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# A recursion formula of local densities with congruence conditions

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In this paper, we give a recursion formula of local densities with congruence conditions. As an application we give another proof for the recursion formula of local densities in (*J. Number Theory*, 64, 1997, 183-210) for a special case. Further we define a certain formal power series which is a generalization of the one in (*Proc. Japan Acad.* 70, 1994, 208-211), and determine an explicit form of its denominator.

Keywords: Local Densities, Recursion Formula

## 1 INTRODUCTION

Local densities of quadratic forms over the  $p$ -adic field are very important invariants in the arithmetic theory of quadratic forms, and various types of recursion formulas for them have been studied by several authors. They all give important<sup>2</sup> information on local densities. Among others, in [Ki1] and [Ki2], Kitaoka have given an explicit formula for a special case. In [Ka1], we have defined the local densities with congruence condition, and in [Ka2], obtained several recursion formulas for them to give the denominator of a certain power series attached to local densities for  $p \neq 2$ . They are very effective to compute the local densities, and give an explicit form of all local densities of quadratic forms over the  $p$ -adic field for  $p \neq 2$  in principle. However they are rather complicated and not all of them can be generalized to all the cases including  $p = 2$  (see the remark at the end of section 3). In this paper, we give a formula expressing a local density with congruence condition as a linear combination of usual local densities for a special but important case (cf. Theorem 3.3). It is rather simpler than the ones in [Ka2], and holds for all  $p$  including 2. As a corollary, we give a new proof to [KH, Theorem 4.1] for a special case.

## 2 LOCAL DENSITIES WITH CONGRUENCE CONDITIONS

In this section, we recall the notion of the local densities with congruence conditions and a recursion formula for them following [Ka2] and [His] with a slight modification. For a commutative ring  $R$ , we denote by  $M_{m,n}(R)$  the set of  $(m, n)$ -matrices with entries in  $R$ . Here we understand  $M_{m,n}(R)$  the set of the empty matrix if  $m = 0$  or  $n = 0$ . We also make the convention that  $\text{diag}(U, V) = V$  if  $U$  is the empty matrix. For an  $(m, n)$  matrix  $X$  and an  $(m, m)$  matrix  $A$ , we write  $A[X] = {}^t XAX$ , where  ${}^t X$  denotes the transposition of  $X$ . Let  $a$  be an element of  $R$ . Then for an element  $X$  of  $M_{m,n}(R)$  we often use the same symbol  $X$  to denote the class of  $X$  mod  $aM_{m,n}(R)$ . Put

$$GL_m(R) = \{A \in M_{m,m}(R); \det A \in R^*\},$$

where  $\det A$  denotes the determinant of a square matrix  $A$ , and  $R^*$  denotes the unit group of  $R$ . Further let  $S_n(R)$  denote the set of symmetric matrices of degree  $n$  with entries in  $R$ . We abbreviate an  $(m, n)$ -matrix whose components are all 1 (resp. 0) as  $1_{mn}$  (resp.  $0_{mn}$ ). We often simply write  $1 = 1_{mn}$  and  $0 = 0_{mn}$  if no confusion arises. Remark that 1 is different from the unit matrix. For square matrices  $X$  and  $Y$  we write  $\text{diag}(X, Y) = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ . Let  $\mathbb{Q}_p$  be the field of  $p$ -adic numbers, and  $\{0, 1\}$  the finite set with two integers 0 and 1. Let  $m, n$  and  $l$  be non-negative integers such that

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$m+l \geq n \geq 1$ . For  $S \in S_m(\mathbb{Z}_p)$ ,  $T \in S_n(\mathbb{Z}_p)$ ,  $U \in S_l(\mathbb{Z}_p)$ ,  $I = (r_{ij}) \in M_{ln}(\{0,1\})$ , and a non-negative integer  $e$ , put

$$\begin{aligned} \mathcal{A}_e(T, (S, U); I) &= \{X = (x_{ij}) \in M_{m+l,n}(\mathbb{Z}_p)/p^e M_{m+l,n}(\mathbb{Z}_p); \\ \text{diag}(S, U)[X] &\equiv T \pmod{p^e} \text{ and } x_{m+i,j} \in p^{r_{ij}}\mathbb{Z}_p \\ &\text{for any } 1 \leq i \leq l, 1 \leq j \leq n\}, \end{aligned}$$

and

$$a_e(T, (S, U); I) = \#\mathcal{A}_e(T, (S, U); I).$$

We then define  $\alpha_p(T, (S, U); I)$  by

$$\alpha_p(T, (S, U); I) = p^{w(I)} \lim_{e \rightarrow \infty} p^{-(m+l)n+n(n+1)/2e} a_e(T, (S, U); I),$$

where for  $I = (r_{ij}) \in M_{ln}(\{0,1\})$  we write

$$w(I) = \sum_{1 \leq i \leq l, 1 \leq j \leq n} r_{ij}$$

If  $\text{diag}(S, U)$  and  $T$  are non-degenerate, the above limit exists. We write  $\alpha_p(T, (S, U); I)$  as  $\alpha_p(T, (S, \emptyset); I)$  or as  $\alpha_p(T, (\emptyset, U); I)$  according as  $\deg U = 0$  or  $\deg S = 0$ . Note that  $\alpha_p(T, (S, U); I)$  coincides with the usual local density  $\alpha_p(T, S)$  if  $l = 0$  and with  $\alpha_p(T, \text{diag}(S, U))$  if  $I = 0_{ln}$ . For an  $(l, n)$ -matrix  $I = (r_{ij})$ , and permutations  $\sigma$  and  $\tau$  of degree  $l$  and  $n$  respectively, we put  $\sigma \circ I \circ \tau = (r_{\sigma(i), \tau(j)})$ . We say that an  $(l, n)$ -matrix  $I$  is standard if

$$I = \begin{pmatrix} n_1 & \cdots & n_{s-1} & n_s \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} \} l_1 \\ \} \\ \} l_{r-1} \\ \} l_r \end{matrix}$$

with non-negative integers  $l_1, \dots, l_{r-1}, l_r, n_1, \dots, n_{s-1}, n_s$  such that  $l_1 + \dots + l_{r-1} + l_r = l$ , and  $n_1 + \dots + n_{s-1} + n_s = n$ . An  $(l, n)$ -matrix  $I$  is called quasi-standard if  $I = \sigma \circ I_0 \circ \tau$  with a standard matrix  $I_0$ , and permutations  $\sigma$  and  $\tau$  of degrees  $l$  and  $n$ , respectively. We denote by  $Sd_{ln}(\{0,1\})$  the subset of  $M_{ln}(\{0,1\})$  consisting of all quasi-standard matrices.

Now let  $A$  be an even unimodular matrix with entries in  $\mathbb{Z}_p$ . That is, let  $A$  be a symmetric unimodular matrix with entries in  $\mathbb{Z}_p$  whose diagonal components belong to  $2\mathbb{Z}_p$ . We remark that it means merely a symmetric unimodular matrix with entries in  $\mathbb{Z}_p$  if  $p \neq 2$ . As is well known,  $A$  is equivalent, over  $\mathbb{Z}_p$ , to a matrix of the following type:

$$\text{diag}(\overbrace{H, \dots, H}^r, U),$$

where  $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , and  $U$  is an anisotropic even unimodular matrix of degree not greater than 2. The above  $r$  is the Witt index of  $A$ , which will be denoted by  $r(A)$ . Here we make the convention that  $\text{diag}(H, \dots, H, U) = U$  or  $= \text{diag}(H, \dots, H)$  according as  $r = 0$  or  $\deg U = 0$ . Then we define  $A^{(k)}$  by

$$A^{(k)} = \text{diag}(\overbrace{H, \dots, H}^{r-k}, U).$$

This  $A^{(k)}$  is uniquely determined only by  $A$  and  $k$  up to equivalence over  $\mathbb{Z}_p$ . As is well known, the value  $\alpha_p(B, A)$

for an even matrix  $B$  of degree  $n$  and  $A = \text{diag}(\overbrace{H, \dots, H}^k)$  is closely connected with the Fourier coefficient of Siegel-Eisenstein series of degree  $n$  and of weight  $k$  (for example, see [IK],[Ka5],[Ka6],[Ki2], and [M].) So it is important problem to find a reasonable expression of  $\alpha_p(B, A)$  when  $A$  is even unimodular.

**PROPOSITION 2.1.** *Let  $m, l, n$  be non-negative integers such that  $m+l \geq n \geq 1$ . Let  $A$  be non-degenerate even unimodular matrix of degree  $m$  with entries in  $\mathbb{Z}_p$ , and let  $b_1, \dots, b_n, c_1, \dots, c_l$  be elements of  $2\mathbb{Z}_p \setminus \{0\}$  such that  $b_1 \in 2p^2\mathbb{Z}_p$ . Further let  $I'$  be an  $(l, n)$ -matrix with entries in  $\{0,1\}$ . Put  $B = \text{diag}(b_1, \dots, b_n)$ ,  $\hat{B} = \text{diag}(b_2, \dots, b_n)$ , and  $C = \text{diag}(c_1, \dots, c_l)$ .*

(1) *Let  $n \geq 2$ . Then we have*

$$\begin{aligned} &\alpha_p(\text{diag}(p^2 b_1, \hat{B}), (A, C); (1_{l1}, I')) \\ &= p^{-m+n+1} \alpha_p(\text{diag}(b_1, \hat{B}), (A, C); (0_{l1}, I')) \\ &+ \beta_p(0, A) \alpha_p(\hat{B}, (\text{diag}(A^{(1)}, -p^2 b_1), C); I'), \end{aligned}$$

where  $(1_{l1}, I')$  is the  $(l, n)$ -matrix whose  $k$ -th column is  $1_{l1}$  or the  $k-1$ -th column of  $I'$  according as  $k = 1$  or not, and others. Here we understand the right-hand side is 0 if the Witt index of  $A$  is 0.

(2) *Let  $n = 1$ . Then we have*

$$\alpha(p^2 b_1, (A, C); 1_{l1}) - p^{-m+2} \alpha_p(b_1, (A, C); 0_{l1}) = \beta_p(0, A).$$

**PROPOSITION 2.2.** *Let  $A, B, C_1$  and  $C_2$  be non-degenerate symmetric matrices of degree  $m, n, l_1$  and  $l_2$ , respectively, with entries in  $\mathbb{Z}_p$  such that  $m+l_1+l_2 \geq n$ , and  $I$  an  $(l_2, n)$ -matrix with entries in  $\{0,1\}$ . Assume that  $B, C_1$  and  $C_2$  are diagonal. Then we have*

$$\begin{aligned} (1) \quad &\alpha_p(B, (\text{diag}(A, p^2 C_1), C_2); I) \\ &= \alpha_p(B, (A, \text{diag}(C_1, C_2)); \begin{pmatrix} 1_{l_1, n} \\ I \end{pmatrix}). \\ (2) \quad &\alpha_p(B, (A, \text{diag}(p^2 C_1, C_2)); \begin{pmatrix} 0_{l_1, n} \\ I \end{pmatrix}) \\ &= \alpha_p(B, (A, \text{diag}(C_1, C_2)); \begin{pmatrix} 1_{l_1, n} \\ I \end{pmatrix}). \end{aligned}$$

Proposition 2.2 for  $p \neq 2$  is nothing but [Ka2, Proposition 3.4] and it also holds for  $p = 2$  without any change. (1) of Proposition 2.1 for  $p \neq 2$  is a special case of [Ka2, Proposition 3.6, 3.7] and it can be proved with slight modification for  $p = 2$ . Now let  $n, l$  be positive integers. Let

$$I = \begin{pmatrix} n_1 & \cdots & n_{s-1} & n_s \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{matrix} \} l_1 \\ \} \\ \} l_{s-1} \\ \} l_s \end{matrix}$$

with non-negative integers  $l_1, \dots, l_{s-1}, l_s, n_1, \dots, n_{s-1}, n_s$  such that  $l_2, \dots, l_s, n_1, \dots, n_{s-1} > 0$  and  $l_1 + \dots + l_{s-1} + l_s = l, n_1 + \dots + n_{s-1} + n_s = n$ . For an integer  $j$  such that  $n_1 + \dots + n_{k-1} + 1 \leq j \leq n_1 + \dots + n_k$  put  $l[j] = l_1 + \dots + l_k$ . Here we understand  $n_0 = 0$ . For each integer  $1 \leq k \leq s$  put  $n[k] = n_1 + \dots + n_k$ . For the above  $I$  and each integers  $1 \leq k \leq s$  define an element  $\tilde{I}_k$  of  $M_m(\{0, 1\})$  by

$$\begin{pmatrix} n_1 & \dots & n_{k-1} & n_k & \dots & n_s \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{matrix} \} l_1 \\ \} \\ \} l_{k-1} \\ \} l_k \\ \} \\ \} l_s \end{matrix}.$$

We note that  $\tilde{I}_1 = I$  if  $l_1 = 0$ . Next, for each integer  $j$  such that  $n[k-1] + 1 \leq j \leq n[k-1] + n_k$ , define an element  $I_j$  of  $M_{l,n}(\{0, 1\})$  by

$$\begin{pmatrix} n_1 & \dots & n_{k-1} & j - n[k-1] & n[k] - j & \dots & n_s \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{matrix} \} l_1 \\ \} \\ \} l_{k-1} \\ \} l_k \\ \} \\ \} l_s \end{matrix}$$

that is,  $I_j$  is the matrix obtained from  $\tilde{I}_k$  by replacing  $(\alpha, \beta)$ -components of  $\tilde{I}_k$  by 0 for  $\alpha = n[k-1] + 1, n[k-1] + 2, \dots, j-1, j$  and  $\beta = 1, \dots, l$ . Further put  $I_0 = I$ . We note that  $I_n = I$ , and  $\tilde{I}_k$  is the matrix obtained from  $I_{n[k-1]}$  by replacing  $(\alpha, \beta)$ -components of  $I_{n[k-1]}$  by 1 for  $\alpha = 1, \dots, n$  and  $\beta = l_1 + \dots + l_{k-1} + 1, \dots, l_1 + \dots + l_{k-1} + l_k$  (see example below).

EXAMPLE. Let  $n = 4, l = 4, s = 3$  and  $n_1 = 2, n_2 = 1, n_3 = 1, l_1 = 1, l_2 = 2, l_3 = 1$ . Then  $n[1] = 2, n[2] = 3, n[3] = 4, l[1] = l[2] = 1, l[3] = 3, l[4] = 4$ , and  $I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ .

By construction we have

$$\tilde{I}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \tilde{I}_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$\tilde{I}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Further we have

$$I_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$I_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

PROPOSITION 2.5. Let  $l, m, n$  be non-negative integers such that  $m + l \geq n \geq 1$ . Let  $A$  be an even unimodular matrix of degree  $m$  with entries in  $\mathbb{Z}_p$  and  $b_1, \dots, b_n \in 2\mathbb{Z}_p \setminus \{0\}$ . Further let  $c_1, \dots, c_l \in 2\mathbb{Z}_p \setminus \{0\}$  and  $I$  be as above if we have  $l > 0$ .

(1) Let  $n \geq 2$ . Then for  $1 \leq i \leq n$  we have

$$\begin{aligned} & \alpha_p(\text{diag}(p^2 b_1, \dots, p^2 b_n), (A, \text{diag}(p^2 c_1, \dots, p^2 c_l)); I) \\ & - p^{i(-m+n+1)} \alpha_p(\text{diag}(b_1, \dots, b_i, p^2 b_{i+1}, \dots, p^2 b_n), (A, C_i); I_i) \\ & = \beta_p(0, A) \sum_{j=1}^i p^{(j-1)(-m+n+1)} \alpha_p(\hat{B}_j, (A^{(1)}, \text{diag}(-p^2 b_j, C_j)); I'_j) \end{aligned}$$

where  $\hat{B}_j = \text{diag}(b_1, \dots, b_{j-1}, p^2 b_{j+1}, \dots, p^2 b_n)$ ,  $C_j = \text{diag}(c_1, \dots, c_{l(j)}, p^2 c_{l(j)+1}, \dots, p^2 c_l)$ , and  $I'_j$  is a certain  $(l+1, n-1)$ -quasi-standard matrix determined by  $I$  and  $j$ . Here we understand the right-hand side of the above equation is 0 if the Witt index of  $A$  is 0.

(2) Let  $n = 1$ . Then we have

$$\begin{aligned} & \alpha_p(p^2 b_1, (A, \text{diag}(p^2 c_1, \dots, p^2 c_l)); I) \\ & = p^{(-m+2)} \alpha_p(b_1, (A, C_1)); I_1 + \beta_p(0, A). \end{aligned}$$

Proof. (1) First let  $l > 0$  and  $j = n_1 + \dots + n_{k-1} + j'$  with  $1 \leq j' \leq n_k$ . By (2) of Proposition 2.2 we have

$$\begin{aligned} & \alpha_p(\text{diag}(p^2 b_1, \dots, p^2 b_n), (A, \text{diag}(p^2 c_1, \dots, p^2 c_l)); I) \\ & = \alpha_p(\text{diag}(p^2 b_1, \dots, p^2 b_n), (A, C_j); \tilde{I}_1). \end{aligned}$$

for any  $1 \leq j \leq n_1$ . Thus the assertion for  $k = 1$  can be proved by using Proposition 2.1 repeatedly. Let  $k \geq 2$  and assume that the assertion holds for  $k-1$ . Then we have

$$\begin{aligned} & \alpha_p(\text{diag}(p^2 b_1, \dots, p^2 b_n), (A, \text{diag}(p^2 c_1, \dots, p^2 c_l)); I) \\ & \quad - p^{n[k-1](-m+n+1)} \\ & \times \alpha_p(\text{diag}(b_1, \dots, b_{n[k-1]}, p^2 b_{n[k-1]+1}, \dots, p^2 b_n), (A, C_{n[k-1]}); I_{n[k-1]}) \\ & = \beta_p(0, A) \sum_{j=1}^{n[k-1]} p^{(j-1)(-m+n+1)} \alpha_p(\hat{B}_j, (A^{(1)}, \text{diag}(-p^2 b_j, C_j)); I'_j). \end{aligned}$$

By (2) of Proposition 2.2 we have

$$\begin{aligned} & \alpha_p(\text{diag}(b_1, \dots, b_{n[k-1]}, p^2 b_{n[k-1]+1}, \dots, p^2 b_n), (A, C_{n[k-1]}); I_{n[k-1]}) \\ & = \alpha_p(\text{diag}(b_1, \dots, b_{n[k-1]}, p^2 b_{n[k-1]+1}, \dots, p^2 b_n), (A, C_j); \tilde{I}_k) \end{aligned}$$

for any  $n[k-1] + 1 \leq j \leq n[k-1] + n_k$ . Thus the assertion for  $k$  can be proved by using (1) of Proposition 2.1 repeatedly. Thus the assertion for  $l > 0$  can be proved by induction, and that for  $l = 0$  can be proved in the same manner.

(2) The assertion can be proved by using (2) of Proposition 2.1 and Proposition 2.2 in the same manner as (1). ■

## 3 PROOF OF MAIN RESULTS

In this section we prove the main result. To do this, we need some preliminaries. For two elements  $u, v \in \mathbb{Z}_p$  we write  $u \sim v$  if there exists an element  $x$  of  $\mathbb{Z}_p^*$  such that  $u = vx^2$ . For two symmetric matrices  $U, V$  with entries in  $\mathbb{Z}_p$  we write  $U \sim V$  if there exists a unimodular matrix  $X$  in  $\mathbb{Z}_p$  such that  $U = V[X]$ .

LEMMA 3.1. *Let  $s, t, n, n_1, \dots, n_s, l$  be non-negative integers such that  $n_1 + \dots + n_s = n, n_1, \dots, n_s > 0$  and  $s \geq t$ . Let  $B_i \in S_{n_i}(\mathbb{Z}_p)$  ( $i = 1, \dots, s$ ),  $Y = (y_{ij})_{1 \leq i \leq t, 1 \leq j \leq n} \in M_{tn}(\mathbb{Z}_p)$  and  $c_1, \dots, c_l \in \mathbb{Z}_p \setminus \{0\}$ . Put  $B = \text{diag}(B_1, B_2, \dots, B_s)$ ,  $B' = \text{diag}(B_1, \dots, B_t)$ ,  $C = \text{diag}(c_1, \dots, c_l)$ ,  $Y_1 = (y_{ij})_{1 \leq i \leq t, 1 \leq j \leq n_1 + \dots + n_t}$ ,  $Y_2 = (y_{ij})_{1 \leq i \leq t, n_1 + \dots + n_t + 1 \leq j \leq n}$ . Assume that for  $i = 1, \dots, s$ ,  $B_i$  is  $p^{r_i}$ -unimodular and  $\text{ord}_p(c_k) \geq r_i + 2e_p + 1$  for  $k = 1, \dots, l, i = t + 1, \dots, s$  (for the definition of  $p^{r_i}$ -unimodular, see [Ki5]). Then there exists a unimodular matrix  $V$  of degree  $l$  independent of  $Y_1$  such that*

$$V \equiv E_l \pmod{p},$$

and

$$B + C[Y] \sim \text{diag}(B' + C[VY_1], B_{t+1}, \dots, B_s).$$

*Proof.* Put  $q = n_1 + \dots + n_{s-1}$ ,  $Z_1 = (y_{ij})_{1 \leq i \leq t, 1 \leq j \leq q}$ ,  $Z_2 = (y_{ij})_{1 \leq i \leq t, q+1 \leq j \leq n}$ ,  $\tilde{B}_1 = \text{diag}(B_1, B_2, \dots, B_{s-1})$  and  $\tilde{B}_2 = B_s + C[Z_2]$ . Then we have

$$B + C[Y] = \begin{pmatrix} \tilde{B}_1 + C[Z_1] & {}^t Z_1 C Z_2 \\ {}^t Z_2 C Z_1 & \tilde{B}_2 \end{pmatrix}.$$

By assumption we have

$$\tilde{B}_2 \equiv B_s \pmod{p^{r_s + 2e_p + 1} M_{n_s, n_s}(\mathbb{Z}_p)}.$$

Thus by [Ki5, Cor 5.4.4], there exists a unimodular matrix  $U$  of degree  $n$ , such that we have

$$(*) \quad \tilde{B}_2[U] = B_s.$$

Thus we have  $\tilde{B}_2^{-1} {}^t Z_2 C Y_1 \in M_{n-q, q}(\mathbb{Z}_p)$  and

$$\begin{aligned} (B + C[Y]) \begin{bmatrix} E_{n-n_s} & O \\ -\tilde{B}_2^{-1} {}^t Z_2 C Z_1 & U \end{bmatrix} \\ = \begin{pmatrix} \tilde{B}_1 + C[Z_1] - \tilde{B}_2^{-1} {}^t Z_2 C Z_1 & O \\ O & B_s \end{pmatrix}. \end{aligned}$$

We have

$$C[Z_1] - \tilde{B}_2^{-1} {}^t Z_2 C Z_1 = (C - \tilde{B}_2^{-1} {}^t Z_2 C)[Z_1].$$

By assumption and (\*), we have

$$C - \tilde{B}_2^{-1} {}^t Z_2 C \equiv C \pmod{p^{M(C) + 2e_p + 1} M_{ll}(\mathbb{Z}_p)},$$

where  $M(C) = \max_i(\text{ord}_p(c_i))$ . Thus, again by [Ki5, Cor. 5.4.4], there exists a unimodular matrix  $V'$  of degree  $l$  independent of  $Z_1$  such that

$$V' \equiv E_l \pmod{p} \text{ and } C - \tilde{B}_2^{-1} {}^t Z_2 C = C[V'].$$

Repeating this process, we complete the proof.  $\blacksquare$

PROPOSITION 3.2. *Let  $m, n, l, q$  be non-negative integers such that  $m + l \geq n$  and  $n \geq q$ . Let  $A \in S_m(\mathbb{Z}_p)$ ,  $B = \text{diag}(b_1, \dots, b_n)$ ,  $C = \text{diag}(c_1, \dots, c_l)$  with  $b_1, \dots, b_n, c_1, \dots, c_l \in \mathbb{Z}_p \setminus \{0\}$ . Let  $I = (\tilde{I}'_q, \tilde{I}''_q)$  with  $I'_q \in M_q(\{0, 1\})$  and  $I''_q \in M_{l, n-q}(\{0, 1\})$ . Assume that  $\text{ord}_p(c_i) \geq \text{ord}_p(b_j) + 2e_p + 1$  for  $i = 1, \dots, l, j = q + 1, \dots, n$ . Then  $\alpha_p(B, (A, C); I)$  does not depend on  $\tilde{I}''_q$ . In particular if  $q = 0$  we have*

$$\alpha_p(B, (A, C); I) = \alpha_p(B, A).$$

*Proof.* For each  $J = (r_{ij}) \in M_{st}(\{0, 1\})$  and  $Z = (z_{ij}) \in M_{st}(\mathbb{Z}_p)$ , we write  $Z \in p^J \mathbb{Z}_p$  if  $z_{ij} \in p^{r_{ij}} \mathbb{Z}_p$  for any  $1 \leq i \leq s, 1 \leq j \leq t$ . Further put  $B' = \text{diag}(b_1, \dots, b_q)$  and  $B'' = \text{diag}(b_{q+1}, \dots, b_n)$ . Then by Lemma 3.1 we have

$$\begin{aligned} a_c(B, (A, C); I) &= \sum_{Y \in M_{tn}(\mathbb{Z}_p)/p^e M_{tn}(\mathbb{Z}_p), Y \in p^I \mathbb{Z}_p} \# \mathcal{A}_c(B + C[Y], A) \\ &= \sum_{Y_2} \sum_{Y_1} \# \mathcal{A}_c(\text{diag}(B' + C[Y_1], B''), A) \\ &= p^{e(l(n-q) - w(I'_q))} \sum_{Y_1} \# \mathcal{A}_c(\text{diag}(B' + C[Y_1], B''), A), \end{aligned}$$

where  $Y_2$  (resp.  $Y_1$ ) runs over elements of  $M_{l-q, n}(\mathbb{Z}_p)/p^e M_{l-q, n}(\mathbb{Z}_p)$  (resp.  $M_{qn}(\mathbb{Z}_p)/p^e M_{qn}(\mathbb{Z}_p)$ ) such that  $Y_2 \in p^{I'_q} \mathbb{Z}_p$  (resp.  $Y_1 \in p^{I''_q} \mathbb{Z}_p$ ). Thus  $p^{e(-(m+l)n + n(n+1)/2) + w(I)} a_c(B, (A, C); I)$  does not depend on  $I''_q$  and therefore nor does  $\alpha_p(B, (A, C); I)$ . Further if  $q = 0$  we have

$$\begin{aligned} p^{e(-(m+l)n + n(n+1)/2) + w(I)} a_c(B, (A, C); I) \\ = p^{e(-mn + n(n+1)/2)} \# \mathcal{A}_c(B, A). \end{aligned}$$

Thus the second assertion holds.  $\blacksquare$

Now for each integers  $i, j, k$  such that  $0 \leq k \leq i$ , put

$$\gamma(i, j, k) = (-1)^k \sum_{0 \leq i_1 < \dots < i_k \leq i-1} p^{(i-i_1)(j+i_1)} \dots p^{(i-i_k)(j+i_k)}.$$

Here we understand that we have  $\gamma(i, j, 0) = 1$ . We remark that  $\gamma(i, j, k)$  is the  $k$ -th coefficient of the polynomial  $\prod_{k=0}^{i-1} (1 - p^{(i-k)(j+k)} x)$  in  $x$ , that is,

$$\prod_{k=0}^{i-1} (1 - p^{(i-k)(j+k)} x) = \sum_{k=0}^i \gamma(i, j, k) x^k.$$

Then our main result is the following, which we can prove by induction using Propositions 2.5 and 3.2.

THEOREM 3.3. *Let  $t, n, m, l$  be non-negative integers such that  $m + l \geq n \geq 1$  and  $n \geq t$ . Let  $B_1 = \text{diag}(b_1, \dots, b_t)$ ,  $B_2 = \text{diag}(b_{t+1}, \dots, b_n)$  with  $b_1, \dots, b_n \in \mathbb{Z}_p \setminus \{0\}$  and  $C = \text{diag}(c_1, \dots, c_l)$  with  $c_1, \dots, c_l \in \mathbb{Z}_p \setminus \{0\}$ . Let  $I \in M_{t, n}(\{0, 1\})$ . Let  $e$  be an integer such that  $e \geq \text{ord}_p(b_j) - \text{ord}_p(b_k) + 2m_0 + 2 + 2e_p$  and  $e \geq \text{ord}_p(b_j) - \text{ord}_p(c_k) + 2m_0 + 3 + 2e_p$  for  $j = t + 1, \dots, n, k = 1, \dots, t, k' = 1, \dots, l$ . Then we have*

$$\sum_{k=0}^{m_0+1} \gamma(t, -m + n + 1, k) \alpha_p(\text{diag}(p^{e-2k} B_1, B_2), (A, p^{e-2k} C); I)$$

$$= \prod_{i=0}^{m_0} \frac{1 - p^{(-m+n+i+1)(n-i)}}{1 - p^{-m+n+i+1}} \alpha_p(B_2, A^{(m_0+1)}) \beta_p(O_{m_0+1}, A).$$

As a special case of the above theorem we have

**COROLLARY.** *Let  $m, n$  and  $t$  be positive integers such that  $m \geq n \geq t$ . Let  $A$  be an even unimodular matrix of degree  $m$  and of Witt index  $r$ . Let  $B_1 = \text{diag}(b_1, \dots, b_t)$  and  $B_2 = \text{diag}(b_{t+1}, \dots, b_n)$  with  $b_i \in \mathbb{Z}_p \setminus \{0\}$ . Put  $e_p = 1$  or  $0$  according as  $p = 2$  or not, and  $m_0 = \min(t-1, r)$ . Further let  $e$  be an integer such that  $e \geq \text{ord}_p(b_j) - \text{ord}_p(b_k) + 2m_0 + 2e_p + 2$  for  $j = t+1, \dots, n, k = 1, \dots, t$ . Then we have*

$$\begin{aligned} & \alpha_p(\text{diag}(p^e B_1, B_2), A) \\ &= - \sum_{i=1}^{m_0+1} \gamma(t, -m+n+1, i) \alpha_p(\text{diag}(p^{e-2i} B_1, B_2), A) \\ &+ \prod_{i=0}^{m_0} \frac{1 - p^{(t-i)(-m+n+i+1)}}{1 - p^{-m+n+i+1}} \beta_p(O_{m_0+1}, A) \alpha_p(B_2, A^{(m_0+1)}), \end{aligned}$$

where  $O_{m_0+1}$  is the zero matrix of degree  $m_0 + 1$ . Here we make the convention that the second term on the right-hand side is 0 if  $m_0 = r$ , and that we have  $\alpha_p(B_2, A^{(m_0+1)}) = 1$  if  $n = t$ .

**REMARK 1.** The above corollary has been proved in [KH] under more general setting. However, in that paper, we have not deal with the local densities with congruence conditions, that is we have not proved Theorem 3.3. Thus our result in this paper is new in this sense.

Now for non-degenerate matrices  $A, B_1, \dots, B_{s-1}$  and  $B_s$  of degree  $m, n_1, \dots, n_{s-1}$  and  $n_s$ , respectively, with entries in  $\mathbb{Z}_p$ , define a formal power series  $R((B_1, \dots, B_s), A; x_1, \dots, x_s)$  by

$$\begin{aligned} & R((B_1, \dots, B_s), A; x_1, \dots, x_s) \\ &= \sum_{e_1 \geq \dots \geq e_s \geq 1} \alpha_p(\text{diag}(p^{e_1} B_1, \dots, p^{e_s} B_s), A) x_1^{e_1} \dots x_s^{e_s}. \end{aligned}$$

Let  $\Delta_p$  be the complete set of representatives of  $\mathbb{Z}_p^*/\mathbb{Z}_p^{*2}$ , and put

$$\Lambda_{n,p} = \{(b_1, \dots, b_n); b_i \in \Delta_p\}$$

or

$$\begin{aligned} &= \{(B_1, \dots, B_s); \deg B_1 + \dots + \deg B_s = n, \\ & B_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \text{ or } \deg B_i = 1 \text{ and } B_i \in \Delta_p\} \end{aligned}$$

according as  $p \neq 2$  or  $= 2$ . Then the set of power series  $\{R((B_1, \dots, B_s), A; x_1, \dots, x_s)\}_{(B_1, \dots, B_s) \in \Lambda_{n,p}}$  gives complete information on the local densities  $\alpha_p(B, A)$  for all  $B$  of degree  $n$  and  $A$ . So it is important to study these power series. The following is a direct consequence of Corollary to Theorem 3.3:

**THEOREM 3.4.** *Assume that  $A$  is even unimodular, and put  $B_i = \text{diag}(b_{n_1+\dots+n_{i-1}+1}, \dots, b_{n_1+\dots+n_i})$  ( $i = 1, \dots, s$ ) with  $b_j \in \mathbb{Z}_p \setminus \{0\}$ . Put  $m_k = \min(n_1 + \dots + n_k - 1, r)$ . Further put  $l_0 = m_1 + e_p$  or  $= m_1 + e_p + \max_{j'=n_1+1, \dots, n} \text{ord}_p(b_{j'}) - \min_{j=1, \dots, n_1} \text{ord}_p(b_j)$  according as  $\max_{j'=n_1+1, \dots, n} \text{ord}_p(b_{j'}) - \min_{j=1, \dots, n_1} \text{ord}_p(b_j) \leq 0$  or not. Then we have*

$$\prod_{i=0}^{m_1} (1 - p^{(n_1-i)(-m+n+i+1)} x_1^2) R((B_1, \dots, B_s), A; x_1, \dots, x_s)$$

$$\begin{aligned} &= \sum_{i=0}^{l_0} x_1^{2i} \sum_{j=0}^i \gamma(n_1, -m+n+1, i-j) \\ &\times R((\text{diag}(p^{2j} B_1, B_2), B_3, \dots, B_s), A; x_1 x_2, x_3, \dots, x_s) \\ &+ \sum_{i=0}^{l_0} x_1^{2i+1} \sum_{j=0}^i \gamma(n_1, -m+n+1, i-j) \\ &\times R((\text{diag}(p^{2j+1} B_1, B_2), B_3, \dots, B_s), A; x_1 x_2, x_3, \dots, x_s) \\ &+ \prod_{i=0}^{m_1} \frac{1 - p^{(n_1-i)(-m+n+i+1)}}{1 - p^{-m+n+i+1}} \beta_p(O_{m_1+1}, A) \frac{x_1^{2l_0+2}}{1-x_1} \\ &\times R((B_2, B_3, \dots, B_s), A^{(m_1+1)}; x_1 x_2, x_3, \dots, x_s). \end{aligned}$$

Here we make the convention that the third term on the right-hand side of the above is 0 if  $r = m_1$ , and that we have  $R((\text{diag}(p^k B_1, B_2), B_3, \dots, B_s), A; x_1 x_2, x_3, \dots, x_s) = \alpha_p(p^k B_1, A)$  for  $k = 1, \dots, 2l_0 + 1$  and  $R((B_2, B_3, \dots, B_s), A^{(m_1+1)}; x_1 x_2, x_3, \dots, x_s) = 1$  if  $s = 1$ .

We remark that if  $s = 1$  and  $n_1 = n$ , our power series coincides with the one defined by Kitaoka [Ki4] and by the above theorem its denominator is

$$\prod_{i=0}^{\min(n-1, r)} (1 - p^{(n-i)(-m+n+i+1)} x^2) (1-x)^{m'},$$

where  $m' = 1$  or  $= 0$  according as  $r \geq n$  or not. This is a certain refinement of the result of [Hir], [Ki4]. Further Theorem 1.2 shows that if  $s \geq 2$ , the series  $R((B_1, \dots, B_s), A; x_1, \dots, x_s) (1-x_1) \prod_{i=0}^{m_1} (1 - p^{(n_1-i)(-m+n+i+1)} x_1^2)$  can be expressed as a  $\mathbb{Q}[x_i]$ -linear combination of the power series in  $s-1$  variables. Using this, by induction, we have

**THEOREM 3.5.** *Let the notation and the assumption be as in Theorem 3.4. Then  $R((B_1, \dots, B_s), A; x_1, \dots, x_s)$  is a rational function of  $x_1, \dots, x_s$  with denominator*

$$\begin{aligned} & \prod_{k=1}^s \prod_{i=0}^{m_k} (1 - p^{(n_1+\dots+n_k-i)(-m+n+i+1)} (x_1 \dots x_k)^2) \\ & \times \prod_{k=1}^s (1 - x_1 \dots x_k)^{m'_k}, \end{aligned}$$

where  $m'_k = 1$  or  $= 0$  according as  $r \geq n_1 + \dots + n_k$  or not. In particular if  $m \geq 2n + 2$ , the denominator of the above power series is

$$\begin{aligned} & \prod_{k=1}^s \prod_{i=0}^{n_1+\dots+n_k-1} (1 - p^{(n_1+\dots+n_k-i)(-m+n+i+1)} (x_1 \dots x_k)^2) \\ & \prod_{k=1}^s (1 - x_1 \dots x_k). \end{aligned}$$

**REMARK 2.** In [Ka3], we have proved the rationality of the power series defined by

$$\begin{aligned} & Q((B_1, \dots, B_s), A; x_1, \dots, x_s) \\ &= \sum_{e_1, \dots, e_s \geq 1} \alpha_p(\text{diag}(p^{e_1} B_1, \dots, p^{e_s} B_s), A) x_1^{e_1} \dots x_s^{e_s} \end{aligned}$$

for arbitrary  $A, B$  and  $p$ . Further, in [Ka2], we have given an explicit form of the denominator of it when  $p \neq 2$ . To do this, we needed a recursion formula similar to Proposition 2.1 for

an arbitrary matrix  $A$  of level  $p$  (cf. [Ka2 Proposition 3.6 (1),(2)]). This type of formula seems difficult to be generalized to the case where  $p = 2$  without any change. So the method of [Ka2] giving an explicit form of the denominator cannot be applied to  $p = 2$  directly.

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合同条件を持つ二次形式の局所密度の漸化式

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## 概要

この論文では合同条件を持った局所密度の間の関係式を考察している。この応用として我々は、(*J. Number Theory*, 64, 1997, 183-210)における局所密度の漸化式の別証明を特別な場合に与える。さらに、(*Proc. Japan Acad.*, 70, 1994, 208-211)における形式べき級数のある一般化を定義して、その分母の形を正確に求める。

キーワード： 局所密度, 漸化式

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