A recursion formula of local densisies with congruence conditions

その他（別言語等）のタイトル	合同条件を持つ二次形式の局所密度の漸化式
著者	KATSURADA Hidenori, HISASUE Masaki
journal or publication title	Memoirs of the Muroran Institute of Technology
volume	47
page range	97-102
year	1997-11-28
URL	http://hdl.handle.net/10258/192

A recursion formula of local densisies with congruence conditions

その他（別言語等）のタイトル	合同条件を持つ二次形式の局所密度の漸化式
著者	KATSURADA Hidenori, HISASUE Masaki
journal or publication title	Memoirs of the Muroran Institute of Technology
volume	47
page range	97-102
year	1997-11-28
URL	http://hdl.handle.net/10258/192

Mem. Muroran Inst. Tech.,47(1997)97～102

A recursion formula of local densities with congruence conditions

Hidenori KATSURADA[1] and Masaki HISASUE[2]

(Received 9 May 1997, Accepted 20 August 1997)

In this paper, we give a recursion formula of local densities with congruence conditions. As an application we give another proof for the recursion formula of local densities in (*J. Number Theory*, 64, 1997, 183-210) for a special case. Further we define a certain formal power series which is a generalization of the one in (*Proc. Japan Acad.* 70, 1994, 208-211), and determine an explicit form of its denominator.

Keywords: Local Densities, Recursion Formula

1 INTRODUCTION

Local densities of quadratic forms over the p-adic field are very important invariants in the arithmetic theory of quadratic forms, and various types of recursion formulas for them have been studied by several authors. They all give important[2] information on local densities. Among others, in [Ki1] and [Ki2], Kitaoka have given an explicit formula for a special case. In [Ka1], we have defined *the local densities with congruence condition*, and in [Ka2], obtained several recursion formulas for them to give the denominator of a certain power series attached to local densities for $p \neq 2$. They are very effective to compute the local densities, and give an explicit form of all local densities of quadartic forms over the p-adic field for $p \neq 2$ in principle. Howerver they are rather complicated and not all of them can be generalized to all the cases including $p = 2$ (see the remark at the end of section 3). In this paper, we give a formula expressing *a local density with congruence condition* as a linear combination of usual local densities for a special but important case (cf. Theorem 3.3). It is rather simpler than the ones in [Ka2], and holds for all p including 2. As a corollary, we give a new proof to [KH, Theorem 4.1] for a special case.

[1]Common Subject Division

[2]Department of Mathematics Faculty of Science, Hokkaido University

2 LOCAL DENSITIES WITH CONGRUENCE CONDITIONS

In this section, we recall the notion of the local densities with congruence conditions and a recursion formula for them following [Ka2] and [His] with a slight modification. For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m,n)-matrices with entries in R. Here we understand $M_{mn}(R)$ the set of the *empty matrix* if $m = 0$ or $n = 0$. We also make the convention that $\operatorname{diag}(U,V) = V$ if U is the empty matrix. For an (m,n) matrix X and an (m,m) matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transposition of X. Let a be an element of R. Then for an element X of $M_{mn}(R)$ we often use the same symbol X to denote the class of X mod $aM_{mn}(R)$. Put

$$GL_m(R) = \{A \in M_{mm}(R); \det A \in R^*\},$$

where $\det A$ denotes the determinant of a square matrix A, and R^* denotes the unit group of R. Further let $S_n(R)$ denote the set of symmetric matrices of degree n with entries in R. We abbreviate an (m,n)-matrix whose components are all 1 (resp. 0) as 1_{mn} (resp. 0_{mn}). We often simply write $1 = 1_{mn}$ and $0 = 0_{mn}$ if no confusion arises. Remark that 1 is different from the unit matrix. For square matrices X and Y we write $\operatorname{diag}(X,Y) = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$. Let $\mathbf{Q}_p$ be the field of p-adic numbers, and $\{0,1\}$ the finite set with two integers 0 and 1. Let m, n and l be non-negative integers such that

$m+l \geq n \geq 1$. For $S \in S_m(\mathbf{Z}_p), T \in S_n(\mathbf{Z}_p), U \in S_l(\mathbf{Z}_p), I = (r_{ij}) \in M_{ln}(\{0,1\})$, and a non-negative integer e, put

$$\mathcal{A}_e(T,(S,U);I) = \{X = (x_{ij}) \in M_{m+l,n}(\mathbf{Z}_p)/p^e M_{m+l,n}(\mathbf{Z}_p);$$
$$\mathrm{diag}(S,U)[X] \equiv T \bmod p^e \text{ and } x_{m+i,j} \in p^{r_{ij}}\mathbf{Z}_p$$
$$\text{for any } 1 \leq i \leq l, 1 \leq j \leq n\},$$

and

$$a_e(T,(S,U);I) = \#\mathcal{A}_e(T,(S,U);I).$$

We then define $\alpha_p(T,(S,U);I)$ by

$$\alpha_p(T,(S,U);I) = p^{w(I)} \lim_{e\to\infty} p^{(-(m+l)n+n(n+1)/2)e} a_e(T,(S,U);I),$$

where for $I = (r_{ij}) \in M_{ln}(\{0,1\})$ we write

$$w(I) = \sum_{1\leq i \leq l, 1 \leq j \leq s} r_{ij}$$

If $\mathrm{diag}(S,U)$ and T are non-degenerate, the above limit exists. We write $\alpha_p(T,(S,U);I)$ as $\alpha_p(T,(S,\emptyset),I)$ or as $\alpha_p(T,(\emptyset,U);I)$ according as $\deg U = 0$ or $\deg S = 0$. Note that $\alpha_p(T,(S,U);I)$ coincides with the usual local density $\alpha_p(T,S)$ if $l=0$ and with $\alpha_p(T,\mathrm{diag}(S,U))$ if $I = 0_{ls}$. For an (l,n)-matrix $I = (r_{ij})$, and permutations σ and τ of degree l and n respectively, we put $\sigma \circ I \circ \tau = (r_{\sigma(i),\tau(j)})$. We say that an $(l,n)-$ matrix I is standard if

$$I = \begin{array}{cc} \begin{array}{cccc} n_1 & \cdots & n_{s-1} & n_s \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} & \begin{array}{l} \}l_1 \\ \}\vdots \\ \}l_{r-1} \\ \}l_r \end{array} \end{array}$$

with non-negative integers $l_1, \cdots, l_{r-1}, l_r, n_1, \cdots, n_{s-1}, n_s$ such that $l_1 + \ldots + l_{r-1} + l_r = l$, and $n_1 + \cdots + n_{s-1} + n_s = n$. An $(l,n)-$ matrix I is called quasi-standard if $I = \sigma \circ I_0 \circ \tau$ with a standard matrix I_0, and permutations σ and τ of degrees l and n, respectively. We denote by $Sd_{ln}(\{0,1\})$ the subset of $M_{ln}(\{0,1\})$ consisting of all quasi-standard matrices.

Now let A be an even unimodular matrix with entries in $\mathbf{Z}_p$. That is, let A be a symmetric unimodular matrix with entries in $\mathbf{Z}_p$ whose diagonal components belong to $2\mathbf{Z}_p$. We remark that it means merely a symmetric unimodular matrix with entries in $\mathbf{Z}_p$ if $p \neq 2$. As is well known, A is equivalent, over $\mathbf{Z}_p$, to a matrix of the following type:

$$\mathrm{diag}(\overbrace{H,...,H}^{r},U),$$

where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and U is an anisotropic even unimodular matrix of degree not greater than 2. The above r is the Witt index of A, which will be denoted by $r(A)$. Here we make the convention that $\mathrm{diag}(H,...,H,U) = U$ or $= \mathrm{diag}(H,...,H)$ according as $r = 0$ or $\deg U = 0$. Then we define $A^{(k)}$ by

$$A^{(k)} = \mathrm{diag}(\overbrace{H,...,H}^{r-k},U).$$

This $A^{(k)}$ is uniquely determined only by A and k up to equivalence over $\mathbf{Z}_p$. As is well known, the value $\alpha_p(B,A)$ for an even matrix B of degree n and $A = \mathrm{diag}(\overbrace{H,...,H}^{k})$ is closely connected with the Fourier coefficient of Siegel-Eisenstein series of degree n and of weight k (for example, see [IK],[Ka5],[Ka6],[Ki2], and [M].) So it is important problem to find a reasonable expression of $\alpha_p(B,A)$ when A is even unimodular.

PROPOSITION 2.1. *Let m,l,n be non-negative integers such that $m+l \geq n \geq 1$. Let A be non-degenerate even unimodular matrix of degree m with entries in $\mathbf{Z}_p$, and let $b_1,...,b_n,c_1,...,c_l$ be elements of $2\mathbf{Z}_p \backslash \{0\}$ such that $b_1 \in 2p^2\mathbf{Z}_p$. Further let I' be an (l,n)-matrix with entries in $\{0,1\}$. Put $B = \mathrm{diag}(b_1,...,b_n), \hat{B} = \mathrm{diag}(b_2,...,b_n)$, and $C = \mathrm{diag}(c_1,...,c_l)$.*
(1) Let $n \geq 2$. Then we have

$$\alpha_p(\mathrm{diag}(p^2b_1,\hat{B}),(A,C);(1_{l1},I'))$$
$$= p^{-m+n+1}\alpha_p(\mathrm{diag}(b_1,\hat{B}),(A,C);(0_{l1},I'))$$
$$+\beta_p(0,A)\alpha_p(\hat{B},(\mathrm{diag}(A^{(1)},-p^2b_1),C);I'),$$

where $(1_{l1},I')$ is the (l,n)-matrix whose k-th column is 1_{l1} or the $k-1$-th column of I' according as $k=1$ or not, and others. Here we understand the right-hand side is 0 if the Witt index of A is 0.
(2) Let $n=1$. Then we have

$$\alpha(p^2b_1,(A,C);1_{l1}) - p^{-m+2}\alpha_p(b_1,(A,C);0_{l1}) = \beta_p(0,A).$$

PROPOSITION 2.2. *Let A,B,C_1 and C_2 be non-degenerate symmetric matrices of degree m,n,l_1 and l_2, respectively, with entries in $\mathbf{Z}_p$ such that $m+l_1+l_2 \geq n$, and I an (l_2,n)-matrix with entries in $\{0,1\}$. Assume that B,C_1 and C_2 are diagonal. Then we have*

(1)
$$\alpha_p(B,(\mathrm{diag}(A,p^2C_1),C_2);I)$$
$$= \alpha_p(B,(A,\mathrm{diag}(C_1,C_2)); \begin{pmatrix} 1_{l_1,n} \\ I \end{pmatrix}).$$

(2)
$$\alpha_p(B,(A,\mathrm{diag}(p^2C_1,C_2); \begin{pmatrix} 0_{l_1,n} \\ I \end{pmatrix}$$
$$= \alpha_p(B,(A,\mathrm{diag}(C_1,C_2)); \begin{pmatrix} 1_{l_1,n} \\ I \end{pmatrix}).$$

Proposition 2.2 for $p \neq 2$ is nothing but [Ka2, Proposition 3.4] and it also holds for $p=2$ without any change. (1) of Proposition 2.1 for $p \neq 2$ is a special case of [Ka2, Proposition 3.6, 3.7] and it can be proved with slight modification for $p=2$. Now let n,l be positive integers. Let

$$I = \begin{array}{cc} \begin{array}{cccc} n_1 & \cdots & n_{s-1} & n_s \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} & \begin{array}{l} \}l_1 \\ \}\vdots \\ \}l_{s-1} \\ \}l_s \end{array} \end{array}$$

with non-negative integers $l_1,\cdots,l_{s-1},l_s,n_1,\cdots,n_{s-1},n_s$ such that $l_2,\dots,l_s,n_1,\dots,n_{s-1}>0$ and $l_1+\ldots+l_{s-1}+l_s=l, n_1+\cdots+n_{s-1}+n_s=n$. For an integer j such that $n_1+\ldots+n_{k-1}+1\le j\le n_1+\ldots+n_k$ put $l[j]=l_1+\ldots+l_k$. Here we understand $n_0=0$. For each integer $1\le k\le s$ put $n[k]=n_1+\ldots+n_k$. For the above I and each integers $1\le k\le s$ define an element $\tilde I_k$ of $M_{nl}(\{0,1\})$ by

$$\begin{array}{c c l}
 & \begin{array}{cccccc} n_1 & \cdots & n_{k-1} & n_k & \cdots & n_s \end{array} & \\
 & \left(\begin{array}{cccccc}
0 & 0 & 0 & 1 & 1 & 1\\
1 & 0 & 0 & 1 & 1 & 1\\
1 & 1 & 0 & 1 & 1 & 1\\
1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 0
\end{array}\right) &
\begin{array}{l} \}l_1\\ \}\\ \}l_{k-1}\\ \}l_k\\ \}\\ \}l_s \end{array}
\end{array}.$$

We note that $\tilde I_1=I$ if $l_1=0$. Next, for each integer j such that $n[k-1]+1\le j\le n[k-1]+n_k$, define an element I_j of $M_{l,n}(\{0,1\})$ by

$$\begin{array}{c c l}
 & \begin{array}{ccccccc} n_1 & \cdots & n_{k-1} & j-n[k-1] & n[k]-j & \cdots & n_s \end{array} & \\
 & \left(\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 1 & 1 & 1\\
1 & 0 & 0 & 0 & 1 & 1 & 1\\
1 & 1 & 0 & 0 & 1 & 1 & 1\\
1 & 1 & 1 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 0
\end{array}\right) &
\begin{array}{l} \}l_1\\ \}\\ \}l_{k-1}\\ \}l_k\\ \}\\ \}l_s \end{array}
\end{array}$$

that is, I_j is the matrix obtained from $\tilde I_k$ by replacing (α,β)-components of $\tilde I_k$ by 0 for $\alpha=n[k-1]+1,n[k-1]+2,\dots,j-1,j$ and $\beta=1,\dots,l$. Further put $I_0=I$. We note that $I_n=I$, and $\tilde I_k$ is the matrix obtained from $I_{n[k-1]}$ by replacing (α,β)-components of $I_{n[k-1]}$ by 1 for $\alpha=1,\dots,n$ and $\beta=l_1+\ldots+l_{k-1}+1,\dots,l_1+\ldots+l_{k-1}+l_k$ (see example below).

EXAMPLE. Let $n=4, l=4, s=3$ and $n_1=2,n_2=1,n_3=1,l_1=1,l_2=2,l_3=1$. Then $n[1]=2,n[2]=3,n[3]=4,l[1]=l[2]=1,l[3]=3,l[4]=4$, and $I=\begin{pmatrix}0&0&0&0\\1&1&0&0\\1&1&0&0\\1&1&1&0\end{pmatrix}$.

By construction we have

$$\tilde I_1=\begin{pmatrix}1&1&1&1\\1&1&0&0\\1&1&0&0\\1&1&1&0\end{pmatrix},\ \tilde I_2=\begin{pmatrix}0&0&1&1\\1&1&1&1\\1&1&1&1\\0&0&1&0\end{pmatrix},$$

and

$$\tilde I_3=\begin{pmatrix}0&0&0&1\\1&1&0&1\\1&1&0&1\\1&1&1&1\end{pmatrix}.$$

Further we have

$$I_1=\begin{pmatrix}0&1&1&1\\0&1&0&0\\0&1&0&0\\0&1&1&0\end{pmatrix},\ I_2=\begin{pmatrix}0&0&1&1\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix},$$

and

$$I_3=\begin{pmatrix}0&0&0&1\\1&1&0&1\\1&1&0&1\\0&0&0&0\end{pmatrix}.$$

PROPOSITION 2.5. *Let l,m,n be non-negative integers such that $m+l\ge n\ge 1$. Let A be an even unimodular matrix of degree m with entries in $\mathbf{Z}_p$ and $b_1,\dots,b_n\in 2\mathbf{Z}_p\backslash\{0\}$. Further let $c_1,\dots,c_l\in 2\mathbf{Z}_p\backslash\{0\}$ and I be as above if we have $l>0$.*

(1) Let $n\ge 2$. Then for $1\le i\le n$ we have

$$\alpha_p(\mathrm{diag}(p^2b_1,\dots,p^2b_n),(A,\mathrm{diag}(p^2c_1,\dots.p^2c_l);I)$$
$$-p^{i(-m+n+1)}\alpha_p(\mathrm{diag}(b_1,\dots,b_i,p^2b_{i+1},\dots,p^2b_n),(A,C_i);I_i)$$
$$=\beta_p(0,A)\sum_{j=1}^{i}p^{(j-1)(-m+n+1)}\alpha_p(\hat B_j,(A^{(1)},\mathrm{diag}(-p^2b_j,C_j));I'_j)$$

where $\hat B_j=\mathrm{diag}(b_1,\dots,b_{j-1},p^2b_{j+1},\dots,p^2b_n)$, $C_j=\mathrm{diag}(c_1,\dots,c_{l[j]},p^2c_{l[j]+1},\dots,p^2c_l)$, and I'_j is a certain $(l+1,n-1)$-quasi-standard matrix determined by I and j. Here we understand the right-hand side of the above equation is 0 if the Witt index of A is 0.

(2) Let $n=1$. Then we have

$$\alpha_p(p^2b_1,(A,\mathrm{diag}(p^2c_1,\dots,p^2c_l));I)$$
$$=p^{(-m+2)}\alpha_p(b_1,(A,C_1));I_1)+\beta_p(0,A).$$

Proof. (1) First let $l>0$ and $j=n_1+\ldots+n_{k-1}+j'$ with $1\le j'\le n_k$. By (2) of Proposition 2.2 we have

$$\alpha_p(\mathrm{diag}(p^2b_1,\dots,p^2b_n),(A,\mathrm{diag}(p^2c_1,\dots.p^2c_l));I)$$
$$=\alpha_p(\mathrm{diag}(p^2b_1,\dots,p^2b_n),(A,C_j);\tilde I_1).$$

for any $1\le j\le n_1$. Thus the assertion for $k=1$ can be proved by using Proposition 2.1 repeatedly. Let $k\ge 2$ and assume that the assertion holds for $k-1$. Then we have

$$\alpha_p(\mathrm{diag}(p^2b_1,\dots,p^2b_n),(A,\mathrm{diag}(p^2c_1,\dots.p^2c_l));I)$$
$$-p^{n[k-1](-m+n+1)}$$
$$\times\alpha_p(\mathrm{diag}(b_1,\dots,b_{n[k-1]},p^2b_{n[k-1]+1},\dots,p^2b_n),(A,C_{n[k-1]});I_{n[k-1]})$$
$$=\beta_p(0,A)\sum_{j=1}^{n[k-1]}p^{(j-1)(-m+n+1)}\alpha_p(\hat B_j,(A^{(1)},\mathrm{diag}(-p^2b_j,C_j));I'_j).$$

By (2) of Proposition 2.2 we have

$$\alpha_p(\mathrm{diag}(b_1,\dots,b_{n[k-1]},p^2b_{n[k-1]+1},\dots,p^2b_n),(A,C_{n[k-1]});I_{n[k-1]})$$
$$=\alpha_p(\mathrm{diag}(b_1,\dots,b_{n[k-1]},p^2b_{n[k-1]+1},\dots,p^2b_n),(A,C_j);\tilde I_k)$$

for any $n[k-1]+1\le j\le n[k-1]+n_k$. Thus the assertion for k can be proved by using (1) of Proposition 2.1 repeatedly. Thus the assertion for $l>0$ can be proved by induction, and that for $l=0$ can be proved in the same manner.

(2) The assertion can be proved by using (2) of Proposition 2.1 and Proposition 2.2 in the same manner as (1). ∎

3 PROOF OF MAIN RESULTS

In this section we prove the main result. To do this, we need some preliminaries. For two elements $u, v \in \mathbf{Z}_p$ we write $u \sim v$ if there exists an element x of $\mathbf{Z}_p^*$ such that $u = vx^2$. For two symmetric matrices U, V with entries in $\mathbf{Z}_p$ we write $U \sim V$ if there exists a unimodular matrix X in $\mathbf{Z}_p$ such that $U = V[X]$.

LEMMA 3.1. *Let $s, t, n, n_1, .., n_s, l$ be non-negative integers such that $n_1 + ... + n_s = n, n_1, ..., n_s > 0$ and $s \geq t$. Let $B_i \in S_{n_i}(\mathbf{Z}_p)$ $(i = 1, ..., s), Y = (y_{ij})_{1 \leq i \leq l, 1 \leq j \leq n} \in M_{ln}(\mathbf{Z}_p)$ and $c_1, ..., c_l \in \mathbf{Z}_p \backslash \{0\}$. Put $B = \mathrm{diag}(B_1, B_2, ..., B_s), B' = \mathrm{diag}(B_1, ..., B_t), C = \mathrm{diag}(c_1, ..., c_l), Y_1 = (y_{ij})_{1 \leq i \leq l, 1 \leq j \leq n_1 + ... + n_t}$, $Y_2 = (y_{ij})_{1 \leq i \leq l, n_1 + ... + n_t + 1 \leq j \leq n}$. Assume that for $i = 1, ..., s$, B_i is p^{r_i}-unimodular and $\mathrm{ord}_p(c_k) \geq r_i + 2e_p + 1$ for $k = 1, ..., l, i = t+1, ..., s$ (for the definition of p^{r_i}-modular, see [Ki5]). Then there exists a unimodular matrix V of degree l independent of Y_1 such that*

$$V \equiv E_l \bmod p,$$

and

$$B + C[Y] \sim \mathrm{diag}(B' + C[VY_1], B_{t+1}, ..., B_s).$$

Proof. Put $q = n_1 + ... + n_{s-1}, Z_1 = (y_{ij})_{1 \leq i \leq l, 1 \leq j \leq q}, Z_2 = (y_{ij})_{1 \leq i \leq l, q+1 \leq j \leq n}, \tilde{B}_1 = \mathrm{diag}(B_1, B_2, ..., B_{s-1})$ and $\tilde{B}_2 = B_s + C[Z_2]$. Then we have

$$B + C[Y] = \begin{pmatrix} \tilde{B}_1 + C[Z_1] & {}^tZ_1CZ_2 \\ {}^tZ_2CZ_1 & \tilde{B}_2 \end{pmatrix}.$$

By assumption we have

$$\tilde{B}_2 \equiv B_s \bmod p^{r_s + 2e_p + 1} M_{n_s n_s}(\mathbf{Z}_p).$$

Thus by [Ki5, Cor 5.4.4], there exsits a unimodular matrix U of degree n_s such that we have

$(*)$ $\quad \tilde{B}_2[U] = B_s.$

Thus we have $\tilde{B}_2^{-1t}Y_2CY_1 \in M_{n-q,q}(\mathbf{Z}_p)$ and

$$(B + C[Y]) \begin{bmatrix} E_{n-n_s} & O \\ -\tilde{B}_2^{-1t}Z_2CZ_1 & U \end{bmatrix}$$

$$= \begin{pmatrix} \tilde{B}_1 + C[Z_1] - \tilde{B}_2^{-1}[{}^tZ_2CZ_1] & O \\ O & B_s \end{pmatrix}.$$

We have

$$C[Z_1] - \tilde{B}_2^{-1}[{}^tZ_2CZ_1] = (C - \tilde{B}_2^{-1}[{}^tZ_2C])[Z_1].$$

By assumption and $(*)$, we have

$$C - \tilde{B}_2^{-1}[{}^tZ_2C] \equiv C \bmod p^{M(C)+2e_p+1} M_{ll}(\mathbf{Z}_p),$$

where $M(C) = \max_i(\mathrm{ord}_p(c_i))$. Thus, again by [Ki5, Cor. 5.4.4], there exists a unimodular matrix V' of degree l independent of Z_1 such that

$$V' \equiv E_l \bmod p \text{ and } C - \tilde{B}_2^{-1}[{}^tZ_2C] = C[V'].$$

Repeating this process, we complete the proof. ∎

PROPOSITION 3.2. *Let m, n, l, q be non-negative integers such that $m + l \geq n$ and $n \geq q$. Let $A \in S_m(\mathbf{Z}_p), B = \mathrm{diag}(b_1, ..., b_n), C = \mathrm{diag}(c_1, ..., c_l)$ with $b_1, ..., b_n, c_1, ..., c_l \in \mathbf{Z}_p \backslash \{0\}$. Let $I = (\tilde{I}'_q, \tilde{I}''_q)$ with $I'_q \in M_{lq}(\{0,1\})$ and $I''_q \in M_{l,n-q}(\{0,1\})$. Assume that $\mathrm{ord}_p(c_i) \geq \mathrm{ord}_p(b_j) + 2e_p + 1$ for $i = 1, ..., l, j = q+1, ..., n$. Then $\alpha_p(B, (A, C); I)$ does not depend on $\tilde{I}''_q$. In particular if $q = 0$ we have*

$$\alpha_p(B, (A, C); I) = \alpha_p(B, A).$$

Proof. For each $J = (r_{ij}) \in M_{st}(\{0,1\})$ and $Z = (z_{ij}) \in M_{st}(\mathbf{Z}_p)$, we write $Z \in p^J\mathbf{Z}_p$ if $z_{ij} \in p^{r_{ij}}\mathbf{Z}_p$ for any $1 \leq i \leq s, 1 \leq j \leq t$. Further put $B' = \mathrm{diag}(b_1, ..., b_q)$ and $B'' = \mathrm{diag}(b_{q+1}, ..., b_n)$. Then by Lemma 3.1 we have

$$a_e(B, (A, C); I) = \sum_{Y \in M_{ln}(\mathbf{Z}_p)/p^e M_{ln}(\mathbf{Z}_p), Y \in p^I\mathbf{Z}_p} \#\mathcal{A}_e(B + C[Y], A)$$

$$= \sum_{Y_2} \sum_{Y_1} \#\mathcal{A}_e(\mathrm{diag}(B' + C[Y_1], B''), A)$$

$$= p^{el(n-q) - w(I''_q)} \sum_{Y_1} \#\mathcal{A}_e(\mathrm{diag}(B' + C[Y_1], B''), A),$$

where Y_2 (resp. Y_1) runs over elements of $M_{l-q,n}(\mathbf{Z}_p)/p^e M_{l-q,n}(\mathbf{Z}_p)$ (resp. $M_{qn}(\mathbf{Z}_p)/p^e M_{qn}(\mathbf{Z}_p)$) such that $Y_2 \in p^{I''_q}\mathbf{Z}_p$ (resp. $Y_1 \in p^{I'_q}\mathbf{Z}_p$). Thus $p^{e(-(m+l)n + n(n+1)/2) + w(I)} a_e(B, (A, C); I)$ does not depend on I''_q and therefore nor does $\alpha_p(B, (A, C); I)$. Further if $q = 0$ we have

$$p^{e(-(m+l)n + n(n+1)/2) + w(I)} a_e(B, (A, C); I)$$

$$= p^{e(-mn + n(n+1)/2)} \#\mathcal{A}_e(B, A).$$

Thus the second assertion holds. ∎

Now for each integers i, j, k such that $0 \leq k \leq i$, put

$$\gamma(i, j, k) = (-1)^k \sum_{0 \leq i_1 < .. < i_k \leq i-1} p^{(i-i_1)(j+i_1)} ... p^{(i-i_k)(j+i_k)}.$$

Here we understand that we have $\gamma(i, j, 0) = 1$. We remark that $\gamma(i, j, k)$ is the k-th coefficient of the polynomial $\prod_{k=0}^{i-1}(1 - p^{(i-k)(j+k)}x)$ in x, that is,

$$\prod_{k=0}^{i-1}(1 - p^{(i-k)(j+k)}x) = \sum_{k=0}^{i} \gamma(i, j, k)x^i.$$

Then our main result is the following, which we can prove by induction using Propositions 2.5 and 3.2.

THEOREM 3.3. *Let t, n, m, l be non-negative integers such that $m + l \geq n \geq 1$ and $n \geq t$. Let $B_1 = \mathrm{diag}(b_1, ..., b_t), B_2 = \mathrm{diag}(b_{t+1}, ..., b_n)$ with $b_1, ..., b_n \in \mathbf{Z}_p \backslash \{0\}$ and $C = \mathrm{diag}(c_1, .., c_l)$ with $c_1, ..., c_l \in \mathbf{Z}_p \backslash \{0\}$. Let $I \in M_{l,n}(\{0,1\})$. Let e be an integer such that $e \geq \mathrm{ord}_p(b_j) - \mathrm{ord}_p(b_k) + 2m_0 + 2 + 2e_p$ and $e \geq \mathrm{ord}_p(b_j) - \mathrm{ord}_p(c_{k'}) + 2m_0 + 3 + 2e_p$ for $j = t+1, ..., n, k = 1, ..., t, k' = 1, ..., l$. Then we have*

$$\sum_{k=0}^{m_0+1} \gamma(t, -m + n + 1, k)\alpha_p(\mathrm{diag}(p^{e-2k}B_1, B_2), (A, p^{e-2k}C); I)$$

$$= \prod_{i=0}^{m_0} \frac{1-p^{(-m+n+i+1)(n-i)}}{1-p^{-m+n+i+1}} \alpha_p(B_2, A^{(m_0+1)})\beta_p(O_{m_0+1}, A).$$

As a special case of the above theorem we have

COROLLARY. *Let m, n and t be positive integers such that $m \geq n \geq t$. Let A be an even unimodular matrix of degree m and of Witt index r. Let $B_1 = \mathrm{diag}(b_1, ..., b_t)$ and $B_2 = \mathrm{diag}(b_{t+1}, ..., b_n)$ with $b_i \in \mathbf{Z}_p \backslash \{0\}$. Put $e_p = 1$ or 0 according as $p = 2$ or not, and $m_0 = \min(t-1, r)$. Further let e be an integer such that $e \geq \mathrm{ord}_p(b_j) - \mathrm{ord}_p(b_k) + 2m_0 + 2e_p + 2$ for $j = t+1, ..., n, k = 1, ..., t$. Then we have*

$$\alpha_p(\mathrm{diag}(p^e B_1, B_2), A)$$

$$= -\sum_{i=1}^{m_0+1} \gamma(t, -m+n+1, i)\alpha_p(\mathrm{diag}(p^{e-2i}B_1, B_2), A)$$
$$+ \prod_{i=0}^{m_0} \frac{1-p^{(t-i)(-m+n+i+1)}}{1-p^{-m+n+i+1}} \beta_p(O_{m_0+1}, A)\alpha_p(B_2, A^{(m_0+1)}),$$

where O_{m_0+1} is the zero matrix of degree m_0+1. Here we make the convention that the second term on the right-hand side is 0 if $m_0 = r$, and that we have $\alpha_p(B_2, A^{(m_0+1)}) = 1$ if $n = t$.

REMARK 1. The above corollary has been proved in [KH] under more general setting. However, in that paper, we have not deal with the local densities with congruence conditions, that is we have not proved Theorem 3.3. Thus our result in this paper is new in this sense.

Now for non-degenerate matrices $A, B_1, ..., B_{s-1}$ and B_s of degree $m, n_1, .., n_{s-1}$ and n_s, respectively, with entries in $\mathbf{Z}_p$, define a formal power series $R((B_1, ..., B_s), A; x_1, ..., x_s)$ by

$$R((B_1, ..., B_s), A; x_1, ..., x_s)$$
$$= \sum_{e_1 \geq ... \geq e_s \geq 1} \alpha_p(\mathrm{diag}(p^{e_1}B_1, ..., p^{e_s}B_s), A)x_1^{e_1}...x_s^{e_s}.$$

Let Δ_p be the complete set of representatives of $\mathbf{Z}_p^*/\mathbf{Z}_p^{*2}$, and put

$$\Lambda_{n,p} = \{(b_1, ..., b_n); b_i \in \Delta_p\}$$

or

$$= \{(B_1, ..., B_s); \deg B_1 + ... + \deg B_s = n,$$
$$B_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \text{ or } \deg B_i = 1 \text{ and} B_i \in \Delta_p\}$$

according as $p \neq 2$ or $= 2$. Then the set of power series $\{R((B_1, ..., B_s), A; x_1, ..., x_s)\}_{(B_1,...,B_s)\in\Lambda_{n,p}}$ gives complete information on the local densities $\alpha_p(B, A)$ for all B of degree n and A. So it is important to study these power series. The following is a direct consequence of Corollary to Theorem 3.3:

THEOREM 3.4. *Assume that A is even unimodular, and put $B_i = \mathrm{diag}(b_{n_1+...+n_{i-1}+1}, ..., b_{n_1+...+n_i})$ $(i = 1, ..., s)$ with $b_j \in \mathbf{Z}_p \backslash \{0\}$. Put $m_k = \min(n_1 + ... + n_k - 1, r)$. Further put $l_0 = m_1 + e_p$ or $= m_1 + e_p + \max_{j'=n_1+1,...,n} \mathrm{ord}_p(b_{j'}) - \min_{j=1,...,n_1} \mathrm{ord}_p(b_j)$ according as $\max_{j'=n_1+1,...,n} \mathrm{ord}_p(b_{j'}) - \min_{j=1,...,n_1} \mathrm{ord}_p(b_j) \leq 0$ or not. Then we have*

$$\prod_{i=0}^{m_1}(1-p^{(n_1-i)(-m+n+i+1)}x_1^2)R((B_1, ..., B_s), A; x_1, ..., x_s)$$

$$= \sum_{i=0}^{l_0} x_1^{2i} \sum_{j=0}^{i} \gamma(n_1, -m+n+1, i-j)$$
$$\times R((\mathrm{diag}(p^{2j}B_1, B_2), B_3, ..., B_s), A; x_1x_2, x_3, ..., x_s)$$
$$+ \sum_{i=0}^{l_0} x_1^{2i+1} \sum_{j=0}^{i} \gamma(n_1, -m+n+1, i-j)$$
$$\times R((\mathrm{diag}(p^{2j+1}B_1, B_2), B_3, ..., B_s), A; x_1x_2, x_3, ..., x_s)$$
$$+ \prod_{i=0}^{m_1} \frac{1-p^{(n_1-i)(-m+n+i+1)}}{1-p^{-m+n+i+1}} \beta_p(O_{m_1+1}, A)\frac{x_1^{2l_0+2}}{1-x_1}$$
$$\times R((B_2, B_3, ..., B_s), A^{(m_1+1)}; x_1x_2, x_3, ..., x_s).$$

Here we make the convention that the third term on the right-hand side of the above is 0 if $r = m_1$, and that we have $R((\mathrm{diag}(p^k B_1, B_2), B_3, ..., B_s), A; x_1x_2, x_3, ..., x_s) = \alpha_p(p^k B_1, A)$ for $k = 1, ..., 2l_0 + 1$ and $R((B_2, B_3, ..., B_s), A^{(m_1+1)}; x_1x_2, x_3, ..., x_s)) = 1$ if $s = 1$.

We remark that if $s = 1$ and $n_1 = n$, our power series coincides with the one defined by Kitaoka [Ki4] and by the above theorem its denominator is

$$\prod_{i=0}^{\min(n-1,r)} (1-p^{(n-i)(-m+n+i+1)}x^2)(1-x)^{m'},$$

where $m' = 1$ or $= 0$ according as $r \geq n$ or not. This is a certain refinement of the result of [Hir],[Ki4]. Further Theorem 1.2 shows that if $s \geq 2$, the series $R((B_1, ..., B_s), A; x_1, ..., x_s)$ $(1-x_1)\prod_{i=0}^{m_1}(1-p^{(n_1-i)(-m+n+i+1)}x_1^2)$ can be expressed as a $\mathbf{Q}[x_1]$-linear combination of the power series in $s-1$ variables. Using this, by induction, we have

THEOREM 3.5. *Let the notation and the assumption be as in Theorem 3.4. Then $R((B_1, ..., B_s), A; x_1, ..., x_s)$ is a rational function of $x_1, ..., x_s$ with denominator*

$$\prod_{k=1}^{s}\prod_{i=0}^{m_k}(1-p^{(n_1+...+n_k-i)(-m+n+i+1)}(x_1...x_k)^2)$$
$$\times \prod_{k=1}^{s}(1-x_1...x_k)^{m'_k},$$

where $m'_k = 1$ or $= 0$ according as $r \geq n_1 + ... + n_k$ or not. In particular if $m \geq 2n+2$, the denominator of the above power series is

$$\prod_{k=1}^{s} \prod_{i=0}^{n_1+...+n_k-1} (1-p^{(n_1+...+n_k-i)(-m+n+i+1)}(x_1...x_k)^2)$$
$$\prod_{k=1}^{s}(1-x_1...x_k).$$

REMARK 2. In [Ka3], we have proved the rationality of the power series defined by

$$Q((B_1, ..., B_s), A; x_1, ..., x_s)$$
$$= \sum_{e_1,...,e_s \geq 1} \alpha_p(\mathrm{diag}(p^{e_1}B_1, ..., p^{e_s}B_s), A)x_1^{e_1}...x_s^{e_s}$$

for arbitrary A, B and p. Further, in [Ka2], we have given an explicit form of the denominator of it when $p \neq 2$. To do this, we needed a recursion formula similar to Proposition 2.1 for an arbitrary matrix A of level p (cf. [Ka2 Proposition 3.6 (1),(2)]). This type of formula seems difficult to be generalized to the case where $p = 2$ without any change. So the method of [Ka2] giving an explicit form of the denominator cannot be applied to $p = 2$ directly.

REFERENCES

[BS] Böcherer, S., Sato, F., Rationality of certain formal power series related to local densities, Commentarii Math Univ. St. Paul., 36(1987), 53-86.

[Hir] Hironaka, Y., On a denominator of Kitaoka's formal power series attached to local densities,Comment. Math. Univ. St. Paul., 37 (1988), 159-171.

[His] Hisasue, M., Rationality of formal power series attached to the 2-adic densities of quadratic forms, Master Thesis, Hokkaido Univ.

[IK] Ibukiyama, T., Katsurada, H., A simple formula for Koecher-Maaß Dirichlet series associated with Siegel-Eisenstein series, preprint.

[Ka1] Katsurada, H., A generalized Igusa local zeta function and local densities of quadratic forms, Tôhoku Math. J., 44 (1992), 211-218.

[Ka2] Katsurada, H., A certain formal power series of several variables attached to local densities of quadratic forms I, J. Number Theory, 51(1995) 169-209.

[Ka3] Katsurada, H., Rationality of the power series attached to local densities, Manuscripta Math., 82(1994), 125-138.

[Ka4] Katsurada, H., A certain formal power series of several variables attached to local densities of quadratic forms II, Proc. Japan Acad., 70(1994) 208-211.

[Ka5] Katsurada, H., An explicit formula for the Fourier coefficients of Siegel-Eisenstein series of degree 3, Nagoya Math. J., 146(1997) 199-223.

[Ka6] Katsurada, H., An explicit formula for Siegel series, preprint.

[KH] Katsurada, H., Hisasue, H., A recursion formula for local densities, J. Number Theory, 64(1997), 183-210.

[Ki1] Kitaoka, Y.: A note on local densities of quadratic forms, Nagoya Math. J., 92(1983), 145-152.

[Ki2] Kitaoka, Y., Fourier coefficients of Eisenstein series of degree 3, Proc. Japan Acad., 60, Ser. A (1984) 259-261.

[Ki3] Kitaoka, Y., Dirichlet series in the theory of quadratic forms, Nagoya math. J., 92 (1984) 73-84.

[Ki4] Kitaoka, Y., Local densities of quadratic forms and Fourier coefficients of Eisenstein series, Nagoya Math. J., 103(1986), 149-160.

[Ki5] Kitaoka, Y.: Arithmetic of quadratic forms (Cambridge Tracts Math. 106) Cambridge: Cambridge Univ. Press, 1993.

[M] Maass, H., Die Fourierkoeffizienten der Eisensteinreihen zweiten Grades, Mat. Fys. Medd. Dan. Vid. Selsk., 34(1964), 1-25.

合同条件を持つ二次形式の局所密度の漸化式

桂田　英典*，久末 正樹**

概要

この論文では合同条件を持った局所密度の間の関係式を考察している．この応用として我々は，(*J. Number Theory*, 64, 1997, 183-210) における局所密度の漸化式の別証明を特別な場合に与える．さらに，(*Proc. Japan Acad.* 70, 1994, 208-211) における形式べき級数のある一般化を定義して，その分母の形を正確に求める．

キーワード：　局所密度，漸化式

*共通講座，　**北海道大学理学部数学科
