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ON A CONFORMAL TRANSFORMATION IN AREAL SPACES

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Abstract

Concept"of "conformal transformation" is introduced in areal spaces. Under tihs transformation, changes of various geometric objects in areal spaces of the general type and of the submetric class are considered.

§ 1. Conformal transformation in the areal space of the general type.

We assume that there be given two metric m-tensors $g_{I,J}$ and $'g_{I,J}$ in the areal space of the general type.* Let us consider a transformation

$$(A_n^{(m)}, g_{I,J}) \rightarrow (A_n^{(m)}, 'g_{I,J}), \tag{1.1}$$

which satisfies the relation;

$$'g_{I,J} = \phi^2 g_{I,J}, \tag{1.2}$$

where ϕ is a scalar function such that

$$\phi = \phi(x^j, p_\alpha^j) > 0\,. \tag{1.3}$$

Differentiating (1.2) by p_γ^k, we have

$$'g_{I,J;}{}^{\gamma}_{k} = 2\phi\phi_{;}{}^{\gamma}_{k} g_{I,J} + \phi^2 g_{I;J}{}^{\gamma}_{k},$$

and contracting by p^I, ws obtain

$$\phi_{;}{}^{\gamma}_{k} = 0\,, \tag{1.4}$$

by means of Iwamoto's theorem 6);** we can insist that the function ϕ in (1.2) does not depend on arguments p_γ^k but depend on only x^i.

On the other hand, if we assume ϕ be in the form such that

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* Latin indices $i,j,k,\ldots$ run over $1,2,\ldots n$; Greek indicee $\alpha,\beta,\gamma,..$ over 1,2,.., m; while Latin capital indices $I,J,K,..$ denote compound indices $i\,[m]$, $j\,[m]$, $k\,[m]$,.. . In what follows, we use the same notations and terminologies as those in papers 1),2),..5).

** Numbers in brackets refer to the references of the end of this paper.

*** In this paper, we use the concept "partial differetiation in p^I ,, such as

$$\Phi_{iI} = \Phi_{;\,i[m]} = m\Phi_{;[i_1}^{\,[1} p_{i_2}^{2} \cdots\cdots p_{im]}^{m]},\quad \Phi_{;i}^{\alpha} = \partial\Phi/\partial p_\alpha^i,$$

for any homogeneous function Φ of order ρ in p^I, cf. 4)

$$\phi = \phi(x^i, p^I) > 0 \tag{1.5}$$

instead of (1.3), then differentiating (1.2) partially.*** we have

$$'g_{I,J;K} = 2\phi\phi_{;K}g_{I,J} + \phi^2 g_{I,J;K},$$

and contracting p^J, we get

$$\phi_{;K} = 0.$$

Hence, we can also insist that the function ϕ in (1.2) does not depend on p^I but depend on only x^i.

An angle θ between m-vectors X^I and Y^I is defined as follows;

$$cos\theta = \frac{g_{I,J}\,X^I\,Y^J}{\|X\|\cdot\|Y\|} \tag{1.6}$$

where $\|X\|$ is a magnitude of X^I such that $\|X\|^2 = \frac{1}{m!}\,g_{I,J}X^I X^J$.

Measuring the angle θ between x^I and y^I with use of $'g_{I,J}$, we have

$$'cos\theta = \frac{'g_{I,J}X^I\,Y^J}{'\|X\|\cdot'\|Y\|} = \frac{'g_{I,J}X^I\,Y^J}{\sqrt{\frac{1}{m!}\,'g_{K,L}X^K X^L}\sqrt{\frac{1}{m!}\,'g_{M,N}\,Y^M\,Y^N}}$$

$$= \frac{\phi^2 g_{I,J}X^I\,Y^J}{\phi\sqrt{\frac{1}{m!}g_{K,L}X^K X^L}\phi\sqrt{\frac{1}{m!}\,g_{M,N}\,Y^M\,Y^N}} = \frac{g_{I,J}X^I\,Y^J}{\|X\|\cdot\|Y\|} = cos\theta$$

Consepuently, the angle θ is invariant transformation (1.1) satisfying (1.2). In such a sense, we call this transformation as conformal transformation.

There is a relation such that

$$g_{I,J}p^I p^J = (m!)^2 F \tag{1.7}$$

between the metric m-tensor $g_{I,J}$ and the fundamental function $F(x^i, p^I)$ of $A_n^{(m)}$, it is to say, $F(x^i, p^I)$ represents the magnitude of m-dimensional area element p^I,

Now, we assume that the fundamental function F is transformed to $'F$ under the conformal transformation. Then;

$$'F = \frac{1}{(m!)^2}\,'g_{I,J}p^I p^J = \frac{1}{(m!)^2}\,\phi^2 g_{I,J}p^I p^J = \phi^2 F,$$

i.e., we obtain

$$'F = \phi F. \tag{1.8}$$

So, we find that (1.8) is hold good when the transformation satisfies (1.2).

Next, let us start from (1,8), conversely. Since there is a function $f(x^i, p^i_\alpha)$ such that $F(x^i, p^I) = f(x^i, p^i_\gamma)$, we may identify f as F unless there is no confusion[1].

Differentiating (1.8), i.e.,

$$'F(x^i,p^i_\alpha)=\phi F(x^i,p^i_\alpha) \tag{1.8'}$$

by p^i_α, and making use of (1.4), we have followings;

$$'F^\alpha_{;j}=\phi F^\alpha_{;i} \tag{1.9}$$

$$'p^\alpha_i=p^\alpha_i,\ (p^\alpha_i=F^{-1}F^\alpha_{;i}), \tag{1.10}$$

$$'L^{\alpha\beta}_{ij}=L^{\alpha\beta}_{ij}, \tag{1.11}$$

where $L^{\alpha\beta}_{ij}$ is a Legendre's form such that $L^{\alpha\beta}_{ij}=p^{\alpha}_{i;j}{}^{\beta}+p^\beta_j p^\alpha_j$.

With the help of $p^\alpha_j, L^{\alpha\beta}_{ij}$, etc., the metric m-tensor $g_{I,J}$ is expressed;[4)]

$$g_{I,J}=\sum_{\lambda=0}^{m}\binom{m}{\lambda}^2\varepsilon_{\alpha_1\cdots\alpha_m}\varepsilon_{\beta_1\cdots\beta_m}F^2L^{[\alpha_1\beta_1}_{[j_1[j_1}\cdots L^{\alpha_\lambda\beta_\lambda}_{i_\lambda j_\lambda}p^{\alpha_{\lambda+1}}_{i_{\lambda+1}}p^{\beta_{\lambda+1}}_{j_{\lambda+1}}\cdots p^{\alpha_m]}_{i_m]}p^{\beta_m]}_{j_m]}. \tag{1.12}$$

Taking account of (1.10) and (1.11), we can conclude the following theorem;

THEOREM. 1. *To what $g_{I,J}$ satisfies* (1.2), *it is equivalent that F satisfies* (1.8).

The contravariant component $g^{I,K}$ of $g_{I,J}$ defined by $g^{I,K}g_{I,J}=\dfrac{1}{m!}\delta^K_J$ is transformed to $'g^{I,K}$ as

$$'g^{I,K}=\phi^{-2}g^{I,K} \tag{1.13}$$

under the conformal transformation.

Next, we have to show how connection coefficients changes. Put

$$g_{ij}{}^{hk}=g_{ii_2\cdots i_m,jj_2\cdots j_m}g^{hh_2\cdots h_m,kk_2\cdots k_m},$$

$$\Lambda_{ij}{}^{hk}=\binom{n-2}{m-1}^{-1}\{((m-1)!)^{-2}g_{(ij)}{}^{hk}-\binom{n-2}{m-1}\delta^h_{(i}\delta^k_{j)}\},$$

$$\Lambda_{ij}{}^{\alpha\beta}=\Lambda_{ij}{}^{hk}p^\alpha_h p^\beta_k,$$

$$\Lambda^*{}_{ij}{}^{\alpha\beta}=F^{\frac{2}{m}}\Lambda_{ij}{}^{\alpha\beta},\ \Lambda^{*ij}{}_{\alpha\beta}=F^{-\frac{2}{m}}\Lambda^{ij}{}_{\alpha\beta},$$

where $\Lambda^{ij}{}_{\alpha\beta}$ is derived from $\Lambda_{ij}{}^{\alpha\beta}\Lambda^{hj}{}_{\alpha\beta}=\delta^h_i\delta^\alpha_\gamma$ under the assumption that the mn-rowed det. $|\Lambda_{ij}{}^{\alpha\beta}|\neq 0$.

Then, we give a covariant differential of a vector V^i in the form;[3)]

$$\delta V^i=dV^i+\Gamma^{*i}_{jk}V^i dx^k+C^{i\delta}_{j,l}V^j\delta p^l_\delta, \tag{1.14}$$

where $\delta p^i_\alpha=\gamma^i_j(dp^j_\alpha+B^j_{\alpha k}dx^k),\ \gamma^i_j=\delta^i_j-p^i_j p^\alpha_j,$

$$\Gamma^{*i}_{jk}=\gamma^i_{jk}-C^{\times}{}^{i\gamma}_{j,h}B^h_{\gamma k}-C^{\times}{}^{i\gamma}_{k,h}B^h_{\gamma j}+C^{\times}{}^{ir\gamma}_{jk,h}B^h_{\gamma r}, \tag{1.15}$$

$$\gamma^i_{jk}=\frac{1}{2m}\Lambda^{*ir}{}_{\alpha\beta}\{\Lambda^*{}_{rk}{}^{\alpha\beta}{}_{,j}+\Lambda^*{}_{jr}{}^{\alpha\beta}{}_{,k}-\Lambda^*{}_{ik}{}^{\alpha\beta}{}_{,r}\}, \tag{1.16}$$

$$C^{\times}{}^{ir\gamma}_{ik,h}=\frac{1}{2m}\Lambda^{*ir}{}_{\alpha\beta}\Lambda^*{}_{ik}{}^{\alpha\beta};{}^{\gamma}_{h},\ C^{\times}{}^{i\gamma}_{j,h}=C^{\times}{}^{ki\gamma}_{kj,h}, \tag{1.17}$$

$$B^h_{\gamma s}=\gamma^j_{jk}p^j_\delta\widetilde{W}^{hk\delta}_{js,\gamma},\ \widetilde{W}^{hk\delta}_{is,\gamma},\ W^{ir\alpha}_{ik,\delta}=\delta^h_\alpha\delta^r_s\delta^\alpha_\gamma, \tag{1.18}$$

$$W^{jr\alpha}_{lk\delta}=(C^{\times i\alpha}_{k,l}\delta^{r}_{j}+C^{\times i\alpha}_{j,l}\ \delta^{j}_{k}-C^{\times jr\alpha}_{j,kl})\,p^{j}_{\delta}+\delta^{i}_{l}\delta^{r}_{k}\delta^{\alpha}_{\gamma},$$

$$C^{i\gamma}_{j,k}=C^{\times i r}_{j,k}-p^{h}_{\alpha}p^{\alpha}_{j}C^{\times i\alpha}_{h,k}. \tag{1.19}$$

Under the conformal transformation, quantities $g_{ij}{}^{hk}$, $\Lambda_{ij}{}^{\alpha\beta}$, $\Lambda^{ij}{}_{\alpha\beta}$ are invariant. Hence, $\Lambda^{*}{}_{ij}{}^{\alpha\beta}$ and $\Lambda^{*ij}{}_{\alpha\beta}$ are transformed as

$$'\Lambda^{*}{}_{ij}{}^{\alpha\beta}=\phi^{\frac{2}{m}}\Lambda^{*}{}_{ij}{}^{\alpha\beta},\quad '\Lambda^{*ij}{}_{\alpha\beta}=\phi^{-\frac{2}{m}}\Lambda^{*ij}{}_{\alpha\beta},$$

which give us follows;

$$'\gamma^{i}_{jk}=\frac{1}{2m}\phi^{-\frac{2}{m}}\Lambda^{*ir}{}_{ir\alpha\beta}\{\phi^{\frac{2}{m}}(\Lambda^{*}{}_{rk}{}^{\alpha\beta}{}_{,j}+\Lambda^{*}{}_{jr}{}^{\alpha\beta}{}_{,k}-\Lambda^{*}{}_{jk}{}^{\alpha\beta}{}_{,r})$$
$$+\frac{2}{m}\phi^{\frac{2}{m}-1}(\phi_{,j}\Lambda^{*}{}_{rk}{}^{\alpha\beta}-\phi_{,r}\Lambda^{*}{}_{jk}{}^{\alpha\beta})\}.$$

With use of the fact that $\Lambda^{*ir}{}_{\alpha\beta}\Lambda^{*}{}_{rj}{}^{\alpha\beta}=m^{2}\delta^{h}_{j}$ and of notation such as $\frac{1}{m^{2}}\Lambda^{*ir}{}_{\alpha\beta}=\Lambda^{*iv}_{jk}$ and $\phi^{-1}\phi_{,r}=\phi_{r}$, we obtain the expression

$$'\gamma^{i}_{jk}=\gamma^{l}_{jk}-(\Lambda^{*ir}_{jk}\phi_{r}-\delta^{i}_{j}\phi_{r}-\delta^{i}_{k}\phi_{i}), \tag{1.20}$$

which give us the transformation law of the Christoffel's symbol of the areal space of the general type.

$\widetilde{W}^{hk\delta}_{is,r}$ being invariant under the confomal transformation, the change of $B^{h}_{\alpha s}$ is given by

$$'B^{h}_{\alpha s}=B^{h}_{\alpha s}-(\Lambda^{*iv}_{jk}\phi_{r}-\delta^{i}_{j}\phi_{k}-\delta^{i}_{k}\phi_{j}-\delta^{i}_{k}\phi_{j})\,p^{j}_{\delta}\widetilde{W}^{hk\delta}_{is,\gamma}, \tag{1.21}$$

by means of (1.18).

On making use of (1.21) and of the invariance of $C^{\times hr\gamma}_{ij,k}$, $C^{\times i\gamma}_{j,k}$ and $\widetilde{W}^{hk\delta}_{is,\gamma}$,we can express the transformation law of Γ^{*j}_{jk} in the form;

$$'\Gamma^{*i}_{jk}=\Gamma^{*i}_{jk}-U^{i}_{jk}, \tag{1.22}$$

where

$$U^{i}_{jk}=\Lambda^{*ir}_{jk}\phi_{r}-\delta^{i}_{j}\phi_{k}-(\Lambda^{*hr}_{bc}\phi_{d}-\delta^{a}_{b}\phi_{d}-\delta^{a}_{c}\phi_{d})$$
$$\times(C^{\times i\gamma}_{k,r}\widetilde{W}^{rc\gamma}_{aj,\gamma}+C^{\times i\gamma}_{j,r}\ \widetilde{W}^{rc\gamma}_{ak,r}-C^{\times ih\gamma}_{jk,r}\widetilde{W}^{rc\alpha}_{ah\gamma}))\,p^{b}_{\alpha}.$$

On account of the invariance of $C^{\times i\gamma}_{j,k}$, we can immediately show, from (1.19), that the connection coefficient $C^{i\gamma}_{j,k}$ is a conformal-invariant, i.e.,

$$'C^{i\gamma}_{j,k}=C^{j\gamma}_{j,k}. \tag{1.24}$$

Accordingly, we have the the following theorem;

THEOREM. 2. *In the areal space of the general type, if there be given a connection* (1.14), *tthe connecion coefficient* Γ^{*i}_{jk} *is transformed such as* (1.22), *while the another coefficient* $C^{i\gamma}_{j,k}$ *is invariant under the conformal transformation.*

§ 2. Conformal transformation in the areal space of the submetric class.

In this section, we take up an areal space of the sudmetric class.

Let us consider a conformal transfomation

$$(A_n^{(m)},\ g_{ij}) \rightarrow (A_n^{(m)},\ 'g_{ij}) \tag{2.1}$$

with

$$'F=\phi F,\ \phi>0\,, \tag{2.2}$$

where g_{ij} is a normalized metric tensor which is given such as

$$g_{ij}=\left(\frac{1}{m}L_{ij}^{\alpha\beta}+p_i^\alpha p_j^\beta\right)g_{\alpha\beta},\ |g_{\alpha\beta}|=F^2, \tag{2.3}$$

$g_{\alpha\beta}$ being a metric tensor in an m-dimensional subspace of $A_n^{(m)}$.

Since $|'g_{\alpha\beta}|=('F)^2=\phi F^2=\phi^2|g_{\alpha\beta}|=|\phi^{\frac{2}{m}}g_{\alpha\beta}|$,

(or $|g_{\alpha\beta}|$ is a polynomial homogeneneous in each $g_{\alpha\beta}$), we have

$$'g_{\alpha\beta}=\phi^{\frac{2}{m}}g_{\alpha\beta},\ \ 'g^{\alpha\gamma}=\phi^{-\frac{2}{m}}g^{\alpha\gamma},\quad g_{\alpha\beta}g^{\alpha\gamma}=\delta_\beta^\gamma. \tag{2.4}$$

Taking (2.3), and making use of (2.4), (1.11) and (1.12), it is evident that $'g_{ij}=\phi^{\frac{2}{m}}g_{ij}$.

On the other hand, the metric m-tensor $g_{I,J}$ decomposes in the form ;

$$g_{I,J}=m!\ g_{[i_1[j_1}g_{i_2j_2}\cdots g_{i_m]j_m]}+J_{i[m],j[m]}, \tag{2.5}$$

in which $J_{i[m],j[m]}$ vanishes in the case that the space is of the metric class.

Taking thought that the expression $'g_{ij}=\Psi^2 g_{ij}$ hold good in. Riemannian, Finsler and Cartan spaces, it is natural that (2.5) would be transformed to

$$'g_{I,J}=m!(\phi^{\frac{2}{m}}g_{[i_1[j_1})(\phi^{\frac{2}{m}}g_{i_2j_2})\cdots(\phi^{\frac{2}{m}}g_{i_m]j_m]})+J_{i[m],j[m]}.$$

Consequently, for the normalizsd metric tesor g_{ij} and g^{ik}, the transformation laws are given such that

$$'g_{ij}=\phi^{\frac{2}{m}}g_{ij},\ \ 'g^{ik}=\phi^{-\frac{2}{m}}g^{ik}. \tag{2.6}$$

(Otherwise, these expressions hold good, since $C_{ij,K}^{\times hl\gamma}=\frac{1}{2}g^{kl}g_{ij;}{}_k^\gamma$ and (1.4)).

For a Christoffel's symbol

$$\left\{ {i \atop jk} \right\}=\frac{1}{2}g^{ih}\{g_{hk,j}+g_{jh,k}-g_{jk,h}\}, \tag{2.7}$$

substituting (2.6), we have

$$'\left\{ {i \atop jk} \right\}=\frac{1}{2}\phi^{-\frac{2}{m}}g^{ih}\{(\phi^{\frac{2}{m}})_{,j}\,g_{hk}+(\phi^{\frac{2}{m}})_{,k}g_{ih}-(\phi^{\frac{2}{m}})_{,h}g_{jk}$$
$$+\phi^{\frac{2}{m}}(g_{hk,j}+g_{jh,}{}_k^k-g_{jk,h})\}.$$

Putting $\phi_j=\frac{1}{m}\phi^{-1}\phi_j,\ \phi^i=g^{ij}\phi_j$, we obtain

$$'\left\{ {i \atop jk} \right\} = \left\{ {i \atop jk} \right\} - (\phi^i g_{jk} - \phi_j \delta^i_k - \phi_k \delta^i_j). \tag{2.8}$$

Under the assumption that the metric tensor g_{ij} is not necessarily real-valued, an affine connection is defined for a contravariant V^i as follows ;[5)]

$$DV^i = dV^i + \Gamma^{*i}_{jk} X^j dx^k + C^{i\delta}_{j,h} X^j Dp^h_\delta, \tag{2.9}$$

where $Dp^i_\alpha = \gamma^i_j dp^j_\alpha + B^i_{\alpha k} dx^k, \quad \gamma^i_j = \delta^i_j - p^i_\alpha p^\alpha_j,$ (2.10)

$$\Gamma^{*i}_{jk} = \left\{ {i \atop jk} \right\} - \frac{1}{2} g^{il} (g_{lk;h}{}^\alpha B^h_{\alpha j} + g_{jl;h}{}^\alpha B^h_{\gamma k} - g_{jk;h}{}^\gamma B^h_{\gamma l}), \tag{2.11}$$

$$B^i_{\alpha k} = \left\{ {p \atop jq} \right\} p^j_\gamma \widetilde{W}^{iq\gamma}_{pk,\alpha}, \quad \widetilde{W}^{iq\gamma}_{pk\alpha} W^{ph\beta}_{jq,\gamma} = \delta^i_j \delta^h_k \delta^\beta_\alpha, \tag{2.12}$$

$$W^{ir\gamma}_{hk,\alpha} = (g^{il} g_{lk;h}{}^\gamma \delta^r_j + g^{il} g_{lj;h}{}^\gamma \delta^r_k - g^{ir} g_{jk;h}{}^\gamma) p^j_\alpha + \delta^\gamma_\alpha \delta^i_h \delta^r_k. \tag{2.13}$$

The connection cofficient $C^{j\gamma}_{j,k}$ is defined by

$$C^{j\alpha}_{j,k} = g^{hi} C_{hj,}{}^\alpha_k, \tag{2.14}$$

$$C_{hj,}{}^\alpha_k = \frac{1}{2} (g_{hj;}{}^\alpha_k - 2 p^l_\beta g_{l[i} L^{*\beta\alpha}_{j]k}), \tag{2.15}$$

where $L^{*\alpha\beta}_{ij}$ is an "ecmetric tensor" which is defined by by A. KAWAGUCHI such that

$$L^{*\alpha\beta}_{ij} = L^{\alpha\beta}_{ij} g''_{ij}, \quad g''_{ij} = \frac{1}{m} L^{\alpha\beta}_{ij} g_{\alpha\beta}, \tag{2.16}$$

and which vanishes in the case that the space is of the metric class.

The ecmetric tensor $L^{*\alpha\beta}_{ij}$ is invariant under the conformal transformation, because $L^{\alpha\beta}_{ij}$ is also invariant.

$C_{hj,}{}^\gamma_k$ in (2.5) is transformed such that

$$'C_{ij,}{}^\gamma_k = \phi^{\frac{2}{m}} C_{ij,}{}^\gamma_k, \tag{2.7}$$

thus, the connection coefficient $C^i_{j,}{}^\gamma_k$ given by (2.14) is conformal-invariant.

The transformation law of the another connection coefficient is derived from (2.11), (2.12) and (2.13), in the same way as that in § 1.

$B^i_{\alpha k}$ is transfomed such as

$$'B^i_{\alpha k} = B^i_{\alpha k} - (\phi^p g_{hl} - \phi_h \delta^p_l - \phi_l \delta^p_h) p^h_\gamma \widetilde{W}^{jl\gamma}_{pk,\alpha}, \tag{2.18}$$

hence, Γ^{*i}_{jk} is transformed in the form ;

$$'\Gamma^{*i}_{jk} = \Gamma^{*i}_{jk} - U^i_{jk}, \tag{2.19}$$

with

$$U^i_{jk} = \frac{1}{2} g^{il} \{ (g_{lk;h}{}^\gamma B^h_{\alpha l} + g_{jl;h}{}^r B^h_{\alpha k} - g_{jk;h}{}^\gamma B^h_{\gamma l}) - (\phi^p g_{st} - \phi_s \delta^p_t - \phi_t \delta^p_s)(g_{lk;h}{}^\gamma \widehat{W}^{ht\delta}_{pj,\gamma} + g_{jl;h}{}^\gamma \widehat{W}^{ht\delta}_{pk\gamma} - g_{jk;h}{}^\gamma \widehat{W}^{ht\delta}_{pl,\gamma}) p^s_\delta. \tag{2.20}$$

Consequently, we have the following theorem ;

THEOREM. 3. *In the areal space of the submetric class, if there be given a connection such as* (2.9), *then, the connection coefficient* Γ^{*i}_{jk} *is transformed as in* (2.19), *while the another connection coefficient* $C^{i\,\gamma}_{j,\,k}$ *is invariant, under the conformal transformation.*

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