On Finite Fourier Sine Series with Respect to Finite Differences

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On Finite Fourier Sine Series with Respect to Finite Differences

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Abstract

The solution of linear Finite Difference Eqaution which is defined for only finite number of points, can be expressed by Finite Sine Series with the same number of terms.

In this paper, the othoganality of the finite sequence of sine function is first given, and several functions which represent the summations regarding the finite sequences of sine function, are introduced from a finite sequence of complex exponential function. The corresponding Finite Difference Equations and their Green Functions, which appear in the structural problems, are discussed.

1. Introduction

The Finite Difference Equation has often been encountered and will surely be so on the field of engineering analysis. Many analitical methods have been found in solving it, and George Boole[1] must be enumarated as the one jwho had first carried out a thorough discussion of the basic principles of the subject, covering nearly all the major theorems and methods with clarity and rigor. It was however not long before that a kind of Finite Fourier Transforms were applied to the analysis of Finite Fourier Transforms were applied to the analysis of Finite Difference E-quation by Ferras[2]. Considering that Fourier Transforms plays an improtant part in the engineering mathematics, this concept may be extended to Finite Difference Equation.

Let then Δ, prefixed to the expression of any function of x, denote the operation of taking the increament of that function corresponding to a given constant increament Δx of the variable x. Then representing the proposal function of x by F_x, we have

$$\Delta F_x = F_{x+\Delta x} - F_x \tag{1}$$

and

$$\frac{\Delta F_x}{\Delta x} = \frac{F_{x+\Delta x} - F_x}{\Delta x} \tag{2}$$

which apparently coinside with dF/dx for the limit $\Delta x \to 0$.

Then, assuming $\Delta x = 1$,

$$\Delta F_x = F_{x+1} - F_x \,. \tag{3}$$

The operation denoted by Δ is able to be repeated. For the difference of a function of x, being itsself a function of x, is subject to operations of the same kind.

$$\Delta\Delta F_x = \Delta^2 F_x$$

similary

$$\Delta\Delta^2 F_x = \Delta^3 F_x$$

and generally

$$\Delta\Delta^{n-1} F_x = \Delta^n F_x$$

which can be written as

$$\left.\begin{aligned} \Delta^n F_x = \Delta^{n-1} F_{x+1} - \Delta^{n-1} F_x = F_{x+n} - {}_nC_1 F_{x+n-1} + {}_nC_2 F_{x+n-2+\cdots} \\ +(-1)^{n-1}{}_nC_{n-1} F_{x+1} + (-1)^n F_x . \end{aligned}\right\} \tag{4}$$

The inverse operation of finite difference is called as the finite Integration which is usually expressed by:
if

$$\Delta Y(x) = F_x \Delta x$$

$$SF_x \Delta x = Y(x) + P(x), \tag{5}$$

in which $P(x)$ is a periodic function by the interval Δx. Therefore, the definite Finite Integration may be written as

$$\overset{b}{\underset{a}{S}} F_x \Delta x = Y(b) - Y(a) = \sum_{x=a}^{b-1} F_x \Delta x . \tag{6}$$

2. Othogonality of Sine Fuction with Respect to Finite Differences

If x is a variable defined only for integer from zero to n, any function including x makes sense only when x takes the prescribed integer. Let now

$$\overset{n}{\underset{0}{S}} \sin \frac{m\pi x}{n} \sin \frac{r\pi x}{n} \Delta x = I$$

where $m,\ r = 1,\ 2,\ 3, \cdots, n$.

By taking $\Delta x = 1$,

$$I = \frac{1}{2} \overset{n}{\underset{0}{S}} \left\{ \cos \frac{\pi x(m-r)}{n} - \cos \frac{\pi x(m+r)}{n} \right\} \Delta x$$

$$= \frac{1}{4} \left[- \frac{\sin \frac{\pi}{n}(m-r)x + \sin \frac{\pi}{n}(m-r)(x-1)}{\sin \frac{\pi}{n}(m-r)} - \frac{\sin \frac{\pi}{n}(m+r)x + \sin \frac{\pi}{n}(m+r)(x-1)}{\sin \frac{\pi}{n}(m+r)} \right]_0^n$$

$$I = \frac{1}{4} \left[\frac{\sin \pi(m-r)}{\sin \frac{\pi}{n}(m-r)} \left\{ 1 + \cos \frac{\pi(m-r)}{n} \right\} - \cos \pi(m-r) + \cos \pi(m+r) \right]$$

So that

$$I=\begin{cases}\dfrac{n}{2} & m=r\\ 0 & m\neq r.\end{cases}$$

The inversion formula is established as follows;

$$\left.\begin{aligned}\sum_{x=0}^{n-1} f(x)\sin\frac{m\pi x}{n}&=\phi(m)\\ \frac{2}{n}\sum_{m=0}^{n-1}\phi(m)\sin\frac{m\pi x}{n}&=f(x).\end{aligned}\right\}\tag{7}$$

3. Hyperbolic Functions

If x represents integer ranging from 0 to $n-1$,

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{(-1)^m\sin\dfrac{m\pi}{n}\sin\dfrac{m\pi x}{n}}{2\left(1-\cos\dfrac{m\pi}{n}\right)+\lambda}=-\frac{\sinh\alpha x}{\sinh n\alpha},\tag{8}$$

where $2\cosh\alpha=2+\lambda$, and $m=1, 2, 3, \cdots, n$.

By letting $\Delta x=1$

$$\overset{n}{\underset{0}{S}}\exp\left\{\alpha+i\frac{m\pi}{n}\right\}x\cdot\Delta x=\sum_{0}^{n-1}\exp\left\{\alpha+i\frac{m\pi}{n}\right\}x$$

the right hand side of which leads to

$$\sum_{0}^{n-1}\exp\left\{\alpha+i\frac{m\pi}{n}\right\}x=\frac{1-\exp\{\alpha n+im\pi\}}{1-\exp\left\{\alpha+i\dfrac{m\pi}{n}\right\}}.\tag{9}$$

The real and imaginary parts of (9) yield

$$\mathrm{Re}\left[\sum_{x=0}^{n-1}\exp\left\{x\left(\alpha+i\frac{m\pi}{n}\right)\right\}\right]=\frac{e^{-\alpha}-e^{\alpha(n-1)}-\cos\dfrac{m\pi}{n}(1-(-1)^m e^{\alpha n})}{2\cosh\alpha-2\cos\dfrac{m\pi}{n}},\tag{10}$$

$$I\left[\sum_{x=0}^{n-1}\exp\left\{\alpha+i\frac{m\pi}{n}\right\}x\right]=\frac{\sin\dfrac{m\pi}{n}(1-(-1)^m e^{\alpha n})}{2\cosh\alpha-2\cos\dfrac{m\pi}{n}}.\tag{11}$$

Substitution of $-\alpha$ for α in the above (11), becomes

$$I\left[\sum_{x=0}^{n-1}\exp\left\{-\alpha+i\frac{m\pi}{n}\right\}x\right]=\frac{\sin\dfrac{m\pi}{n}(1-(-1)^m e^{-\alpha n})}{2\cosh\alpha-2\cos\dfrac{m\pi}{n}}\tag{12}$$

Subtraction (12) from (11) becomes

$$\sum_{x=0}^{n-1} \sinh \alpha x \cdot \sin \frac{\pi m x}{n} = (-1)^m \frac{\sin \frac{\pi m}{n} \cdot \sinh \alpha n}{2 \cosh \alpha - 2 \cos \frac{m\pi}{n}} \tag{13}$$

which is by Formulas (7) transformed into

$$\frac{2}{n} \sum_{m=0}^{n-1} \frac{(-1)^m \sin \frac{m\pi}{n} \sin \frac{m\pi x}{n}}{2\left(1 - \cos \frac{m\pi}{n}\right) + \lambda} = -\frac{\sinh \alpha x}{\sinh \alpha n}. \qquad (x \neq n) \tag{14}$$

Replacement of x by $n-x$, produces

$$\frac{2}{n} \sum_{m=0}^{n-1} \frac{\sin \frac{m\pi}{n} \sin \frac{m\pi x}{n}}{2\left(1 - \cos \frac{m\pi}{n}\right) + \lambda} = \frac{\sinh \alpha (n-x)}{\sinh \alpha n}. \qquad (x \neq 0) \tag{15}$$

4. In Case of a Complex Variable for λ, α

Let $\lambda = \xi + \eta i$ and $\alpha = \beta + \gamma i$, then $2 + \xi + \eta i = 2 \cosh (\beta + \gamma i)$ because of $\cosh \alpha = 1 + \frac{\lambda}{2}$. It follows:

$$1 + \frac{\xi}{2} = \cosh \beta \cos \gamma ,$$

$$\frac{\eta}{2} = \sinh \beta \sin \gamma ,$$

from which

$$\sqrt{\left(2 + \frac{\xi}{2}\right)^2 + \left(\frac{\eta}{2}\right)^2} \pm \sqrt{\left(\frac{\xi}{2}\right)^2 + \left(\frac{\eta}{2}\right)^2} = \begin{cases} 2 \cosh \beta , \\ 2 \cos \gamma . \end{cases} \tag{16}$$

On the other hand the conformal mapping between $(\xi \cdot \eta)$ and $(\beta \cdot \gamma)$ planes, reveals

$$\left.\begin{array}{lll} 1 + \frac{\xi}{2} \geqq 0 & \frac{\eta}{2} > 0 & 0 < \gamma \leqq \frac{\pi}{2} \\ 1 + \frac{\xi}{2} \leqq 0 & \frac{\eta}{2} > 0 & \frac{\pi}{2} \leqq \gamma < \pi \\ 1 + \frac{\xi}{2} \leqq 0 & \frac{\eta}{2} < 0 & \pi < \gamma \leqq \frac{3}{2} \pi \\ 1 + \frac{\xi}{2} \geqq 0 & \frac{\eta}{2} < 0 & \frac{3}{2} \pi \leqq \gamma < 2\pi \end{array}\right\} \tag{17}$$

excluding $\eta = 0$ line.

The real parts of (14) and (15) yield another group of formulas as follows:

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{(-1)^m\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2+\eta^2}\sin\frac{m\pi}{n}\sin\frac{m\pi x}{n}$$
$$=\frac{\cosh\beta(n-x)\cdot\cos\gamma(n+x)-\cosh\beta(n+x)\cos\gamma(n-x)}{\cosh 2\beta n-\cos 2\gamma n} \tag{18}$$

for $1+\frac{\xi}{2}\geqq 0,\quad \frac{\eta}{2}>0$ and $1+\frac{\xi}{2}\geqq 0,\quad \frac{\eta}{2}<0,\quad x\neq n.$

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{(-1)^m\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2+\eta^2}\sin\frac{m\pi}{n}\sin\frac{m\pi x}{n}$$
$$=\frac{(-1)^{n+x}\cosh\beta(n-x)\cdot\cos\gamma(n+x)-(-1)^{n-x}\cosh\beta(n+x)\cdot\cos\gamma(n-x)}{\cosh 2\beta n-\cos 2\gamma n} \tag{19}$$

for $1+\frac{\xi}{2}\leqq 0,\quad \frac{\eta}{2}>0$ and $\frac{\eta}{2}<0,\quad x\neq n.$

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{2\left(1-\cos\frac{m\pi}{n}\right)+\xi}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2+\eta^2}\sin\frac{m\pi}{n}\sin\frac{m\pi x}{n}$$
$$=\frac{-\cosh\beta x\cdot\cos\gamma(2n-x)+\cosh\beta(2n-x)\cdot\cos\gamma x}{\cosh 2\beta n-\cos 2\gamma n} \tag{20}$$

for $x\neq 0,\quad 1+\frac{\xi}{2}\geqq 0$ and $\eta\neq 0.$

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{2\left(1-\cos\frac{m\pi}{n}\right)+\xi}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2+\eta^2}\sin\frac{m\pi}{n}\sin\frac{m\pi x}{n}$$
$$=\frac{-(-1)^x\cosh\beta x\cdot\cos\gamma(2n-x)+(-1)^x\cosh\beta(2n-x)\cdot\cos\gamma x}{\cosh 2\beta n-\cos 2\gamma n} \tag{21}$$

for $1+\frac{\xi}{2}\leqq 0$ and $\eta\neq 0,\quad x\neq 0.$

In like manner, the imaginary parts of (14) and (15) are expressed by the formulas:

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{(-1)^m\eta}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2+\eta^2}\sin\frac{m\pi}{n}\sin\frac{m\pi x}{n}$$
$$=\pm\frac{\sinh\beta(n-x)\cdot\sin\gamma(n+x)-\sinh\beta(n+x)\cdot\sin\gamma(n-x)}{\cosh 2\beta n-\cos 2\gamma n} \tag{22}$$

for $1+\frac{\xi}{2}\geqq 0,\quad \frac{\eta}{2}>0;\ 1+\frac{\xi}{2}\geqq 0,\quad \frac{\eta}{2}<0$

respectively.

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{(-1)^m\eta}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2+\eta^2}\sin\frac{m\pi}{n}\sin\frac{m\pi x}{n}$$
$$=\begin{matrix}+\\-\end{matrix}\Big\}\frac{(-1)^{n+x}\sinh\beta(n-x)\sin\gamma(n+x)-(-1)^{n-x}\sinh\beta(n+x)\cdot\sin\gamma(n-x)}{\cosh 2\beta n-\cos 2\gamma n} \tag{23}$$

for $1+\frac{\xi}{2}\leqq 0,\quad \frac{\eta}{2}>0;\ 1+\frac{\xi}{2}\leqq 0,\quad \frac{\eta}{2}<0$

respectively.

$$\frac{2}{n}\sum_{m=0}^{n-0}\frac{\eta}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2+\eta^2}\sin\frac{m\pi}{n}\sin\frac{m\pi x}{n}$$
$$=\begin{matrix}-\\+\end{matrix}\Big\}\frac{\sinh\beta x\cdot\sin\gamma(2n-x)-\sinh\beta(2n-x)\sin\gamma x}{\cosh 2\beta n-\cos 2\gamma n} \tag{24}$$

for $1+\frac{\xi}{2}\geqq 0,\quad \frac{\eta}{2}>0;\ 1+\frac{\xi}{2}\geqq 0,\quad \frac{\eta}{2}<0;$

respectively.

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{\eta}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2+\eta^2}\sin\frac{m\pi}{n}\sin\frac{m\pi x}{n}$$
$$=\begin{matrix}-\\+\end{matrix}\Big\}\frac{(-1)^x\sinh\beta x\cdot\sin\gamma(2n-x)-(-1)^x\sinh\beta(2n-x)\sin\gamma x}{\cosh 2\beta n-\cos 2\gamma n} \tag{25}$$

for $1+\frac{\xi}{2}\leqq 0,\quad \frac{\eta}{2}>0;\ 1+\frac{\xi}{2}\leqq 0,\quad \frac{\eta}{2}<0$

respectively.

5. Formulas for $\eta=0$

The equation (16) is rewritten as

$$\sqrt{\left(2+\frac{\xi}{2}\right)^2+\left(\frac{\eta}{2}\right)^2}-\sqrt{\left(\frac{\xi}{2}\right)^2+\left(\frac{\eta}{2}\right)^2}=2\cos\gamma,$$

from which the limit when η tends to zero, yields for the left side

$$\lim_{\eta\to 0}\left\{\sqrt{2+\frac{\xi}{2}\Big)^2+\left(\frac{\eta}{2}\right)^2}-\sqrt{\left(\frac{\xi}{2}\right)^2+\left(\frac{\eta}{2}\right)^2}\right\}\to 2-\frac{\eta^2}{\xi(4+\xi)}$$

$$\lim_{\eta\to 0}\ 2\cos\to 2-\gamma^2$$

Therefore it follows

$$\lim_{\eta\to 0} \gamma = \frac{\eta}{\sqrt{\xi(4+\xi)}}, \qquad \lim_{\eta\to 0} \sin\gamma = \frac{\eta}{\sqrt{\xi(4+\xi)}} \tag{26}$$

Adding to it cosh β varies along $\eta=0$ line, with the variation of ξ, as follows:

$$\begin{aligned} &\beta\sim|\beta|\times(+1) && \text{for} \quad \xi>0 \\ &\beta\sim|\beta|\times i && \text{for} \quad 0>\xi>-2 \\ &\beta\sim\pi i-|\beta|\times i && \text{for} \quad -2>\xi>-4 \\ &\beta\sim\pi i-|\beta|\times(+1) && \text{for} \quad -4>\xi\,. \end{aligned} \tag{27}$$

Substituting (26) into the formulas from (18) to (21), and considering the relation (27), we have

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{(-1)^m \sin\frac{m\pi}{n}\sin\frac{m\pi x}{n}}{2\left(1-\cos\frac{m\pi}{n}\right)+\xi}$$

$$=\frac{\cosh\beta(n-x)-\cosh\beta(n+x)}{\cosh 2\beta n-1} \tag{28}$$

for $\xi\geqq 0\,;\ x\neq n\,,$

$$=\frac{\cos\beta(n-x)-\cos\beta(n+x)}{\cos 2\beta n-1} \tag{29}$$

for $0\geqq\xi\geqq-2\,;\ x\neq n\,,$

$$=\frac{(-1)^{n-x}\cos\beta(n-x)-(-1)^{n+x}\cos\beta(n+x)}{\cos 2\beta n-1} \tag{30}$$

for $-2\geqq\xi\geqq-4\,;\ x\neq n\,,$

or

$$=\frac{(-1)^{n-x}\cosh\beta(n-x)-(-1)^{n+x}\cosh\beta(n+x)}{\cosh 2\beta n-1} \tag{31}$$

for $-4\geqq\xi\,;\ x\neq n\,.$

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{\sin\frac{m\pi}{n}\sin\frac{m\pi x}{n}}{2\left(1-\cos\frac{m\pi}{n}\right)+\xi}$$

$$=\frac{\cosh\beta(2n-x)-\cosh\beta x}{\cosh 2\beta n-1} \tag{32}$$

for $\xi\geqq 0\,;\ x\neq 0\,,$

$$=\frac{\cos\beta(2n-x)-\cos\beta x}{\cos 2\beta n-1} \tag{33}$$

for $0\geqq\xi\geqq-2\,;\ x\neq 0\,,$

$$= \frac{(-1)^x \{\cos \beta (2n-x) - \cos \beta x\}}{\cos 2\beta n - 1} \tag{34}$$

for $-2 \geqq \xi \geqq -4\,;\; x \neq 0\,,$

or

$$= (-1)^x \frac{\cosh \beta (2n-x) - \cosh \beta x}{\cosh 2\beta n - 1} \tag{35}$$

for $-4 \geqq \xi, \quad x \neq 0\,,$

and from (22) to (25) we also have

$$\frac{2}{n} \sum_{m=0}^{n-1} \frac{(-1)^m \sin \dfrac{m\pi}{n} \sin \dfrac{m\pi x}{n}}{\left\{2\left(1-\cos \dfrac{m\pi}{n}\right) + \xi\right\}^2}$$

$$= \frac{(n+x) \sinh \beta (n-x) - (n-x) \sinh \beta (n+x)}{\sqrt{\xi (4+\xi)}\,(\cosh 2\beta n - 1)} \tag{36}$$

for $\xi \geqq 0\,,$

$$= -\frac{(n+x) \sin \beta (n-x) - (n-x) \sin \beta (n+x)}{\sqrt{\xi (4+\xi)}\,(\cos 2\beta n - 1)} \tag{37}$$

for $0 \geqq \xi \geqq -2\,,$

$$= (-1)^{n-x} \frac{(n+x) \sin \beta (n-x) - (n-x) \sin \beta (n+x)}{\sqrt{\xi (4+\xi)}\,(\cos 2\beta n - 1)} \tag{38}$$

for $-2 \geqq \xi \geqq -4\,,$

$$= -(-1)^{n-x} \frac{(n+x) \sinh \beta (n-x) - (n-x) \sinh \beta (n+x)}{\sqrt{\xi (4+\xi)}\,(\cosh 2\beta n - 1)} \tag{38}$$

for $-4 \geqq \xi\,;$

which, by puttin $\xi = 0$, are transformed into

$$\frac{2}{n} \sum_{m=0}^{n-1} \frac{(-1)^m \sin \dfrac{m\pi}{n} \sin \dfrac{m\pi x}{n}}{\left\{2\left(1-\cos \dfrac{m\pi}{n}\right)\right\}^2} = -\frac{nx\,(n+x)\,(n-x)}{6n^2}\,, \tag{39}$$

and

$$\frac{2}{n} \sum_{m=0}^{n-1} \frac{\sin \dfrac{m\pi}{n} \sin \dfrac{m\pi x}{n}}{\left\{2\left(1-\cos \dfrac{m\pi}{n}\right)^2 + \xi\right\}^2}$$

$$= \frac{x \sinh \beta (2n-x) - (2n-x) \sinh \beta x}{\sqrt{\xi (4+\xi)}\,(\cosh 2\beta n - 1)} \tag{40}$$

for $$\xi \geqq 0,$$

$$= -\frac{x \sin \beta (2n-x) - (2n-x) \sin \beta x}{\sqrt{\xi (4+\xi)} (\cos 2\beta n - 1)} \tag{41}$$

for $$0 \geqq \xi \geqq -2,$$

$$= (-1)^x \frac{x \sin \beta (2n-x) - (2n-x) \sin \beta x}{\sqrt{\xi (4+\xi)} (\cos 2\beta n - 1)} \tag{42}$$

for $$-2 \geqq \xi \geqq -4,$$

$$= -(-1)^x \frac{x \sinh \beta (2n-x) - (2n-x) \sinh \beta x}{\sqrt{\xi (4+\xi)} (\cosh 2\beta n - 1)} \tag{43}$$

for $$-4 \geqq \xi,$$

which, by putting $\xi = 0$, are transformed into

$$\frac{2}{n} \sum_{m=0}^{n-1} \frac{\sin \frac{m\pi}{n} \sin \frac{m\pi x}{n}}{4\left(1-\cos \frac{m\pi}{n}\right)^2} = \frac{nx(n-x)(2n-x)}{6n^2}, \tag{44}$$

6. The Corresponding Green Functions

If F_x is a function satisfying

$$\Delta^2 F_{x-1} - \lambda F_x = \begin{cases} 0, & (x = c) \\ 1, & (x = c) \end{cases} \tag{45}$$

then F_x may be called Green Function with respect to the finite difference on the left hand side of (45).

In order to apart F_x on $0<x<c$ from the one on $c<x<n$, F'_x will represent the latter one. Thus,

$$F_x = A \frac{\sinh \alpha x}{\sinh \alpha n} \qquad (x \leqq c)$$

$$F'_x = B \frac{\sinh \alpha (n-x)}{\sinh \alpha n} \qquad (x \geqq c)$$

where $2\cosh \alpha = \lambda + 2$.

The integration constant A and B are obainable by the condition of continuity

$$F_c = F'_c$$

$$F'_{c+1} + F_{c-1} - (2+\lambda) F_c = 1,$$

that is

$$A \sinh c = B \sinh (n-c)$$

$$A\{\sinh\alpha(c-1)-\sinh\alpha c\}-B\{\sinh\alpha(n-c)-\sinh\alpha(n-c+1)\}$$
$$+B\sinh\alpha(n-c)=\sinh\alpha n\,,$$

from which

$$B=-\frac{\sinh\alpha c}{\sinh\alpha}$$

$$A=-\frac{\sinh\alpha(n-c)}{\sinh\alpha}\,.$$

Consequently, the Green Function for $\Delta^2F_{x-1}-\lambda F_x$ can be written as

$$F_x=\begin{cases}-\dfrac{\sinh\alpha(n-c)\sinh\alpha x}{\sinh\alpha\cdot\sinh n\alpha}, & (x<c)\\[2ex] -\dfrac{\sinh\alpha c\sinh\alpha(n-x)}{\sinh\alpha\cdot\sinh n\alpha}, & (x>c)\end{cases}\tag{46}$$

On the other hand, if the equation (45) is solved by means of the finite Fourier Serise,

$$F_x=\sum_{m=0}^{n-1}\overline{F}_m\sin\frac{m\pi x}{n}\,,\tag{47}$$

which being put into (45), there follows

$$\sum_{m=0}^{n}\overline{F}_m\left\{2\left(\cos\frac{m\pi}{n}-1\right)-\lambda\right\}\sin\frac{m\pi x}{n}=\frac{2}{n}\sum_{m=0}^{n}\sin\frac{m\pi c}{n}\sin\frac{m\pi x}{n}$$

and

$$F_x=-\frac{2}{n}\sum_{m=0}^{n-1}\frac{\sin\dfrac{\pi mc}{n}\sin\dfrac{m\pi x}{n}}{2\left(1-\cos\dfrac{m\pi}{n}\right)+\lambda}\,.\tag{48}$$

Compare this with Formula (46), obviously it can be concluded that

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{\sin\dfrac{m\pi c}{n}\sin\dfrac{m\pi x}{n}}{2\left(1-\cos\dfrac{m\pi}{n}\right)+\lambda}=\begin{cases}\dfrac{\sinh\alpha(n-c)\cdot\sinh\alpha x}{\sinh\alpha\cdot\sinh n\alpha} & (x<c)\\[2ex] \dfrac{\sinh\alpha c\cdot\sinh\alpha(n-x)}{\sinh\alpha\cdot\sinh n\alpha} & (x>c)\end{cases}\tag{49}$$

If λ and α are complex numbers such as $\lambda=\xi+\eta i$, $\alpha=\beta+\gamma i$, the real and imaginary parts of Formula 49 give other Green Funtions.

(A) Real part for $1+\dfrac{\xi}{2}>0$, $\eta\neq0$, becomes

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{\left\{2\left(1-\cos\dfrac{m\pi}{n}\right)+\xi\right\}\sin\dfrac{m\pi c}{n}}{\left\{2\left(1-\cos\dfrac{m\pi}{n}\right)+\xi\right\}^2+\eta^2}\sin\frac{m\pi x}{n}$$

$$
\left.\begin{aligned}
&= \frac{1}{(\cosh 2\beta - \cos 2\gamma)(\cosh 2n\beta - \cos 2n\gamma)} \\
&\quad \times [\{\cosh \beta (c+1) \cos \gamma (c-1) - \cosh \beta (c-1) \cos \gamma (c+1)\} \\
&\quad \times \{\cosh \beta (2n-x) \cos \gamma x - \cosh \beta x \cos \gamma (2n-x)\} \\
&\quad + \{\sinh \beta (c+1) \sin \gamma (c-1) - \sinh \beta (c-1) \sin \gamma (c+1)\} \\
&\quad \times \{\sinh \beta (2n-x) \sin \gamma x - \sinh \beta x \sin \gamma (2n-x)\}] \\
&\hspace{20em} x > c \\
&= \frac{1}{(\cosh 2\beta - \cos 2\gamma)(\cosh 2n\beta - \cos 2n\gamma)} \\
&\quad \times [\{\cosh \beta (x+1) \cos \gamma (x-1) - \cosh \beta (x-1) \cos \gamma (x+1)\} \\
&\quad \times \{\cosh \beta (2n-c) \cos \gamma c - \cosh \beta c \cos \gamma (2n-c)\} \\
&\quad + \{\sinh \beta (x+1) \sin \gamma (x-1) - \sinh \beta (x-1) \sin \gamma (x+1)\} \\
&\quad \times \{\sinh \beta (2n-c) \sin \gamma c - \sinh \beta c \cdot \sin \gamma (2n-c)\}] \\
&\hspace{20em} c > x,
\end{aligned}\right\} \quad (50)
$$

and the real part for $1+\frac{\xi}{2}<0$, $\eta \neq 0$, yields

$$
\frac{2}{n}\sum_{m=0}^{n-1} \frac{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}\sin\frac{m\pi c}{n}}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2+\eta^2} \sin\frac{m\pi x}{n}
$$

$$
\left.\begin{aligned}
&= \frac{(-1)^{c+x+1}}{(\cosh 2\beta - \cos 2\gamma)(\cosh 2n\beta - \cos 2n\gamma)} \\
&\quad \times [(-1)^{c+x}\{\cosh \beta (c+1) \cos \gamma (c-1) - \cosh \beta (c-1) \cos \gamma (c+1)\} \\
&\quad \times \{\cosh \beta (2n-x) \cos \gamma x - \cosh \beta x \cos \gamma (2n-x)\} \\
&\quad + \{\sinh \beta (c+1) \sin \gamma (c-1) - \sinh \beta (c-1) \sin \gamma (c+1)\} \\
&\quad \times \{\sinh \beta (2n-x) \sin \gamma x - \sinh \beta x \sin \gamma (2n-x)\}] \\
&\hspace{20em} x > c \\
&= \frac{(-1)^{c+x+1}}{(\cosh 2\beta - \cos 2\gamma)(\cosh 2n\beta - \cos 2n\gamma)} \\
&\quad \times [\{\cosh \beta (x+1) \cos \gamma (x-1) - \cosh \beta (x-1) \cos \gamma (x+1)\} \\
&\quad \times \{\cosh \beta (2n-c) \cos \gamma c - \cosh \beta c \cos \gamma (2n-c)\} \\
&\quad + \{\sinh \beta (x+1) \sin \gamma (x-1) - \sinh \beta (x-1) \sin \gamma (x+1)\} \\
&\quad \times \{\sinh \beta (2n-c) \sin \gamma c - \sinh \beta c \cdot \sin \gamma (2n-c)\}] \\
&\hspace{20em} x < c.
\end{aligned}\right\} \quad (51)
$$

The above functions from the real part, satisfy the following problem

$$
\Delta^4 F_{x-2} - 2\xi \Delta^2 F_{x-1} + (\xi^2+\eta^2) F_x = \begin{cases} 1 & (x = c-1) \\ -2 & (x = c) \\ 1 & (x = c+1) \end{cases}
$$

for any value else of x

$$\Delta^4 F_{x-2} - 2\xi\Delta^2 F_{x-1} + (\xi^2+\eta^2) F_x = 0\,,$$

with the boundary conditions

$$\left.\begin{aligned} F_x &= 0 \quad \text{for} \quad x=0\,,\ n\,,\\ \Delta^2 F_{x-1} &= 0 \quad \text{for} \quad x=0\,,\ n\,. \end{aligned}\right\} \quad (52)$$

From the viewpoint of operating procedure, $\Delta^2 F_{x-1}$ for $x=0$, n do not make any sense because there are not F_{-1} and F_{n+1}, $\Delta^2 F_{x-1}$ for $x=0$, n are to represent the boundary values of $\Delta^2 F_{x-1}$ in the above expression.

(B) The imaninary part of (49), for $1+\frac{\xi}{2}>0$, becomes

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{\sin\frac{m\pi c}{n}\sin\frac{m\pi x}{n}}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2+\eta^2}$$

$$\left.\begin{aligned}
&= \frac{\mp 1}{\eta(\cosh 2\beta-\cos 2\gamma)(\cosh 2n\beta-\cos 2n\gamma)}\\
&\quad\times[\{\cosh\beta(2n-x)\cos\gamma x-\cosh\beta x\cdot\cos\gamma(2n-x)\}\\
&\quad\times\{\sinh\beta(c+1)\sin\gamma(c-1)-\sinh\beta(c-1)\cdot\sin\gamma(c+1)\}\\
&\quad-\{\cosh\beta(c+1)\cos\gamma(c-1)-\cosh\beta(c-1)\cdot\cos\gamma(c+1)\}\\
&\quad\times\{\sinh\beta(2n-x)\cdot\sin\gamma x-\sinh\beta x\sin\gamma(2n-x)\}]\\
&\qquad\qquad\qquad \eta\gtrless 0\,,\quad x>c\,,\\
&= \frac{\mp 1}{\eta(\cosh 2\beta-\cos 2\gamma)(\cosh 2n\beta-\cos 2n\gamma)}\\
&\quad\times[\{\cosh\beta(2n-c)\cos\gamma c-\cosh\beta c\cdot\cos\gamma(2n-c)\}\\
&\quad\times\{\sinh\beta(x+1)\sin\gamma(x-1)-\sinh(x-1)\sin\gamma(x+1)\}\\
&\quad-\{\cosh\beta(x+1)\cos\gamma(x-1)-\cosh\beta(x-1)\cos\gamma(x+1)\}\\
&\quad\times\{\sinh\beta(2n-c)\cdot\sin\gamma c-\sinh\beta c\cdot\sin\gamma(2n-c)\}]\\
&\qquad\qquad\qquad \eta\gtrless 0\,,\quad c>x\,,
\end{aligned}\right\} \quad (53)$$

and the imaginary part for $1+\frac{\xi}{2}<0$, yields

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{\sin\frac{m\pi c}{n}\sin\frac{m\pi x}{n}}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2+\eta^2}$$

$$\begin{aligned}
&= \frac{\mp(-1)^{x+c+1}}{\eta(\cosh 2\beta-\cos 2\gamma)(\cosh 2n\beta-\cos 2n\gamma)}\\
&\quad\times[\{\cosh\beta(2n-x)\cos\gamma x-\cosh\beta x\cdot\cos\gamma(2n-x)\}\\
&\quad\times\{\sinh\beta(c+1)\sin\gamma(c-1)-\sinh\beta(c-1)\sin\gamma(c+1)\}
\end{aligned}$$

$$
\left.
\begin{aligned}
&-\{\cosh\beta(c+1)\cos\gamma(c-1)-\cosh\beta(c-1)\cos\gamma(c+1)\}\\
&\times\{\sinh\beta(2n-x)\cdot\sin\gamma x-\sinh\beta x\cdot\sin\gamma(2n-x)\}]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \eta\gtrless 0,\quad x>c,\\
&=\frac{\mp(-1)^{x+c+1}}{\eta(\cosh 2\beta-\cos 2\gamma)(\cosh 2n\beta-\cos 2n\gamma)}\\
&\times[\{\cosh\beta(2n-c)\cdot\cos\gamma c-\cosh\beta c\cdot\cos\gamma(2n-c)\}\\
&\times\{\sinh\beta(x+1)\sin\gamma(x-1)-\sinh(x-1)\sin\gamma(x+1)\}\\
&-\{\cosh\beta(x+1)\cos\gamma(x-1)-\cosh\beta(x-1)\cos\gamma(x+1)\}\\
&\times\{\sinh\beta(2n-c)\cdot\sin\gamma c-\sinh\beta c\cdot\sin\gamma(2n-c)\}]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \eta\gtrless 0,\quad c>x.
\end{aligned}
\right\} \tag{54}
$$

which are the solution of the problem:

$$
\Delta^4F_{x-2}-2\xi\Delta^2F_{x-1}+(\xi^2+\eta^2)F_x=\begin{cases}1 & x=c\\ 0 & x\neq c\end{cases}
$$

with the boundary conditions

$$
\left.
\begin{aligned}
F_x&=0, &&\text{for}\quad x=0,\quad x=n\\
\Delta^2F_{x-1}&=0 &&\text{for}\quad x=0,\quad x=n.
\end{aligned}
\right\} \tag{55}
$$

In the particular case $\eta=0$, another group of Green Functions can be found from the prescribed formulas.

(C) The real part leads us to

$$
\left.
\begin{aligned}
\frac{2}{n}\sum_{m=0}^{n-1}\frac{\sin\frac{m\pi c}{n}\sin\frac{m\pi x}{n}}{2\left(1-\cos\frac{m\pi}{n}\right)+\xi}
&=\begin{cases}\dfrac{\sinh\beta(n-c)\cdot\sinh\beta x}{\sinh\beta\cdot\sinh n\beta} & (x<c)\\[2ex] \dfrac{\sinh\beta c\cdot\sinh\beta(n-x)}{\sinh\beta\cdot\sinh n\beta} & (x>c)\end{cases}\\
&\qquad\qquad\text{for}\quad \xi\geqq 0,\\
&=\begin{cases}\dfrac{\sin\beta(n-c)\cdot\sin\beta x}{\sin\beta\cdot\sin n\beta} & (x<c)\\[2ex] \dfrac{\sin\beta c\cdot\sin\beta(n-x)}{\sin\beta\cdot\sin n\beta} & (x>c)\end{cases}\\
&\qquad\qquad\text{for}\quad 0\geqq\xi\geqq -2,\\
&=\begin{cases}(-1)^{x+c-1}\dfrac{\sin\beta(n-c)\cdot\sin\beta x}{\sin\beta\cdot\sin n\beta} & (x<c)\\[2ex] (-1)^{x+c-1}\dfrac{\sin\beta c\times\sin\beta(n-x)}{\sin\beta\cdot\sin n\beta} & (x>c)\end{cases}\\
&\qquad\qquad\text{for}\quad -2\geqq\xi\geqq -4,\\
&=\begin{cases}(-1)^{x+c-1}\dfrac{\sinh\beta(n-c)\cdot\sinh\beta x}{\sinh\beta\cdot\sinh n\beta} & (x<c)\\[2ex] (-1)^{x+c-1}\dfrac{\sinh\beta c\cdot\sinh\beta(n-x)}{\sinh\beta\cdot\sinh n\beta} & (x>c)\end{cases}\\
&\qquad\qquad\text{for}\quad -4\geqq\xi,
\end{aligned}
\right\} \tag{56}
$$

from which substitution $\xi=0$ yields

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{\sin\frac{m\pi c}{n}\sin\frac{m\pi x}{n}}{2\left(1-\cos\frac{m\pi}{n}\right)}=\begin{cases}\frac{x\cdot(n-c)}{n} & (x<c)\\ \frac{c\cdot(n-x)}{n} & (x>c)\,.\end{cases} \tag{57}$$

(D) The imaginary part turns out to be

$$\left.\begin{aligned}&\frac{2}{n}\sum_{m=0}^{n-1}\frac{\sin\frac{m\pi c}{n}\sin\frac{m\pi x}{n}}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2}\\
&=\frac{-1}{\sqrt{\xi(4+\xi)}\,(\cosh 2\beta-1)(\cosh 2n\beta-1)}\\
&\times[\{\cosh\beta(2n-x)-\cosh\beta x\}\\
&\times\{(c-1)\sinh\beta(c+1)-(c+1)\sinh\beta(c-1)\}\\
&-\{\cosh\beta(c+1)-\cosh\beta(c-1)\}\\
&\times\{x\sinh\beta(2n-x)-(2n-x)\sinh\beta x\}]\\
&\hspace{20em} x>c\\
&=\frac{-1}{\sqrt{\xi(4+\xi)}\,(\cosh 2\beta-1)(\cosh 2n\beta-1)}\\
&\times[\{\cosh\beta(2n-c)-\cosh\beta c\}\\
&\times\{(x-1)\sinh\beta(x+1)-(x+1)\sinh\beta(x-1)\}\\
&-\{\cosh\beta(x+1)-\cosh\beta(x-1)\}\\
&\times\{c\sinh\beta(2n-c)-(2n-c)\sinh\beta c\}]\\
&\hspace{20em} c>x\,,\\
&\hspace{17em}\text{for}\quad \xi\geqq 0\,,\end{aligned}\right\} \tag{58}$$

$$\left.\begin{aligned}&\frac{2}{n}\sum_{m=0}^{n-1}\frac{\sin\frac{m\pi c}{n}\sin\frac{m\pi x}{n}}{\left\{2\left(1-\cos\frac{m\pi}{n}\right)+\xi\right\}^2}\\
&=\frac{-1}{\sqrt{\xi(4+\xi)}\,(\cos 2\beta-1)(\cos 2n\beta-1)}\\
&\times[\{\cos\beta(2n-x)-\cos\beta x\}\,\{(c-1)\sin\beta(c+1)-(c+1)\sin\beta(c-1)\}\\
&-\{\cos\beta(c+1)-\cos\beta(c-1)\}\,\{x\sin\beta(2n-x)-(2n-x)\sin\beta x\}]\\
&\hspace{20em} x>c\\
&=\frac{+1}{\sqrt{\xi(4+\xi)}\,(\cos 2\beta-1)(\cos 2n\beta-1)}\\
&\times[\cos\beta(2n-c)-\cos\beta c\}\,\{(x-1)\sin\beta(x+1)-(x+1)\sin\beta(x-1)\}\\
&-\cos\beta(x+1)-\cos\beta(x-1)\}\,\{c\sin\beta(2n-c)-(2n-c)\sin\beta c\}]\\
&\hspace{20em} c>x\\
&\hspace{15em}\text{for}\quad 0\geqq\xi\geqq-2\end{aligned}\right\} \tag{59}$$

$$\begin{aligned}
&\frac{2}{n}\sum_{m=0}^{n-1}\frac{\sin\dfrac{m\pi c}{n}\sin\dfrac{m\pi x}{n}}{\left\{2\left(1-\cos\dfrac{m\pi}{n}\right)+\xi\right\}^2} \\
&=\frac{(-1)^{x+c+1}}{\sqrt{\xi(4+\xi)}\,(\cos 2\beta-1)(\cos 2n\beta-1)} \\
&\times[\{\cos\beta(2n-x)-\cos\beta x\}\,\{(c-1)\sin\beta(c+1)-(c+1)\sin\beta(c-1)\} \\
&-\{\cos\beta(c+1)-\cos\beta(c-1)\}\,\{x\sin\beta(2n-x)-(2n-x)\sin\beta x\}] \\
&\qquad\qquad\qquad\qquad x>c \\
&=\frac{(-1)^{x+c+1}}{\sqrt{\xi(4+\xi)}\,(\cos 2\beta-1)(\cos 2n\beta-1)} \\
&\times[\{\cos\beta(2n-c)-\cos\beta c\}\,\{(x-1)\sin\beta(x+1)-(x+1)\sin\beta(x-1)\} \\
&-\{\cos\beta(x+1)-\cos\beta(x-1)\}\,\{c\sin\beta(2n-c)-(2n-c)\sin\beta c\}] \\
&\qquad\qquad\qquad\qquad c>x \\
&\qquad\qquad\qquad\qquad \text{for}\quad -2\geqq\xi\geqq-4
\end{aligned}\tag{60}$$

$$\begin{aligned}
&\frac{2}{n}\sum_{m=0}^{n-1}\frac{\sin\dfrac{m\pi c}{n}\sin\dfrac{m\pi x}{n}}{\left\{2\left(1-\cos\dfrac{m\pi}{n}\right)+\xi\right\}^2} \\
&=\frac{(-1)^{x+c+1}}{\sqrt{\xi(4+\xi)}\,(\cosh 2\beta-1)(\cosh 2n\beta-1)} \\
&\times[\{\cosh\beta(2n-x)-\cosh\beta x\} \\
&\times\{(c-1)\sinh\beta(c+1)-(c+1)\sinh\beta(c-1)\} \\
&-\{\cosh\beta(c+1)-\cosh\beta(c-1)\} \\
&\times\{x\sinh\beta(2n-x)-(2n-x)\sinh\beta x\}] \\
&\qquad\qquad\qquad\qquad x>c \\
&=\frac{(-1)^{x+c+1}}{\sqrt{\xi(4+\xi)}\,(\cosh 2\beta-2)(\cosh 2n\beta-1)} \\
&\times[\{\cosh\beta(2n-c)-\cosh\beta c\} \\
&\times\{(x-1)\sinh\beta(x+1)-(x+1)\sinh\beta(x-1)\} \\
&-\{\cosh\beta(x+1)-\cosh\beta(x-1)\} \\
&\times\{c\sinh\beta(2n-c)-(2n-c)\sinh\beta c\}] \\
&\qquad\qquad\qquad\qquad c>x \\
&\qquad\qquad\qquad\qquad \text{for}\quad -4\geqq\xi,
\end{aligned}\tag{61}$$

which are transformed, by $\xi=0$, into

$$\frac{2}{n}\sum_{m=0}^{n-1}\frac{\sin\dfrac{m\pi c}{n}\sin\dfrac{m\pi x}{n}}{4\left(1-\cos\dfrac{m\pi}{n}\right)^2}$$

$$= \left\{ \begin{array}{ll} \dfrac{nc(n-x)}{6n^2}(x(2n-x)-c^2+1) & x>c \\ \dfrac{nx(n-c)}{6n^2}(c(2n-c)-x^2+1) & x<c\,. \end{array} \right\} \qquad (62)$$

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