [14, Corollary] that

$$N_P(u) = N_M(u-1).$$

Hence if $v \leq k \leq u$, then by the inductive assumption,

$$N_P(k) = 1+p+\cdots+p^v.$$

Since $N_P(k) = N_P(u+v-k)$, the assertion of the proposition follows. □

It is easy to prove the following.

Lemma 5.4 *Let k be a positive integer. Then the number of cyclic subgroups of order p^k in P is $p^{k-1}+p^k$ if $0 < k \leq v$, and is p^v if $v < k \leq u$.*

The next result is a consequence of Proposition 5.3 and Lemma 5.4.

Proposition 5.5 *We have*

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}/p^u\mathbb{Z} \times \mathbb{Z}/p^v\mathbb{Z}, S_n)|}{p^n n!} X^n$$
$$= \exp\left(\sum_{k=0}^{v-1} \frac{1+\cdots+p^k}{p^k} X^{p^k} + \sum_{k=v}^{u} \frac{1+\cdots+p^v}{p^k} X^{p^k} + \sum_{k=u+1}^{u+v} \frac{1+\cdots+p^{u+v-k}}{p^k} X^{p^k}\right)$$

and

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}/p^u\mathbb{Z} \times \mathbb{Z}/p^v\mathbb{Z}, (\mathbb{Z}/p\mathbb{Z}) \wr S_n)|}{p^n n!} X^n$$
$$= \exp\left(\sum_{k=0}^{v-1} \frac{p+\cdots+p^{k+1}}{p^k} X^{p^k} + \sum_{k=v}^{u-1} \frac{p+\cdots+p^v}{p^k} X^{p^k} + \sum_{k=v}^{u-1} \frac{p^v}{p^k} X^{p^k} + \sum_{k=u}^{u+v-1} \frac{p+\cdots+p^{u+v-k}}{p^k} X^{p^k} + \sum_{k=u}^{u+v} \frac{p^{u+v-k-1}}{p^k} X^{p^k}\right).$$

We are now in position to determine the form of $E_P(X/p : \{G(p,p,n)\}_0^{\infty})$, $E_P(X/2 : \{L(2,2,n)\}_0^{\infty})$, and $E_P(X : \{A_n\}_0^{\infty})$.

Theorem 5.6 *We have*

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}/p^u\mathbb{Z} \times \mathbb{Z}/p^v\mathbb{Z}, G(p,p,n))|}{p^n n!} X^n$$
$$= \frac{1}{p^2} \exp\left(\sum_{k=1}^{v-1} \frac{p+\cdots+p^k}{p^k} X^{p^k} + \sum_{k=v}^{u-1} \frac{p+\cdots+p^v}{p^k} X^{p^k} + \sum_{k=u}^{u+v-1} \frac{p+\cdots+p^{u+v-k}}{p^k} X^{p^k} + \sum_{k=u}^{u+v} \frac{p^{u+v-k-1}}{p^k} X^{p^k}\right)$$
$$\times \left\{\exp\left(\sum_{k=0}^{v-1} pX^{p^k} + \sum_{k=v}^{u-1} \frac{p^v}{p^k} X^{p^k}\right) + (p-1)\exp\left(\sum_{k=v}^{u-1} \frac{p^v}{p^k} X^{p^k}\right) + p(p-1)\right\},$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}/2^u\mathbb{Z} \times \mathbb{Z}/2^v\mathbb{Z}, L(2,2,n))|}{2^n n!} X^n \\
&\quad = \frac{1}{2^2} \exp\left(\sum_{k=1}^{v-1} \frac{2+\cdots+2^k}{2^k} X^{2^k} + \sum_{k=v}^{u-1} \frac{2+\cdots+2^v}{2^k} X^{2^k} - \sum_{k=v}^{u-1} \frac{2^v}{2^k} X^{2^k} \right. \\
&\qquad\qquad \left. - \frac{2}{2^u} X^{2^u} + \sum_{k=u+1}^{u+v-1} \frac{2+\cdots+2^{u+v-k}}{2^k} X^{2^k} + \sum_{k=u}^{u+v} \frac{2^{u+v-k-1}}{2^k} X^{2^k} \right) \\
&\qquad \times \left\{ \exp\left(\sum_{k=0}^{v} 2X^{2^k} + \sum_{k=v+1}^{u} \frac{2^{v+1}}{2^k} X^{2^k} \right) + \exp\left(\sum_{k=v+1}^{u} \frac{2^{v+1}}{2^k} X^{2^k} \right) + 2 \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}/2^u\mathbb{Z} \times \mathbb{Z}/2^v\mathbb{Z}, A_n)|}{n!} X^n \\
&\quad = \frac{1}{2^2} \exp\left(X - \sum_{k=1}^{u} \frac{1}{2^k} X^{2^k} + \sum_{k=u+1}^{u+v} \frac{1+\cdots+2^{u+v-k}}{2^k} X^{2^k} \right) \\
&\qquad \times \left\{ \exp\left(\sum_{k=1}^{v} 2X^{2^k} + \sum_{k=v+1}^{u} \frac{2^{v+1}}{2^k} X^{2^k} \right) + \exp\left(\sum_{k=v+1}^{u} \frac{2^{v+1}}{2^k} X^{2^k} \right) + 2 \right\}.
\end{aligned}
$$

Remark 5.7 In [15, Exapmle 6.2], the formula of $E_P(X/2 : \{W(D_n)\}_0^\infty)$, where $P = \mathbb{Z}/2^u\mathbb{Z} \times \mathbb{Z}/2^v\mathbb{Z}$, is not correct, and neither is the formula of $E_P(X : \{A_n\}_0^\infty)$; either of them has a wrong term.

6. The additive group of p-adic integers

Let $\mathbb{Z}_p$ be the additive group of p-adic integers. The subgroups of finite index in $\mathbb{Z}_p$ are $p^k\mathbb{Z}_p$, $k = 0, 1, 2, \ldots$. Moreover, $\mathbb{Z}_p/p^k\mathbb{Z}_p \cong \mathbb{Z}/p^k\mathbb{Z}$ for each nonnegative integer k. In [6] Dress and Yoshida pointed out that

$$
\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}_p, S_n)|}{n!} X^n = \exp\left(\sum_{k=0}^{\infty} \frac{1}{p^k} X^{p^k} \right);
$$

this is called the Artin-Hasse exponential. We conclude this paper with a presentation of the following consequences of Theorem 3.2 :

$$
\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}_2, A_n)|}{n!} X^n = \frac{1}{2} \exp\left(\sum_{k=0}^{\infty} \frac{1}{2^k} X^{2^k} \right) + \frac{1}{2} \exp\left(X - \sum_{k=1}^{\infty} \frac{1}{2^k} X^{2^k} \right);
$$

$$
\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(\mathbb{Z}_p, G(p,p,n))|}{p^n n!} X^n = \frac{1}{p} \exp\left(\sum_{k=0}^{\infty} \frac{1}{p^k} X^{p^k} \right) + \frac{p-1}{p}.
$$

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