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著者	IWATA Kazuo
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Kazuo Iwata*

Abstract

Here the author both in a pre-Hilbert space and in an algebraic dual discusses four types of definite linear dependence of finite members. Besides, he deals with the Taussky's inequality in positive semi-definite case.

Introduction. In the preceeding paper 13) §1 (with respect to real inner product space R and to such) by means of the properties** (P), (P.O) and (P.N) we have discussed the linear dependence with positive coefficients (*generalizations of Stiemke*[2)]*-Carver*[3)]*-Dines*[4),5)] *theorems*, algebraically, with no use of separation theorem). And used it on a totally ordered linear space.

In the present paper, in §1, we apply the above to pre-Hilbert space and to algebraic dual, and thereby in each case we characterize four types of definite linear dependence. Next in §2, in connection with §1, we deal with the inequalities of Hadamard-Fischer[1)]-Taussky*** type.

§1. Characterization of definite linear Dependences

Let Q be a quaternionic inner product space**** (*pre-Hilbert space*) where vectors are to be multiplied by scalars on the left, and the inner product of vectors x and y will be written by (x, y). Besides, let S be a quaternionic linear

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** The author, ibid. gave as follows.

Definition. *With respect to a given system* $\{a_\nu\}$ *of* n *members of* R, *we define*:

(P): *There exist* $\rho_\nu>0$ $(\nu=1,2,\cdots,n)$ *with* $\left(a_i, \sum_1^n \rho_\nu a_\nu\right)>0$ $(i=1,2,\cdots,n)$.

(P.O): *There exist* $\rho_\nu>0$ $(\nu=1,2,\cdots,n)$ *with* $\left(a_i, \sum_1^n \rho_\nu a_\nu\right)\geqslant 0$ $(i=1,2,\cdots,n)$ *and not all zero.*

(P.N): *Non*-(P.O).

And he did also the others with respect to *linear functionals*. For the details, see 13) §1.

*** O. Taussky says in his 12), p. 312 as follows: *For the decomposition of a positive definite Hermitian matrix* $H=A+iB$, *where* A, B *are real*;

$$\det H \leqslant \det A \quad \textit{holds.}$$

**** Briefly speaking, the *quaternionic inner product space* Q is defined in parallel with the *complex inner product space* C with exceptions that *scalars* and *inner product* are *quaternions* and *quaternionic-valued* respectively. Here

$$(\alpha x, y)=\alpha(x, y), \quad (x, \beta y)=(x, y)\bar{\beta},$$

where $\bar{\beta}=a-ib-jc-kd$ for *quaternion* $\beta=a+ib+jc+kd$. In the following we mean also $\Re\beta=a$, $\mathfrak{C}\beta=a+ib$.

space. And let $\tilde{S}$ (*algebraic dual**) be the set of all the quaternionic-valued linear functionals defined on S.

Let a system of n vectors $a_1, a_2, \cdots, a_n \in Q\,(1 \leqslant n)$ be given (such a system in an inner product space will be denoted by A_n throughout), and let a system of vectors $f_1, f_2, \cdots, f_n \in \tilde{S}\,(1 \leqslant n)$ be given.

Now let us apply the results of 13) §1 to both Q and $\tilde{S}$. To do this, for the former we proceed by the use of associated real inner product space Q_R, and for the latter by the uses of real linear functionals $\Re f_\nu$ $(\nu=1,2,\cdots n)$ and associated real linear space S_R.

In view of this, we shall get** at once the present Theorems and Corollaries corresponding to those of 13) §1 respectively. Then at first, these are including the *characterization of definite linear dependence with positive coefficients*. In fact, for example, Theorem 1′ is applicable in part in the proof of Riesz representation theorem. Next here, corresponding to the Corollary 3 (resp. 3′***) of 13) §1, remarkably there are existing three kinds by the present Theorem 2 (resp. 2′):

Corollary 3. *Let a system A_n in Q be given. It is indep**** in Q_R (resp. Q_C, Q) iff every system $\{\delta_\nu a_\nu : \nu=1,2,\cdots,n\}$ (resp. $\{e^{i\varphi_\nu}a_\nu\}$, $\{\alpha_\nu a_\nu\}$) has* (P), *where each δ_ν $(\nu=1,2,\cdots,n)$ (resp. φ_ν, α_ν) stands for* 1 *or* -1 *(resp. arbitrary real, non-zero but arbitrary quaternion) individually.*

Corollary 3′ will be omitted (for this, also achievable via 8) part I).

By the way, for their determinant uses, for a given system A_n, let us put as follows:

$$G_n = \det\Big(\mathfrak{C}(a_i, a_j)\Big), \qquad G_n^{\divideontimes} = \det\Big(\Re(a_i, a_j)\Big).$$

As regards the Gram's theorem (*stepwise*), modifications:

$$G_n = \frac{1}{A_{nn}} \begin{vmatrix} \mathfrak{C}(a_1, a_1) \cdots \mathfrak{C}(a_1, a_{n-1})\ \mathfrak{C}\Big(a_1, \sum\limits_1^n \rho_\nu A_{\nu n} a_\nu\Big) \\ \vdots \\ \mathfrak{C}(a_n, a_1) \cdots \mathfrak{C}(a_n, a_{n-1})\ \mathfrak{C}\Big(a_n, \sum\limits_1^n \rho_\nu A_{\nu n} a_\nu\Big) \end{vmatrix},$$

where each $A_{\nu n}$ $(\nu=1,2,\cdots,n)$ *is the cofactor* of $\mathfrak{C}(a_\nu, a_n)$ *in* G_n; $\rho_n = 1$ *and* ρ_ν(*suitably*)>0 $(\nu=1,2,\cdots,n-1)$,

$$G_n^{\divideontimes} = \textit{the analogue of the above};$$

and Corollary 3 enable us to get the proofs at once. That is to say, Corollary 3 just means a sort of *characterization of definite linear dependence with real*

* $\tilde{S} \ni \beta f\,(f \in \tilde{S})$ is defined by $(\beta f)(x) = (f(x))\bar{\beta}$ for $x \in S$.

** Cf. $f(x) = \Re f(x) - i\Re f(ix) - j\Re f(jx) - k\Re f(kx)$ for $f \in \tilde{S}$, $x \in S$.

*** Corollaries 2′, 3′ in 13) §1 may be naturally considered.

**** In the following, for brevity, we use "*indep in Q_C*" and "*dep in C_R*", etc. in short for "*linearly independent over the complex field in Q*" and "*linearly dependent over the real field in C*", etc. respectively.

(resp. *complex, quaternionic) coefficients in Q.*

And so is Corollary 3′ in $\tilde{S}$.

From these points of view, further we shall treat the *Gramians* themselves in §2.

§2. Inequalities of Gram-Hadamard-Fischer[1)]-Taussky[12)] Type

Let C be a complex inner product space.

Theorem. *Let a system A_n in C, where $a_\nu \neq 0$ $(\nu=1,2,\cdots,n)$ and $1\leqslant n$, be given. Then we have*

$$0\leqslant G_n\leqslant G_n^{\divideontimes}\leqslant \|a_1\|^2\|a_2\|^2\cdots\|a_n\|^2,$$

where the 1st equality holds iff A_n is dep in C;

the 2nd equality holds iff either A_n is dep in C_R,

or A_n is indep in C and $\mathfrak{I}(a_i, a_j)=0$ for $1\leqslant i, j\leqslant n$;

the final equality holds iff $\mathfrak{R}(a_i, a_j)=0$ for $i\neq j$.

Proof.

(2. 1). *Proof of $G_n\leqslant G_n^{\divideontimes}$.* (For the sake of (2. 2).)

Let A_n be *indep* in *C*. For $n=2$, it is clear. Suppose it is true for $n-1$. As it is usually given:

$$b_n=\begin{vmatrix}(a_1, a_1)\cdots(a_1, a_{n-1}) & a_1\\ \vdots & \\ (a_n, a_1)\cdots(a_n, a_{n-1}) & a_n\end{vmatrix},\qquad b_n'=\begin{vmatrix}\mathfrak{R}(a_1, a_1)\cdots\mathfrak{R}(a_1, a_{n-1}) & a_1\\ \vdots & \\ \mathfrak{R}(a_n, a_1)\cdots\mathfrak{R}(a_n, a_{n-1}) & a_n\end{vmatrix};$$

and letting A_{in} and A_{in}' be the cofactors of a_i $(i=1,2,\cdots,n)$ respectively, we have

$$(b_n, b_n)=\overline{A_{nn}}G_n=G_{n-1}G_n,$$

$$(b_n, b_n)=\overline{A_{nn}'}G_n=G_{n-1}^{\divideontimes}G_n$$

and

$$(b_n', b_n')=\mathfrak{R}(b_n', b_n')$$

$$=\overline{A_{nn}'}\cdot\mathfrak{R}\begin{vmatrix}\mathfrak{R}(a_1, a_1)\cdots\mathfrak{R}(a_1, a_{n-1})(a_1, a_n)\\ \vdots\\ \mathfrak{R}(a_n, a_1)\cdots\mathfrak{R}(a_n, a_{n-1})(a_n, a_n)\end{vmatrix}=G_{n-1}^{\divideontimes}G_n^{\divideontimes}.$$

Hence the induction hypothesis $G_{n-1}\leqslant G_{n-1}^{\divideontimes}$ leads to

$$\|b_n\|^2\leqslant(b_n, b_n')\leqslant\|b_n\|\,\|b_n'\|.$$

Hence

$$(b_n, b_n')\leqslant\|b_n\|\,\|b_n'\|\leqslant\|b_n'\|^2.$$

And so

$$G_n\leqslant G_n^{\divideontimes}.$$

(2. 2). *Proof of the 2nd equality condition.*

Let A_n be *indep* in *C*. For $n=2$, it is true. Suppose it is true for $n-1$.

Now, from (2. 1) we obtain that $G_n = G_n^*$ holds if and only if not only $G_{n-1} = G_{n-1}^*$ holds but also b_n, b'_n are *dep* in C. That is equivalent to both

$$A_{nn} = A'_{nn}$$

and

$$A_{in} - A'_{in} = 0 \quad (i = 1, 2, \cdots, n-1)$$

hold. Then, upon subtracting the left-hand side of the latter formulas, we get the simultaneous linear equations in unknowns $(a_n, a_j) - \Re(a_n, a_j)$ $(j = 1, 2, \cdots, n-1)$; where the determinant of the coefficients exactly coincides with the adjugate matrix of A'_{nn}. Hence by $A'_{nn} > 0$, our equality holds if and only if

$$(a_n, a_j) - \Re(a_n, a_j) = 0 \quad (j = 1, 2, \cdots, n-1)$$

together with

$$\Im(a_i, a_j) = 0 \quad (1 \leqslant i, j \leqslant n-1),$$

and this completes (2. 2).

(3). The final inequality is the Fischer's[1] case.

Now, combining the present Theorem with the generalized* Fischer's case, let us make use of them in two directions.

That is, on the one hand:

Corollary 1. *Let a system A_n in C, where $a_\nu \neq 0$ $(\nu = 1, 2, \cdots, n)$, $2 \leqslant n$ be given, Then*

$$0 \leqslant G_n \leqslant G_n^* \leqslant G_r^*(a_1, a_2, \cdots, a_r) G_{n-r}^*(a_{r+1}, \cdots, a_n) \leqslant \|a_1\|^2 \|a_2\|^2 \cdots \|a_n\|^2,$$

where the 3rd equality holds if either and only if either A_r or

A_{n-r} is dep in C_R, or A_n is indep in C_R and

$$\Re(a_i, a_j) = 0 \quad (i = 1, 2, \cdots, r;\ j = r+1, \cdots, n);$$

the final equality holds iff

$$\Re(a_i\ a_j) = 0 \quad \text{for} \quad 1 \leqslant i, j \leqslant r \quad \& \quad r+1 \leqslant i, j \leqslant n, \quad \text{where} \quad i \neq j.$$

Here, A_{n-r} is the complementary subsystem of A_r in A_n. (And so forth.)

And on the other hand:

Corollary 2. *Let a system A_n in C, where $a_\nu \neq 0$ $(\nu = 1, 2, \cdots, n)$, $2 \leqslant n$ be given. Then*

$$0 \leqslant G_n \leqslant G_r(a_1, a_2, \cdots, a_r) G_{n-r}(a_{r+1}, \cdots, a_n)$$
$$\leqslant G_r^*(a_1, a_2, \cdots, a_r) G_{n-r}^*(a_{r+1}, \cdots, a_n) \leqslant \|a_1\|^2 \|a_2\|^2 \cdots \|a_n\|^2,$$

where the 2nd equality holds if either and only if either A_r or A_{n-r} is dep in C, or A_n is indep in C and

$$(a_i, a_j) = 0 \quad (i = 1, 2, \cdots, r;\ j = r+1, \cdots, n);$$

* Refer to 11), No. 1, p. 472, from where we get easily.

the 3rd equality holds if either and only if either A_r or A_{n-r} is dep in C_R or both A_r and A_{n-r} are indep in C and

$$\Im(a_i, a_j)=0 \quad \text{for} \quad 1\leqslant i, j\leqslant r \ \& \ r+1\leqslant i, j\leqslant n.$$

These proofs are easy by the foregoing statement.

Remark. Needless to say, in the above, changing slightly the final equality conditions alone, *zero vectors* may be used.

Here, applying these to the *n-dimensional unitary space*, we note.

Note. We enumerate the results here. But we omit the *equality conditions.*

(I). Inequalities 1, 2 (*On Hadamard's inequality*). *Let an $n\times n$ matrix (α_{ij}) be given. We have*

$$0\leqslant\left|\det(\alpha_{ij})\right|\leqslant\left\{\begin{array}{c}\left\{\det\left(\sum_{\nu=1}^{n}(\Re\alpha_{i\nu}\Re\alpha_{j\nu}+\Im\alpha_{i\nu}\Im\alpha_{j\nu})\right)\right\}^{\frac{1}{2}}\\ \left\{\sum_{1\leqslant p_1<\cdots<p_r\leqslant n}\left|\det(\alpha_{sp_\nu})\right|^2\right\}^{\frac{1}{2}}\cdot\left\{\sum_{1\leqslant q_1<\cdots<q_{n-r}\leqslant n}\left|\det(\alpha_{tq_\nu})\right|^2\right\}^{\frac{1}{2}}\end{array}\right\}$$

$$\leqslant\left[\sum_{1\leqslant p_1<\cdots<p_r\leqslant 2n}\left\{\det(\theta_{sp_\nu})\right\}^2\right]^{\frac{1}{2}}\cdot\left[\sum_{1\leqslant q_1<\cdots<q_{n-r}\leqslant 2n}\left\{\det(\theta_{tq_\nu})\right\}^2\right]^{\frac{1}{2}}\leqslant\|a_1\|\,\|a_2\|\cdots\|a_n\|,$$

where s runs form 1 through r, and t does from $r+1$ through n; and

$$\theta_{ik}=\Re\alpha_{ik}(1\leqslant k\leqslant n),\quad \theta_{i\,n+k}=\Im\alpha_{ik}(1\leqslant k\leqslant n),\quad a_i=(\alpha_{i1},\alpha_{i2},\cdots,\alpha_{in})$$

for $i=1,2,\cdots,n$.

Next, as a special case:

(II). Inequalities 3, 4 (*On Fischer*[1]*-Taussky*[12] *inequality*). *Let an n-square positive semi-definite Hermitian matrix (α_{ij}) be given. Still we have (since it coincides with a suitable Gram matrix* of order n):*

$$0\leqslant\begin{vmatrix}\alpha_{11}&\cdots&\\ \vdots&&\\ &&\alpha_{nn}\end{vmatrix}\leqslant\left\{\begin{array}{c}\begin{vmatrix}\Re\alpha_{11}&\cdots&\\ \vdots&\Re\alpha_{ij}&\\ &&\Re\alpha_{nn}\end{vmatrix}\\ \begin{vmatrix}\alpha_{11}&\cdots&\\ \vdots&&\\ &&\alpha_{rr}\end{vmatrix}\cdot\begin{vmatrix}\alpha_{r+1\,r+1}&\cdots&\\ \vdots&&\\ &&\alpha_{nn}\end{vmatrix}\end{array}\right\}$$

$$\leqslant\begin{vmatrix}\Re\alpha_{11}&\cdots&\\ \vdots&\Re\alpha_{ij}&\\ &&\Re\alpha_{rr}\end{vmatrix}\cdot\begin{vmatrix}\Re\alpha_{r+1\,r+1}&\cdots&\\ \vdots&\Re\alpha_{ij}&\\ &&\Re\alpha_{nn}\end{vmatrix}\leqslant\alpha_{11}\alpha_{22}\cdots\alpha_{nn}.$$

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* 9), l. c. p. 266-270. And such.

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