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S.Sasaki introduced a Riemannian metric on tangent bundle of Riemannian space.Instead of Riemannian space, we treat subspace of Finsler space. Then, we can construct three Riemannian metrics which are related with E.Cartan's symmetric connection coefficients. We shall investigate relations between them and get conditions of these coincidences.

Let N be an $n-$dimensional Finsler space with fundamental function F. Hereafter, indices h, i, j, k, l run the range $\{1, 2, \cdots, n\}$ and we adopt summation convention for repeated index. At point with local coordinate (x^h), we consider a tangent vector $x^{*h}\partial/\partial x^h$. Then, $(x^H)=(x^h, x^{*h})$ yields local coordinate system on tangent bundle TN of N, where indices H, I, J run the range $\{1,2,\cdots,n,*1,*2,\cdots,*n\}$. Fundamental function F is continuous on TN and differentiable on $UN=TN\setminus F^{-1}(0)$. Moreover, $F=F(x,x^*)$ is positive on UN and positively homogeneous of degree 1 in x^*. We put

$$g_{ji}=\frac{1}{2}\cdot\frac{\partial^2 F^2}{\partial x^{*j}\,\partial x^{*i}}$$

at point (x,x^*) on UN. Then F satisfies that the matrix (g_{ji}) is positive definite. We denote its inverse matrix by $(g^{ji})=(g_{ji})^{-1}$. The quantities g_{ji} are components of metric tensor on Finsler space N. Now we employ the following notations :

$$C_{kji}=\frac{1}{2}\cdot\frac{\partial g_{ji}}{\partial x^{*k}}\ ,\quad r_{ji}{}^h=\frac{1}{2}\cdot g^{kh}\left(\frac{\partial g_{ki}}{\partial x^j}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ji}}{\partial x^k}\right)\ ,$$

$$\Gamma_i{}^h=\frac{1}{2}\cdot\frac{\partial}{\partial x^{*i}}(r_{kj}{}^h\, x^{*k}x^{*j}),$$

$$\Gamma^{*}_{ji}{}^{h}=r_{ji}{}^{h}-g^{kh}(C_{lki}\Gamma_{j}{}^{l}+C_{ljk}\Gamma_{i}{}^{l}-C_{lji}\Gamma_{k}{}^{l}) .$$

The quantities $\Gamma^{*}_{ji}{}^{h}$ are E.Cartan's symmetric connection coefficients. We shall often use the well known identities

$$C_{kji}x^{*k}=0 , \quad \Gamma_{i}{}^{h}=\Gamma^{*}_{ji}{}^{h}x^{*j} , \quad \frac{\partial\Gamma^{*}_{ji}{}^{h}}{\partial x^{*k}}x^{*j}x^{*i}=0$$

and others.

Let us consider an m − dimensional subspace M of N. Space M with restricted function F is also a Finsler space. We assume that indices $\alpha,\beta,\gamma,\delta,\varepsilon$ run the range $\{1, 2, \cdots, m\}$ and that indices A , B run the range $\{1, 2, \cdots, m, *1, *2, \cdots, *m\}$. We introduce local coordinate (u^{a}) on subspace M and attendant local coordinate $(u^{A})=(u^{a},u^{*a})$ on tangent bundle TM . Now, subspace M of N is locally expressed by $x^{h}=x^{h}(u)$. We denote the components of projection tensor on M by

$$B_{a}{}^{h}=\frac{\partial x^{h}}{\partial u^{a}} .$$

Matrix $(B_{a}{}^{h})$ has maximal rank m. Therefore, we can define $C_{i}{}^{\lambda}$ such that

$$C_{i}{}^{\lambda}B_{a}{}^{i}=0$$

and matrix $(C_{i}{}^{\lambda})$ has rank $n-m$, where indices λ,μ run the range $\{m+1, m+2, \cdots, n\}$. Metric tensor on subspace M is defined from fundamental function F and coincides with induced metric tensor, that is, its components are given by

$$g_{\beta a}=\frac{1}{2}\cdot\frac{\partial^{2}F^{2}}{\partial u^{*\beta}\partial u^{*a}}=g_{ji}B_{\beta}{}^{j}B_{a}{}^{i} .$$

Now, intrinsically, we define $g^{\beta a}$, $C_{\gamma\beta a}$, Γ_{β}^{a} and E.Cartan's symmetric connection coefficients $\Gamma^{*}_{\gamma\beta}{}^{a}$ of subspace M. At point (u,u^{*}) on $UM=TM\setminus F^{-1}(0)$, we put

$$g^{\mu\lambda}=g^{ji}C_{j}{}^{\mu}C_{i}{}^{\lambda} , \quad (g_{\mu\lambda})=(g^{\mu\lambda})^{-1} , \quad C_{\lambda}{}^{h}=g_{\mu\lambda}C_{i}{}^{\mu}g^{ih},$$
$$B_{i}{}^{a}=g_{ji}B_{\beta}{}^{j}g^{\beta a}.$$

Subspace TM of TN is locally expressed by $x^{H}=x^{H}(u,u^{*})$ i. e.

$$\begin{cases} x^h = x^h(u) \\ x^{*h} = B_a{}^h u^{*a} \end{cases}$$

and components

$$D_A{}^H = \frac{\partial x^H}{\partial u^A}$$

of projection tensor on the subspace TM are represented as

$$D_a{}^h = B_a{}^h \ , \quad D_{*a}{}^h = 0 \ ,$$

$$D_a{}^{*h} = \frac{\partial B_\beta{}^h}{\partial u^a} u^{*\beta} \ , \quad D_{*a}{}^{*h} = B_a{}^h \ .$$

Its restriction on UM constructs projection tensor on subspace UM of UN .

From E.Cartan's symmetric connection coefficients of N, we shall induce another connection coefficients of M . We define these by

$$\bar{\Gamma}^*{}_{\gamma\beta}{}^a = B_h{}^a \left(\frac{\partial B_\beta{}^h}{\partial u^\gamma} + \Gamma^*{}_{ji}{}^h B_\gamma{}^j B_\beta{}^i \right) .$$

Using $\Gamma^*{}_{ji}{}^h$ and $\bar{\Gamma}^*{}_{\gamma\beta}{}^a$, mixed type covariant differentiation of $B_a{}^h$ is given by

$$H_{\beta a}{}^h = \frac{\partial B_a{}^h}{\partial u^\beta} + \Gamma^*{}_{ji}{}^h B_\beta{}^j B_a{}^i - \bar{\Gamma}^*{}_{\beta a}{}^\gamma B_\gamma{}^h \ .$$

These are components of Euler－Schouten's tensor. Because of $B_i{}^a B_a{}^h + C_i{}^\lambda C_\lambda{}^h = \delta_i{}^h$, we obtain another expression :

$$H_{\beta a}{}^h = C_\lambda{}^h C_k{}^\lambda \left(\frac{\partial B_a{}^k}{\partial u^\beta} + \Gamma^*{}_{ji}{}^k B_\beta{}^j B_a{}^i \right).$$

To describe the difference between two connections, we put

$$H^h = H_{\beta a}{}^h u^{*\beta} u^{*a} \ , \ P_{\gamma\beta a} = C_{kji}(B_\gamma{}^k B_a{}^j H_{\delta\beta}{}^i u^{*\delta} - B_\beta{}^k H^j B_\delta{}^i g^{\delta\varepsilon} C_{\varepsilon\gamma a}).$$

Then, after somewhat long calculations, we shall find

$$\Gamma^{*}{}_{\gamma\beta}{}^{\alpha}-\overline{\Gamma}^{*}{}_{\gamma\beta}{}^{\alpha}=g^{\delta\alpha}(P_{\gamma\beta\delta}+P_{\beta\gamma\delta}-P_{\gamma\delta\beta}).$$

When we put

$$P_{\beta\alpha}=P_{\beta\gamma\alpha}u^{*\gamma},\ \overline{\Gamma}_{\beta}{}^{\alpha}=\overline{\Gamma}^{*}{}_{\gamma\beta}{}^{\alpha}u^{*\gamma},$$

we have

$$P_{\beta\alpha}=C_{kji}H^{k}B_{\beta}{}^{j}B_{\alpha}{}^{i}\ ,\quad \Gamma_{\beta}{}^{\alpha}-\overline{\Gamma}_{\beta}{}^{\alpha}=g^{\gamma\alpha}P_{\gamma\beta}\ .$$

We consider a curve on N which is given by $x^{h}=x^{h}(s)$ where we suppose parameter s to satisfy $F(x(s),x'(s))=1$, that is, parameter s is an arc length. If the curve satisfies a system of equations

$$x^{h''}+\Gamma^{*}{}_{ji}{}^{h}(x,x')x^{j'}x^{i'}=0\ ,$$

it is called geodesic on N. Let us consider a geodesic on subspace M which is given by $u^{\alpha}=u^{\alpha}(s)$ with arc length parameter s. According to the identity $\Gamma^{*}{}_{\gamma\beta}{}^{\alpha}u^{*\gamma}u^{*\beta}=\overline{\Gamma}^{*}{}_{\gamma\beta}{}^{\alpha}u^{*\gamma}u^{*\beta}$, it satisfies a system of equations

$$u^{\alpha''}+\overline{\Gamma}^{*}{}_{\gamma\beta}{}^{\alpha}(u,u')u^{\gamma'}u^{\beta'}=0\ .$$

For our geodesic $x^{h}=x^{h}(u(s))$ on subspace M, we have

$$x^{h''}+\Gamma^{*}{}_{ji}{}^{h}(x,x')x^{j'}x^{i'}$$

$$=B_{\alpha}{}^{h}u^{\alpha''}+\left(\frac{\partial B_{\alpha}{}^{h}}{\partial u^{\beta}}+\Gamma^{*}{}_{ji}{}^{h}(x,x')B_{\beta}{}^{j}B_{\alpha}{}^{i}\right)u^{\beta'}u^{\alpha'}$$

$$=B_{\alpha}{}^{h}(u^{\alpha''}+\overline{\Gamma}^{*}{}_{\gamma\beta}{}^{\alpha}(u,u')u^{\gamma'}u^{\beta'})+H_{\beta\alpha}{}^{h}(u,u')u^{\beta'}u^{\alpha'}$$

$$=H^{h}(u,u').$$

It is geodesic on N if and only if system of equations $H^{h}(u,u')=0$ is established for each s. If all geodesics on M are also geodesics on N, M is said to be totally geodesic. Our considerations show that the necessary and sufficient condition for M to be totally geodesic is $H^h(u,u^*)=0$ for $F(u,u^*)=1$. But, according to homogeneity of H^h in u^*, presupposed condition $F(u,u^*)=1$ can be removed. Now, we know that M is totally geodesic if and only if $H^h=0$ for each point on UM.

We shall attempt to describe this property in another way. From $H^h=0$, we get $C_h{}^\lambda H^h=0$, that is,

$$C_h{}^\lambda\left(\frac{\partial B_\alpha{}^h}{\partial u^\beta}+\Gamma^{*}{}_{ji}{}^h B_\beta{}^j B_\alpha{}^i\right)u^{*\beta}u^{*\alpha}=0 .$$

Because of $B_\alpha{}^h=B_\alpha{}^h(u)$ and $C_h{}^\lambda=C_h{}^\lambda(u)$, partial differentiation with respect to u^* turns to

$$C_h{}^\lambda\left(\frac{\partial B_\alpha{}^h}{\partial u^\beta}+\Gamma^{*}{}_{ji}{}^h B_\beta{}^j B_\alpha{}^i\right)u^{*\beta}=0 .$$

By contraction with $C_\lambda{}^k$, we obtain $H_{\beta\alpha}{}^h u^{*\beta}=0$. Once more, partially differentiating with respect to u^* and contracting with $C_\lambda{}^k$, we get

$$H_{\beta\alpha}{}^h=-C_\lambda{}^h C_k{}^\lambda\frac{\partial\Gamma^{*}{}_{ji}{}^k}{\partial x^{*l}}x^{*j}B_\beta{}^i B_\alpha{}^l$$

and then, evidently, system of above equations gives $H^h=0$.

THEOREM. Following (1), (2) and (3) are equivalent each other.

(1) M is a totally geodesic subspace of N.

(2) $H_{\beta\alpha}{}^h u^{*\beta}=0$.

(3) $H_{\beta\alpha}{}^h=-C_\lambda{}^h C_k{}^\lambda\dfrac{\partial\Gamma^{*}{}_{ji}{}^k}{\partial x^{*l}}x^{*j}B_\beta{}^i B_\alpha{}^l$.

If $H_{\beta\alpha}{}^h u^{*\beta}=0$, then $P_{\gamma\beta\alpha}=0$ and we obtain the following property.

THEOREM. If M is a totally geodesic subspace of N, intrinsic connection on M coincides with induced one, that is, $\Gamma^{*}{}_{\gamma\beta}{}^\alpha=\bar{\Gamma}^{*}{}_{\gamma\beta}{}^\alpha$.

We shall constitute a Riemannian metric tensor on UN which was introduced by

S.Sasaki. For two tangent vectors $X=X^H\partial/\partial x^H$ and $Y=Y^H\partial/\partial x^H$ on UN , we consider the following four tangent vectors on N :

$$X^h\partial/\partial x^h\ ,\ (X^{*h}+\Gamma_i{}^hX^i)\partial/\partial x^h\ ,$$
$$Y^h\partial/\partial x^h\ ,\ (Y^{*h}+\Gamma_i{}^hY^i)\partial/\partial x^h\ .$$

Now, we can define an inner product of X and Y by

$$G(X,Y)=g_{ji}X^jY^i+g_{ji}(X^{*j}+\Gamma_l{}^jX^l)(Y^{*i}+\Gamma_k{}^iY^k)\ .$$

It gives a Riemannian metric tensor G on UN and its components $G_{JI}=G(\partial/\partial x^J,\partial/\partial x^I)$ are given by

$$G_{ji}=g_{ji}+g_{lk}\Gamma_j{}^l\Gamma_i{}^k\ ,\ G_{*ji}=G_{i*j}=g_{jk}\Gamma_i^k\ ,\ G_{*j*i}=g_{ji}\ .$$

Using $\overline{\Gamma}_\beta{}^\alpha$, we similarly define a Riemannian metric tensor $\overline{G}$ on UM . Then, components $\overline{G}_{BA}=\overline{G}(\partial/\partial u^B,\partial/\partial u^A)$ of this tensor are given by

$$\overline{G}_{\beta\alpha}=g_{\beta\alpha}+g_{\delta\gamma}\overline{\Gamma}_\beta{}^\delta\overline{\Gamma}_\alpha{}^\gamma\ ,\ \overline{G}_{*\beta\alpha}=\overline{G}_{\alpha*\beta}=g_{\gamma\beta}\overline{\Gamma}_\alpha^\gamma\ ,\ \overline{G}_{*\beta*\alpha}=g_{\beta\alpha}\ .$$

On the other hand, induced metric tensor G on subspace UM of UN may also be existing. Its components are defined by

$$G_{BA}=G(\partial/\partial u^B,\partial/\partial u^A)=G_{JI}D_B{}^JD_A{}^I\ .$$

We shall search out the relation between G and $\overline{G}$.

$$G_{\beta\alpha}=G_{ji}B_\beta{}^jB_\alpha{}^i+G_{*ji}\frac{\partial B_\gamma^j}{\partial u^\beta}u^{*\gamma}B_\alpha^i+G_{j*i}B_\beta^j\frac{\partial B_\gamma^i}{\partial u^\alpha}u^{*\gamma}+$$
$$+G_{*j*i}\frac{\partial B_\delta^j}{\partial u^\beta}u^{*\delta}\frac{\partial B_\gamma^i}{\partial u^\alpha}u^{*\gamma}$$
$$=g_{\beta\alpha}+g_{ji}\left(\frac{\partial B_\delta{}^j}{\partial u^\beta}u^{*\delta}+\Gamma_l{}^jB_\beta^l\right)\left(\frac{\partial B_\gamma{}^i}{\partial u^\alpha}u^{*\gamma}+\Gamma_k{}^iB_\alpha{}^k\right)$$
$$=g_{\beta\alpha}+g_{ji}(H_{\delta\beta}^ju^{*\delta}+\overline{\Gamma}_\beta{}^\delta B_\delta{}^j)(H_{\gamma\alpha}^iu^{*\gamma}+\overline{\Gamma}_\alpha{}^\gamma B_\gamma{}^i)$$

$$=\overline{G}_{\beta\alpha}+g_{ji}(H_{\delta\beta}{}^{j}u^{*\delta})(H_{\gamma\alpha}{}^{i}u^{*\gamma}) \ ,$$

$$G_{*\beta\alpha}=G_{*ji}B_{\beta}{}^{j}B_{\alpha}{}^{i}+G_{*j*i}B_{\beta}{}^{j}\frac{\partial B_{\gamma}{}^{i}}{\partial u^{\alpha}}u^{*\gamma}$$

$$=g_{ji}B_{\beta}{}^{j}(H_{\gamma\alpha}{}^{i}u^{*\gamma}+\overline{\Gamma}_{\alpha}{}^{\gamma}B_{\gamma}{}^{i})=g_{\gamma\beta}\overline{\Gamma}_{\alpha}{}^{\gamma}=\overline{G}_{*\beta\alpha} \ ,$$

$$G_{*\beta*\alpha}=G_{*j*i}B_{\beta}{}^{j}B_{\alpha}{}^{i}=g_{\beta\alpha}=\overline{G}_{*\beta*\alpha} \ .$$

Remarking the fact that system of equations $g_{ji}(H_{\delta\beta}{}^{j}u^{*\delta})(H_{\gamma\alpha}{}^{i}u^{*\gamma})=0$ is equivalent to system of equations $H_{\beta\alpha}{}^{h}u^{*\beta}=0$, we know that the subspace M of N is totally geodesic if and only if $G=\overline{G}$ on UM .

Instead of $\overline{\Gamma}_{\beta}{}^{\alpha}$, we will make use of $\Gamma_{\beta}{}^{\alpha}$. Then, another Riemannian metric tensor $\widetilde{G}$ on UM can be defined and its components $\widetilde{G}_{BA}=\widetilde{G}(\partial/\partial u^{B},\partial/\partial u^{A})$ are given by

$$\widetilde{G}_{\beta\alpha}=g_{\beta\alpha}+g_{\delta\gamma}\Gamma_{\beta}{}^{\delta}\Gamma_{\alpha}{}^{\gamma} \ , \ \widetilde{G}_{*\beta\alpha}=\widetilde{G}_{\alpha*\beta}=g_{\gamma\beta}\Gamma_{\alpha}{}^{\gamma} \ , \ \widetilde{G}_{*\beta*\alpha}=g_{\beta\alpha} \ .$$

Mentioning $\widetilde{G}_{*\beta\alpha}-\overline{G}_{*\beta\alpha}=g_{\gamma\beta}(\Gamma_{\alpha}{}^{\gamma}-\overline{\Gamma}_{\alpha}{}^{\gamma})$, we obtain the condition of $\widetilde{G}=\overline{G}$.

THEOREM. Metric tensor $\widetilde{G}$ on UM coincides with $\overline{G}$ if and only if $\Gamma_{\beta}{}^{\alpha}=\overline{\Gamma}_{\beta}{}^{\alpha}$, that is, $P_{\beta\alpha}=0$.

By straight forward calculations, we obtain the relation between $\widetilde{G}$ and $\overline{G}$.

$$\widetilde{G}_{\beta\alpha}=\overline{G}_{\beta\alpha}+\overline{\Gamma}_{\beta}{}^{\gamma}P_{\gamma\alpha}+\overline{\Gamma}_{\alpha}{}^{\gamma}P_{\gamma\beta}+g^{\delta\gamma}P_{\delta\beta}P_{\gamma\alpha} \ ,$$

$$\widetilde{G}_{*\beta\alpha}=\widetilde{G}_{\alpha*\beta}=\overline{G}_{*\beta\alpha}+P_{\beta\alpha} \ , \ \widetilde{G}_{*\beta*\alpha}=\overline{G}_{*\beta*\alpha} \ .$$

If $\widetilde{G}=G$, equation $\widetilde{G}_{*\beta\alpha}=\overline{G}_{*\beta\alpha}+P_{\beta\alpha}=G_{*\beta\alpha}+P_{\beta\alpha}$ gives $P_{\beta\alpha}=0$ and then we conclude $G=\widetilde{G}=\overline{G}$. Now, our investigations may be summarized as follows.

THEOREM. Following (1), (2) and (3) are equivalent each other.

(1) M is a totally geodesic subspace of N.

(2) Metric tensor $\overline{G}$ on UM coincides with G.

(3) Metric tensor $\widetilde{G}$ on UM coincides with G.

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