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The units of a partial Burnside ring relative to the Young subgroups of a symmetric group

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Abstract

The unit group of a partial Burnside ring relative to the Young subgroups of the symmetric group S_n on n letters is included in the image by the tom Dieck homomorphism. As a consequence of this fact, the alternating character ν_n of S_n is expressed explicitly as a \mathbb{Z} -linear combinations of permutation characters associated with finite left S_n -sets S_n/Y for the Young subgroups Y .

1 Introduction

Let G be a finite group, and let $\text{Cl}(G)$ be a full set of non-conjugate subgroups of G . For each $H \leq G$, G/H denotes the set of left cosets gH , $g \in G$, of H in G . The Burnside ring $\Omega(G)$ of G is the commutative ring consisting of all formal \mathbb{Z} -linear combinations of symbols $[G/H]$ corresponding to G/H , $H \in \text{Cl}(G)$, with multiplication given by

$$[G/H] \cdot [G/U] = \sum_{HgU \in H \backslash G/U} [G/(H \cap {}^gU)]$$

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for all $H, U \in \text{Cl}(G)$, where ${}^gU = gUg^{-1}$ and $[G/(H \cap {}^gU)] = [G/K]$ for a conjugate $K \in \text{Cl}(G)$ of $H \cap {}^gU$ (see, *e.g.*, [5], [13, §2.1]). The identity of $\Omega(G)$ is $[G/G]$. For shortness' sake, we usually write $1 = [G/G]$. While the unit group of $\Omega(G)$ is an elementary abelian 2-group (cf. §2), it is quite interesting to analyze units of $\Omega(G)$.

Let S_n be the symmetric group on n letters $[n] := \{1, 2, \dots, n\}$, and let \mathfrak{Y}_n be the set of Young subgroups of S_n . Since the Young subgroups are closed under intersection and conjugation, there exists a subring of $\Omega(S_n)$ consisting of all formal \mathbb{Z} -linear combinations of symbols $[S_n/Y]$ for $Y \in \mathfrak{Y}_n$, which is denoted by $\Omega(S_n, \mathfrak{Y}_n)$ (see also [2]). (In short, this subring is a partial Burnside ring relative to the Young subgroups of a symmetric group.) Let $R(S_n)$ be the character ring of S_n . Then it is well-known that $\Omega(S_n, \mathfrak{Y}_n) \cong R(S_n)$ (see, *e.g.*, [13, Proposition 7.2]) and the unit group of $R(S_n)$ consists of $\pm 1_{S_n}, \pm \nu_n$, where 1_{S_n} is the trivial character of S_n and ν_n is the alternating character of S_n (see, *e.g.*, [11, Example 2]). In particular, the unit group of $\Omega(S_n, \mathfrak{Y}_n)$ is isomorphic to the Klein four-group.

Recently, in [7], Idei and the first author have given a formula of a non-identity unit of $\Omega(S_n, \mathfrak{Y}_n)$ which is described in terms of the Möbius function $\mu_{\mathfrak{Y}_n}$ on the poset (\mathfrak{Y}_n, \leq) (see Eq. (2)). Such a unit is also a unit of $\Omega(S_n)$, and there seems to be some characterization of it as a unit of $\Omega(S_n)$. In general, however, there are many units of $\Omega(S_n)$ (see [3]). The purpose of this paper is to characterize a non-identity unit of $\Omega(S_n, \mathfrak{Y}_n)$ in terms of the tom Dieck homomorphism (see §2). Consequently, we have shown that the unit group of $\Omega(S_n, \mathfrak{Y}_n)$ is included in the image by the tom Dieck homomorphism. In the sequel, ν_n is expressed explicitly as a \mathbb{Z} -linear combinations of permutation characters associated with finite left S_n -sets S_n/Y for $Y \in \mathfrak{Y}_n$ (cf. Theorem 4.4), which is also a consequence of [6, Proposition 2.3.8, Exercise 3.15]. In order to show such a result, we see that each image of a permutation character associated with a finite left G -set by the tom Dieck homomorphism is a reduced Lefschetz invariant of a certain G -poset, which is essentially given in [10].

2 The tom Dieck homomorphism

Given $H \leq G$ and a finite left G -set X , we set

$$\text{inv}_H(X) = \{x \in X \mid hx = x \text{ for all } h \in H\}.$$

By [5, Proposition 1.2.2], the map $\varphi : \Omega(G) \rightarrow \tilde{\Omega}(G) := \prod_{H \in \text{Cl}(G)} \mathbb{Z}$ given by

$$[G/U] \mapsto (\#\text{inv}_H(G/U))_{H \in \text{Cl}(G)}$$

for all $U \in \text{Cl}(G)$ is an injective ring homomorphism, which is called the Burnside homomorphism or the mark homomorphism. Obviously, the unit group of $\tilde{\Omega}(G)$ is $\prod_{H \in \text{Cl}(G)} \langle -1 \rangle$, whence the unit group of $\Omega(G)$ is an elementary abelian 2-group.

We denote by $R_{\mathbb{R}}(G)$ the real representation ring of G , and denote by $\Omega(G)^{\times}$ the unit group of $\Omega(G)$. For each element x of $\Omega(G)$ with $\varphi(x) = (x_H)_{H \in \text{Cl}(G)}$,

we write $x = \varphi^{-1}((x_H)_{H \in \text{Cl}(G)})$. By [5, Proposition 5.5.9], there exists a group homomorphism $u = u_G : R_{\mathbb{R}}(G) \rightarrow \Omega(G)^\times$ such that

$$M \mapsto \varphi^{-1}(((-1)^{\dim M^H})_{H \in \text{Cl}(G)})$$

for all $\mathbb{R}G$ -module M , where M^H is the space of H -invariants of M .

Let $H \leq G$. We set $W_G(H) = N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G . Assume that a finitely generated left $\mathbb{C}G$ -module M affords a \mathbb{C} -character χ of G . For each $H \leq G$, M^H is viewed as a $\mathbb{C}W_G(H)$ -module, which affords the \mathbb{C} -character $\bar{\chi}$ of $W_G(H)$ given by

$$\bar{\chi}(gH) = \frac{1}{|H|} \sum_{h \in H} \chi(gh)$$

for all $gH \in W_G(H)$ (see, e.g., [1, Lemma 3.1]). In particular, $\dim M^H$ is equal to the inner product $\langle \chi|_H, 1_H \rangle_H$ of the \mathbb{C} -character $\chi|_H$ of H and the trivial character 1_H of H , where $\chi|_H$ is the restriction of χ into H .

Let $\bar{R}_{\mathbb{R}}(G)$ be the ring of real valued virtual \mathbb{C} -characters of G . Then it follows from the preceding argument and [12, Theorem A] that $u : R_{\mathbb{R}}(G) \rightarrow \Omega(G)^\times$ is extended to the group homomorphism $\bar{u} = \bar{u}_G : \bar{R}_{\mathbb{R}}(G) \rightarrow \Omega(G)^\times$ given by

$$\chi \mapsto \varphi^{-1}(((-1)^{\langle \chi|_H, 1_H \rangle_H})_{H \in \text{Cl}(G)})$$

for all $\chi \in \bar{R}_{\mathbb{R}}(G)$, which is called the tom Dieck homomorphism. According to [13, Corollary 4.3], we have

$$\bar{u}(\chi) = \sum_{U \in \text{Cl}(G)} \frac{1}{|W_G(U)|} \left(\sum_{H \leq G} \mu(U, H) (-1)^{\langle \chi|_H, 1_H \rangle_H} \right) [G/U] \quad (1)$$

for all $\chi \in \bar{R}_{\mathbb{R}}(G)$, where μ is the Möbius function on the poset $(\mathfrak{S}(G), \leq)$ of all subgroups of G .

Example 2.1 Obviously, $\bar{u}(1_G) = -1$.

Example 2.2 Let A_n be the alternating group on $[n]$. Then $1 - [S_n/A_n]$ is a unit of $\Omega(S_n)$ and is the image of ν_n by the tom Dieck homomorphism.

Remark 2.3 If G is not solvable, then by [9, Theorem 5.4], $u : R_{\mathbb{R}}(G) \rightarrow \Omega(G)^\times$ is not surjective. In particular, if $n \geq 4$, then $\bar{u} : \bar{R}_{\mathbb{R}}(S_n) \rightarrow \Omega(S_n)^\times$ is not surjective; \bar{u}_{S_2} and \bar{u}_{S_3} are surjective, however (see [9]). (Note that $2|\text{Im } \bar{u}_{S_4}| = |\Omega(S_4)^\times| = 2^6$.)

3 The units of $\Omega(S_n, \mathfrak{Y}_n)$

We denote by $R(G)$ the character ring of G . The permutation character π_X associated with a finite left G -set X is given by

$$\pi_X(g) = \#\{x \in X \mid gx = x\}$$

for all $g \in G$. We define a ring homomorphism $\text{char}_G : \Omega(G) \rightarrow R(G)$ by

$$[X] \mapsto \pi_X$$

for all finite left G -sets X (cf. [12, §6]).

By [13, Proposition 7.2], the ring homomorphism $\text{char}_{S_n} : \Omega(S_n) \rightarrow R(S_n)$ induces an isomorphism

$$\overline{\text{char}_{S_n}} : \Omega(S_n, \mathfrak{Y}_n) \rightarrow R(S_n)$$

(see also [8, 2.3]). Hence there exists a unique unit, say α , of $\Omega(S_n, \mathfrak{Y}_n)$ satisfying $\overline{\text{char}_{S_n}}(\alpha) = \nu_n$. The unit group of $\Omega(S_n, \mathfrak{Y}_n)$ consists of $\pm 1, \pm \alpha$, which are also units of $\Omega(S_n)$.

For each $H \leq S_n$, we define a Young subgroup Y_H to be the intersection of all Young subgroups containing H . Each Young subgroup Y of S_n with respect to a partition (n_1, n_2, \dots, n_r) includes a product σ_Y of pairwise disjoint n_i -cycles for $i = 1, 2, \dots, r$ satisfying $Y = Y_{\langle \sigma_Y \rangle}$. Under these notations, we are now in a position to state the following lemma (cf. [13, §7.1]).

Lemma 3.1 *If $\varphi(\alpha) = (\alpha_H)_{H \in \text{Cl}(S_n)}$, then $\alpha_H = \alpha_{Y'_H}$ for all $H \in \text{Cl}(S_n)$, where $Y'_H \in \text{Cl}(S_n)$ is a conjugate of Y_H , and $\alpha_Y = \nu_n(\sigma_Y)$ for all $Y \in \text{Cl}(S_n) \cap \mathfrak{Y}_n$.*

Proof. Suppose that $\alpha = \sum_{j=1}^s a_j [S_n/Y_j]$ with $a_j \in \mathbb{Z}$ and $Y_j \in \text{Cl}(S_n) \cap \mathfrak{Y}_n$. If $H \leq S_n$, then by the definition of Y_H ,

$$\text{inv}_{Y_H}(S_n/Y_j) = \{\sigma Y_j \mid Y_H \leq \sigma Y_j\} = \{\sigma Y_j \mid H \leq \sigma Y_j\} = \text{inv}_H(S_n/Y_j)$$

for $j = 1, 2, \dots, s$. Hence it turns out that $\alpha_H = \alpha_{Y'_H}$ for all $H \in \text{Cl}(S_n)$. If $Y \in \mathfrak{Y}_n$, then by assumption, $Y = Y_{\langle \sigma_Y \rangle}$, whence $\text{inv}_Y(S_n/Y_j) = \text{inv}_{\langle \sigma_Y \rangle}(S_n/Y_j)$ for $j = 1, 2, \dots, s$. Since $\overline{\text{char}_{S_n}}(\alpha) = \nu_n$, we conclude that for each $Y \in \text{Cl}(S_n) \cap \mathfrak{Y}_n$,

$$\nu_n(\sigma_Y) = \sum_{j=1}^s a_j \pi_{[S_n/Y_j]}(\sigma_Y) = \sum_{j=1}^s a_j \#\text{inv}_{\langle \sigma_Y \rangle}(S_n/Y_j) = \sum_{j=1}^s a_j \#\text{inv}_Y(S_n/Y_j) = \alpha_Y.$$

This completes the proof. \square

By using [13, Corollary 4.3] and Lemma 3.1, α is expressed in the form

$$\alpha = \sum_{Y \in \text{Cl}(S_n) \cap \mathfrak{Y}_n} \frac{1}{|W_{S_n}(Y)|} \left(\sum_{H \in \mathfrak{Y}_n} \mu_{\mathfrak{Y}_n}(Y, H) \nu_n(\sigma_H) \right) [S_n/Y], \quad (2)$$

where $\sigma_H \in S_n$ with $H = Y_{\langle \sigma_H \rangle}$. This formula is presented in [7, Corollary 5.2].

We aim to show that α is included in the image by the tom Dieck homomorphism. The permutation character $\pi_{[n]}$ associated with the S_n -set $[n]$ is given by

$$\sigma \mapsto \#\{k \in [n] \mid \sigma(k) = k\}$$

for all $\sigma \in S_n$. For each $H \leq S_n$, let $\text{Orb}_H([n])$ be the set of H -orbits in $[n]$. By the Cauchy-Frobenius lemma (see, *e.g.*, [13, Lemma 2.7]), $\langle \pi_{[n]}|_H, 1_H \rangle_H = \#\text{Orb}_H([n])$ for all $H \leq S_n$. Set $\chi_n = \pi_{[n]} - 1_{S_n}$. Then it is easily verified that χ_n is an irreducible \mathbb{C} -character of S_n . Obviously, $\pi_{[n]} \in \overline{R}_{\mathbb{R}}(S_n)$. We define a unit β of $\Omega(S_n)$ by

$$\begin{aligned} \varphi(\beta) &= ((-1)^{\#\text{Orb}_H([n])})_{H \in \text{Cl}(S_n)} = ((-1)^{\langle \pi_{[n]}|_H, 1_H \rangle_H})_{H \in \text{Cl}(S_n)} \\ &= -((-1)^{\langle \chi_n|_H, 1_H \rangle_H})_{H \in \text{Cl}(S_n)}, \end{aligned}$$

so that β is the image of $\pi_{[n]}$ by the tom Dieck homomorphism. The fact that $\alpha = (-1)^n \beta$ (cf. Theorem 3.4) is obtained by a combination of Lemma 3.1 and the following lemmas.

Lemma 3.2 *For each $H \leq S_n$, $\#\text{Orb}_H([n]) = \#\text{Orb}_{Y_H}([n])$. In particular, for each $Y \in \mathfrak{Y}_n$, $\#\text{Orb}_Y([n]) = \#\text{Orb}_{\langle \sigma_Y \rangle}([n])$, where $\sigma_Y \in S_n$ with $Y = Y_{\langle \sigma_Y \rangle}$.*

Proof. Evidently, the H -orbits in $[n]$ coincide with the Y_H -orbits in $[n]$, completing the proof. \square

Lemma 3.3 *For each $\sigma \in S_n$, $(-1)^{\#\text{Orb}_{\langle \sigma \rangle}([n])} = (-1)^n \nu_n(\sigma)$.*

Proof. Assume that σ is a product of pairwise disjoint n_i -cycles for $i = 1, 2, \dots, r$ such that $\sum_i n_i = n$. Then it is obvious that $\#\text{Orb}_{\langle \sigma \rangle}([n]) = r$. On the other hand, if $\ell = \#\{i \mid n_i \text{ is odd}\}$, then $\nu_n(\sigma) = (-1)^{r-\ell} = (-1)^{r+n}$, because $\ell \equiv n \pmod{2}$, completing the proof. \square

We are now successful in characterizing the units of $\Omega(S_n, \mathfrak{Y}_n)$ in terms of the tom Dieck homomorphism.

Theorem 3.4 *The unit group of $\Omega(S_n, \mathfrak{Y}_n)$ is included in the image by the tom Dieck homomorphism. In particular, $\alpha = (-1)^n \beta$.*

Proof. Since $-1 = \overline{u}(1_{S_n})$, it suffices to verify that $\alpha = (-1)^n \beta$. Suppose now that $\varphi(\alpha) = (\alpha_H)_{H \in \text{Cl}(S_n)}$ and $\varphi(\beta) = (\beta_H)_{H \in \text{Cl}(S_n)}$. If $\alpha_Y = (-1)^n \beta_Y$ for all $Y \in \text{Cl}(S_n) \cap \mathfrak{Y}_n$, then by Lemmas 3.1 and 3.2, $\alpha_H = (-1)^n \beta_H$ for all $H \in \text{Cl}(S_n)$, whence $\alpha = (-1)^n \beta$. Now let $Y \in \text{Cl}(S_n) \cap \mathfrak{Y}_n$. Then by virtue of Lemma 3.1, $\alpha_Y = \nu_n(\sigma_Y)$, where $\sigma_Y \in S_n$ with $Y = Y_{\langle \sigma_Y \rangle}$. Hence it follows from Lemmas 3.2 and 3.3 that

$$\alpha_Y = (-1)^{\#\text{Orb}_{\langle \sigma_Y \rangle}([n])+n} = (-1)^{\#\text{Orb}_Y([n])+n} = (-1)^n \beta_Y.$$

We have thus completed the proof. \square

4 The reduced Lefschetz invariant of a G -poset

There is a valuable application of Theorem 3.4. The expression of ν_n as a \mathbb{Z} -linear combinations of permutation characters $\pi_{S_n/Y}$ for $Y \in \mathfrak{Y}_n$ is implicit in Eq. (2), while it is worth studying the explicit descriptions.

A finite left G -set equipped with order relation \leq is called a G -poset if \leq is invariant under the action of G . Let P be a G -poset, and let $Sd_i(P)$ be the set of chains $x_0 < x_1 < \cdots < x_i$ of elements of P of cardinality $i + 1$. Recall that $\Omega(G)$ is the Grothendieck group of the category of finite left G -sets and is an abelian group generated by the isomorphism classes $[X]$ of finite left G -sets X (cf. [5, 13]). The Lefschetz invariant Λ_P of P is defined by

$$\Lambda_P = \sum_{i=0}^{\infty} (-1)^i [Sd_i(P)] \in \Omega(G),$$

and the reduced Lefschetz invariant $\tilde{\Lambda}_P$ of P is defined by $\tilde{\Lambda}_P = \Lambda_P - 1$ (cf. [4, 10]). In particular, for the poset $P(X)$ consisting of nonempty and proper subsets of a finite left G -set X , the K -component of $\varphi(\Lambda_{P(X)})$ with $K \in \text{Cl}(G)$ is equal to the reduced Euler-Poincaré characteristic of $P(X)^K (= \text{inv}_K(P(X)))$:

$$\sum_{i=0}^{\infty} (-1)^i |Sd_i(P(X)^K)| - 1.$$

We next give a combinatorial proof of the following proposition, which is essentially proved by [10, Proposition 5.1].

Proposition 4.1 *Let X be a finite left G -set. The reduced Lefschetz invariant $\tilde{\Lambda}_{P(X)}$ of $P(X)$ is the image of π_X by the tom Dieck homomorphism.*

To prove Proposition 4.1, we require the following combinatorial lemma.

Lemma 4.2 *For each positive integer j , set*

$$c_j = \sum_{i=1}^j (-1)^i \sum_{(n_1, n_2, \dots, n_i) \in A(i, j)} \binom{j}{n_1, n_2, \dots, n_i},$$

where $A(i, j) = \{(n_1, n_2, \dots, n_i) \mid \sum_k n_k = j \text{ and } n_1, n_2, \dots, n_i \in \mathbb{N}\}$ and

$$\binom{j}{n_1, n_2, \dots, n_i} = \frac{j!}{n_1! n_2! \cdots n_i!} \quad (\text{multinomial coefficients}).$$

Then $c_j = (-1)^j$ for any positive integers j .

Proof. We use induction on j . Obviously, $c_1 = -1$. Assume that $j \geq 2$ and

$$c_\ell = \sum_{i=1}^{\ell} (-1)^i \sum_{(n_1, n_2, \dots, n_i) \in A(i, \ell)} \binom{\ell}{n_1, n_2, \dots, n_i} = (-1)^\ell$$

for any positive integer ℓ less than j . Clearly, there exists a bijection

$$\bigcup_{\ell=i, \dots, j-1} A(i, \ell) \rightarrow A(i+1, j), (n_1, n_2, \dots, n_i) \mapsto (n_1, n_2, \dots, n_i, j-\ell)$$

for each positive integer i less than j . Hence the inductive assumption yields

$$\begin{aligned} c_j &= -\binom{j}{j} + \sum_{i=2}^j (-1)^i \sum_{\ell=i-1}^{j-1} \sum_{(n_1, n_2, \dots, n_{i-1}) \in A(i-1, \ell)} \frac{j!}{n_1! n_2! \cdots n_{i-1}! (j-\ell)!} \\ &= -\binom{j}{j} - \sum_{\ell=1}^{j-1} \binom{j}{\ell, j-\ell} c_\ell \\ &= -1 - (1-1)^j + 1 + (-1)^j = (-1)^j, \end{aligned}$$

as desired. This completes the proof. \square

Proof of Proposition 4.1. For each $K \leq G$, we denote by m_K the number of K -orbits in X . Then it follows from the Cauchy-Frobenius lemma that $m_K = \langle \pi_X|_K, \mathbf{1}_K \rangle_K$. Hence the assertion is equivalent to the equality $\varphi(\tilde{\Lambda}_{P(X)}) = ((-1)^{m_K})_{K \in \text{Cl}(G)}$. Let $K \in \text{Cl}(G)$. Every chain $x_0 < x_1 < \cdots < x_i$ of elements of $P(X)^K$ is built on pairwise disjoint unions y_j of ℓ_j K -orbits for $j = 0, 1, \dots, i$ such that $\sum_j \ell_j < m_K$ and $x_k = \dot{\cup}_{j=0}^k y_j$ for $k = 0, 1, \dots, i$. Hence a simple observation enables us to get

$$|Sd_i(P(X)^K)| = \sum_{(m_1, m_2, \dots, m_{i+2}) \in A(i+2, m_K)} \binom{m_K}{m_1, m_2, \dots, m_{i+2}}$$

for each integer i with $0 \leq i \leq m_K - 2$. This, together with Lemma 4.2, shows that

$$\begin{aligned} \sum_{i=0}^{\infty} (-1)^i |Sd_i(P(X)^K)| &= \sum_{i=2}^{m_K} (-1)^i \sum_{(m_1, m_2, \dots, m_i) \in A(i, m_K)} \binom{m_K}{m_1, m_2, \dots, m_i} \\ &= (-1)^{m_K} + 1. \end{aligned}$$

Thus the reduced Euler-Poincaré characteristic of $P(X)^K$ is $(-1)^{m_K}$ (see also [10, Proposition 5.1]). Consequently, we have $\varphi(\tilde{\Lambda}_{P(X)}) = ((-1)^{m_K})_{K \in \text{Cl}(G)}$. This completes the proof. \square

We turn to the study of an explicit description of β (cf. Eq. (3)).

Corollary 4.3 *The reduced Lefschetz invariant $\tilde{\Lambda}_{P([n])}$ of $P([n])$ coincides with β .*

Proof. By definition, β is the image of $\pi_{[n]}$ by the tom Dieck homomorphism. Hence the assertion is an immediate consequence of Proposition 4.1 with $G = S_n$ and $X = [n]$. \square

Given $H \leq G$, we denote by 1_H^G the \mathbb{C} -character of G induced from 1_H , which coincides with $\pi_{G/H}$.

For each cycle type $\lambda = (1^{m_1}, \dots, n^{m_n})$ of a permutation on $[n]$, let S_λ be a Young subgroup of S_n isomorphic to $S_1^{(m_1)} \times \dots \times S_n^{(m_n)}$, where each $S_j^{(m_j)}$ is the direct product of m_j copies of S_j .

We are now ready to express ν_n explicitly as a \mathbb{Z} -linear combinations of $1_Y^{S_n}$ for $Y \in \mathfrak{Y}_n$. The following result simplifies the expression of ν_n in [8, Theorem 2.3.15], and is also a consequence of [6, Proposition 2.3.8, Exercise 3.15].

Theorem 4.4 *The alternating character ν_n of S_n is expressed explicitly in the form*

$$\nu_n = \sum_{\lambda=(1^{m_1}, \dots, n^{m_n})} (-1)^{n+m_1+\dots+m_n} \frac{(m_1 + \dots + m_n)!}{m_1! \dots m_n!} 1_{S_\lambda}^{S_n},$$

where the sum runs over all cycle types of permutations on $[n]$.

Proof. Since $\beta = (-1)^n \alpha$ by Theorem 3.4 and $\nu_n = \overline{\text{char}}_{S_n}(\alpha)$, the assertion is equivalent to the formula

$$((-1)^n \alpha =) \beta = \sum_{\lambda=(1^{m_1}, \dots, n^{m_n})} (-1)^{m_1+\dots+m_n} \frac{(m_1 + \dots + m_n)!}{m_1! \dots m_n!} [S_n/S_\lambda]. \quad (3)$$

Let i be a nonnegative integer. Then $[Sd_i(P([n]))] = \sum_t [O_t]$, where O_t 's are the S_n -orbits in $Sd_i(P([n]))$. Observe that each representative $x_0 < x_1 < \dots < x_i$ of an S_n -orbit O_t determines $\{y_0, y_1, \dots, y_i, y_{i+1}\} \subset P([n])$ such that $x_k = \dot{\cup}_{j=0}^k y_j$ for $k = 0, 1, \dots, i$ and $[n] = \dot{\cup}_{j=0}^{i+1} y_j$, which corresponds to a cycle type $(1^{m_1}, \dots, n^{m_n})$ with $m_\ell = \#\{k \mid \#y_k = \ell\}$ for $\ell = 1, \dots, n$. Conversely, if $\lambda = (1^{m_1}, \dots, n^{m_n})$ with $\sum_\ell m_\ell = i + 2 \geq 2$ is a cycle type of a permutation on $[n]$, then we can make up

$$\frac{(m_1 + \dots + m_n)!}{m_1! \dots m_n!}$$

S_n -orbits in $Sd_i(P([n]))$, which are isomorphic to S_n/S_λ as S_n -sets. Hence Eq. (3) follows from Corollary 4.3. This completes the proof. \square

Remark 4.5 By the Cauchy-Frobenius lemma, $\langle 1_H^G, 1_G \rangle_G = \langle \pi_{G/H}, 1_G \rangle_G = 1$ for all $H \leq G$. Hence it follows from Theorem 4.4 that

$$\sum_{\lambda=(1^{m_1}, \dots, n^{m_n})} (-1)^{m_1+\dots+m_n} \frac{(m_1 + \dots + m_n)!}{m_1! \dots m_n!} = (-1)^n \langle \nu_n, 1_{S_n} \rangle_{S_n} = 0.$$

Example 4.6 By Eq. (3), the coefficients of $[S_n/U]$ in Eq. (1) with $G = S_n$ and $\chi = \pi_{[n]}$ are completely determined, and so are the coefficients of $[S_n/Y]$ in Eq. (2). If $n = 3$, then

$$\alpha = [S_3/S_{(1^3)}] - 2[S_3/S_{(1^1, 2^1)}] + [S_3/S_{(3^1)}]$$

(see also [8, pp. 41–42 and Theorem 2.3.15]). If $n = 4$, then

$$\alpha = [S_4/S_{(1^4)}] - 3[S_4/S_{(1^2, 2^1)}] + [S_4/S_{(2^2)}] + 2[S_4/S_{(1^1, 3^1)}] - [S_4/S_{(4^1)}].$$

5 Concluding remarks

If e is an idempotent of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(S_n, \mathfrak{Y}_n)$ and if $2e \in \Omega(S_n, \mathfrak{Y}_n)$, then $1 - 2e$ and $-1 + 2e$ are units of $\Omega(S_n, \mathfrak{Y}_n)$. Since α is a unit of $\Omega(S_n, \mathfrak{Y}_n)$, it follows that $(1 - \alpha)/2$ is an idempotent of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(S_n, \mathfrak{Y}_n)$. However, $(1 - \alpha)/2 \notin \Omega(S_n, \mathfrak{Y}_n)$, because $(1_{S_n} - \nu_n)/2 \notin R(S_n)$. Thus we have the following.

Proposition 5.1 *The idempotents of $\Omega(S_n, \mathfrak{Y}_n)$ are only 0, 1.*

What about *Proof* of Proposition 5.1 in terms of S_n -sets? Of course, in Eq. (3), the coefficient of $[S_n/S_{(1^n)}]$ is always $(-1)^n$, which implies that $(1 - \alpha)/2 \notin \Omega(S_n)$. This fact is also a consequence of Theorem 3.4, because it follows from [9, (5.4.1)] that, if e is a non-trivial idempotent, *i.e.*, $e \neq 0, 1$, of $\Omega(G)$, then the unit $1 - 2e$ is not included in the image by the tom Dieck homomorphism.

The following lemma is [12, Lemma 2.1] (see also [5, Proposition 1.3.5]), which is used for the proof of existence of the tom Dieck homomorphism in [12].

Lemma 5.2 *An element $(x_H)_{H \in \text{Cl}(G)}$ of $\widetilde{\Omega}(G)$ is included in the image $\text{Im } \varphi$ by the Burnside homomorphism if and only if*

$$\sum_{gU \in W_G(U)} x_{\langle g \rangle U} \equiv 0 \pmod{|W_G(U)|}$$

for all $U \in \text{Cl}(G)$, where $x_{\langle g \rangle U} = x_K$ for a conjugate $K \in \text{Cl}(G)$ of $\langle g \rangle U$.

We end this paper by giving a direct proof of the fact that $(1 - \alpha)/2 \notin \Omega(S_n)$.

Proof of Proposition 5.1. Suppose that $\varphi(\alpha) = (\alpha_H)_{H \in \text{Cl}(S_n)}$. Then it follows from Lemma 3.3 and Theorem 3.4 (or the definition of α) that

$$\sum_{\sigma \in S_n} \frac{1 - \alpha_{\langle \sigma \rangle}}{2} = \sum_{\sigma \in S_n} \frac{1 - \nu_n(\sigma)}{2} = |A_n| \not\equiv 0 \pmod{|S_n|},$$

where $\alpha_{\langle \sigma \rangle} = \alpha_{\langle \tau \rangle}$ if σ is a conjugate of $\tau \in S_n$ with $\langle \tau \rangle \in \text{Cl}(S_n)$. This, combined with Lemma 5.2, shows that $((1 - \alpha_H)/2)_{H \in \text{Cl}(S_n)} \notin \text{Im } \varphi$. Thus $(1 - \alpha)/2 \notin \Omega(S_n)$, completing the proof. \square

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