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Upper Bounds on Percolation Probabilities for Oriented Bond Percolation

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This paper concerns the estimation of percolation probability for oriented bond percolation in two dimensions. First we present a more general class of processes (discrete-time growth models) which contains the oriented bond percolation as a special case. We then recall a result due to Ted Harris, which allows one to obtain an upper bound $\theta^{(1)}(p)$ on the percolation probability starting from the origin for processes from this class (here p is a parameter whose meaning is the probability of a bond to be open in the considered models). We then present a method based on the Harris' result, which gives a sequence $\theta^{(n)}(p)$ $(n \geq 1)$ that converges monotonicaly to the true value of this percolation probability, from above. Furthermore we obatin explicit forms of $\theta^{(n)}(p)$ for $n = 1, \ldots, 9$. In particular, these bounds indicate a particular property of the percolation probability starting from the origin when considered as a function of p, exactly to say, they indicate a presence of an inflection point close to $p = 0.561821$.

Keywords: Oriented Bond Percolation, Percolation Probability, Upper Bound, Harris Lemma

1 INTRODUCTION

In this paper we will construct a sequence of upper bounds on percolation probability in oriented bond percolation by using a new method based on the Harris lemma. First we consider a class of discrete-time growth models which contains oriented bond percolation as a spetial case as we will show below. The discrete-time growth model starting from $A \subset \mathbf{Z}^1$, is the name for the discrete-time Markov chain $\xi_n^A, n \in \mathbf{N}$, whose state space on Y, the collection of all finite subsets of $\mathbf{Z}^1$, such that the initial state is A, that is $\xi_0^A = A$, and the dynamics is given by the follwoing rule. Write ξ_n^A, the state of the process at time n, as a union of maximal subintervals

$$\xi_n^A = \cup_{i=1}^k I_i,$$

where $I_i = \{m_i + 1, m_i + 2, \ldots, n_i\}$ and $m_i < n_i < m_{i+1}$. Then ξ_{n+1}^A is obtained by choosing points in $\{m_i + 1, m_i + 2, \ldots, n_i - 1\}$ each with probability q, and points m_i and n_i each with probability p. The choices are made independently. Throughout this paper, we assume that

$$0 \leq p \leq q \leq 1,$$

so this process is attractive; that is, if $\xi_n^A \subset \xi_n^B$, then we can guarantee that $\xi_{n+1}^A \subset \xi_{n+1}^B$ by using an appropriate coupling. Note that if $q = p(2-p)$ (resp. $q = p$) then this process becomes what is called the oriented bond (resp. site) percolation model in two dimensions. Concerning oriented percolation models, see Durrett,(1) for example.

We define percolation (or survival) probability starting from $A \in Y$ by

$$\sigma(A) = P(\xi_n^A \neq \phi \ \text{ for all } n \geq 0).$$

Furthermore, $\sigma(\{0\})$, the percolation probability starting from the origin 0, will be denoted by $\rho(p, q)$. For given q, define the critical value $p_c(q)$ by

$$p_c(q) = \inf\{p \geq 0 : \rho(p, q) > 0\}.$$

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Concerning bounds on percolation probability and critical values for discrete-time growth models, see Chapter 7 of Konno,[2] for example.

The main purpose of this paper is *(1)* to present a new method which gives a systematic sequence of rigorous upper bounds $\theta^{(n)}(p)$ (which are independent of q) for percolation probability of the discrete-time growth models by using the Harris lemma and *(2)* to give an explicit form of these bounds for $n = 1, \dots, 9$.

This paper is organized as follows. In Chapter 2, we will give a version of the Harris lemma for the type of processes considered in this paper. This lemma is a basic ingredient for our method. Chapter 3 treats our new results. Chapter 4 is devoted to conclusions.

2 HARRIS LEMMA

Here we present a discrete-time version of the Harris lemma[3] whose presentation here follows Konno.[4] Let Y^* denote the set of all [0,1]-valued measurable functions on Y.

Lemma 2.1. *(Harris lemma) Let $h \in Y^*$ with*

(1) $$h(\phi) = 0,$$

(2) $$0 < h(A) \le 1 \quad \text{for any } A \in Y \text{ with } A \ne \phi.$$

Assume that for any $\varepsilon > 0$, there is an $N \ge 1$ such that if $|A| \ge N$, then

(3) $$E\left(h(\xi_1^A)\right) \ge 1 - \varepsilon.$$

Assume also that for any $A \in Y$,

(4) $$E\left(h(\xi_1^A)\right) \le h(A).$$

Then

(5) $$\sigma(A) \le h(A) \quad \text{for any } A \in Y.$$

In particular,

(6) $$\rho(p, q) \le h(\{0\}),$$

where 0 is the origin.

The proof of Lemma 2.1 will give us a systematic sequence of upper bounds for percolation probability starting from A of discrete-time growth models. So, for the convenience of readers, we review the proof of this lemma. In the rest of this section, we assume that $p \le q < 1$. When $q = 1$, the proof is almost trivial, so we will omit it.

To prove Lemma 2.1 we shall need Lemmas 2.2 and 2.3 which we shall now present. Write A as a union of maximal subintervals

$$A = \cup_{i=1}^k I_i, \tag{1}$$

where $I_i = \{m_i + 1, m_i + 2, \dots, n_i\}$ and $m_i < n_i < m_{i+1}$.

Define

$$L = |\{m_1 + 1, \dots, n_1 - 1, m_2 + 1, \dots, n_2 - 1, \dots, m_k + 1, \dots, n_k - 1\}|,$$
$$M = |\{m_1, n_1, m_2, n_2, \dots, m_k, n_k\}|.$$

The definitions of L and M give

$$L + M = |A| + k, \tag{2}$$

$$M = 2k. \tag{3}$$

Then the following is easily shown by the property of binomial distribution.

Lemma 2.2. *For any $A \in Y$ and $n \in \{0, 1, \dots, |A| + k\}$,*

$$P(|\xi_1^A| = n) = \sum_{l=0}^{L} \sum_{m=0}^{M} 1_n(l+m) \binom{L}{l} q^l (1-q)^{L-l} \times \binom{M}{m} p^m (1-p)^{M-m},$$

where $1_x(y) = 1$ if $y = x$, and $= 0$ otherwise and

$$\binom{i}{j} = \frac{i!}{j!(i-j)!} \quad \text{for} \quad 0 \le j \le i.$$

Furthermore, we shall need the following result.

Lemma 2.3. *For any $A \in Y$ and $N \ge 1$,*

$$\lim_{n \to \infty} P(0 < |\xi_n^A| \le N) = 0.$$

Proof. It is enough to show that for any $A \in Y$ and $r \ge 1$,

$$\lim_{n \to \infty} P(|\xi_n^A| = r) = 0.$$

By the Markov property,

$$\begin{aligned} P(|\xi_{n+1}^A| = r - 1) &= \sum_{m=0}^{\infty} E(P(|\xi_1^{\xi_n^A}| = r - 1) : |\xi_n^A| = m) \\ &\ge E(P(|\xi_1^{\xi_n^A}| = r - 1) : |\xi_n^A| = r) \\ &\ge c(r) P(|\xi_n^A| = r), \end{aligned}$$

where

$$c(r) = \inf_{B:|B|=r} P(|\xi_1^B| = r - 1) > 0.$$

Note that the positivity of $c(r)$ follows from Lemma 2.2. Therefore it suffices to prove that for any $A \in Y$,

$$\lim_{n \to \infty} P(|\xi_n^A| = 1) = 0.$$

To do so, we will show that

$$\sum_{n=1}^{\infty} P(|\xi_n^A| = 1) < \infty.$$

By the Markov property,

$$\begin{aligned}
&P(|\xi_{n+1}^A|=0)-P(|\xi_n^A|=0)\\
&=P(|\xi_{n+1}^A|=0)-P(|\xi_n^A|=0,|\xi_{n+1}^A|=0)\\
&=\sum_{k=1}^{\infty}P(|\xi_n^A|=k,|\xi_{n+1}^A|=0)\\
&\geq P(|\xi_n^A|=1,|\xi_{n+1}^A|=0)\\
&=E(P(|\xi_1^{\xi_n^A}|=0):|\xi_n^A|=1)\\
&=(1-p)^2P(|\xi_n^A|=1).
\end{aligned}$$

Then we see that $p<1$ gives

$$\sum_{n=1}^{\infty}P(|\xi_n^A|=1)\leq\frac{1}{(1-p)^2}<\infty.$$

Thus the proof is complete.

Proof of Lemma 2.1. For any $A\in Y$ and $N\geq 1$, Lemma 2.3 gives

$$\begin{aligned}
\sigma(A)&=\lim_{n\to\infty}P(\xi_n^A\neq\phi)\\
&=\lim_{n\to\infty}P(|\xi_n^A|>N)+\lim_{n\to\infty}P(0<|\xi_n^A|\leq N)\\
&=\lim_{n\to\infty}P(|\xi_n^A|>N).
\end{aligned}\tag{4}$$

From the Markov property and condition (1),

$$\begin{aligned}
&E(h(\xi_{n+1}^A))\\
&=E(E(h(\xi_1^{\xi_n^A})))\\
&=E(E(h(\xi_1^{\xi_n^A})):|\xi_n^A|>N)\\
&\qquad+E(E(h(\xi_1^{\xi_n^A})):0<|\xi_n^A|\leq N).
\end{aligned}\tag{5}$$

By using $h(A)\leq 1$ for any $A\in Y$ and Lemma 2.3, we have

$$\lim_{n\to\infty}E(E(h(\xi_1^{\xi_n^A})):0<|\xi_n^A|\leq N)=0.$$

Using this result and Eq.(5), we have

$$\liminf_{n\to\infty}E(h(\xi_{n+1}^A))=\liminf_{n\to\infty}E(E(h(\xi_1^{\xi_n^A})):|\xi_n^A|>N).\tag{6}$$

Therefore combination of Eqs.(4), (6) and condition (3) implies that for any $\varepsilon>0$, there is an $N\geq 1$ such that

$$\begin{aligned}
\liminf_{n\to\infty}E(h(\xi_{n+1}^A))&\geq(1-\varepsilon)\liminf_{n\to\infty}P(|\xi_n^A|>N)\\
&=(1-\varepsilon)\sigma(A).
\end{aligned}\tag{7}$$

By using Eq.(7), $h(\phi)=0, h(A)\leq 1$ for any $A\in Y$ and the definition of $\sigma(A)$, we see that for any $\varepsilon>0$,

$$\begin{aligned}
(1-\varepsilon)\sigma(A)&\leq\liminf_{n\to\infty}E(h(\xi_n^A))\\
&=\liminf_{n\to\infty}E(h(\xi_n^A):\xi_n^A\neq\phi)\\
&\leq\limsup_{n\to\infty}E(h(\xi_n^A):\xi_n^A\neq\phi)\\
&\leq\lim_{n\to\infty}P(\xi_n^A\neq\phi)\\
&=\sigma(A).
\end{aligned}$$

Thus it follows that

$$\sigma(A)=\lim_{n\to\infty}E(h(\xi_n^A)).\tag{8}$$

From the Markov property and condition (4), we obtain

$$E(h(\xi_2^A))=E(E(h(\xi_1^{\xi_1^A})))\leq E(h(\xi_1^A))\leq h(A).$$

Using a similar argument repeatedly, we see that for any $n\geq 1$,

$$E(h(\xi_n^A))\leq h(A).\tag{9}$$

Combining Eqs.(8) and (9) gives

$$\sigma(A)\leq h(A),$$

for any $A\in Y$. Thus the proof of part (5) in Lemma 2.1 is complete. Part (6) follows from taking $A=\{0\}$ in part (5).

3 RESULTS

In this chapter we give our new results. First we see that the following main theorem comes from the last part of proof of the Harris lemma. This result is important in our paper. Because if we find a suitable h which satisfies conditions (1)-(4) in the Harris lemma, then we can obtain a new systematic sequence of upper bounds $E(h(\xi_n^A))$ on percolation probability $\sigma(A)$ starting from $A\in Y$ for the discrete-time growth models. Furthermore, as a special case, this sequence gives upper bounds on percolation probability $\sigma(A)$ starting from A for oriented bond percolation.

Theorem 3.1. *If $h\in Y^*$ satisfies conditions (1)-(4) in the Harris lemma, then for any $A\in Y$,*

(1) $$E(h(\xi_{n+1}^A))\leq E(h(\xi_n^A))\qquad(n\geq 0),$$

(2) $$E(h(\xi_n^A))\searrow\sigma(A)\qquad(n\to\infty).$$

Let $|A|$ be the cardinality of $A\in Y$. In fact if we take

$$h(A)=1-\left(\frac{1-p}{p}\right)^{2|A|},$$

then this h satisfiies the conditions (1)-(4) in the Harris lemma. The proof appeared in Konno,[4] so we will omit it here. From now on, we focus on the case of oriented bond percolation (i.e. $q=p(2-p)$ for the discrete-time growth models). We define the percolation probability starting from the origin for oriented bond percolation as

$$\rho(p)=\sigma(\{0\})=P(\xi_n^0\neq\phi\text{ for all }n\geq 0),$$

where 0 is the origin. Then by using Theorem 3.1, we have the follwing result.

Corollary 3.2. *In the case of oriented bond percolation, we define* $\theta^{(n)}(p) = E(h(\xi_n^{\{0\}}))$. *Then*

$$\theta^{(n)}(p) \searrow \rho(p) \qquad \text{as} \quad n \to \infty,$$

where

$$h(A) = 1 - \left(\frac{1-p}{p}\right)^{2|A|}.$$

From Corollary 3.2, we get the following explicit forms of upper bounds $\theta^{(n)}(p)$ $(n = 1, \dots 9)$ for percolation probability $\rho(p)$ of oriented bond prcolation.

$$\theta^{(1)}(p) = -\frac{1}{p^2} + \frac{2}{p},$$

$$\theta^{(2)}(p) = -\frac{4}{p} + 14 - 16p + 9p^2 - 2p^3,$$

$$\begin{aligned}\theta^{(3)}(p) = &-9 + 22p + 40p^2 - 206p^3 + 331p^4 - 282p^5 \\ &+ 137p^6 - 36p^7 + 4p^8,\end{aligned}$$

$$\begin{aligned}\theta^{(4)}(p) = &-18p + 35p^2 + 94p^3 - 215p^4 + 10p^5 - 805p^6 \\ &+ 5122p^7 - 11983p^8 + 15986p^9 13789p^{10} + 8022p^{11} \\ &- 3148p^{12} + 802p^{13} - 120p^{14} + 8p^{15},\end{aligned}$$

$$\begin{aligned}\theta^{(5)}(p) = &-36p^2 + 70p^3 + 146p^4 - 286p^5 + 449p^6 - 3284p^7 \\ &+ 7206p^8 - 6268p^9 + 10789p^{10} - 11598p^{11} - 122560p^{12} \\ &+ 579570p^{13} - 1301118p^{14} + 1877202p^{15} - 1916931p^{16} \\ &+ 1444832p^{17} - 817731p^{18} + 347992p^{19} - 109951p^{20} \\ &+ 25044p^{21} - 3888p^{22} + 368p^{23} - 16p^{24},\end{aligned}$$

$$\begin{aligned}\theta^{(6)}(p) = &-72p^3 + 140p^4 + 310p^5 - 831p^6 + 1708p^7 - 5205p^8 + 6798p^9 - 14748p^{10} \\ &+ 50904p^{11} - 34575p^{12} - 156762p^{13} + 532173p^{14} - 1469866p^{15} \\ &+ 3181230p^{16} - 3337010p^{17} + 2242520p^{18} - 20449868p^{19} \\ &+ 101210422p^{20} - 268024840p^{21} + 473241652p^{22} - 613179448p^{23} \\ &+ 611682098p^{24} - 482133156p^{25} + 304413972p^{26} - 154830240p^{27} \\ &+ 63387844p^{28} - 20747336p^{29} + 5355799p^{30} - 1066096p^{31} \\ &+ 157804p^{32} - 16344p^{33} + 1056p^{34} - 32p^{35},\end{aligned}$$

$$\begin{aligned}\theta^{(7)}(p) = &-144p^4 + 280p^5 + 656p^6 - 1696p^7 + 2150p^8 - 5782p^9 + 17095p^{10} - 62604p^{11} \\ &+ 108811p^{12} + 40228p^{13} - 476906p^{14} + 1565028p^{15} - 4230840p^{16} \\ &+ 7075054p^{17} - 7352359p^{18} + 8415008p^{19} - 22096375p^{20} \\ &+ 91177550p^{21} - 270871273p^{22} + 350414414p^{23} + 277783879p^{24} \\ &- 1590875352p^{25} + 2374616022p^{26} - 5019288596p^{27} + 24342890976p^{28} \\ &- 86114724564p^{29} + 204622753576p^{30} - 356124984550p^{31} + 481193510857p^{32} \\ &- 522896766360p^{33} + 467043587932p^{34} - 347574971704p^{35} + 217285574009p^{36} \\ &- 114583533678p^{37} + 51020611974p^{38} - 19145517698p^{39} + 6025408815p^{40} \\ &- 1577565268p^{41} + 339473740p^{42} - 58996004p^{43} + 8071236p^{44} \\ &- 836384p^{45} + 61664p^{46} - 2880p^{47} + 64p^{48},\end{aligned}$$

$$
\begin{aligned}
\theta^{(8)}(p) = & -288p^5 + 560p^6 + 1384p^7 - 3460p^8 + 3868p^9 - 15524p^{10} + 53168p^{11} \\
& - 108889p^{12} + 52680p^{13} + 192708p^{14} - 619812p^{15} + 2875638p^{16} \\
& - 8675434p^{17} + 13351166p^{18} - 7513582p^{19} - 12680870p^{20} \\
& - 10012396p^{21} + 278226110p^{22} - 941463796p^{23} + 1667292285p^{24} \\
& - 1753873054p^{25} + 1019405305p^{26} - 772624922p^{27} - 2111379902p^{28} \\
& + 37392590206p^{29} - 157992585684p^{30} + 342323568232p^{31} \\
& - 471254137710p^{32} + 709541249646p^{33} - 1809998049067p^{34} \\
& + 4142720438866p^{35} - 6798468563094p^{36} + 12086521793044p^{37} \\
& - 36652815053037p^{38} + 116503839132030p^{39} - 288306749394660p^{40} \\
& + 550991436671158p^{41} - 841727999796970p^{42} + 1058219144184362p^{43} \\
& - 1117741947587496p^{44} + 1006248814765292p^{45} - 779774836819158p^{46} \\
& - 523654041894484p^{47} - 306050843087175p^{48} + 156041549645810p^{49} \\
& - 69453200574360p^{50} + 26965122013356p^{51} - 9111290957004p^{52} \\
& + 2669046993642p^{53} - 674061979608p^{54} + 145635281244p^{55} \\
& - 26641324881p^{56} + 4069267598p^{57} - 509203296^{p}58 + 50823832p^{59} \\
& - 3889296p^{60} + 214176p^{61} - 7552p^{62} + 128p^{63},
\end{aligned}
$$

$$
\begin{aligned}
\theta^{(9)}(p) = & -576p^6 + 1120p^7 + 2912p^8 - 7056p^9 + 6836p^{10} - 30078p^{11} \\
& + 87444p^{12} - 128996p^{13} + 141078p^{14} - 287842p^{15} \\
& - 699633p^{16} + 6844326p^{17} - 16471207p^{18} + 20608128p^{19} \\
& - 19847269p^{20} + 24751286p^{21} - 110601178p^{22} + 512135230p^{23} \\
& - 1416494591p^{24} + 2067305734p^{25} + 471585995p^{26} - 8053480128p^{27} \\
& + 13316622691p^{28} - 15272814914p^{29} + 71239696897p^{30} - 296358497876p^{31} \\
& + 730459107448p^{32} - 1293303009656p^{33} + 2328082200618p^{34} \\
& - 4997784855240p^{35} + 8415233465845p^{36} - 6653945306268p^{37} \\
& + 2452363328518p^{38} - 26939047621642p^{39} + 91897777627579p^{40} \\
& - 94045741987764p^{41} - 84075175798288p^{42} + 342385059025806p^{43} \\
& - 899911472485776p^{44} + 3722528711327042p^{45} - 12081847226227191p^{46} \\
& + 26822305168376020p^{47} - 48484573188285044p^{48} + 100692788994288998p^{49} \\
& - 268660367389354934p^{50} + 706341834272616372p^{51} \\
& - 1561889633883578957p^{52} + 2840027488043618376p^{53} \\
& - 4307289153557295499p^{54} + 5546720939028297690p^{55} \\
& - 6155559273486481443p^{56} + 5954433996273587632p^{57} \\
& - 5063566849719437586p^{58} + 3809374638823206528p^{59} \\
& - 2546952358298713564p^{60} + 1518273653906179274p^{61} \\
& - 808613850015164342p^{62} + 385186427098629550p^{63} \\
& - 164148233059897686p^{64} + 62542571224061790p^{65} \\
& - 21275254293806300p^{66} + 6446919762647916p^{67} - 1734723829162845p^{68} \\
& + 412745518361084p^{69} - 86368160701648p^{70} + 15784982726428p^{71} \\
& - 2497665186969p^{72} + 338316580628p^{73} - 38656411584p^{74} \\
& + 3653293264p^{75} - 277877600p^{76} + 16343488p^{77} \\
& - 697344p^{78} + 19200p^{79} - 256p^{80}
\end{aligned}
$$

In Fig.1, we show the graphs of $\theta^{(n)}(p)$ as functions of p, for $n=1$ and $n=9$. Observe that $\theta^{(1)}(\cdot)$ is concave while $\theta^{(9)}(\cdot)$ possesses an inflection point. This motivated us to study the second derivative of the functions $\theta^{(n)}(p)$, $p \in [1/2, 1]$, $n = 1, \ldots, 9$. Their second derivatives are presented in Fig.2. This figure shows that there is an inflection point $p^{(n)}$ between 1/2 and 1 for $n = 6, 7, 8, 9$. This fact suggests that $\rho(p)$ also has an inflection point between 1/2 and 1. Our results provide the following estimates of its abscissa:

$$p^{(6)} = 0.511774$$
$$p^{(7)} = 0.536978$$
$$p^{(8)} = 0.553858$$
$$p^{(9)} = 0.561821$$

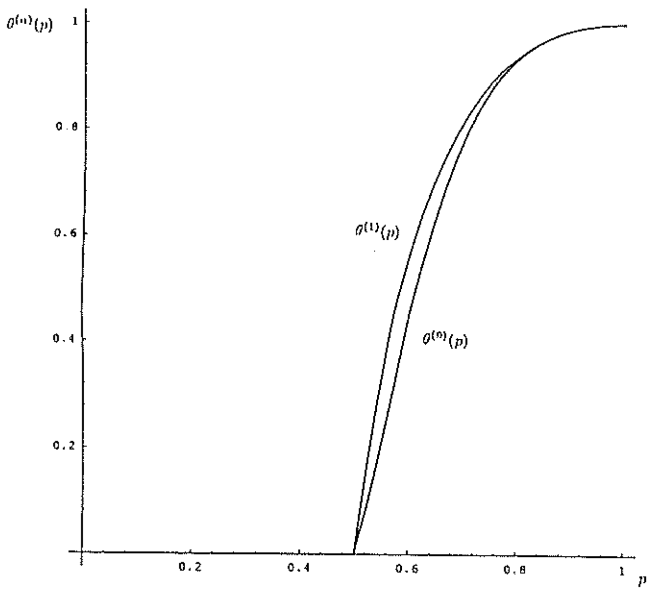

Fig.1. The graphs of $\theta^{(n)}(p)$ for $n = 1$ and $n = 9$.

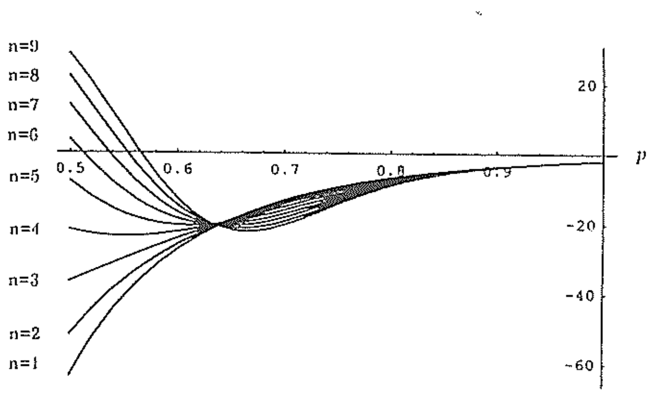

Fig.2. The second derivative of $\theta^{(n)}(p)$ for $n = 1, 2, \ldots, 9$.

4 CONCLUSIONS

In this paper we study the percolation probability starting from $A \in Y$, for a class of discrete-time growth models which contains the oriented bond percolation as a particular case. We present a new method which gives a systematic sequence of upper bounds $\theta^{(n)}(p)$ $(n \geq 1)$ for this probability, basing on a corollary of the Harris lemma. The first of these bounds was given by the Harris lemma directly. These bounds converge to the true value, as $n \to \infty$. We obtain explicit forms of $\theta^{(n)}(p)$ for $n = 1, 2, \ldots, 9$. These forms suggest that the percolation probability in the oriented bond percolation starting from the origin, possesses an inflection point, for certain value of $p \in [1/2, 1]$ which we estimate.

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方向性のあるボンドパーコレーションの浸透確率の上限

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概要

本論文では、方向性のあるボンドパーコレーションの浸透確率について考える。議論を見通しよくする為に、方向性のあるボンドパーコレーションを特殊な場合として含む一般のクラスを導入する。ハリスの補題によりこのクラスの浸透確率の上限の第1近似$\theta^{(1)}(p)$ が与えられることは知られているが、本論文ではさらに系統的な上限の列 $\theta^{(n)}(p)$ $(n \geq 1)$ を与える新しい手法を提案し、具体的に計算を行った。

キーワード： 方向性のあるボンドパーコレーション、浸透確率、上限、ハリスの補題

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