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<td>雑誌名</td>
<td>Memoirs of the Muroran Institute of Technology. Science and engineering</td>
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ON AREAL SPACES BASED ON THE FUNDAMENTAL FUNCTION $F = \alpha^2 / \beta$ (I)

by

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Abstract

We consider the necessary and sufficient condition for a sufficient for a special areal space $A_{m}^{(0)}$ to belong to the semi-metric class.

§ 0. INTRODUCTION. In the Finsler geometry, a Finsler space with $(\alpha, \beta)$-metric is, as well known, a space of which fundamental function is given in the form

$$(0.1) \quad F(x, p) = f(\alpha, \beta), \quad \alpha = [\det (a_{ij}(x)y^{i}y^{j})]^{1/2}, \quad \beta = b_{i}(x)y^{i}$$

where $a_{ij}(x)$ is a Riemannian metric and $b_{i}(x)$ is non-zero covariant vector.

We know, as typical $(\alpha, \beta)$-metrics, so-called Randers' metric $F = \alpha + \beta$

$[1]^{(*)}$, and Kropina's metric $F = \alpha^{2} / \beta[2]$. On areal spaces $A_{n}^{(m)}$, G. T. Bollis [3] gave metric $F = \alpha + \beta, \quad \alpha = [\det (g_{ij}(x)y^{i}y^{j})]^{1/2}, \quad \beta = b_{i}(x)y^{i}$, where $g_{ij}(x)$ is a Riemannian metric and $b_{i}(x)$ is a skew-symmetric tensor.

Recently, the author [4] treated an areal space $A_{m}^{(n)}$ equipped a fundamental function in the form

$$(0.2) \quad F = \alpha^{2} / \beta, \quad \alpha = [\det (a_{ij}(x))]^{1/2}, \quad a_{\lambda \mu} = a_{ij}(x)y^{i}y^{j}p_{\lambda}p_{\mu}, \quad a_{0i} = a_{i0}, \quad \beta = \epsilon \lambda \mu b_{\lambda \mu} / 2, \quad b_{\lambda \mu} = b_{ij}(x)y^{i}y^{j}p_{\lambda}p_{\mu}, \quad b_{ij} = -b_{ji}$$. In that paper, the main result which we obtained is such that

THEOREM. When a fundamental function of an area space $A_{m}^{(n)}$ is given by $(0.2)$, then the following two conditions are equivalent:

(i). $A_{m}^{(n)}$ is of semi-metric class.

(ii). The relation $(\rho_{i}^{(a)} - \sigma_{i}^{(a)}) (\rho_{j}^{(b)} - \sigma_{j}^{(b)}) = 0$ holds good.

However, it was found that the above theorem holds good, even if we rewrite $\beta$ as $\beta = [\det (b_{\lambda \mu})]^{1/2}$, what we give from now on.

§ 1. PRELIMINARY. We consider an n-dimensional areal space $A_{m}^{(n)}$ based on the notion of the m-dimensional surface-element $p$.

Let $(x')$ be local coordinates and $(p')$ be local representations of $p$. In this paper, Latin indices

$\ast$) Number in brackets refer to the references at the end of the paper.
run over 1, 2, ..., n; Greek indices over 1, 2, ..., \( m \); where \( 1 < m < n \), and we adopt the Einstein’s summation convention. Other notations and terminologies are employed as same as those of the work of A. Kawaguchi [5].

We put a fundamental function of \( A^{(m)} \) as

\[
F(x, p) = \frac{a^2}{\beta}
\]

(1.1)

Next, we define a Legendre’s form of a function \( \varphi (x, p) \) as follows;

\[
L_{i}^\alpha (\varphi) = (\ln \varphi)_i^\alpha - (\ln \varphi)_i^\alpha + (\ln \varphi)_i^\alpha
\]

(1.3)

where the notation \( ;^\alpha \) means the partial differentiation with respect to \( p^\alpha \).

Differentiating (1.2) by \( p^\beta \), we have

\[
\alpha_i^\beta = (1/2) a_a^\lambda b_{\lambda \mu} a_{\mu i}^\beta
\]

(1.4)

and analogously on \( \beta_i^\alpha \).

PROPOSITION 1.

\[
\rho_i^\alpha = a_{\alpha \beta} a_{ib} p^b_i, \quad \sigma_i^\alpha = b_{\alpha \beta} b_{ib} p^b_i.
\]

Proof. From (1.4), it follows

\[
\rho_i^\alpha = (1/2) a_a^\lambda b_{\lambda \mu} a_{\mu i}^\beta
\]

= (1/2) \( a_a^\lambda a_{hk} p^b_i p^b_k a_{\lambda i}^\beta \)

\[
= a_{\alpha \beta} a_{ib} p^b_i.
\]

PROPOSITION 2.

\[
\sigma_i^\alpha = -a_{\alpha \beta} a_{ib} p^b_i
\]

(1.6)

Proof. It is sufficient that we do with \( \rho_i^\alpha, \sigma_i^\alpha \). Differentiating \( \rho_i^\alpha \) by \( p^b_i \) partially, we have

\[
\rho_{ij}^\alpha = (a_{\eta \xi} a_{ib} p^b_j)_{\eta} = a_{\eta \xi} a_{ib} p^b_j + a_{\eta \xi} a_{ib} p^b_j a_{\xi p} \delta_{ij} + a_{\eta \xi} a_{ib} p^b_j = \delta a_{\eta \xi} a_{ib} p^b_j + a_{\eta \xi} a_{ib} p^b_j a_{\xi p} \delta_{ij}.
\]

substituting the relation

\[
a_{\eta \xi} a_{ib} p^b_j = (a_{\eta \xi} a_{ib} p^b_j)_{\eta}^{-1} a_{\eta \xi} a_{ib} p^b_j = (a_{\eta \xi} a_{ib} p^b_j)_{\eta}^{-1} a_{\eta \xi} a_{ib} p^b_j.
\]

into the above representation, we can rewrite as follows:

\[
\rho_{ij}^\alpha = -a_{\eta \xi} a_{ib} p^b_j a_{\xi p} \delta_{ij} = -a_{ib} p^b_j a_{\beta \gamma} \delta_{ij} = -a_{ib} p^b_j a_{\beta \gamma} \delta_{ij}.
\]

About \( \sigma_i^\alpha \), we can obtain the right hand analogously. Q.E.D.

Then, with use of Proposition 1 and 2, we can represent the Legendre’s forms of \( \alpha \) and \( \beta \) such that
ON AREAL SPACES BASED ON THE FUNDAMENTAL FUNCTION $F = a^2/\beta$ (II)

(1.7) $L_{ij}^{\alpha\beta} |\alpha| = (l_{ij})^{\alpha\beta} + (l_{ij})^{\alpha\beta} = \rho^{\alpha\beta} + \rho^{\alpha\beta} a$

If we define tensors $a_{ij}(x, p)$ and $b_{ij}(x, p)$ as

(1.9) $a_{ij} = a_{ii} - a_{i} a_{j} \sigma^{\gamma} \rho_{j} \hat{\sigma} \rho_{j}$, rank$(a_{ij}) = n - m$,

then we have:

PROPOSITION 3. Legendre's form of $a$ and $\beta$ are given in the form such that

$L_{ij}^{\alpha\beta} |\alpha| = a^{\alpha\beta} a_{ij}$, $L_{ij}^{\alpha\beta} |\beta| = b^{\alpha\beta} b_{ij}$.

§ 2. RESULTS. First of all, we show:

PROPOSITION 4. The Legendre's form of the fundamental fundamental function given by

(1.1) together with (1.2) is

$L_{ij}^{\alpha\beta} [F] = 2 L_{ij}^{\alpha\beta} [a] - L_{ij}^{\alpha\beta} [\beta] + 2 (\rho^{\alpha\beta} \sigma^{\gamma}) (\rho_{j}^{\gamma} - \rho_{j}^{\gamma})$.

Proof. Starting from $F_{ij}^{\alpha\beta} = (a^{2}/\beta)^{\alpha\beta}$, we rewrite the quantity $p_{i}^{\alpha\beta}$ defined by $p_{i}^{\alpha\beta} = (\ln F)^{\alpha\beta}$ as

(2.1) $p_{i}^{\alpha\beta} = p_{i}^{\alpha\beta} = 2 a^{\alpha\beta} a_{ij} - 2 a^{\alpha\beta} a_{ij} - 2 (\rho^{\alpha\beta} - \rho_{i}^{\alpha\beta})$.

by means of (2.3). Applying (2.6) to the fundamentenal fundamental function $F$, we have the Legendre's form of $F$ such that $L_{ij}^{\alpha\beta} [F] = p_{ij}^{\alpha\beta} + p_{ij}^{\alpha\beta} p_{ij}^{\alpha\beta}$, to which we substitute (2.1), then it follows;

(2.2) $L_{ij}^{\alpha\beta} [F] = 2 \rho^{\alpha\beta} - 2 (\rho^{\alpha\beta} - \rho_{i}^{\alpha\beta}) + 2 (\rho^{\alpha\beta} - \rho_{j}^{\alpha\beta})$.

With use of (2.2) and Proposition 3, we can conclude this proposition. Q.E.D.

By means of the symmetry of $a^{\alpha\beta}$ and (1.7) (respectively by means of antisymmetry of $b^{\alpha\beta}$ and (1.8)), we obtain:

PROPOSITION 5. The symmetric part of $a$ (resp. $\beta$) satisfies the relation

$L_{ij}^{\alpha\beta} [a] = a^{\alpha\beta} a_{ij}$, (resp. $L_{ij}^{\alpha\beta} [\beta] = 0$).

From this proposition, it yields:

PROPOSITION 6. The symmetric part of the Legendre's form of $F$ satisfies the relation

$L_{ij}^{\alpha\beta} [F] = 2 a^{\alpha\beta} a_{ij} + 2 (\rho^{\alpha\beta} - \rho_{i}^{\alpha\beta}) (\rho_{j}^{\alpha\beta} - \rho_{j}^{\alpha\beta})$.

An areal space in which the relation $L^{\alpha\beta}_{ij} [F] = g^{\alpha\beta} g_{ij}$ holds good is said to be of "semi-metric class", where $g_{ij} = a_{ij} - a_{i} a_{j} \rho_{i}^{\alpha\beta} \rho_{j}^{\alpha\beta}$, rank$(g_{ij}) = n - m$, and $g^{\alpha\beta}$ is symmetric.

Now, in conclusion, we obtain the following theorem which is the same in appearence as the theorem in [4].

THEOREM. When the fundamental function of an areal space $A^{(m)}_{n}$ is given by (1.1) together with
then following two conditions are equivalent.

(i). \( A^{(m)} \) belongs to the semi-metric class.

(ii). The relation \( (\rho_i^{\alpha} - \sigma_i^{\alpha})(\rho_j^\beta - \sigma_j^\beta) = 0 \) holds good.

Especially we have

**COROLLARY.** When the fundamental function of an areal space \( A^{(m)} \) is given by (1.1) together with (1.2), in addition, when the relation, when the relation \( \rho_i^{\alpha} = \sigma_i^{\alpha} \) holds good, then the space \( A^{(m)} \) belongs to the metric class and it is conformal to the Riemannian space whose metric is \( a_{ij}(x) \).

Proof. Substituting the relation \( \rho_i^{\alpha} = \sigma_i^{\alpha} \) into (2.2), we have \( L_i^\beta \ L^\alpha_j = 2 a^{\alpha\beta} \ a^{\nu\nu} \) what explains that \( A^{(m)} \) belongs metric class. Moreover, from \( \rho_i^{\alpha} - \sigma_i^{\alpha} = (\ln \alpha / \beta) \), \( \sigma_i^{\alpha} = 0 \), it yields \( \ln(\alpha / \beta) = c(x) \).

Putting \( c_0(x) = \exp(c(x)) \), we have \( F = a^{2/\beta} = c_0(x) \ a = c_0(x) \ \det(a_{ij}(x) \ p^i \ p^j)^{1/2} = \det(\tilde{a}_{ij}(x) \ p^i \ p^j)^{1/2} \), where \( \tilde{a}_{ij}(x) = \exp((2/m)c(x))a_{ij}(x) \), it shows the conformality.

**REFERENCES**


