Basal Relativities in the Space X

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journal or publication title	Memoirs of the Muroran Institute of Technology
volume	4
number	3
page range	797-816
year	1964-06-30
URL	http://hdl.handle.net/10258/3214

Basal Relativities in the Space X

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Abstract

The space X, expounded in the provious papers[1),2)], is reinvestigated from the viewpoint of '*hypothetical objectification*' so as to establish a renovated system of logical rudiments. '*Trans-induction*' is a specially important notion, which is posited as a renovated modification of 'transfinite induction'. Basis of a subspace is classified according to several properties, by which relativities among subspaces may be distinctly observed. It may be most characteristic of our theory that we do not use any state to be satisfied 'almost everywhere'.

1. Hypothetical Objectification of the Space X

The space X is given as a linear space, which is the aggregation of such vectors that

$$x=(x(\xi))_{\xi\in\Xi}$$

$x(\xi)$ being ξ-component of x, complex-valued. Ξ is called the (*coordinate*) *indication of* X, and is thought as a metric space provided with a normal measure $\tilde{\mu}$, by which $\tilde{\mu}\Xi=1$. Density of the set Ξ is firstly assumed to be larger than enumerability.

If p_ξ is the probability that

$$0\leqslant \arg x(\xi)<2\left(1-\frac{1}{n}\right)\pi \tag{1. 1}$$

then

$$p_\xi=1-\frac{1}{n}$$

provided that $0\leqslant \arg x(\xi)<2\pi$. Therefore, if p is the probability that (1. 1) is observed for all $\xi\in\Xi$, we may have

$$0\leqslant p\leqslant \lim_{\nu\to\infty}\left(1-\frac{1}{n}\right)^{\nu}\equiv \text{\textcircled{$\triangle$}}_1.$$

Moreover, it may be written as

$$\log p\leqslant \mathfrak{m}_\Xi\cdot\log \text{\textcircled{$\triangle$}}_1 \tag{1. 2}$$

denoting by $\mathfrak{m}_\Xi$ the density of Ξ. As $\mathfrak{m}_\xi$ is larger than enumerability, the quantity p cannot be taken as a practically realizable one to make a probability, when we deny that

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$$p = \circledcirc \text{ (empty null}^{2),3)}).$$

(1. 2) is consequently thought to stand on the same ground as the non-practical meaning implied in *Zermelo's* axion of choice. Thus it seems that we must give up defining the space $\mathbf{X}$ within the practical observation.

We firstly define $\boldsymbol{X}$ as such that

$$\boldsymbol{X} \ni x \bowtie x = \mathfrak{S} x(\xi)\, \partial_\xi \tag{1. 3}$$

∂_ξ being the characteristic function of the point set $\{\xi\}$. Such a definition may be taken as a *hypothetical objectification* of the space $\boldsymbol{X}$. (1. 3) may mean that $\boldsymbol{X}$ is the aggregation of vectors of which all the components are finite complex numbers, and that may answer our purpose well. However, it may be said that we here restrict the axiom of choice to be but once used in the formulation (1. 3). So, the hypothetical objectification is, in effect, a sort of pure formularism.

When we test computations to build up a course of analysis on the space $\boldsymbol{X}$ provided with the fundamental formulation (1. 3), we shall find that many things are still left undecided. On the ground of metrization in the space Ξ, we introduce a scalar product $(\;|\;)$ in the form

$$(x|y) = \mathfrak{S} x(\xi)\, \overline{y(\xi)}\, \mu_\xi \; (\mu_\xi = \mu), \tag{1. 4}$$

which naturally accompanies the norm $\|\;\|$ by the relation

$$\|x\|^2 = (x|x).$$

Then, the following three cases may be cited for our criticism: (i) $(x|y) = \infty$, (ii) $(x|y) = \triangle\!\!\!\!\bigcirc$ (infinitesimal) and (iii) $(x|y) =$ indefinite. From our viewpoint, the case (i) is not thought critical, as it gives but a very natural state in which the value of $(x|y)$ is computed as larger than any finite stretch. It will make a characteristic point of our theory that we regard the infinitesimal quantity $\triangle\!\!\!\!\bigcirc$ as but a basic element of computation. In effect, we compute it as

$$\|\partial_\xi\|^2 = \mu \; (= \triangle\!\!\!\!\bigcirc).$$

So, we cannot identify the quantity $\triangle\!\!\!\!\bigcirc$ with the usual zero, and this is one of the reasons why we introduced the symbol $\circledcirc$, say *empty null*[2),3)], which will be used instead of zero, in our analysis. Lastly, only the case (iii) is left to make a truly singular case. Thus gradually, the course of our analysis is proceeded and this will make the way of our hypothetical objectification, too.

With respect to a family of vectors $(z_\lambda)_{\lambda \in \Lambda}$ in $\boldsymbol{X}$, if a vector z defined as

$$z = \mathfrak{S} z_\lambda \tag{1. 5}$$

is really a vector in $\boldsymbol{X}$,

$$\mathfrak{S}_{\lambda \in \Lambda} z_\lambda(\xi) \tag{1. 6}$$

must be a finite complex number for any $\xi \in \Xi$, because $z(\xi)$ must be so for any

ξ. Besides, the summation (1.6) should generally be understood as of random proceed. So, we shall naturally conform to the riemannian law i.e.:

$$\mathfrak{S}_{\lambda\in\Lambda}|z_\lambda(\xi)| \textit{ be convergent}.$$

Such being the case, we may reach the following conclusion.

Proposition. 1.1. *For the case that the vector* z *defined by* (1.5) *is a vector in* $\boldsymbol{X}$, *it is necessary and sufficient that the set*

$$\Lambda_{(\xi)}=\{\lambda: z_\lambda(\xi)\neq 0\}$$

is at most enumerable for any $\xi\in\Xi$ *and the summation*

$$\sum_{\lambda\in\Lambda_{(\xi)}}|z_\lambda(\xi)|$$

is convergent for any $\xi\in\Xi$.

$(y_\lambda)_{\lambda\in\Lambda}$ be a family of vectors in $\boldsymbol{X}$. If

$$\mathfrak{S}c_\lambda y_\lambda=0 \triangleright c_\lambda=0 \text{ for each } \lambda$$

(c_λ) being a family of finite complex numbers, (y_λ) is called a (*linearly independent*) *system of vectors* (in $\boldsymbol{X}$). Orthogonality between two vectors x and y

$$x\perp y$$

is defined by the relation

$$(x|y)=\circledcirc^*.$$

If a system (y_λ) implies the relation

$$y_\lambda\perp y_\kappa$$

whenever $\lambda\neq\kappa$, then (y_λ) is an *orthogonal system.*

When we write it as

$$0<q<\infty$$

q cannot be an infinitesimal. But, when we write it as

$$\circledcirc<q=\triangle$$

q is a positive infinitesimal; and when

$$q=0$$

it means that

$$q=\circledcirc.$$

The integral

$$\tilde{\mu}(f)=\mathfrak{S}f(\xi)\,\mu_\xi=\mathfrak{S}_{f(\xi)<0}f(\xi)\,\mu_\xi+\mathfrak{S}_{f(\xi)>0}f(\xi)\,\mu_\xi$$

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* In this relation x or y or both of them may be of infinite norm.

lays a foundation for our analysis. This integral is not exactly of a practical category. For any partition of Ξ (Ξ_λ), the integral decomposition

$$\tilde{\mu}(f) = \underset{\lambda}{\mathfrak{S}}(\underset{\Xi_\lambda}{\mathfrak{S}} f(\xi)\mu_\xi)$$

is generally assumed to be possible. As it is, this integral is essentially of a hypothetical sort of category so that, if any contradiction occurs, the use of $\tilde{\mu}(f)$ shall be thereby stopped.

If $\boldsymbol{Y}$ is a subset of $\boldsymbol{X}$ and

$$x,\ y \in \boldsymbol{Y} \triangleright \alpha x + \beta y \in \boldsymbol{Y}$$

for any pair of complex numbers α and β, then $\boldsymbol{Y}$ is called a (*vector*) *subspace* in $\boldsymbol{X}$. If $\boldsymbol{Y}$ is a subspace in $\boldsymbol{X}$ and

$$\boldsymbol{Y} \ni y \bowtie y = \mathfrak{S} y(\lambda)\rho_\lambda$$

$(y(\lambda))$ being a family of (finite) complex numbers, then (ρ_λ) is called a *basis** of $\boldsymbol{Y}$. This is an analogous formulation to (1. 3). A subspace does not always have its basis.

If (z_λ) is a family of vectors and

$$z = \mathfrak{S} z_\lambda \ \&\ z \in \boldsymbol{X},$$

then we generally assume that the relation

$$(z|x) = \mathfrak{S}(z_\lambda|x)$$

effects for any $x \in \boldsymbol{X}$; this is called the *component law*. Then, by some computations it may be proved that:

Proposition 1. 2. *For the case that a system (ρ_λ) is a basis of a subspace, it is necessary and sufficient that the set*

$$\Lambda_{(\xi)} = \{\lambda :\ \lambda \in \Lambda\ \&\ \rho_\lambda(\xi) \neq 0\}$$

is a finite set for any $\xi \in \Xi$.

By the component law and Proposition 1. 2 it may be easily proved that:

Proposition 1. 3. *When $(\rho_\lambda)_{\lambda\in\Lambda}$ is a basis af $\boldsymbol{Y}$,*

$$(\boldsymbol{Y} \ni y\ \&\ \boldsymbol{X} \ni x) \triangleright (y|x) = \mathfrak{S} y(\lambda)(\rho_\lambda|x)\,; \qquad (1.\ 7)_1$$

$$(\boldsymbol{Y} \ni y,\ z) \triangleright (y|z) = \underset{\lambda}{\mathfrak{S}}\underset{\lambda'}{\mathfrak{S}} y(\lambda)\overline{z(\lambda')}(\rho_\lambda|\rho_{\lambda'}) \qquad (1.\ 7)_2$$

$$= \underset{\lambda'}{\mathfrak{S}}\underset{\lambda}{\mathfrak{S}} y(\lambda)\overline{z(\lambda')}(\rho_\lambda|\rho_{\lambda'})\,. \qquad (1.\ 7)_3$$

$(1.\ 7)_k$ $(k=1, 2, 3)$ are also called *component laws.*

Thus far, we have illustrated the outline of the hypothetical objectification. Finally it is notable that, if any practical objects are taken to be tested on our analysis, no contradiction is to be found in computation. This is but an expecta-

* This definition is notably stricter than the one commonly used in classical books.

tion on our side, but makes a radical ground of estimation, though hypothetical. If we really meet a contradiction, we will make any correction to it and continue the study.

2. Reaxile Extension and Local Projectivity

The set

$$\Xi_x = \{\xi : \xi \in \Xi \ \& \ x(\xi) \neq 0)$$

is called the (*defining*) *support of vector* x, and the set

$$\Xi_\xi \,(=\Xi_\xi(\boldsymbol{Y})) = \cap \, \Xi_x \, (x \in \boldsymbol{Y} \ \& \ x(\xi) \neq 0)$$

is called a (*supporting*) *scale of a subspace* $\boldsymbol{Y}$. The scales are either identically equal or disjoint with each other[1], so that we may assume the set of distinct scales to be given in the indexed form

$$(\Xi_\iota)_{\iota \in I}.$$

Then, it is proved that[1] there exist vectors e_ι of which the supports are Ξ_ι respectively such that

$$\boldsymbol{Y} \subseteq \vee \, \boldsymbol{Y}_\iota \ \& \ \boldsymbol{Y}_\iota = \langle\!\langle e_\iota \rangle\!\rangle^*.$$

The family of vectors

$$\widetilde{B}(\boldsymbol{Y}) = (e_\iota)_{\iota \in I}$$

is called the *reaxilization of the subspace* $\boldsymbol{Y}$. We will denote the span of $\widetilde{B}(\boldsymbol{Y})$ by

$$\widetilde{\boldsymbol{Y}},$$

then we have

$$\boldsymbol{Y} \subseteq \widetilde{\boldsymbol{Y}},$$

but the equality does not always occur. So we call the subspace $\widetilde{\boldsymbol{Y}}$ the *reaxile extension of* $\boldsymbol{Y}$. When $\boldsymbol{Y} = \widetilde{\boldsymbol{Y}}$, we say $\boldsymbol{Y}$ is *base-separable* and when $\boldsymbol{Y} \neq \widetilde{\boldsymbol{Y}}$, *chain-based*. When $\boldsymbol{Y}$ is chain-based, any basis of $\boldsymbol{Y}$ is called a *chain-basis***.

The vector

$$P_x y = \frac{(y|x)}{\|x\|^2} x$$

is the (*proper*) *projection of a vector* y *on a vector* x. When $P_x y \in \boldsymbol{X}$, i.e. $(y|x)/\|x\|^2$ is a finite complex number, $P_x y$ is said to be *possible* and, otherwise, to be *impossible*. If $P_x y$ is possible, x is a y-*projective vector*, and if $P_x y$ is possible for all $y \in \boldsymbol{Y}$, then x is a $\boldsymbol{Y}$-*projective vector*. If $(\rho_\lambda)_{\lambda \in \Lambda}$ is a basis of $\boldsymbol{Y}$

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* $\langle\!\langle v \rangle\!\rangle$ indicates the space generated by a single vector v, i.e. the set of vectors cv, c being finite complex numbers.

** In the previous papers, the contents have been restricted within but the base-separable case, while, in this paper, the general case is taken up.

and each ρ_λ are $\boldsymbol{Y}$-projective, (ρ_λ) is called *a $\boldsymbol{Y}$-projective basis* and will be denoted as

$$B_\pi^*(\boldsymbol{Y}) = (\rho_\lambda)_{\lambda\in\Lambda}, \tag{2. 1}$$

if no confusion is expected. In case of (2. 1)

$$P_Y y = \mathfrak{S} P_\lambda y \quad (P_\lambda \equiv P_\rho)$$

is the *projection of y on $\boldsymbol{Y}$.*

In the theory of linear operator, if

$$L^2 = L$$

L is a *projector*, but, in our theory, such operators will be called *converters*, in distinction. Hereafter, we will use the term 'projectivity' instead of '$\boldsymbol{X}$-projectivity'. When $\boldsymbol{Y}\subset\boldsymbol{X}$, $\boldsymbol{Y}$-projectivity is, as it were, a *local projectivity*. When $(\rho_\lambda)_{\lambda\in\Lambda}$ and $(\rho_\nu)_{\nu\in N}$ are $\boldsymbol{Y}$-projective bases both, we may take it for granted that

$$\Lambda = N. \tag{2. 2}$$

But we use (2. 2) as a convention of notation, on condition that we may not use it as a basal relation to any demonstration. In this case,

$$z\in\boldsymbol{Y} \triangleright \frac{(z|\delta_\nu)}{\|\delta_\nu\|^2} = \mathfrak{S} z(\lambda)\frac{(\rho_\lambda|\delta_\nu)}{\|\delta_\nu\|^2}$$

and we can choose such z as

$$z(\lambda) = \frac{\|\delta_\nu\|^2}{(\rho_\lambda|\delta_\nu)} \quad \text{when } \delta_\nu \not\perp \rho_\lambda,$$

so that we may have

$$\frac{(z|\delta_\nu)}{\|\delta_\nu\|^2} = \mathfrak{S}1. \tag{2. 3}$$

If the summation on the right hand of (2. 3) is of infinite terms, the value is ∞. This is a contradiction, because $(z|\delta_\nu)/\|\delta_\nu\|^2$ must be the δ_ν-component of z with respect to δ_ν and therefore must be of finite value. Such being the case, we conclude:

Proposition 2. 1. *When $(\rho_\lambda)_{\lambda\in\Lambda}$ and (δ_ν) are $\boldsymbol{Y}$-projective bases both, the set*

$$\Lambda_\nu = \{\lambda:\ \lambda\in\Lambda\ \&\ \rho_\lambda \not\perp \delta_\nu\}$$

is a finite set for each $\nu\in\Lambda$.

By almost similar computations, we may have:

Proposition 2. 2. *If $B_\pi^*(\boldsymbol{Y}) = (\rho_\lambda)_{\lambda\in\Lambda}$ and $\tilde{B}(\boldsymbol{Y}) = (e_\iota)_{\iota\in I}$, the set*

$$\Lambda_\iota = \{\lambda:\ \lambda\in\Lambda\ \&\ \rho_\lambda \not\perp e_\iota\}$$

is a finite set for each $\iota\in I$.

When $\rho_\lambda,\ \rho_\nu\in B_\pi^*(\boldsymbol{Y})$ and $\rho_\lambda \not\perp \rho_\nu$,

$$\frac{(\rho_\lambda|\rho_\nu)}{\|\rho_\nu\|^2}=\overline{\left(\frac{(\rho_\nu|\rho_\lambda)}{\|\rho_\lambda\|^2}\right)}\cdot\frac{\|\rho_\lambda\|^2}{\|\rho_\nu\|^2}=a \text{ finite complex number} \neq 0.$$

Then, since $(\rho_\nu|\rho_\lambda)/\|\rho_\lambda\|^2$ must be also a finite complex number which$\neq 0$, we may conclude:

Proposition 2.3. *When* $B_\pi^*(\boldsymbol{Y})=(\rho_\lambda)_{\lambda\in\Lambda}$ and λ, $\nu\in\Lambda$,

$$\rho_\lambda \not\perp \rho_\nu \rhd 0<\|\rho_\lambda\|/\|\rho_\nu\|<\infty .$$

The aggregation of such vectors z that a fixed vector y is z-projective, apparently makes a subspace in $\boldsymbol{X}$; let it be denoted by

$$\boldsymbol{X}_{(y)} .$$

Then, assuming that $\boldsymbol{X}_{(y)}$ has a basis (y_λ), we have

$$\boldsymbol{X}_{(y)}\ni z \rhd z=\mathfrak{S}z(\lambda)y_\lambda \rhd \frac{(z|y)}{\|y\|^2}=\mathfrak{S}z(\lambda)\frac{(y_\lambda|y)}{\|y\|^2}. \tag{2.4}$$

Then, since all of $(z|y)/\|y\|^2$ and $(y_\lambda|y)/\|y\|^2$ are finite complex numbers, we may choose z such as

$$z(\lambda)=\frac{\|y\|^2}{(y_\lambda|y)} \quad \text{whenever } y_\lambda \not\perp y ,$$

so that the right-most hand of (2.4) takes the form

$$\mathfrak{S}1 ,$$

and therefore diverges except when the set

$$\Lambda_y=\{\lambda:\ \lambda\in\Lambda \ \&\ y_\lambda\not\perp y\} \tag{2.5}$$

is a finite set. On the other hand, if the support of y Ξ_y is an infinite set, for any $\xi\in\Xi_y$ we can choose z such as

$$\xi\in\Xi_z\subset\Xi_y$$

and

$$(z|y)/\|y\|^2=1/2 .$$

Then, about the vector

$$z_1=z-z(\xi)\partial_\xi$$

we have

$$(z_1|y)/\|y\|^2=\frac{(z|y)-z(\xi)\overline{y(\xi)}\,\|\partial_\xi\|^2}{\|y\|^2}=1/2$$

because $z(\xi)y(\xi)\|\partial_\xi\|^2/\|y\|^2=\triangle$. Hence it must be that

$$z,\ z_1\in\boldsymbol{X}_{(y)}$$

so that

$$\partial_\xi=\frac{z-z_1}{z(\xi)}\in\boldsymbol{X}_{(y)} .$$

Thus we have

$$\Xi_y \ni \xi \triangleright \partial_\xi \in \boldsymbol{X}_{(y)}\,,$$

but

$$(\partial_\xi|y) = \overline{y(\xi)}\,\|\partial_\xi\|^2 \neq \circledcirc$$

i.e.

$$\partial_\xi \not\perp y\,.$$

In this case the family $(\partial_\xi)_{\xi\in\Xi_y}$ can apparently be taken as a part of a basis of $\boldsymbol{X}_{(y)}$, so that we may have a contradiction between finiteness of the set (2. 5) and infiniteness of the set Ξ_y. Consequently, we have :

Proposition 2. 4. *For the case that $\boldsymbol{X}_{(y)}$ has a basis, it is necessary and sufficient that the suppport of y is a finite set* (i.e. *y is a projective vector*).

3. Trans-induction

Analysis on the space $\boldsymbol{X}$ may find its primitive steps in the set and measure theories devised on $\boldsymbol{X}$, which give formulations and make inferences possible among them to reach decisions, under the superintendence of '*consistency*'. Our theory is promised to be proceeded in such construction, on the ground of hypothetical objectification. By the way, it is thought very efficient to apply certain categorical devices named '*modes*', as motives to promote the logical reasoning. In this paper, one of them, the trans-inductive mode, will be illustrated to reach some important results. To attempt a general presentation of such a device may possibly accompany some leaks on practical criticism. So, in this paper, we restrict it to be stated within a subapace of $\boldsymbol{X}$ exactly conditioned.

When an individual (or a set of individuals, or every individual element of a set) P has a property $\mathfrak{p}$, we denote it as

$$P \subset \mathfrak{p}\,.$$

If it is generally observed that

$$(M_1 \subset M_2 \text{ and } M_2 \subset \mathfrak{p}) \triangleright (M_1 \subset \mathfrak{p})$$

then the property $\mathfrak{p}$ is called a *regressive* one. For a regressive property $\mathfrak{p}$ let us suppose that

$$M_0 \subset \mathfrak{p}$$
$$P_1 \bar{\in} M_0 \ \&\ P_1 \subset \mathfrak{p}$$
$$\cdots \quad \cdots \quad \cdots$$

and

$$P_{\eta+1} \bar{\in} M_0 \bigcup_{\iota\leqq\eta} P_\iota \subset \mathfrak{p} \ \&\ P_{\eta+1} \subset \mathfrak{p}$$

and so on. Then η may be thought as an ordinal number ; but we will not emphasize this point because we are rather taking a manner to avoid the theory of ordinal numbers. As it is, if the property of a simple ordered set will not be practically used except in the after part of P_1, we may symbolically let the elements of M_0 slip into the initial part

$$P_1,\ P_2, \cdots,$$

on condition that only the part of P_η and $P_{\eta+1}$ is to be taken as the principal part. In this meaning we will use the conventional representation

$$P_1,\ P_2, \cdots, P_\eta,\ P_{\eta+1}, \cdots \tag{3. 1}$$

instead of

$$M_0, P_1, \cdots, P_\eta, P_{\eta+1}, \cdots.$$

(3. 1) is called an *indicial disposition with respect to the regressive property* $\mathfrak{p}$. In regard to this composition let it be denoted as

$$\widetilde{P}_\eta = \bigcup_{\lambda<\eta} P_\lambda.$$

We will at times use the following disposition instead of (3. 1)

$$(M_0 \equiv)\ \widetilde{P}_1 \subset \widetilde{P}_2 \subset \cdots \subset \widetilde{P}_\eta \subset \widetilde{P}_{\eta+1} \subset \cdots. \tag{3. 1}'$$

Through all the stages above-mentioned, we assume the following conditions are satisfied:

(1) $$\bigvee_{\lambda<\eta} \langle\!\langle P_\lambda \rangle\!\rangle \subseteq \boldsymbol{Y} \text{ for each } \eta;$$

(2) $$P_\eta \in \boldsymbol{Y} - \bigvee_{\lambda<\eta} \langle\!\langle P_\lambda \rangle\!\rangle.$$

Now we posit two types of constructions to give rise to a method of induction:

[I] On denoting as

$$\overset{*}{\boldsymbol{Y}}_\eta = \boldsymbol{Y} - \bigvee_{\lambda<\eta} \langle\!\langle P_\lambda \rangle\!\rangle,$$

for any vector Q_η in $\overset{*}{\boldsymbol{Y}}_\eta$, by an exact method dependent on Q_η we can get a vector P_η in $\boldsymbol{Y}$ such that

$$Q_\eta \bar{\in} \overset{*}{\boldsymbol{Y}}_{\eta+1}.$$

[II] When a set S is found in $\boldsymbol{Y}$ such that

$$\bigvee_{P \in S} \langle\!\langle P \rangle\!\rangle = \boldsymbol{Y},$$

on denoting as

$$\overset{*}{S}_\eta = S \cap \overset{*}{\boldsymbol{Y}}_\eta,$$

for any vector Q_η in $\overset{*}{S}_\eta$, by an exact method dependent on Q_η we can get a vector P_η in $\boldsymbol{Y}$ such that

$$Q_\eta \bar{\in} \overset{*}{S}_{\eta+1}.$$

[I] or [II] is called *exhaustibility*. If the indicial disposition (3. 1) or (3. 1)′ with respect to $\mathfrak{p}$ is provided with exhaustibility [I] or [II], then (3. 1) or (3. 1)′ is said to give a *trans-inductive mode*; and, in this case, we assert the conclusion

$$\tilde{P} = \cup \tilde{P}_\gamma \subset \mathfrak{p} \text{ and } \boldsymbol{Y} = \vee \langle\langle P_\gamma \rangle\rangle . \tag{3. 2}$$

This process of conclusion is called *trans-induction*. However, it must be noted that there are some other cases than the above-mentioned ones, which are also said to give trans-inductive modes. The above-mentioned mode is, as it is, *of progressive type*. About the trans-induction *of regressive type*, we will later on state with examples.

Any system (i.e. any indepentent family of vectors) in $\boldsymbol{X}$ cannot always be a basis of a subspace in $\boldsymbol{X}$. So we will call a *basal system* a system satisfying the conditions of Proposition 1.2. Then, if a system (ρ_λ) is not basal, there exists at least one family of (finite) complex numbers $(x(\lambda))$ for at least one ξ such that

$$\underset{\lambda}{\mathfrak{S}} |x(\lambda)\rho_\lambda(\xi)| = \infty .$$

This being so, it may be understood that a $\boldsymbol{Y}$-projective or an orthogonal system cannot always be basal. When two basal systems generate the same subspace in $\boldsymbol{X}$, they are called (*mutually*) *equivalent*. If $(\rho_\lambda)_{\lambda\in\Lambda}$ and $(\delta_\nu)_{\nu\in N}$ are equivalent, we use the symbolical convention

$$\Lambda = N$$

on the same condition mentioned about (2. 2). When Λ is a finite set, this relation is surely true.

When $S=(\rho_\lambda)_{\lambda\in\Lambda}$ is a $\boldsymbol{Y}$-projective basal system in $\boldsymbol{Y}$ which contains an orthogonal system

$$S_0 = (\rho_\lambda)_{\lambda\in\Lambda_0} ,$$

if we fix ν_1 from $\Lambda-\Lambda_0$ and define a vector

$$\hat{\rho}_{\nu_1} = \rho_{\nu_1} - \underset{\lambda\in\Lambda_0}{\mathfrak{S}} P_\lambda \rho_{\nu_1} \quad (P_\lambda \equiv P_{\rho_\lambda}) ,$$

then, by Proposition 2. 1, the summation in the left hand is of finite terms, and moreover, as will be proved later on in Lemma 3. 1, the vector $\hat{\rho}_{\nu_1}$ is also $\boldsymbol{Y}$-projective. Besides,

$$\begin{aligned}(\hat{\rho}_{\nu_1}|\rho_{\lambda_0}) &= (\rho_{\nu_1} - \sum P_\lambda \rho_{\nu_1}|\rho_{\lambda_0}) \\ &= (\rho_{\nu_1}|\rho_{\lambda_0}) - \sum \frac{(\rho_{\nu_1}|\rho_\lambda)}{\|\rho_\lambda\|^2} (\rho_\lambda|\rho_{\lambda_0}) ,\end{aligned}$$

where by assumption we have $\rho_\lambda \perp \rho_{\lambda_0}$, i.e. $(\rho_\lambda|\rho_{\lambda_0}) = ◎$ if λ, $\lambda_0 \in \Lambda_0$ and $\lambda \neq \lambda_0$, so that then

$$\begin{aligned}&= (\rho_{\nu_1}|\rho_{\lambda_0}) - \frac{(\rho_{\nu_1}|\rho_{\lambda_0})}{\|\rho_{\lambda_0}\|^2} \|\rho_{\lambda_0}\|^2 \\ &= (\rho_{\nu_1}|\rho_{\lambda_0}) - (\rho_{\nu_1}|\rho_{\lambda_0}) .\end{aligned}$$

In this proceeding we are applying no other substitution than by means of definition formulas and nor other vanish than by the relation '$=\circledcirc$', so that we may conclude that the computation proceeded in the above results in

$$=\circledcirc .$$

Consequently we have

$$\hat{\rho}_{\nu_1} \perp \rho_{\lambda_0} \text{ for each } \lambda_0 \in \Lambda_0 .$$

Now let us take the following Λ_1 and S_1 instead of Λ_0 and S_0:

$$\Lambda_1 = \Lambda_0 \cup \{\nu_1\}$$

and

$$S_1 = (\hat{\rho}_\lambda)_{\lambda \in \Lambda_1}$$

on denoting as $\hat{\rho}_\lambda \equiv \rho_\lambda$ for each $\lambda \in \Lambda_0$. Then S_1 is a $\boldsymbol{Y}$-projective orthogonal basal system and apparently

$$\bigvee_{\lambda \in \Lambda_1} \langle\!\langle \hat{\rho}_\lambda \rangle\!\rangle \subseteq \boldsymbol{Y} .$$

So, let us repeat the same process as the above-mentioned, on letting any ν_2 from $\Lambda - \Lambda_1$ be fixed. For the general step, denoting as

$$\Lambda_\kappa = (\bigcup_{\iota < \kappa} \Lambda_\iota) \cup \{\nu_\kappa\}$$

and

$$\bigcup_{\iota < \kappa} \Lambda_\iota = \Lambda_{\kappa(-1)} ,$$

let us define a vector $\hat{\rho}_{\nu_\kappa}$ in the form

$$\hat{\rho}_{\nu_\kappa} = \rho_{\nu_\kappa} - \sum_{\lambda \in \Lambda_{\kappa(-1)}} \hat{P}_\lambda \rho_{\nu_\kappa} \quad (\hat{P}_\lambda \equiv P_{\hat{\rho}_\lambda})$$

and define such as

$$S_\kappa = (\hat{\rho}_\lambda)_{\lambda \in \Lambda_\kappa} .$$

Then, by quite the same way with the above-stated, it is concluded that

$$\hat{\rho}_\lambda \perp \hat{\rho}_{\nu_\kappa} \text{ for each } \lambda \in \Lambda_{\kappa(-1)}$$

and

$$\bigvee_{\lambda \in \Lambda_\kappa} \langle\!\langle \hat{\rho}_\lambda \rangle\!\rangle \subseteq \boldsymbol{Y} .$$

On repeating this process, we may gain an indicial disposition

$$S_0 \subset S_1 \subset \cdots \subset S_\kappa \subset S_{\kappa+1} \subset \cdots \tag{3. 3}$$

with respect to the property that

$$S_\kappa \subset \mathfrak{p} \text{ or } \hat{\rho}_{\nu_\kappa} \subset \mathfrak{p}$$

means that S_κ is an orthogonal $\boldsymbol{Y}$-projective basal system. Since the increase of (3. 3) does not stop till $\Lambda - \cup \Lambda_\kappa$ becomes empty, we have a trans-inductive mode by (3. 3) provided with exhaustibility of type [II]. So it seems that we may conclude that

$$\tilde{S} = \cup S_\kappa = (\hat{\rho}_\lambda)_{\lambda\in\Lambda}$$

is an orthogonal basal system in $\boldsymbol{Y}$ and equivalent to $(\rho_\lambda)_{\lambda\in\Lambda}$. But, to tell the truth, it still needs demonstration that $\tilde{S}$ is really a basal system.

Let us define $\kappa(\nu)$ such that

$$\kappa = \kappa(\nu)$$

when $\nu_\kappa = \nu$; then by definition $\hat{\rho}_\nu$ may be expressed in the form

$$\hat{\rho}_\nu = \rho_\nu - \sum_{\lambda\in\Lambda_{\kappa(\nu)(-1)}} c_{\nu\lambda}(\rho_\nu|\rho_\lambda)\rho_\lambda \tag{3. 4}$$

and also in the form

$$\hat{\rho}_\nu = \mathfrak{S}_{\lambda\in\Lambda_{\kappa(\nu)}} \hat{\rho}_\nu(\lambda)\rho_\lambda,$$

so that we may have

$$\hat{\rho}_\nu(\nu) = 1,\ \hat{\rho}_\nu(\lambda) = -c_{\nu\lambda}(\rho_\nu|\rho_\lambda) \text{ if } \lambda\in\Lambda_{\kappa(\nu)(-1)}$$

and

$$\hat{\rho}_\nu(\lambda) = 0 \text{ otherwise.}$$

If there simultaneously exists a value of ξ and a family of (finite) complex numbers $(z(\nu))_{\nu\in\Lambda}$ such that $\mathfrak{S}_\nu z(\nu)\hat{\rho}_\nu(\xi) = \infty$, then by (3. 4) we have

$$\infty = \mathfrak{S}_\nu z(\nu)\rho_\nu(\xi) - \mathfrak{S}_\nu\sum_\lambda z(\nu)c_{\nu\lambda}(\rho_\nu|\rho_\lambda)\rho_\lambda(\xi).$$

We may use the letter λ instead of ν on the first summation of the right hand, so that

$$= \mathfrak{S}_\lambda z(\lambda)\rho_\lambda(\xi) - \mathfrak{S}_\nu\sum_\lambda z(\nu)c_{\nu\lambda}(\rho_\nu|\rho_\lambda)\rho_\lambda(\xi)$$

$$= \mathfrak{S}_\lambda\big(z(\lambda) - \mathfrak{S}_\nu z(\nu)c_{\nu\lambda}(\rho_\nu|\rho_\lambda)\big)\rho_\lambda(\xi).$$

Then since, by Proposition 2. 1, the indices ν for which $(\rho_\nu|\rho_\lambda)\neq$ ◎ make at most a finite set, it may be written in the form

$$= \mathfrak{S}_\lambda\big(z(\lambda) - \sum_\nu z(\nu)c_{\nu\lambda}(\rho_\nu|\rho_\lambda)\big)\rho_\lambda(\xi),$$

where $c_{\nu\lambda}(\rho_\nu|\rho_\lambda)$, $z(\lambda)$ and $z(\nu)$ are all finite numbers, so that the numbers definined by

$$z'(\lambda) = z(\lambda) - \sum_\nu z(\nu)c_{\nu\lambda}(\rho_\nu|\rho_\lambda)$$

are all finite (complex) numbers. Consequently we have

$$\infty = \mathfrak{S}_\lambda z'(\lambda)\rho_\lambda(\xi)$$

for the family of finite complex numbers $(z'(\lambda))$, which gives a contradiction to the fact that (ρ_λ) is a basal system; hence it must be that $(\hat{\rho}_\lambda)$ is a basal system. Thus finishing the demonstration we conclude:

Proposition 3. 1. *If S is a $\boldsymbol{Y}$-projective basal system in $\boldsymbol{Y}$, there exists an orthogonal $\boldsymbol{Y}$-projective basal system S' equivalent to S. In addition, if S contains an orthogonal system S_0 within, then S' can be made to contain S_0.*

Lemma 3. 1. *If a $\boldsymbol{Y}$-projective basal system $(\rho_\lambda)_{\lambda\in\Lambda}$ generates a subspace in $\boldsymbol{Y}$, which contains a subspace $\boldsymbol{Z}$ generated by an orthogonal $\boldsymbol{Y}$-projective basis $(\delta_\kappa)_{\kappa\in K}$, then if*

$$\hat{\rho}_\lambda = \rho_\lambda - P_Z\rho_\lambda$$

it must be either $\hat{\rho}_\lambda=0$ or $\hat{\rho}_\lambda$ is a $\boldsymbol{Y}$-projective vector.

Demonstration. By Proposition 2. 1 it can be written as

$$\hat{\rho}_\lambda = \rho_\lambda - \sum_\kappa P_\kappa\rho_\lambda \quad (P_\kappa \equiv P_{\delta_\kappa})$$

and the summation in the right hand is of finite terms, so the following two relations may be directly verified:

$$(\hat{\rho}_\lambda|\rho_\lambda)/\|\rho_\lambda\|^2 = 1 - \sum_\kappa \frac{|(\rho_\lambda|\delta_\kappa)|^2}{\|\rho_\lambda\|^2\|\delta_\kappa\|^2}$$

and

$$\|\hat{\rho}_\lambda\|^2/\|\rho_\lambda\|^2 = 1 - \sum_\kappa \frac{|(\rho_\lambda|\delta_\kappa)|^2}{\|\rho_\lambda\|^2\|\delta_\kappa\|^2}\,. \tag{3. 5}$$

In these computations, we are applying no other substitutions than by means of formulations of component law and nor other vanish than by the relation '$=$◎', so that the value of (3. 5) should be either a finite complex number or $=$◎ (i.e. $\|\hat{\rho}_\lambda\|^2=$◎). In the latter case it is namely that $\rho_\lambda = P_Z\rho_\lambda$. Therefore, it must be the case that the value of (3. 5) is a finite complex number when $\rho_\lambda \bar{\in} \boldsymbol{Z}$. In this case, for any $y\in\boldsymbol{Y}$ we have

$$\begin{aligned}\frac{(y|\hat{\rho}_\lambda)}{\|\hat{\rho}_\lambda\|^2} &= \left(\frac{(y|\rho_\lambda)}{\|\rho_\lambda\|^2} - \sum_\kappa \frac{\overline{(\rho_\lambda|\delta_\kappa)}}{\|\delta_\kappa\|^2}\cdot\frac{(y|\delta_\kappa)}{\|\rho_\lambda\|^2}\right)\frac{\|\rho_\lambda\|^2}{\|\hat{\rho}_\lambda\|^2}\\ &= \left(\frac{(y|\rho_\lambda)}{\|\rho_\lambda\|^2} - \sum_\kappa \frac{(y|\delta_\kappa)}{\|\delta_\kappa\|^2}\cdot\frac{(\delta_\kappa|\rho_\lambda)}{\|\rho_\lambda\|^2}\right)\left(1-\sum_\kappa\frac{|(\rho_\lambda|\delta_\kappa)|^2}{\|\rho_\lambda\|^2\,\|\delta_\kappa\|^2}\right)^{-1}\end{aligned}$$

which appartly gives a finite complex value. Besides, values of c_κ defined by

$$c_\kappa = \frac{(y|\delta_\kappa)}{\|\delta_\kappa\|^2}$$

cannot be other than finite complex numbers, and when they vanish it is that they are $=$◎. Then the value

$$\frac{(y|\rho_\lambda)}{\|\rho_\lambda\|^2} - \sum \frac{(y|\delta_\kappa)}{\|\delta_\kappa\|^2}\cdot\frac{(\delta_\kappa|\rho_\lambda)}{\|\rho_\lambda\|^2} = (y - \sum_\kappa c_\kappa\delta_\kappa|\rho_\lambda)/\|\rho_\lambda\|^2$$

is $=$◎ if it vanishes, because ρ_λ is, by assumption, $\boldsymbol{Y}$-projective and $y-\sum c_\kappa\delta_\kappa\in\boldsymbol{Y}$. Thus we see $(y|\hat{\rho}_\lambda)/\|\hat{\rho}_\lambda\|^2$ is $=$◎ if it vanishes. The demonstration is hereby finished.

An orthogonal basal system holds a strong property. If $\boldsymbol{Y}$ is the span of an orthogonal basis (δ_ν)

$$\boldsymbol{Y} \in y \triangleright y = \mathfrak{S} y(\lambda)\delta_\lambda \triangleright (y|\delta_\nu) = y(\nu)\|\delta_\nu\|^2$$
$$\triangleright P_\nu y = y(\nu) \quad (P_\nu \equiv P_{\delta_\nu}),$$

hence δ_ν are all $\boldsymbol{Y}$-projective. Thus we have:

Lemma 3.2. *If $\boldsymbol{Y}$ is the span of an orthogonal basal system A, then A is a $\boldsymbol{Y}$-projective basis of $\boldsymbol{Y}$.*

When $\boldsymbol{Y}$ has a basis and S_0 is a basal system in $\boldsymbol{Y}$, it comes into question if $\boldsymbol{Y}$ may have a basis containing S_0. In fact, if $\boldsymbol{Y} = \bigvee_{k=1}^{\infty} \langle\langle \partial_{\xi_k} \rangle\rangle$ and

$$\rho_0 = \sum_{k=1}^{\infty} \partial_{\xi_k},$$
$$\cdots \cdots \cdots$$

the vectors

$$\rho_0, \partial_{\xi_1}, \cdots, \partial_{\xi_n}$$

apparently give an independent system when n is finite, but the infinite set of vectors

$$\rho_0, \partial_{\xi_1}, \partial_{\xi_2}, \cdots, \partial_{\xi_n}, \partial_{\xi_{n+1}}, \cdots$$

may not do so.

When there is a set of vectors $M = (z_\iota)_{\iota \in I}$ which does not give an independent system, if for any set of finite complex numbers $(y(\iota))$ it is observed that

$$y = \mathfrak{S} y(\iota) z_\iota \in \boldsymbol{Y}$$

and inversely for any vector $y \in \boldsymbol{Y}$ exists at least one set of finite complex numbers $(y(\iota))$ such that

$$y = \mathfrak{S} y(\iota) z_\iota,$$

then M is called a *super-basis* of $\boldsymbol{Y}$. If M is a super-basis of $\boldsymbol{Y}$, then there must exist at least one suffix κ in $\boldsymbol{I}$ such that z_κ may be composed in the form

$$z_\kappa = \mathfrak{S}_{\iota \neq \kappa} z_\kappa(\iota) z_\iota .$$

We remove one such suffix κ from I and define I_1 as

$$I_1 = I - \{\kappa\},$$

and if there still exists another suffix κ_1 such that

$$z_{\kappa_1} = \mathfrak{S}_{\substack{\iota \neq \kappa_1 \\ \iota \in I_1}} z_{\kappa_1}(\iota) z_\iota$$

then we remove κ_1 from I_1 and define I_2 as

$$I_2 = I_1 - \{\kappa_1\}.$$

On reiteration of this process we gain a disposition

$$I \supset I_1 \supset I_2 \supset \cdots \supset I_\eta \supset I_{\eta+1} \supset \cdots . \tag{3.6}$$

This succession shall not stop as long as I_η is found to be a super-basis. Thus we deem (3.6) to make a trans-inductive mode of regressive type, and we conclude:

Proposition 3.2. *If M is a super-basis of a subspace $\boldsymbol{Y}$ in $\boldsymbol{X}$, there exists a basis of $\boldsymbol{Y}$ contained in M.*

By a certain device, we can similarly reach the following result by trans-induction (of regressive type).

Proposition 3.3. *If a subspace $\boldsymbol{Y}$ has a basis and B_0 is a basal system contained in $\boldsymbol{Y}$, there exists a basis of $\boldsymbol{Y}$ containing B_0.*

4. Relative Supplement

For a vector z and a set M in $\boldsymbol{X}$, if

$$z \bar{\in} M \text{ and } z \perp y \text{ for all } y \in M,$$

z is said to be *orthogonal to* M and it is denoted as

$$M \perp z \text{ or } z \perp M.$$

And for two sets M and N in $\boldsymbol{X}$, if

$$M \cap N = \text{void and } M \perp z \text{ for all } z \in N,$$

M is said to be *orthogonal to* N and it is denoted as

$$M \perp N \text{ or } N \perp M$$

(the proof of the commutative relation being omitted).

If a subspace $\boldsymbol{Y}$ contains a $\boldsymbol{Y}$-projective basal system $(\rho_\kappa)_{\kappa \in K}$ and

$$\boldsymbol{Z} = \vee \langle\!\langle \rho_\kappa \rangle\!\rangle ,$$

by Proposition 3.1 there exists a $\boldsymbol{Y}$-projective orthogonal basis $(\delta_\kappa)_{\kappa \in K}$ of $\boldsymbol{Z}$. Then, if we make a corresponding vector y' to any vector $y \in \boldsymbol{Y} - \boldsymbol{Z}$ by

$$y' = y - \mathfrak{S} P_\kappa y \quad (P_\kappa \equiv P_{\delta_\kappa}) ,$$

we have:

$$\begin{aligned} \boldsymbol{Z} \in z \,\rhd\, z &= \mathfrak{S} z(\kappa) \delta_\kappa \\ \rhd\, (y'|z) &= (y|z) - \underset{\iota}{\mathfrak{S}}\underset{\kappa}{\mathfrak{S}} (P_\iota y | z(\kappa) \delta_\kappa) \\ &= (y|z) - \underset{\kappa}{\mathfrak{S}}\underset{\iota}{\mathfrak{S}} \left(\frac{(y|\delta_\iota)}{\|\delta_\iota\|^2} \delta_\iota \Big| z(\kappa) \delta_\kappa \right) \\ &= (y|z) - \underset{\kappa}{\mathfrak{S}} \frac{(y|\delta_\kappa)}{\|\delta_\kappa\|^2} \overline{z(\kappa)} \|\delta_\kappa\|^2 \\ &= (y|z) - \underset{\kappa}{\mathfrak{S}} (y|\delta_\kappa) \overline{z(\kappa)} \\ &= (y|z) - (y | \underset{\kappa}{\mathfrak{S}} z(\kappa) \delta_\kappa) \\ &= (y|z) - (y|z) = \circledcirc . \end{aligned}$$

In this computation the commutation of summations $\underset{\iota}{\mathfrak{S}}$ and $\underset{\kappa}{\mathfrak{S}}$ needs no special inquiry, because these summations are essentially regarded as of random proceed and are used in the formal meaning of formulation irrespective of convergence. Thus we may conclude that

$$y' \perp \boldsymbol{Z}.$$

If $\boldsymbol{Z}'$ is the aggregation of such y', $\boldsymbol{Z}'$ is evidently disjoint with $\boldsymbol{Z}$, so that we consequently have

$$\boldsymbol{Z} \perp \boldsymbol{Z}'. \tag{4.1}$$

Moreover, $\boldsymbol{Z}'$ is apparently a subspace in $\boldsymbol{X}$ and stands in the relation

$$\boldsymbol{Y} = \boldsymbol{Z} \vee \boldsymbol{Z}'. \tag{4.2}$$

When (4. 1) and (4. 2) are simultaneously effected, $\boldsymbol{Z}'$ is called the (*orthogonal*) *$\boldsymbol{Y}$-supplement of $\boldsymbol{Z}$* and is denoted as

$$\boldsymbol{Z}' = \boldsymbol{Z}_{Y}{}^{\perp}.$$

Then, lastly we have:

Proposition 4. 1. *If $\boldsymbol{Z}$ is a subspace in a subspace $\boldsymbol{Y}$ generated by a $\boldsymbol{Y}$-projective basal system, there exists the orthogonal $\boldsymbol{Y}$-supplement $\boldsymbol{Z}_{Y}^{\perp}$.*

If $B_{\pi}^{*}(\boldsymbol{Y})$ exists, $\boldsymbol{Y}$ is said to *bear an immantent projectivity*, or simply to be *immanently projective*.

Proposition 4. 2. *When $\boldsymbol{Y}$ bears an immanent projectivity and $\boldsymbol{Z}_{Y}{}^{\perp}$ is a $\boldsymbol{Y}$-projective subspace in $\boldsymbol{Y}$, $\boldsymbol{Z}_{Y}^{\perp}$ is also a $\boldsymbol{Y}$-projective subspace in $\boldsymbol{Y}$.*

Demonstration. Let it be that $B_{\pi}^{*}(\boldsymbol{Y}) = (\rho_{\lambda})_{\lambda\in\Lambda}$. By proposition 3. 1 $\boldsymbol{Z}$ may be assumed to have a $\boldsymbol{Y}$-projective orthogonal basis $(\delta_{\kappa})_{\kappa\in K}$. Then, on defining as

$$\hat{\rho}_{\lambda} = \rho_{\lambda} - \mathfrak{S} P_{\kappa}\rho_{\lambda} \quad (P_{\kappa} \equiv P_{\delta_{\kappa}})$$

by Lemma 3. 1 we see that $\hat{\rho}_{\lambda}$ is a $\boldsymbol{Y}$-projective vector and the summation $\underset{\kappa}{\mathfrak{S}}$ is, by Proposition 2. 1, of finite terms. So we have:

$$\begin{aligned}
\boldsymbol{Z} \ni z \rhd z &= \mathfrak{S} z(\kappa')\delta_{\kappa'} \\
\rhd (z|\hat{\rho}_{\lambda}) &= \underset{\kappa'}{\mathfrak{S}} z(\kappa')(\delta_{\kappa'}|\hat{\rho}_{\lambda}) \\
&= \underset{\kappa'}{\mathfrak{S}} z(\kappa')\Big(\delta_{\kappa'}|\rho_{\lambda} - \sum_{\kappa} \frac{(\rho_{\lambda}|\delta_{\kappa})}{\|\delta_{\kappa}\|^{2}}\,\delta_{\kappa}\Big) \\
&= \underset{\kappa'}{\mathfrak{S}} z(\kappa')(\delta_{\kappa'}|\rho_{\lambda}) - \underset{\kappa'}{\mathfrak{S}} z(\kappa') \frac{(\rho_{\lambda}|\delta_{\kappa'})}{\|\delta_{\kappa'}\|^{2}}\,\|\delta_{\kappa'}\|^{2} \\
&= \circledcirc,
\end{aligned}$$

so that

$$(\hat{\rho}_{\lambda})_{\lambda\in\Lambda} \perp \boldsymbol{Z}.$$

Hence it is evident that $(\hat{\rho}_{\lambda})_{\lambda\in\Lambda}$ is either a basis or a super-basis of $\boldsymbol{Z}_{Y}{}^{\perp}$. In the latter case, by Proposition 3.2, we may take a subset of $(\hat{\rho}_\lambda)$ just as to make a $\boldsymbol{Y}$-projective basis of $\boldsymbol{Z}_Y{}^\perp$. Thus the demonstration is finished. If $\boldsymbol{Y}=\boldsymbol{X}$ and $\boldsymbol{Y}$ is used to indicate $\boldsymbol{Z}$ (in the statement of Proposition 4.2), we gain the following theorem previously known.

Corollary. *If $\boldsymbol{Y}$ is a projective subspace in $\boldsymbol{X}$, the orthogonal supplement $\boldsymbol{Y}^\perp$ is also projective.*

5. On Basal Decomposition

If $\boldsymbol{Y}$ is a base-separable subspace in $\boldsymbol{X}$, we have

$$\boldsymbol{Y}=\widetilde{\boldsymbol{Y}},$$

so that we may adopt its reaxilization

$$\widetilde{B}(\boldsymbol{Y})=(e_\iota)_{\iota\in I}$$

as its basis. In this case, if $(\rho_\lambda)_{\lambda\in\Lambda_0}$ is a $\boldsymbol{Y}$-projective basal system which generates subspace $\boldsymbol{Z}$, by Proposition 4.2 $\boldsymbol{Z}_Y{}^\perp$ is also $\boldsymbol{Y}$-projective since $\widetilde{B}(\boldsymbol{Y})$ is then a $\boldsymbol{Y}$-projective basis too. Hence, there exists a $\boldsymbol{Y}$-projective basis of $\boldsymbol{Z}_Y{}^\perp$, which may be denoted as $(\rho_\lambda)_{\lambda\in\Lambda_1}$. Then, if

$$\Lambda=\Lambda_0\cup\Lambda_1,$$

$(\rho_\lambda)_{\lambda\in\Lambda}$ is evidently a $B_\pi^*(\boldsymbol{Y})$ ($\boldsymbol{Y}$-projective basis of $\boldsymbol{Y}$). This being so, $(\rho_\lambda)_{\lambda\in\Lambda}$ and $\widetilde{B}(\boldsymbol{Y})$ are found to be equivalent, so that, by Proposition 2.1, the set

$$I_{(\lambda)}=\{\iota:\ \iota\in I\ \&\ e_\iota\perp\rho_\lambda\}$$

must be a finite set for all $\lambda\in\Lambda$. Therefore, ρ_λ is expressed in the form

$$\rho_\lambda=\sum_\iota\rho_\lambda(\iota)e_\iota \tag{5.1}$$

of which the summation of the right hand is of finite terms. This gives a necessary condition for the case

$$B=(\rho_\lambda)_{\lambda\in\Lambda_0}$$

is a $\boldsymbol{Y}$-projective basal system in the above-stated base-separable subspace $\boldsymbol{Y}$, but does not make a sufficient one. In effect, if

$$\boldsymbol{Y}=\langle\langle\partial_\xi\rangle\rangle\vee\langle\langle\partial_\eta\rangle\rangle\vee\langle\langle z\rangle\rangle$$

where $\xi\neq\eta$, $z(\xi)=z(\eta)=0$ and $\|z\|>0$, and if

$$B=\{\partial_\xi+\partial_\eta,\ \partial_\xi-\partial_\eta+z\},$$

then B is an orthogonal system which satisfies the condition (5.1), but is not $\boldsymbol{Y}$-projective. Because, if

$$\rho=\partial_\xi-\partial_\eta+z$$

we have

$$\frac{(\partial_\xi|\rho)}{\|\rho\|^2}=\frac{\|\partial_\xi\|^2}{\|\rho\|^2}=\frac{\mu}{\|z\|^2}=\text{\textcircled{$\triangle$}}$$

so that $(\partial_\xi|\rho)/\|\rho\|^2$ connot be a finite complex number.

The above-mentioned sufficient condition is realized by simultaneous connection of (5. 1) and the condition of Proposition 2. 3 :

$$\rho_\lambda(\iota)\,\rho_\lambda(\kappa)\neq 0 \triangleright 0<\|e_\iota\|/\|e_\kappa\|<\infty\ .$$

The proof may be easily gained. When (ρ_λ) is a basis of $\boldsymbol{Y}$, by definition

$$\boldsymbol{Y}\ni y \bowtie y=\mathfrak{S}y(\lambda)\rho_\lambda$$

$(y(\lambda))$ being a set of finite complex numbers. In this connection the set

$$\Lambda_{(y)}=\{\lambda:\ \lambda\in\Lambda\ \&\ y(\lambda)\neq 0\}$$

is called the *Λ-support of y.*

Propositiou 5.1. *For the case that a basal system $S=(\rho_\lambda)_{\lambda\in\Lambda_0}$ in a base-separable subspace $\boldsymbol{Y}$ in $\boldsymbol{X}$, of which $\widetilde{B}(\boldsymbol{Y})=(e_\iota)_{\iota\in I}$ is the reaxilization, is $\boldsymbol{Y}$-projective, it is necessary and sufficient that the I-support of ρ_λ is a finite set and*

$$\rho_\lambda(\iota)\,\rho_\lambda(\kappa)\neq 0 \triangleright 0<\|e_\iota\|/\|e_\kappa\|<\infty \tag{5. 2}$$

for all $\lambda\in\Lambda_0$.

In this theorem, if $\boldsymbol{Y}=\boldsymbol{X}$ $(e_\iota)_{\iota\in I}$ is not other than $(\partial_\xi)_{\xi\in\Xi}$. So, we reach the following theorem which has been verified in a previous paper[2].

Corollary. *For the case that a basal system $S=(\rho_\lambda)_{\lambda\in\Lambda}$ is a projective system, it is necessary and sufficient that the support of ρ_λ is a finite set for all $\lambda\in\Lambda$.*

In connection with Proposition 2. 3 or the condition (5. 2), as it were, a sort of hardness of a vector may be associated to compare vectors, though it is thought rather earlier to test an exact definition of such notion. In the primitive stage, if the supports of positive vectors x and y are Ξ_x and Ξ_y respectively and

$$\Xi_x\supseteqq\Xi_y\ ,$$

then it may be considered that y is not harder than x. In connection with such matters, the following theorem may be remarked.

Proposition 5.2. *If $B_\pi^*(\boldsymbol{Y})=(\rho_\lambda)_{\lambda\in\Lambda}$, it is observed that*

$$(\rho_\nu|y)/\|y\|^2=\text{\textcircled{$\triangle$}} \tag{5. 3}$$

whenever $P_y\rho_\nu$ is impossible for $y\in\boldsymbol{Y}$.

Demonstration. By Proposition 3. 1, there exists a $\boldsymbol{Y}$-projective orthogonal basis S containing a fixed vector ρ_ν. So we may assume

$$S=B_\pi^*(\boldsymbol{Y})$$

to prove the relation (5. 3). Then, for any vector $y\in\boldsymbol{Y}$

$$\|y\|^2=|y(\nu)|^2\|\rho_\nu\|^2+\underset{\lambda\neq\nu}{\mathfrak{S}}|y(\lambda)|^2\|\rho_\lambda\|^2\ ,$$

hence

$$\frac{\|y\|^2}{\|\rho_\nu\|^2} = |y(\nu)|^2 + \underset{\lambda\neq\nu}{\mathfrak{S}} |y(\lambda)|^2 \frac{\|\rho_\lambda\|^2}{\|\rho_\nu\|^2} > |y(\nu)|^2 . \tag{5. 4}$$

On the other hand,

$$\frac{(\rho_\nu|y)}{\|y\|^2} = \frac{\overline{y(\nu)}\,\|\rho_\nu\|^2}{\|y\|^2} = \overline{y(\nu)}\Big/\frac{\|y\|^2}{\|\rho_\nu\|^2} ,$$

hence, by (5. 4), we gain the inequalty

$$\left|\frac{(\rho_\nu|y)}{\|y\|^2}\right| < \frac{|y(\nu)|}{|y(\nu)|^2} = \frac{1}{|y(\nu)|} . \tag{5. 5}$$

In case $y(\nu)=0$, it is evident that $y \perp \rho_\nu$, i.e.

$$P_y\rho_\nu = 0 ,$$

so that $P_y\rho_\nu$ is possible. Therefore, for the case that $P_y\rho_\nu$ is impossible, it must be that $y(\nu)\neq 0$. Then by (5. 5)

$$\left|\frac{(\rho_\nu|y)}{\|y\|^2}\right| \neq \infty .$$

Since the value $(\rho_\nu|y)/\|y\|^2$ must not be a finite (complex) number when $P_y\rho_\nu$ is impossible, there is left only the case (5. 3), Q.E.D.

Partly as an example of a chain-based case, we will show here the following fact:

Proposition 5. 3. *If $\boldsymbol{Y}$ is a base-separable projective subspace (in $\boldsymbol{X}$) of which the reaxilization is $\widetilde{B}(\boldsymbol{Y})=(e_\iota)_{\iota\in I}$ and if there exists at least one suffix $\kappa\in I$ such that size of the support of e_κ is not less than 3, then $\boldsymbol{Y}^\perp$ is chain-based.*

Demonstration. Let Ξ_ι be the support of e_ι and

$$\Xi_Y = \bigcup \Xi_\iota .$$

In addition, let S be a set of vectors such that

$$\Xi - \Xi_Y \ni \eta \triangleright \partial_\eta \in S \tag{5. 5}$$

and

$$\langle\!\langle e_\iota\rangle\!\rangle \ni \partial_{\xi_0} + c_1\partial_{\xi_1} + \cdots + c_n\partial_{\xi_n}{}^{*}$$

$$\triangleright S \ni \delta_{\iota,k} = \partial_{\xi_0} + c_1\partial_{\xi_1} + \cdots + c_{k-1}\partial_{\xi_{k-1}} - \frac{1+|c_1|^2+\cdots+|c_{k-1}|^2}{\overline{c_k}}\,\partial_{\xi_k} \qquad (k=1, 2, \cdots, n) \tag{5. 6}$$

and S does not contain other vectors than described by (5. 5) or (5. 6). Then it is evident that S makes an orthogonal system and the subspace generated by S is just $\boldsymbol{Y}^\perp$.

Now let $\boldsymbol{Z}_\iota$ be the subspace generated by

$$(\delta_{\iota,k})_{k=1,2,\cdots,n} .$$

* Since $\boldsymbol{Y}$ is a projective subspace in $\boldsymbol{X}$, size of the support of e_ι is finite; $=n+1$ in this case.

Then it may easily be seen that

$$Z_\iota \ni \bar{c}_1\partial_{\xi_0}-\partial_{\xi_1}\ ,\ \bar{c}_2\partial_{\xi_1}-\bar{c}_1\partial_{\xi_2}\ ,\ \cdots\ ,\bar{c}_n\partial_{\xi_{n-1}}-\bar{c}_{n-1}\partial_{\xi_n}\ ,\ \partial_{\xi_n}-\bar{c}_n\partial_{\xi_0}\ .$$

Hence, in case $n\geqslant 2$, we have

$$\widetilde{B}(Z_\iota)=\{\partial_{\xi_0},\ \partial_{\xi_1},\ \cdots,\ \partial_{\xi_n}\}$$

so that

$$\widetilde{Z}_\iota \ni e_\iota\ .$$

Then, since it is evident that

$$Z_\iota \not\ni e_\iota$$

it must be that

$$\widetilde{Z}_\iota \neq Z_\iota$$

(i.e. Z_ι is chain-based). In case $n=1$, we have

$$\widetilde{Z}_\iota = \langle\!\langle \bar{c}_1\partial_{\xi_0}-\partial_{\xi_1}\rangle\!\rangle$$

so that

$$\widetilde{Z}_\iota = Z_\iota\ .$$

In case $n=0$, Z_ι is void. Such being the caces, since

$$\Xi_\iota \cap \Xi_\kappa = \text{void when } \iota \neq \kappa\ ,$$

we consequently conclude that S is chain-based when and only when there exists at least one ι for which $n\geqslant 2$.

Mathematical Seminar in the Muroran Inst. Tech., Hokkaido

(Received Apr. 9, 1964)

Inference

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