Some Advancements in the Structural Theory of Integral

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journal or publication title	Memoirs of the Muroran Institute of Technology. Science and engineering
volume	9
number	2
page range	581-588
year	1977-12-10
URL	http://hdl.handle.net/10258/3672

Some Advancements in the Structural Theory of Integrals

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Abstract

To establish an epistemo-geometrical interpretation of the integration process to be based on the a priori measure, we meet some difficulties. Especially, an important classical theorem does not hold in this theory of integrals. However, through some renovations, relations are found in refreshed fashions.

0. Introduction

When we look into a euclidean space (of finite dimension) E, it is found requisite that the arrangement of its points is forced to have its geometrical form to conform to the coordinate system thereto given. So, we may specifically associate the points themselves of E with their forms. Moreover, sizes of the points are accordingly considered to be associated with them. For instance, if we adopt the polar coordinate system, the size of a point must accordingly be considered to be the larger as its distance from the origin increases. We denote by $[p]$ the spatial occupation of a point p in E associated with its gemetrical form and size such as abstracted in the above and posit such that

$$\mu_p = \tilde{m}[p], \tag{0.1}$$

$\tilde{m}$ being the a priori measure. Then, μ_p will be taken as an abstract measure of a point p.

Using μ_p, for a set A in E we may have the integral expression of $\tilde{m}A$ in the form

$$\tilde{m}A = \int_{p \in A} \mu_p = \int_{p \in A} dp. \tag{0.2}$$

However, there is an important criticism on this construction. For instance, if A is a closed circular disk, for a boundary point p of A, it may be considered natural that

$$\tilde{m}[p] \cap A = \frac{1}{2}\tilde{m}[p]. \tag{0.3}$$

So then, in the integration of (0.2), to such a p $\frac{1}{2}\mu_p$ will rather be taken to be assigned instead of μ_p. However, to avoid such a complexity, we will find it better if we apply instead of (0.2) the expression

$$\tilde{m}A = \nu(A) \cdot \mu \tag{0.4}$$

on condition that all the points of E are assumed to be of the same size measured as μ. In this relation $\nu(A)$ is called the *inversion number* of A in respect to μ.

If $U(p)$ is a neighborhood of p, in case of a circular disk A, we may, with regard to the

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formula (0.3), have

$$\tilde{m}U(p)\cap A=\frac{\nu(U(p)\cap A)}{\nu(U(p))}\tilde{m}U(p).$$

Thus, if the diameter of $U(p)$ tends to zero, the right hand tends to

$$\frac{1}{2}\lim\tilde{m}U(p),$$

which will give a duplicate version of (0.3). Incidentally, if we take μ_p in the relation (0.2) as a primitive summand simply corresponding to the spatial position of p which is not directly connected with any limiting process as lim $\tilde{m}U(p)$, then the preference of $\frac{1}{2}\mu_p$ may not necessarily be claimed, because the relation (0.2) then, instead of the construction

$$\frac{\tilde{m}[p]\cap A}{\tilde{m}[p]}=\frac{1}{2},$$

simply suggest that the density of the points of A at the point p is equal to $\frac{1}{2}$. However, if we particularly insist on this version, the definition (0.1) is thereby to meet a contradiction. So, we shall henceforth renounce the expression (0.3). We may thus eventually regard the formulas (0.2) and (0.4) are telling the same meaning in case $\mu_p=\mu$ for every point in E.

A similar thing to the above-stated correlation is observed on the limiting process of a function $f(x)$ of a real variable x. By G. Cantor was adopted the conventional version that

$$1=0.999\cdots$$

This is considered as based on the admission that

$$1=1-0. \tag{0.5}$$

However, the mere formula (0.5) apparently meets a contradiction when we have

$$f(1)\neq f(1-0). \tag{0.6}$$

In this context, we may regard (0.5) is, as it were, a static expression about the point 1 whereas (0.6) is a sort of kinetic relation between the values of $f(x)$. So, also in the above-stated case, we may regard $[p]$ is the static notion of the point-occupation whereas

$$\lim U(p)$$

is the kinetic notion of the practical limit.

The integral

$$\int_A f(p)dp \tag{0.7}$$

is primarily defined as the limit of the summation

$$\sum_k\frac{k}{2^n}\tilde{m}\left\{p\in A\,\middle|\,\frac{k-1}{2^n}<f(p)\leqslant\frac{k}{2^n}\right\}$$

for $n\to\infty$, and thus we have the relation

$$\int_A f(p)dp=E(f,A)\tilde{m}A,$$

$E(f,A)$ being the mathematical expectation of the values of f over a set A. (0.7) may be referred as an *integral by the Lebesgue process*. But, since $\tilde{m}$ is a generalized extension of the

2. Stieltjes Construction

Since we may write

$$\int_{p\in A} f(p)dp=\int_{p\in A} f(p)\mu_p=\int_{p\in A} f(p)\tilde{m}[p],$$

we denote this integral by $m(f,A)$, and since

$$H(A)=\int_{p\in A} h(p)dp,$$

regarding H as a measure to be applied instead of $\tilde{m}$, we may possibly have an integral of Stieltjes type

$$H(f,A)=\int_{p\in A} f(p)H([p]).$$

In effect, we can define this integral by the following Lebesgue-Stieltjes process :

$$H(f,A)=\lim_n\sum_k\frac{k}{2^n}H(A_{(n,k)}) \tag{2.1}$$

with
$$A_{(n,k)}=\left\{p\in A\,\middle|\,\frac{k-1}{2^n}<f(p)\leqslant\frac{k}{2^n}\right\}.$$

We decompose h in the form

$$h(p)=h_{(+)}(p)-h_{(-)}(p),$$

where $h_{(+)}$ and $h_{(-)}$ are defined such that

$$h_{(+)}(p)=h(p) \text{ and } h_{(-)}(p)=0 \text{ when } h(p)\geqslant 0$$

and
$$h_{(+)}(p)=0 \text{ and } h_{(-)}(p)=-h(p) \text{ when } h(p)<0.$$

Then $H(f,A)$ may correspondingly be decomposed as

$$H(f,A)=H_{(+)}(f,A)-H_{(-)}(f,A). \tag{2.2}$$

Since (in the bounded case) both of the limitations

$$\lim\sum\frac{k}{2^n}H_{(+)}(A_{(n,k)}) \text{ and } \lim\sum\frac{k}{2^n}H_{(-)}(A_{(n,k)})$$

are easily ascertained to be convergent, the relation (2.1) is found adoptable as a definition, provided that h and f are both bounded in A.

Now, having regard to the composition (2.2), let us assume that $h(p)>0$ everywhere in A. Then, for each $A_{(n,k)}$ we have

$$H(A_{(n,k)})\geqslant 0,$$

hence
$$\frac{k-1}{2^n}H(A_{(n,k)})\leqslant\frac{k}{2^n}H(A_{(n,k)}) \tag{2.3}$$

Then, by the definition of $A_{(n,k)}$, we have

$$\frac{k-1}{2^n}H(A_{(n,k)})=\int_{A(n,k)}\frac{k-1}{2^n}h(p)dp\leqslant\int_{A(n,k)}f(p)h(p)dp$$

and
$$\int_{A(n,k)}f(p)h(p)\leqslant\int_{A(n,k)}\frac{k}{2^n}h(p)dp=\frac{k}{2^n}H(A_{(n,k)}),$$

so that, interpolating these relations in (2.3), we have

$$\frac{k-1}{2^n}H(A_{(n,k)})\leqslant\int_{A(n,k)}f(p)dp\leqslant\frac{k}{2^n}H(A_{(n,k)}).$$

Thus, from the definition (2.1), we conclude the following theorem.

Theorem D. *If $\tilde{m}A \neq \infty$ and if one-valued functions f and h are both bounded in A, then we have*

$$H(f,A)=F(h,A)=\int_A f(p)h(p)dp.$$

By the way, if A is a subset of an open set G and at almost every point of A has the determinate density $h\ (p)$ and if $f\ (p)$ is bounded in G, it is notable that we may then have the relation

$$m(f,A) = H(f,G)$$

on extension of $h\ (p)$ such that $h\ (p)=0$ when $p \notin A$. In addition, this case can be regarded as the one where it is almost everywhere in G observed that $h^*(p) = h(p)$.

3. Relative Expectance

We define $E_H\ (f,A)$ as

$$E_H(f,A)=H\ (f,A)/H\ (A)$$

on condition $H\ (A) \neq 0$, and refer to it as the *expectation* of f in A *with respect to* H or the *H-expectation* of f in A. If the value $f_H^*(p)$ defined as

$$f_H^*(p)=\lim\ E_H(f,U\ (p)\ )$$

where the neighborhood $U\ (p)$ of p is let to tend to the singleton $\{p\}$, does not vary with the choice of the tending behavior of $U\ (p)$ except for the condition that the diameter of $U\ (p)$ tends to zero, then we say f is *strongly expectant in respect of* H and refer to f_H^* as the *relative expectance* of f to H or the H-expectance of f.

When f and h are functions bounded in an open set G, if f is strongly expectant in respect of H almost everywhere in G and yet if h is strongly expectant almost everywhere in G, then we may, at almost every point p of G, have

$$\lim\frac{H(f,U(p))}{\tilde{m}U(p)} = \lim\frac{H(f,U(p))}{H(U(p))}\cdot\frac{H(U(p))}{\tilde{m}U(p)} = f_H^*(p)h^*(p).$$

Therefore, the function $f\cdot h$ is found to be strongly expectant almost everywhere in G, because, by Theorem D, $H\ (f,U\ (p)\ )= \tilde{m}\ (f\cdot h,U\ (p)\ )$. This being so, by virtue of the relation (1.1), we then have the relation

$$H(f,G)=(c)\int_{G_1} f_H^*(p)h^*(p)dp$$

G_1 being the largest subdomain of G where $f\ (p)\ h\ (p)$ is found to be strongly expectant.

If we take up an application (or a general additive function of a set) γ instead of an integral $H\ (A)= \tilde{m}\ (h,A)$ in (2.1), we may define a general integral by the Lebesgue process

$$\gamma(f,A)=\int_A f(p)\gamma_p \tag{3.1}$$

with

$$\gamma_p=\gamma[p].$$

Lebesgue measure m, the integral (0.7) is sometimes found to meet unexpected critical conditions which have never been met in case of m.

1. Strong Expectance

When the integral is produced by the Lebesgue process in respect of the a priori measure $\tilde{m}$, the following well-known theorem[1] does not generally hold :

If Φ is the indefinite integral of a bounded measurable function φ, then $\Phi'_s(x)=\varphi(x)$ at almost every point x, when Φ'_s means the strong derivative of Φ.*

In this therem 'measurable' means 'Lebesgue measurable' whereas we intend to mean '$\tilde{m}$ measurable'. Inconsistency of this theorem can be shown by the following counter-example : If A is a subset of an interval I and has everywhere in I constant density λ ($\neq 0$, and <1) and if $\varphi(x)$ is the characteristic function of A (that is, $=1$ for $x\in A$ and $=0$ otherwise), then denoting by I_x the set $\{y\in I \mid y<x\}$ we have

$$\Phi(x)=c+\int_{I_x}\varphi(y)dy=c+\lambda\cdot\tilde{m}I_x$$

(c being an arbitrary constant) so that

$$\Phi'_s(x)=\lambda\neq\varphi(x) \quad \text{for every point } x \text{ of } I.$$

When $f(p)$ is a one-valued real-valued function of a variable point p in a finite dimensional euclidean space E, by the capital letter of f we indicate the integral which is a set function such that

$$F(A)=\int_{p\in A}f(p)dp,$$

A being an arbitrary subset of the domain of f. Then the derivation of F is closely related to the expectation of f, because

$$F(U(p))=E(f,U(p))\cdot\tilde{m}U(p).$$

If the value of

$$\lim\frac{F(U(p))}{\tilde{m}U(p)}=\lim E(f,U(p))$$

is uniquely determined whenever the diameter of the neighborhood $U(p)$ of the point p tends to zero, then f is said to be *strongly expectant* at the point p and is indicated such that

$$f^*(p)=\lim E(f,U(p)).$$

As it is, this $f^*(p)$ may be regarded as the strong derivative of $F(A)$, though we emphasize its relation to $f(p)$ itself and refer to f^* as the (*strong*) *expectance* of f at the point p.

Now let us assume $f(p)$ is strongly expectant almost everywhere in a bounded open set G in E. For the sake of simplicity, we take E as of two dimensions and provided with rectangular coordinates. We draw x-lines $y=k/2^n$, y-lines $x=k/2^n$ ($k=0, \pm 2, \cdots$; $n=0, 1,$

* $\lim\dfrac{\Phi(I)}{\tilde{m}I}$ if exists, I being intervals which contain p and tend to p.

2,$\cdots$) and denote by G_0 the remained part of G after the removal of all these x- and y-lines from G. Then it is easily seen that $\tilde{m}G_0=\tilde{m}G$.

Since f is strongly expectant everywhere in G and therefore in G_0, for almost every point p of G_0 and for any given positive real number ε there must be found an open square cell Q (p) which satisfies the following conditions :

(i) Q (p) is enclosed by four lines out of the above-stated x- and y-lines for the same n ;

(ii) $p\in Q(p)\subseteqq G$;

(iii) $|F(Q(p))-f^*(p)\cdot \tilde{m}Q(p)|<\varepsilon\cdot \tilde{m}Q(p)$.

If G_1 is the remained part of G_0 after the removal of all points at which f is not strongly expectant, then evidently $\tilde{m}G_1=\tilde{m}G_0=\tilde{m}G$ and the family of the cells Q (p) $(p\in G_1$ and $n=1,2,\cdots$ if possible, i. e., on restriction that at least one Q (p) exists for n) obviously gives an open covering of G. Thus, by virtue of the Lindelöf theorem[2], there must be an enumerable covering $(Q$ (p_k) $)$ $(k=1,2,\cdots)$ of G_1.

Now, about the cells Q (p_k) $(k=1, 2,\cdots)$, it may be easily seen that if Q (p_j) $\neq$ Q (p_k) we have

$$Q(p_j)\cap Q(p_k)=\varnothing.\ \vee.\ Q(p_j)\subseteqq Q(p_k).\ \vee.\ Q(p_k)\subseteqq Q(p_j).$$

So we may eventually suppose that the sequence $(Q(p_k))$ satisfy the condition that if $k\neq j$ then

$$Q(p_j)\cap Q(p_k)=\varnothing.$$

Thus we consequently have the relation :

$$\begin{aligned}|F(G)-\sum_{k=1}^{\infty}f^*(p_k)\tilde{m}Q(p_k)| &<\varepsilon\sum\tilde{m}Q(p_k)\\ &=\varepsilon\cdot\tilde{m}G_1\\ &=\varepsilon\cdot\tilde{m}G.\end{aligned}$$

Then, letting ε tend to zero, we have

$$F(G)=\lim\sum f^*(p_k)\tilde{m}Q(p_k).$$

The right side of this relation may be regarded as a kind of integral. So we denote it by

$$(c)\int_{G_1}f^*(p)dp$$

and refer to it as an *integral by the covering process*. Then we have

$$F(G)=(c)\int_{G_1}f^*(p)dp. \tag{1.1}$$

However, on the above-stated discourse, it should be noted that the integral by the covering process on the right side of (1.1) cannot always be constituted if the domain of integration G is not given as an open set.

In this case too, the set A will be decomposed into two parts, $A_{(+)}$ where $\gamma_p > 0$ and $A_{(-)}$ where $\gamma_p < 0$. But the most important point is that we may possibly have

$$\gamma A \neq 0$$

even when $\bar{m}A=0$. Thus the value of $\gamma(f,A)$ may possibly not vanish even when $\bar{m}A=0$.

The relative expectance f_r^* of a function f to the application γ will analogously be defined by the formula

$$f_\gamma^*(p)=\lim\frac{\gamma(f,U(p))}{\gamma U(p)}$$

It should, among other things, be noted that, even when f has at almost every point of an open set G the γ-expectance to vanish, we may possibly have

$$\gamma(f,G) \neq 0.$$

4. Incompetence of a General System of Neighborhoods

In constructing an integration of any sort so far discoursed, a general system of neighborhoods may not always be found adoptable, because it may possibly be incompetent to restrict our eyes toward the specific sightviewing around a single point. Particularly, we may, in a euclidean space E, have a system of neighborhoods which may not make E separable. That is, if N is a general system of neighborhoods, for some two points p and q there may possibly exist two sequences (U_k) and (V_k) $(k=1, 2, \cdots)$ from N such that

$$\cap U_k=\{p\} \quad \text{and} \quad \cap V_k=\{q\},$$

but, for every $k=1, 2, \cdots$, we have

$$U_k\cap V_k\neq\varnothing.$$

In effect, on defining $B(p, \rho)$ as

$$B(p.\rho)=\{x|\,|x-p|<\rho\}$$

indicating by $|x-p|$ the distance between the points x and p, if we construct a system of neighborhoods $(U(p,\rho))$ $(\rho > 0, p\in E)$ such that

$$U(p,\rho)=B(p,\rho) \quad \text{for } p\neq q$$

and

$$U(q,\rho)=B(q,\rho)\cup B(p_o+\rho,\rho)$$

where q and p_0 are different fixed points and $p_0+\rho$ means the point $(x_{01}+\rho, x_{02}, \cdots, x_{on})$ when $p_0=(x_{01}, x_{02}, \cdots, x_{0n})$, then we have

$$\bigcup_{\rho>0} U(p,\rho)=\{p\}$$

and

$$\bigcup_{\rho>0} U(q,\rho)=\{q\}.$$

However, for any positive real numbers ρ and ρ', we identically have

$$U(q,\rho)\cap U(p_o,\rho')\neq\varnothing.$$

Mathematical Seminar of the Muroran Inst. Tch., Hokkaido

(Received Apr. 20, 1977)

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