Totally Ordered Linear Space Structures and Extension Theorems

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Abstract

Let E be a partially ordered linear or partially ordered linear topological space over the real field **R**. By copying 20), from the viewpoint of *totally ordered linear space structures* of the product linear space $E \times \mathbf{R}$, the author synthetically improves Cotlar-Cignoli [21), III, § 1.2] type extension theorems in the direction of Anger-Lembcke [22), § 1, § 2].

Introduction. By means of [20), Ths. 1, 2] —applying our new (for the author) views 18) — we have been concerned with the Hahn-Banach extension theorem* in some detail**. As a sequel, let us now introduce our short approach 20) to the arguments of Cotlar-Cignoli [21), III, § 1.2] and of Anger-Lembcke [22), § 1, § 2]. Now that things have come to this pass, both problems of Krein type extensions*** and of Hahn-Banach type extensions (even if in the sense of 22)) are unified with and are answered simultaneously. In this article, all these particulars are given as Theorems 3 and 4, the former for real linear spaces, the latter for real linear topological spaces.

Needless to say, our present results are self-contained. For instance, for Theorem 1, it suffices to put together the proofs of Theorem 1 itself, [20), Th. 1], and [18), Th. 4].

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Preliminaries. In this paper, let E $(\neq \{0\})$ denote a linear space over the real field **R**. We sometimes abbreviate a real linear topolgical space to l.t.s. To introduce "hypolinear functional", we abopt the follwing.

DEFINITION (cf. Klee [6), § 12]). Let $\bar{\boldsymbol{R}} = \boldsymbol{R} \cup \{\infty\}$ with $\xi + \infty = \infty$ for each ξ, $\xi \cdot \infty = \infty$ for $\xi > 0$, and $0 \cdot \infty = 0$. A *hypolinear functional* h on a pointed convex cone K is a positively homogeneous and subadditive functinal on K to $\bar{\boldsymbol{R}}$.

In particular, p (resp. q) denotes a *gauge function* on E (resp. on a pointed convex cone K in E).

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* By this the author quotes [14), § 17, 3. (1)].

** Recently, Anger-Lembcke [22), existence Theorem (1. 8), Theorem (2. 4), etc.] were announced. Our result 20) (although our method is quite different from them) are closely related to them. The whole circumstances will be read in the present article.

*** To learn the Krein's, the Krein-Rutman, and the Bauer-Namioka extension theorems, the author relied upon [15), (V, 5)] (and others) instead of their originals. Our present work follows the wake of the Krein's extension theorem.

In addition, for convenience, notations and terminology employed in 18), 19), and 20) are available unless otherwise specified. Especially, e.g., $(E\times \boldsymbol{R}, \mathcal{R})$ signifies a *totally ordered linear space stucture* (for short, t.o.l.s.) of underling linear space $E\times \boldsymbol{R}$ with respect to a binary relation $\mathcal{R}$. As already indicated in the title of this paper, these structures are important in our discussion.

Statement of the results. Let us introduce our short approach 20) to the argument of the literatures*. Indebted to these literatures for the subject, first we can draw the following.

THEOREM 1. *Let E be a partially ordered linear space with positive cone C. Let K be a pointed convex cone in E, h a hypolinear functional on K. Let X be a pointed convex cone in E, and f a linear functional on X. A necessary and sufficient condition that there exists a linear form F on E extending f and satisfying $F(y)\leqslant h(y+c)$ for all $y+c\in K$ with $c\in C\cup\{0\}$ is that there exists a t.o.l.s. $(L, \mathcal{R})$ with the following properties:*

(i) $B_{\bar{f}}\cup C_c\subset(L, \mathcal{R})^+$;

(ii) $(L, \mathcal{R})^+$ *is absorbing at* $(0, 1)$ *for* L;

where L is the product linear space $E\times \boldsymbol{R}$ and $B_{\bar{f}}=\{(x_1-x_2, \xi):f(x_1)-f(x_2)<\xi,\ x_1, x_2\in X\}$, $C_c=\{(y, \eta)$: *there exists* $c\in C\cup\{0\}$ *such that* $y+c\in K$ *with* $h(y+c)<\eta\}$ *in* L.

PROOF. To begin with, extending f to the unique linear form $\bar{f}$ on (X), where (X) is the linear hull of X, $B_{\bar{f}}$ is none other than $\{(x, \xi):\bar{f}(x)<\xi, x\in(X)\}$. h is a hypolinear functional on K. Besides, each hypothesis in question assures that convex cone C_c $(\neq\varnothing)$ does not contain the origin. In these circumstances, treating $B_{\bar{f}}$, C_c as B_f, C_q, one may now finish the proof as in [20), Th. 1].

For reference, the simplest examples are

EXAMPLES. Let E be $\boldsymbol{R}^2$. Set $C=\{(\alpha, \beta):\alpha>0$, or $\alpha=0$ and $\beta>0\}$ (that is, maximal positive cone in E). Taking $K=\{(0, \beta):\beta\geqslant 0\}$ (or $K=\{(0, 0)\}$) in E, define h on K to mean $h(0, \beta)=\beta$. With this

(1) let $X=\{(\alpha, 0):\alpha\geqslant 0\}$ and define f on X by $f(\alpha, 0)=\alpha$;

(2) let X be $\{(0, 0)\}$ and define $f(0, 0)=0$;

(3) let X be the β-axis and define f on X by $f(0, \beta)=\beta$;

respectively. Then in case of (1) (resp. (2)), notwithstanding $B_{\bar{f}}\cup C_c$ is not absorbing at $((0, 0), 1)$ (resp. at any point of $X\times \boldsymbol{R}$) for L, the sufficient condition of Theorem 1 is met enough. While in case of (3), although $f(x)\leqslant h(x+c)$ $(x\in X, x+c\in K, c\in C\cup\{0\})$ holds, f fails to have desired extension. This comes from the identity $\rho((0, 0), 1)+((1, 0), -1)+((0, -\rho-1), -\rho)+((-1, \rho+1), 1)=0$ in L.

To return to the subject, notice that, in the direction of Cotlar-Cignoli [21), III, § 1. 2, 4 and 6 ((c) implies (a))], there are made some specializations about Theorem

* By these the author means [6), § 12], [21), III, § 1, 2], and [22), § 1, § 2].

1. Aside from this, of course, the same is made for the sake of subspace X. Thus as promised before, Theorem 1 reproduces e.g. the Krein's extension theorem simultaneously with the Hahn-Banach extension theorem:

COROLLARY 1. *Each statement of Theorem 1 with*

(1) $K=\{0\}$ *(where,* $h(0)=0$*) (i.e., as a consequence, this statement under subspace* X is consistent with* 〔*15*〕, (*V, 5. 4*), *Cor.1* (*Bauer-Namioka*)]),

(2) *C is a maximal positive cone in E and* $K=\{0\}$ $(h(0)=0)$,

(3) $C=\varnothing$ *(i.e., this statement may* be consistent *with*** *non-topological aspects of Anger-Lembcke* 〔*22*〕, *extension Th.* (*1. 8*), *Th.* (*2. 4*)〕),

(4) $K=E$, and

(5) *both* $C=\varnothing$ *and* $K=E$ *(i.e., this statement under subspace* X *immediately generalizes the Hahn-Banach extension theorem for linear spaces)*
gives 〔18〕, *Th. 4*(*2*)〕, *the non-topological portion of* 〔19〕, *Cor. to Th. 3*〕, 〔*20*〕, *Th. 1*〕, 〔*21*〕, *III, §1.2.3* (*qua strict cone*)〕, *and the non-topological portion of* 〔*14*〕, *§17, 3.*(*1*) (*Satz von* HAHN-BANACH)〕 *respectively.*

PROOFS. Let X be a subspace of E. By the first assertion, to the purpose, is asserted that unless f is identically-zero, both conditions submitted are directly equivalent. For such a sake, to this end, positive independence of $B_f \cup C_c$ is inherited to $A\cup C$ (esp. vice versa). Suppose that any $(E, \mathscr{R})^+$ which contains $A\cup C$ were not absorbing at $a_1\in X$ $(f(a_1)>0)$. Then there would exist $u_0\in E$ with the following properties : corresponding to $\rho>0$, there exist both finite many respective vectors $a_r\in A$, $c_s\in C$, and corresponding scalars $\alpha_r>0$ $(\alpha_1=1)$, $\beta_s\geqslant 0$ such that $\sum\alpha_r a_r+\sum\beta_s c_s+\rho u_0=0$. This entails that there exist $(\sum\alpha_r(-a_r), \xi)\in B_f$ $(f(\sum\alpha_r(-a_r)\leqslant f(-a_1)<\xi<0)$, $(-c_s, \eta_s)\in C_c$ $(\eta_s>0)$, and $(0,\varepsilon)\in B_f$ (ε fixed for any $\rho>0$) such that

$$(*)\qquad (\sum\alpha_r(-a_r), \xi)+\sum\beta_s(-c_s, \eta_s)+\{(\rho(-u_0), 0)+(0, \varepsilon)\}=(0,0),$$

which yields the first implication. For the converse, we can make use of the eq. (*) in a modified form†. The second assertion is likewise carried out by the above. The third is self-evident. For the fourth (resp. for the fifth), indeed to this purpose, $B_f \cup C_c$ is positively independent and $(0, 1)\in C_p\subset C_c$ (resp. $(0, 1)\in C_p=C_c$) (h being a gauge function) holds. Hence the assertion is met (a fortiori) by the "if" part of Theorem 1. Thus Corollary 1 is proved.

In this connection, an extreme case of Theorem 1 with $C=\varnothing$, $K=\{0\}$ $(h(0)=0)$ and X being a subspace of E corresponds to a problem of simple extensions. Of course, in view of the modified eq. (*)

COROLLARY 2††. *Such an extension is always possible.*

Returning to the subject, our concern is also

* See 〔19〕, Suppl. to Th.3〕. (Alternatively, see Rem. 2 below.)

** Cf. Corollary 3 to Theorem 1, Corollaries 2, 4 to Theorem 2.

† Take $\rho(-u_0, \tau)$ instead of $(\rho(-u_0), 0)$ thereof.

†† Alternatively, this is done using Hamel basis for E.

COROLLARY 3. *Our condition* (i) *plus* (ii) *of Theorem 1 is equivalent to that there exists a convex absorbing (at the origin) subset W of E such that $B_{\bar{f}} \cup C_c \cup (W \times \{1\})$ is positively independent in L.*

PROOF. For the necessity of the condition, take $W = \{x \in E : |F(x)| < 1\}$ in view of Theorem 1. For the converse, apply [18), Lemma 1] combining [20), Rem. 2].

In this context, as a topological version of Theorem 1, we can state and prove a criterion of the form

THEOREM 2. *Let E be an ordered linear topological space with positive cone C, and let K, h, X, f be as in the statement of Theorem 1. A necessary and sufficient condition that there exists a continuous linear form F on E extending f and satisfying $F(y) \leqslant h(y+c)$ for all $y+c \in K$ with $c \in C \cup \{0\}$ is that there exists a convex 0-neighdourhood U in E such that*

$$B_{\bar{f}} \cup C_c \cup (U \times \{1\})$$

is positively independent in L; where $B_{\bar{f}}$, C_c and L are the same as in Theorem 1.

PROOF. For the necessity of the condition, it suffices to take $U = \{x \in E : |F(x)| < 1\}$. To prove the sufficiency, appealing to [20), Rem. 2] and [18), Lemma 1], consult the proof of [20), Th. 2]. This leads up to the conclusion.

Notice also that there are several specializations about Theorem 2. At this point, we have

COROLLARY 1. *Each statement of Theorem 2 with*

(1) $K = \{0\}$ *(where, $h(0) = 0$) (i.e., as a consequence, this statement under subspace X is consistent with* [15), Th. (V,5. 4) (Bauer-Namioka)] and generalizes** the Krein-Rutman extension theorem),*

(2) *C is a maximal positive cone in E and $K = \{0\}$ ($h(0) = 0$),*

(3) *$C = \varnothing$ (i.e., this statement under locally convex space E is consistent with† Anger-Lembcke [22), extension Th. (1. 8), Th. (2. 4)]),*

(4) $K = E$

(5) *both $C = \varnothing$ and $K = E$ (i.e.,' this statement under subspace X immediately generalizes the Hahn-Banach extension theorem for l. t. s.), and*

(6) *both* $C = \varnothing$ and $X \supset K$ with $f(x) \leqslant h(x)$ $(x \in K)$

gives [19), Th. 3], the topological portion of [19), Cor. to Th. 3], [20), Th. 2], the topological version†† of [21), III, § 1. 2. 3 (qua strict cone)], the topological portion of [14), § 17, 3. (1)], and the Hahn-Banach theorem in [17), p.598] respectively.

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* See [19), Suppl. to Th.3]. (See also Rem. 2 below.)

** For details cf. also [19), p. 47].

† See Corollaries 2, 4 below.

†† The present author says: *Let E be an ordered l. t. s. with positive cone C, and h a gauge function on E. Let M be a linear subspace of E and f a linear-form on M such that $f(x) \leqslant h(x+c)$ whenever $x \in M$, $c \in C \cup \{0\}$. If h is continuous at the origin, there exists a continuous linear form F on E extending f and satisfying $F(y) \leqslant h(y+c)$ for all $y \in E$, $c \in C \cup \{0\}$.*

PROOFS. It suffices to take X to be a subspace of E. For the first and second assertions, proceed as preceding Corollary 1 and observe the eq. (*) in the form : $(\Sigma\alpha_r(-a_r), \xi)+\Sigma\beta_s(-c_s, \eta_s)+(\rho(-u_0), \varepsilon)=(0, 0)$. For the next, [20), Th. 2] is immediately paraphrased from our Theorem 2. For the fourth and fifth assertions, to these purposes, $B_f\cup C_c$ is positively independent and h (gauge function) gets continuous on E. Hence it suffices to take $U=\{y\in E: h(y)<1)\}$ for the "if" part of Theorem 2. For the last, since $C_h=\{(y, \eta): h(y)<\eta\}\subset B_f$, it suffices to refer to [20), Cor. 4].

By the way, if we are concerned with Hahn-Banach type theorems for l. t. s., as alluded before, the third statement above is very close to the results of [22), §1, §2]. That is

COROLLARY 2. *Let E be a l. t. s. Let K be a pointed convex cone in E, h a hypolinear functional on K. The condition [22), (6) of Th. (1. 8)] is mutually equivalent to that* there exists a convex symmetric 0-neighbourhood U in E such that $C_h\cup(U\times\{1\})$ is positively independent in L, where L is the product linear space $E\times\boldsymbol{R}$ and $C_h=\{(y, \eta): h(y)<\eta, y\in K\}$ in L.*

PROOF. For the necessity, let U be a convex symmetric 0-neighbourhood with $h(y)\geqslant -1$ for all $y\in K\cap U$. As is our custom, taking finite many respective vectors $(y_s, \eta_s)\in C_h$ and $(u_t, 1)\in U\times\{1\}$, suppose that $\Sigma\beta_s\eta_s+\Sigma\gamma_t u_t=0$ for some corresponding scalars $\beta_s>0$, $\gamma_t>0$ where $\Sigma\gamma_t=1$. Whereas, by hypothesis, we must have $0\leqslant h(\Sigma\beta_s y_s)+1\leqslant \Sigma\beta_s h(y_s)+1<\Sigma\beta_s\eta_s+\Sigma\gamma_t$, which proves the assertion. For the converse, let $y\in K\cap(\varepsilon U)$ $(\varepsilon>0)$ Then, even if $h(y)$ is finite, in view of $y+(-y)=0$, $\eta+\varepsilon=0$ implies $\eta\leqslant h(y)$. Whence $h(y)\geqslant -\varepsilon$ holds for all $y\in K\cap(\varepsilon U)$.

As will soon be shown, [22), Th. (1. 12)] is also treated after our own fashion. Prior to this, all the same

COROLLARY 3. *Let E be a l. t. s., and let I, J be disjoint index sets* with $I\cup J\neq\varnothing$. *For each $\lambda\in I\cup J$, let K_λ be a pointed convex cone in E and h_λ a hypolinear functional on K_λ. The substance of the condition [22), (2) of Th. (1. 12)] is mutually equivalent to that there exists a convex symmetric 0-neighbourhood U in E such that $(\bigcup_{i\in I}C_{\tilde{h_i}})\cup(\bigcup_{j\in J}C_{h_j})\cup(U\times\{1\})$ is positively independent in L, where L is the product linear space $E\times\boldsymbol{R}$, $C_{\tilde{h_i}}=\{(-y, \eta): h_i(y)<\eta, y\in K_i\}$ for $i\in I$, and $C_{h_j}=\{(y, \eta): h_j(y)<\eta, y\in K_j\}$ for $j\in J$.*

PROOF. For the necessity, by hypothesis, anyway $(\bigcup_{i\in I}C_{\tilde{h_i}})\cup(\bigcup_{j\in J}C_{h_j})$ is positively independent in L. For the sufficiency, for reference $\sum_{\nu=1}^{n}\eta'_\nu+1<0$ implies that there exist η_ν such that $\sum_{\nu=1}^{n}\eta_\nu+1=0$ with $\eta'_\nu<\eta_\nu$ $(\nu=1, 2, \cdots, n)$. At this point, the rest of the proof is analogous to that of Corollary 2.

COROLLARY 4. *Let E be a l. t. s. Let K be a pointed convex cone in E, h a hypoli-*

* On our part, take account of an extra f with $B_f\subset C_h$.

near functional on K. Let X be a pointed convex cone in E and f a linear functional on X. The substance of the condition [22], *(4) of Th. (2. 4)*] *is mutually equivalent to that there exists a convex symmetric 0-neighbourhood U in E such that $B_{\bar{f}} \cup C_h \cup (U \times \{1\})$ is positively independent in L, where L and $B_{\bar{f}}$ are as in Theorem 1 and $C_h = \{(y, \eta): h(y) < \eta\}$ in L.*

PROOF. (Alternatively, apply Corollary 3.) For the necessity, let U be a convex symmetric 0-neighbourhood such that $f(x_1) - f(x_2) + h(y) \geqslant -1$ for $x_1, x_2 \in X$, $y \in K$, $x_1 - x_2 + y \in U$. $\bar{f}$ is majorized by h on K is at once. Hence taking finite many respective vectors $(x_{1r} - x_{2r}, \xi_r) \in B_{\bar{f}}$, $(y_s, \eta_s) \in C_h$ and $(u_t, 1) \in U \times \{1\}$, suppose that $\Sigma \alpha_r (x_{1r} - x_{2r}) + \Sigma \beta_s y_s + \Sigma \gamma_t u_t = 0$ for $\alpha_r > 0$, $\beta_s \geqslant 0$, $\gamma_t > 0$ (or $\alpha_r = 0$, $\beta_s > 0$, $\gamma_t > 0$) where $\Sigma \gamma_t = 1$. Whereas it comes that $0 \leqslant f(\sum \alpha_r x_{1r}) - f(\Sigma \alpha_r x_{2r}) + h(\Sigma \beta_s y_s) + 1 < \Sigma \alpha_r \xi_r + \Sigma \beta_s \eta_s + \Sigma \gamma_t$, which proves the assertion. For the converse, let $x_1, x_2 \in X$, $y \in K$, $u = x_1 - x_2 + y \in U$. Then, even if $h(y)$ is finite, in view of $u + (-u) = 0$, hypothesis deduces $f(x_1) - f(x_2) + h(y) \geqslant -1$, which completes the proof.

In view of this, needless to say, letting U be a convex 0-neighbourhood in E, the assumption of local convexity in [22], (1. 8), (1. 12), (2. 4)] may be dropped.

Now, influenced by [22], § 1] (esp.), we can generalize Theorems 1 and 2 as follows. The proofs are nearly as before (cf. for the necessity, $\Phi(x, \xi) = -F(x) + \xi$ is positive (> 0) on every positive cone $C_{\tilde{h}_i C_i} (i \in I)$, $C_{h_j C_j} (j \in J)$; and for the sufficiency, $(0, 1) \in (X) \times \boldsymbol{R}$). In these criteria, e.g., $I = \varnothing$ corresponds to $\bigcup_{i \in I} C_{\tilde{h}_i C_i} = \varnothing$; $B_{\bar{f}}$ may be replaced by $C_{\tilde{f}} \cup C_f$.

THEOREM 3. *Let E be a linear space, and let X, f be as in Theorem 1. Let I, J, K_λ, h_λ be as in Corollary 3, and for each $\lambda \in I \cup J$ let C_λ be a positive cone in E. A necessary and sufficient condition that there exists a linear form F on E extending f and satisfying*

(a) $-h_i(y-c) \leqslant F(y)$ *whenever* $y - c \in K_i$, $c \in C_i \cup \{0\}$ *for* $i \in I$;

(b) $F(y) \leqslant h_j(y+c)$ *whenever* $y + c \in K_j$, $c \in C_j \cup \{0\}$ *for* $j \in J$;

is that there exists a t. o. l. s. $(L, \mathfrak{R})$ with the following properties:

(i) $B_{\bar{f}} \cup (\bigcup_{i \in I} C_{\tilde{h}_i C_i}) \cup (\bigcup_{j \in J} C_{h_j C_j}) \subset (L, \mathfrak{R})^+$;

(ii) $(L, \mathfrak{R})^+$ *is absorbing at* $(0, 1)$ *for* L;

where L, $B_{\bar{f}}$ are as in Theorem 1, and $C_{\tilde{h}_i C_i} = \{(-y, \eta)$: there exists $c \in C_i \cup \{0\}$ such that $y - c \in K_i$ with $h_i(y-c) < \eta\}$ for $i \in I$, $C_{h_j C_j} = \{(y, \eta)$: there exists $c \in C_j \cup \{0\}$ such that $y + c \in K_j$ with $h_j(y+c) < \eta\}$ for $j \in J$.

THEOREM 4. *Let E be a l. t. s., and let $X, f, I, J, K_\lambda, h_\lambda, C_\lambda$ be as in Theorem 3. A necessary and sufficient condition that there exists a continuous linear form F as in the statement of Theorem 3 is that ther exists a convex 0-neighbourhood U in E such that*

$$B_{\bar{f}} \cup (\bigcup_{i \in I} C_{\tilde{h}_i c_i}) \cup (\bigcup_{j \in J} C_{h_j c_j}) \cup (U \times \{1\})$$

is positively independent in L *; where* $B_{\bar{f}}$, $C_{\tilde{h}_i c_i}$, $C_{h_j c_j}$, *and* L *are as in Theorem 3.*

REMARK 1. Let F be such that as desired in Theorem 3 (or 4). This implies $F(c) \geqslant 0$ for all $c \in (\bigcup_{i \in I} C_i) \cup (\bigcup_{j \in J} C_j)$.

By the way, upon reconsidering the matter

REMARK 2. We are thus as well in a position to restate our Theorems 1-4 in terms of [22) §1, §2]. These resulting versions (which generalize [21), III, §1. 2. 3]) immediately give the Bauer-Namioka extension theorem (real case, qua preordered) and some of Anger-Lembcke [22), Theorems (1.8), (1.12), (2.4) etc.] simultaneously.

Finally we put the following.

REMARK 3. Let in particular $C = (E, \mathscr{R})^+$ in Theorem 1 (or 2). If there exists a desired extension F, the fact is $C = (E, F(\mathscr{R}))^+$. Thus if two non-zero linear forms should be the case, they are at most positive scalar multiples each of the other.

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