CHARACTERISTICS IN AN RF SUPERCONDUCTING QUANTUM INTERFERENCE DEVICE AS A FUNCTION OF APPLIED MAGNETIC FLUX: SYSTEMATIC CALCULATIONS

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Memoirs of the Muroran Institute of Technology. Science and engineering

Volume 42

Page Range 45-66

Year 1992

URL http://hdl.handle.net/10258/591
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CHARACTERISTICS IN AN RF SUPERCONDUCTING QUANTUM INTERFERENCE DEVICE AS A FUNCTION OF APPLIED MAGNETIC FLUX: SYSTEMATIC CALCULATIONS II

Tsuyoshi AOCHI, Shuji EBISU and Shoichi NAGATA

Abstract

Characteristic feature in superconducting quantum interference device (rf - SQUID) is shown on the basis of the analysis of the foregoing paper. The behavior will be given in detail. The parameter $\beta = (2 \pi L I_0) / \Phi_0$ changes gradually the characteristic feature, here $I_0$ is the critical current of the junction, $L$ is the self-inductance of the ring and $\Phi_0$ is the flux quantum. Abrupt transitions between two adjacent quantum states are clearly shown in the regime $\beta > 1$. The results of the systematic calculations of the characteristics in the rf - SQUID are presented over the range of $\beta = 0.20$ to $2 \pi$.

1. Introduction

The superconducting quantum interference device (rf - SQUID) is based on the two physical pillars. The first is fluxoid quantization and the second is Josephson effect. Figure 1 shows a superconducting ring with a single Josephson weak link. We shall make the simplification that the ideal Josephson junction area is small enough for the current density to be uniform, and that it never contains a significant fraction of a flux quantum. The internal magnetic flux $\Phi$ passing through the ring includes the magnetic flux $LI_s$ generated by the current $I_s$ circulating in the ring, where $L$ is the self-inductance of the ring. As shown in Fig. 1, the internal flux $\Phi$ threading the ring is then related to the applied flux $\Phi_x$ by $\Phi = \Phi_x - LI_s$, where $\Phi_x$ is the applied flux intercepted by the ring, and $LI_s$ is the screening flux generated by the induced supercurrent.

In the present paper, many physical quantities have been calculated as a function of applied magnetic flux $\Phi_x$. Their behavior depends on the dimensionless parameter $\beta = (2 \pi L I_0) / \Phi_0$, where $I_0$ is the critical current of the junction and $\Phi_0$ is the flux quantum.

Our numerical calculations have been carried out for values of $\beta$ from 0.20 to $2 \pi$. The present work is concerned with systematic computer calculations of the static behavior of the rf - SQUID, which is based on the theoretical investigation given in the previous paper of this volume. Here we will present further detailed characteristics of the rf - SQUID.
2. Basic Equations

The basic equations are summarized and are described below. The main characteristics of the rf-SQUID are the behaviors of the internal flux $\Phi$ and of the screening circulating current $I_s$ as a function of the external flux $\Phi_x$. They are derived from the next equations.

$$\Phi = \Phi_x - LI_s, \quad (1)$$

$$\theta = 2\pi \left( \frac{\Phi}{\Phi_0} + n \right). \quad (2)$$

$$I_s = I_0 \sin \theta. \quad (3)$$

Equations (1), (2) and (3) are linked equations for the three unknown quantities $\Phi, I_s$ and $\theta$ in terms of the applied flux $\Phi_x$. Here we introduce dimensionless parameter $\beta$, defined as

$$\beta = \frac{(2\pi LI_0)}{\Phi_0}. \quad (4)$$

where $\beta$ depends on the value of $LI_0$. The limiting forms of the equations are $\Phi = \Phi_x$ for $LI_0 = 0$, which corresponds to an open ring, and complete flux quantization $\Phi = n\Phi_0$ for $LI_0 \geq \Phi_0$, which corresponds to a closed ring with no weak link. Making the substitution of eqs. (2) and (3) into eq. (1), we get a next relation,

$$\Phi = \Phi_x - LI_0 \sin \left( 2\pi \frac{\Phi}{\Phi_0} \right). \quad (5)$$

Substituting eqs. (1) and (2) into eq. (3) gives

$$I_s = I_0 \sin \left( 2\pi \frac{\Phi}{\Phi_0} \right). \quad (6)$$

For the ring with a junction the energy of the system is given by

$$U = \frac{1}{2L} (\Phi - \Phi_x)^2 - E_0 \cos \left( \frac{2\pi \Phi}{\Phi_0} \right). \quad (7)$$

3. Numerical Computer Calculations for the Characteristics in rf-SQUID

We have investigated the following problems on the basis of the theoretical analysis of the foregoing our paper: \(^1\)

1. The system energy $U(\Phi, \Phi_x)$
2. The junction coupling energy $E_j$ vs. external flux $\Phi_x$
3. The magnetic energy $E_m$ vs. external flux $\Phi_x$
4. $E_J$, $E_m$ vs. phase difference $\theta$
5. Internal flux $\Phi$ vs. external flux $\Phi_x$
6. Induced flux $LI_s$ vs. external flux $\Phi_x$
7. Phase difference $\theta$ vs. external flux $\Phi_x$
8. Fluxoid vs. external flux $\Phi_x$

The results of the systematic calculations are shown in Figs 2 to 35.

4. Summary

Static characteristics of an rf - SQUID are described on the basis of numerical computer calculations. Systematic changes in the behavior of a superconducting ring are found when the parameter $\beta$ varies from 0.20 to $2\pi$.

When $\beta > 1$, the internal flux $\Phi$ and the screening current $I_s$ are continuous single valued functions of the external flux $\Phi_x$. There are no sudden transitions, the superconducting ring can go continuously from one quantum state to the next.

For $\beta > 1$, the transitions between two quantum states are irreversible. The transition to successive fluxoid takes place at $\theta = \cos^{-1}(-1 / \beta)$. The maximum in the system energy $U(\theta)$ corresponds to the critical external flux $\Phi_{xc}$ at which the internal flux $\Phi$ and the screening current $I_s$ have an infinite slope as a function of the external flux $\Phi_x$. From the energy view point of $U(\Phi, \Phi_x)$, $\Phi_{xc}$ corresponds to the value at which the system changes from metastable state to the stable state.

Acknowledgment

The author (S. N.) would like to thank Professor M. Ocio for critical reading of these manuscripts.

Reference

Fig. 1 Superconducting ring with a ideal Josephson junction denoted by $J$. The contour used for integration is shown by the broken line. Internal magnetic flux $\Phi$, circulating current $I_s$, self-inductance $L$ and applied magnetic flux $\Phi_x$ are related by $\Phi = \Phi_x - LI_s$. Typical values are $L = 5$ nH, and $I_0 = 1 \mu$ A. The junction resistance in the normal state is $R = 10 \Omega$, and the diameter of the ring is about 2 mm.
Fig. 2 A demonstration of a flux jump in a potential surface in the case of $\beta = 6\pi$. The system potential $U(\Phi, \Phi_x)$ surface for $\Phi_x = 0$ to $3\Phi_o$ and $\Phi = \Phi_o$ to $-4\Phi_o$ is shown. When $\Phi_x = 0$, the system is trapped around a minimum such as point A in the potential well associated with a fluxoid quantum state. The system is constrained by a potential barrier at B. As $\Phi_x$ is increased, the potential energy increases along the valley $A - A'$ and the system can transfer from point $A'$, where $\Phi_x = \Phi_{x'}$ and $\Delta U = 0$, to point C.
Fig. 3 Potential $U(\Phi, \Phi_x)$ surface for $\Phi_x = 0$ to $2\Phi_0$ and $\Phi = 3\Phi_0$ to $-3\Phi_0$ in the case where $\beta = 0.50$. The sharp transition can not occur between the two adjacent quantum states.

Fig. 4 Potential $U(\Phi, \Phi_x)$ surface for $\Phi_x = 0$ to $2\Phi_0$ and $\Phi = 3\Phi_0$ to $-3\Phi_0$ in the case where $\beta = 1.50$. The sharp transitions can occur between the two adjacent quantum states at $\Phi_x = 0.544\Phi_0$ and $0.456\Phi_0$ in the irreversible process, see Fig. 9.
Fig. 5 Potential $U(\Phi, \Phi_\phi)$ surface for $\Phi_\phi = 0$ to $2\Phi_0$ and $\Phi = 3\Phi_0$ to $-3\Phi_0$ in the case where $\beta = 2\pi$. The sharp transitions can occur between the two adjacent quantum states, see Fig. 11.
Fig. 6 System energy $U$ for $\beta = 0.20$ as a function of $\Phi$. The energy minimum shifts gradually from a flux quantum state to a neighbor state when the external magnetic flux $\Phi_x$ changes. The value of $\Phi_x$ denoted in each graph is normalized by $\Phi_0$.

Fig. 7 System energy $U$ for $\beta = 0.50$ as a function of $\Phi$. The value of $\Phi_x$ changes from 0.0 to 1.10.
Fig. 8 System energy $U$ for $\beta = 1.00$ as a function of $\Phi$. The value of $\Phi_x$ changes from 0.0 to 1.20.

Fig. 9 Behavior of hysteresis in $U$ for $\beta = 1.50$. With increasing external flux $\Phi_x$, the superconducting ring stays at the minimum point up to $\Phi_x / \Phi_0 = 0.500$. From 0.500 to 0.544 the system remains in the metastable state and the transition takes place at 0.544. On the other hand, with decreasing $\Phi_x$, the transition occurs at 0.456. For simple illustration, the solid circles indicate the flux in increasing process and the open circles show the flux in decreasing process.
Fig. 10 Behavior of hysteresis in $U$ for the case of $\beta = 3.00$. The hysteresis appears in the same way shown in Fig. 9. The value of $\Phi_z$ changes from 0.0 to 1.10.

Fig. 11 Behavior of hysteresis in $U$ for the case of $\beta = 2\pi$. The hysteresis appears in the same way shown in Fig. 9. The value of $\Phi_z$ changes from $-0.30$ to 2.70.
Fig. 12 Junction coupling energy $E_J$, magnetic energy $E_m$ and system energy $U$ as a function of the external flux $\Phi_x$ for $\beta = 0.20$. The value of $U$ corresponds to the minimum value in Fig. 6.

Fig. 13 $E_J$, $E_m$ and $U$ as a function of $\Phi_x$ for $\beta = 0.50$. The value of $U$ corresponds to the minimum value in Fig. 7.
Fig. 14  \( E_l, E_m \) and \( U \) as a function of \( \Phi_x \) for \( \beta = 1.00 \). The value of \( U \) corresponds to the minimum value in Fig. 8.

Fig. 15  \( E_l, E_m \) and \( U \) as a function of \( \Phi_x \) for \( \beta = 1.50 \). The hysteresis with transitions at different \( \Phi_x \) is indicated by arrows. The hysteresis behavior can be understood by considering the correspondence between Fig. 9 and Fig. 15. The value of \( U \) corresponds to the minimum or maximum value in Fig. 9.
Fig. 16 $E_\nu, E_m$ and $U$ as a function of $\Phi_x$ for $\beta = 3.00$. The hysteresis behavior can be understood by considering the correspondence between Fig. 10 and Fig. 16. The value of $U$ corresponds to the minimum or maximum value in Fig. 10.

Fig. 17 $E_\nu, E_m$ and $U$ as a function of $\Phi_x$ for $\beta = 2\pi$. The hysteresis behavior can be understood by considering the correspondence between Fig. 11 and Fig. 17. The value of $U$ corresponds to the minimum or maximum value in Fig. 11.
Fig. 18 Junction coupling energy $E_J$, magnetic energy $E_m$ and system energy $U$ as a function of the phase difference $\theta$ across the junction for $\beta = 0.20$.

Fig. 19 $E_J$, $E_m$ and system energy $U$ as a function of $\theta$ for $\beta = 0.50$. 
Fig. 20 $E_J$, $E_m$ and system energy $U$ as a function of $\theta$ for $\beta = 1.00$.

Fig. 21 $E_J$, $E_m$ and system energy $U$ as a function of $\theta$ for $\beta = 1.50$. 
Fig. 22 $E_{01}$, $E_{m}$ and system energy $U$ as a function of $\theta$ for $\beta = 3.00$.

Fig. 23 $E_{01}$, $E_{m}$ and system energy $U$ as a function of $\theta$ for $\beta = 2\pi$. 
Fig. 24 Internal flux $\Phi$ and the flux $L_I$ induced by screening current as a function of the external flux $\Phi_x$ for $\beta = 0.20$.

Fig. 25 Internal flux $\Phi$ and the induced flux $L_I$ as a function of $\Phi_x$ for $\beta = 0.50$. 
Fig. 26 Internal flux $\Phi$ and the induced flux $I_{\text{ind}}$ as a function of $\Phi_x$ for $\beta = 1.00$.

Fig. 27 Internal flux $\Phi$ and the induced flux $I_{\text{ind}}$ as a function of $\Phi_x$ for $\beta = 1.50$. The hysteresis with transitions at different $\Phi_x$ is indicated by arrows. The hysteresis behavior corresponds to that in Fig. 9.
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**Fig. 28** Internal flux $\Phi$ and the induced flux $LI_s$ as a function of $\Phi_x$ for $\beta = 3.00$. The hysteresis behavior corresponds to that in Fig. 10.

**Fig. 29** Internal flux $\Phi$ and the induced flux $LI_s$ as a function of $\Phi_x$ for $\beta = 2\pi$. The hysteresis behavior corresponds to that in Fig. 11.
Fig. 30 Phase difference $\theta$ across the junction and fluxoid as a function of the external flux $\Phi_x$ for $\beta = 0.20$.

Fig. 31 Phase difference $\theta$ and fluxoid as a function of $\Phi_x$ for $\beta = 0.50$. 
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**Fig. 32** Phase difference $\theta$ and fluxoid as a function of $\Phi_x$ for $\beta = 1.00$. The hysteresis with transitions at different $\Phi_x$ is indicated by arrows. The hysteresis feature corresponds to that in Fig. 9.

**Fig. 33** Phase difference $\theta$ and fluxoid as a function of $\Phi_x$ for $\beta = 1.50$. The hysteresis with transitions at different $\Phi_x$ is indicated by arrows. The hysteresis feature corresponds to that in Fig. 9.
Fig. 34 Phase difference $\theta$ and fluxoid as a function of $\Phi_x$ for $\beta = 3.00$. The hysteresis feature corresponds to that in Fig. 10.

Fig. 35 Phase difference $\theta$ and fluxoid as a function of $\Phi_x$ for $\beta = 2\pi$. The hysteresis feature corresponds to that in Fig. 11.