

# Envelopes of Legendre Curves in the Unit Tangent Bundle over the Euclidean Plane

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# Envelopes of Legendre curves in the unit tangent bundle over the Euclidean plane

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## Abstract

For singular plane curves, the classical definitions of envelopes are vague. In order to define envelopes for singular plane curves, we introduce a one-parameter family of Legendre curves in the unit tangent bundle over the Euclidean plane and the curvature. Then we give a definition of an envelope for the one-parameter family of Legendre curves. We investigate properties of the envelopes. For instance, the envelope is also a Legendre curve. Moreover, we consider bi-Legendre curves and give a relationship between envelopes.

## 1 Introduction

Envelopes are classical object in the differential geometry. There are many applications of envelopes to differential geometry, differential equations and physics, for instance [4, 5, 7, 9, 10, 15, 16, 18, 20]. An envelope of a family of curves in the plane is a curve that is "tangent" to each member of the family at some point. If the curves are regular, then the tangent is well-defined. However, the definitions of envelopes are vague for singular plane curves (smooth curves with singular points). In this paper, we would like to clarify the definition of the envelope for a family of singular curves. As singular curves, we consider Legendre curves in the unit tangent bundle over  $\mathbb{R}^2$ , see Appendix A (cf. [8]). The basic results on the singularity theory see [2, 4, 14, 17]. In §2, we quickly review on the definitions of envelopes which are given by implicit functions [3, 4, 12] and parametric curves [11, 19]. In §3, we consider one-parameter families of Legendre curves. We give a moving frame and the curvature of the one-parameter family of Legendre curves. Then we show that the existence and uniqueness theorem for one-parameter families of Legendre curves. In §4, we define an envelope of a one-parameter family of Legendre curves. Then the envelope is also a Legendre curve and hence we give a curvature of the envelope as a Legendre curve. Moreover, we give relationships between the envelopes given by implicit functions and one-parameter family of Legendre curves. In §5, we define a bi-Legendre curve as a special class of one-parameter family of Legendre curves and give a relationship between envelopes.

All maps and manifolds considered here are differential of class  $C^\infty$ .

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## 2 Previous results

Let  $\mathbb{R}^2$  be the Euclidean plane equipped with the inner product  $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$ , where  $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$ .

We review two definitions of envelopes for one-parameter family of plane curves. These are given by implicit functions and parametrized curves. Here we denote these envelopes by  $E_I$  and  $E_P$  respectively. Other related definitions of envelopes see [3, 19, 21].

Let  $F : V \times \Lambda \rightarrow \mathbb{R}, (x, y, \lambda) \mapsto F(x, y, \lambda)$  be a smooth function, where  $V$  is a domain in  $\mathbb{R}^2$ , and  $\Lambda$  is an interval or  $\mathbb{R}$ . A family of curves in the plane is given by  $\Gamma_\lambda = \{(x, y) \in V \mid F(x, y, \lambda) = 0\}$  for each  $\lambda \in \Lambda$ . Then one of the classical definition of the envelope is as follows, see for instance [3, 4]:

**Definition 2.1** The *envelope* of the family  $F$  is the set  $E_I$  of points given by

$$E_I = \left\{ (x, y) \in V \mid \text{for some } \lambda \in \Lambda, F(x, y, \lambda) = \frac{\partial F}{\partial \lambda}(x, y, \lambda) = 0 \right\}.$$

If  $F(x, y, \lambda) = (\partial F / \partial \lambda)(x, y, \lambda) = 0$ , we say that  $(x, y) \in E_I$  *with respect to*  $\lambda$ .

On the other hand, let  $\gamma : I \times \Lambda \rightarrow \mathbb{R}^2$  be a one-parameter family of smooth parametrized curves, and let  $e_p : U \rightarrow I \times \Lambda, e_p(u) = (t(u), \lambda(u))$  be a regular curve, where  $I, \Lambda$  and  $U$  are intervals or  $\mathbb{R}$ . We denote  $\Gamma_\lambda(t) = \gamma(t, \lambda)$  and  $E_P(u) = \gamma \circ e_p(u)$ .

**Definition 2.2** ([11, Page 138]) We call  $E_P$  an *envelope* (and  $e_p$  a *pre-envelope*) for the family  $\gamma$ , when the following conditions are satisfied.

- (i) The function  $\lambda$  is non-constant on any non-trivial subinterval of  $U$ . (The Variability Condition.)
- (ii) For all  $u$ , the curve  $E_P$  is tangent at  $u$  to the curve  $\Gamma_{\lambda(u)}$  at the parameter  $t(u)$ , meaning that the tangent vectors  $E'_P(u) = (dE_P/du)(u)$  and  $\dot{\Gamma}_{\lambda(u)}(t(u)) = (d\Gamma_{\lambda(u)}/dt)(t(u))$  are linearly dependent. (The Tangency Condition.)

We say that the *singular set* of  $\gamma : I \times \Lambda \rightarrow \mathbb{R}^2, \gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda))$  is the subset of the domain  $I \times \Lambda$  defined by

$$\det(\gamma_t(t, \lambda), \gamma_\lambda(t, \lambda)) = \det \begin{pmatrix} x_t(t, \lambda) & y_t(t, \lambda) \\ x_\lambda(t, \lambda) & y_\lambda(t, \lambda) \end{pmatrix} = 0. \quad (1)$$

Here we denote  $\gamma_t(t, \lambda) = (\partial \gamma / \partial t)(t, \lambda) = (x_t(t, \lambda), y_t(t, \lambda))$  and  $\gamma_\lambda(t, \lambda) = (\partial \gamma / \partial \lambda)(t, \lambda) = (x_\lambda(t, \lambda), y_\lambda(t, \lambda))$ . Then the envelope theorem is as follows:

**Theorem 2.3** ([11, Page 140]) *Let  $\gamma : I \times \Lambda \rightarrow \mathbb{R}^2$  be a family of parametrized curves, and let  $e_p : U \rightarrow I \times \Lambda$  be a regular curve satisfying the variability condition. Then  $e_p$  is a pre-envelope of  $\gamma$  (and  $E_P$  is an envelope) if and only if the trace of  $e_p$  lies in the singular set of  $\gamma$ .*

We consider one-parameter families of 3/2-cusps as examples. Other examples see [3, 11].

**Example 2.4** Let  $F : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}, F(x, y, \lambda) = (x - \lambda)^3 - y^2$ . Since  $(\partial F / \partial \lambda)(x, y, \lambda) = -3(x - \lambda)^2$ , the envelope is given by  $E_I = \{(\lambda, 0) | \lambda \in \mathbb{R}\}$ .

Let  $\gamma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2, \gamma(t, \lambda) = (t^2 + \lambda, t^3)$ . Since (1), we have  $-3t^2 = 0$ . By Theorem 2.3, the pre-envelope and the envelope are given by  $e_p : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, e_p(u) = (0, u)$  and  $E_P : \mathbb{R} \rightarrow \mathbb{R}^2, E(u) = (u, 0)$ .

Both cases, the envelopes are given by the  $x$ -axis, see Figure 1.

**Example 2.5** Let  $F : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}, F(x, y, \lambda) = x^3 - (y - \lambda)^2$ . Since  $(\partial F / \partial \lambda)(x, y, \lambda) = -2(y - \lambda)$ , the envelope is given by  $E_I = \{(0, \lambda) | \lambda \in \mathbb{R}\}$ .

Let  $\gamma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2, \gamma(t, \lambda) = (t^2, t^3 + \lambda)$ . Since (1), we have  $2t = 0$ . By Theorem 2.3, the pre-envelope and the envelope are given by  $e_p : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, e_p(u) = (0, u)$  and  $E_P : \mathbb{R} \rightarrow \mathbb{R}^2, E_P(u) = (0, u)$ .

Both cases, the envelopes are given by the  $y$ -axis, see Figure 2. However, in the sense of limit tangent of the 3/2-cusp,  $y$ -axis is not tangent to the 3/2-cusps. Moreover, as a solution of differential equations, the  $x$ -axis in Figure 1 is a singular solution of the ODE  $-y + ((2/3)y')^3 = 0$  and  $y$ -axis in Figure 2 is not a singular solution of the ODE  $-x + ((2/3)y')^2 = 0$  (cf. [15, 16, 20]). We would like to distinguish as envelopes, see Examples 4.2 and 4.3 below.

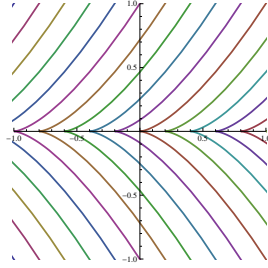


Figure 1.

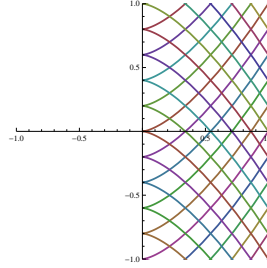


Figure 2.

### 3 One parameter families of Legendre curves

In this section, we consider one-parameter families of Legendre curves in the unit tangent bundle  $T_1S^2 = \mathbb{R}^2 \times S^1$  over  $\mathbb{R}^2$ . The fundamental results for Legendre curves in the unit tangent bundle over  $\mathbb{R}^2$  see the Appendix or [8].

**Definition 3.1** Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a smooth mapping. We say that  $(\gamma, \nu)$  is a *one-parameter family of Legendre curves* if  $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$  for all  $(t, \lambda) \in I \times \Lambda$ .

Then  $(\gamma(\cdot, \lambda), \nu(\cdot, \lambda)) : I \rightarrow \mathbb{R}^2 \times S^1$  is a Legendre curve for each fixed parameter  $\lambda \in \Lambda$ , that is,  $(\gamma(\cdot, \lambda), \nu(\cdot, \lambda))$  is an integrable curve with respect to the canonical contact 1-form on  $\mathbb{R}^2 \times S^1$ . Therefore,  $\gamma : I \times \Lambda \rightarrow \mathbb{R}^2$  is a one-parameter family of frontals.

We denote  $J(\mathbf{a}) = (-a_2, a_1)$  the anticlockwise rotation by  $\pi/2$  of a vector  $\mathbf{a} = (a_1, a_2)$ . We define  $\boldsymbol{\mu}(t, \lambda) = J(\nu(t, \lambda))$ . Since  $\{\nu(t, \lambda), \boldsymbol{\mu}(t, \lambda)\}$  is a moving frame along  $\gamma(t, \lambda)$  on  $\mathbb{R}^2$ , we have the Frenet type formula.

$$\begin{aligned} \begin{pmatrix} \nu_t(t, \lambda) \\ \boldsymbol{\mu}_t(t, \lambda) \end{pmatrix} &= \begin{pmatrix} 0 & \ell(t, \lambda) \\ -\ell(t, \lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t, \lambda) \\ \boldsymbol{\mu}(t, \lambda) \end{pmatrix}, \\ \begin{pmatrix} \nu_\lambda(t, \lambda) \\ \boldsymbol{\mu}_\lambda(t, \lambda) \end{pmatrix} &= \begin{pmatrix} 0 & m(t, \lambda) \\ -m(t, \lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t, \lambda) \\ \boldsymbol{\mu}(t, \lambda) \end{pmatrix} \end{aligned}$$

and

$$\gamma_t(t, \lambda) = \beta(t, \lambda)\boldsymbol{\mu}(t, \lambda),$$

where  $\ell(t, \lambda) = \nu_t(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda)$ ,  $m(t, \lambda) = \nu_\lambda(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda)$  and  $\beta(t, \lambda) = \gamma_t(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda)$ . By the integrability condition  $\nu_{t\lambda}(t, \lambda) = \nu_{\lambda t}(t, \lambda)$ ,  $\ell$  and  $m$  satisfies the condition

$$\ell_\lambda(t, \lambda) = m_t(t, \lambda) \tag{2}$$

for all  $(t, \lambda) \in I \times \Lambda$ . We call the pair  $(\ell, m, \beta)$  with the integrability condition (2) a *curvature of the one-parameter family of Legendre curves*  $(\gamma, \nu)$ .

**Remark 3.2** Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a one-parameter family of Legendre curves with the curvature  $(\ell, m, \beta)$ . Then  $(\gamma, -\nu)$  is also a one-parameter family of Legendre curves with the curvature  $(\ell, m, -\beta)$ .

**Example 3.3** (Example 2.4) Let  $(\gamma, \nu) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2 \times S^1$ ,  $\gamma(t, \lambda) = (t^2 + \lambda, t^3)$ ,  $\nu(t, \lambda) = (4 + 9t^2)^{-1/2}(-3t, 2)$ . Since  $\gamma_t(t, \lambda) = (2t, 3t^2)$ ,  $\nu_t(t, \lambda) = 6(4 + 9t^2)^{-3/2}(-2, -3t)$ ,  $\nu_\lambda(t, \lambda) = 0$  and  $\boldsymbol{\mu}(t, \lambda) = (4 + 9t^2)^{-1/2}(-2, -3t)$ ,  $(\gamma, \nu)$  is a one-parameter family of Legendre curves with the curvature  $(\ell(t, \lambda), m(t, \lambda), \beta(t, \lambda)) = (6(4 + 9t^2)^{-1}, 0, -t(4 + 9t^2)^{1/2})$ .

**Example 3.4** (Example 2.5) Let  $(\gamma, \nu) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2 \times S^1$ ,  $\gamma(t, \lambda) = (t^2, t^3 + \lambda)$ ,  $\nu(t, \lambda) = (4 + 9t^2)^{-1/2}(-3t, 2)$ . Since  $\gamma_t(t, \lambda) = (2t, 3t^2)$ ,  $\nu_t(t, \lambda) = 6(4 + 9t^2)^{-3/2}(-2, -3t)$ ,  $\nu_\lambda(t, \lambda) = 0$  and  $\boldsymbol{\mu}(t, \lambda) = (4 + 9t^2)^{-1/2}(-2, -3t)$ ,  $(\gamma, \nu)$  is a one-parameter family of Legendre curves with the curvature  $(\ell(t, \lambda), m(t, \lambda), \beta(t, \lambda)) = (6(4 + 9t^2)^{-1}, 0, -t(4 + 9t^2)^{1/2})$ .

**Definition 3.5** Let  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu}) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be one-parameter families of Legendre curves. We say that  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu})$  are *congruent as one-parameter family of Legendre curves* if there exist a constant rotation  $A \in SO(2)$  and a smooth translation mapping  $\mathbf{a} : \Lambda \rightarrow \mathbb{R}^2$  such that  $\tilde{\gamma}(t, \lambda) = A(\gamma(t, \lambda)) + \mathbf{a}(\lambda)$  and  $\tilde{\nu}(t, \lambda) = A(\nu(t, \lambda))$  for all  $(t, \lambda) \in I \times \Lambda$ .

We give the existence and uniqueness theorems for one-parameter families of Legendre curves.

**Theorem 3.6** (The Existence Theorem for one-parameter families of Legendre curves.) *Let  $(\ell, m, \beta) : I \times \Lambda \rightarrow \mathbb{R}^3$  be a smooth mapping with the integrability condition. There exists a one-parameter family of Legendre curves  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  whose associated curvature is  $(\ell, m, \beta)$ .*

*Proof.* Let  $(t_0, \lambda_0) \in I \times \Lambda$  be fixed. We define a smooth mapping  $\theta : I \times \Lambda \rightarrow \mathbb{R}$  by

$$\theta(t, \lambda) = \int_{t_0}^t \ell(t, \lambda) dt + \int_{\lambda_0}^\lambda m(t_0, \lambda) d\lambda.$$

Then  $\theta$  satisfy the conditions  $\theta_t(t, \lambda) = \ell(t, \lambda)$  and  $\theta_\lambda(t, \lambda) = m(t, \lambda)$ . We define a smooth mapping  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  by

$$\begin{aligned} \gamma(t, \lambda) &= \left( - \int \beta(t, \lambda) \sin \theta(t, \lambda) dt, \int \beta(t, \lambda) \cos \theta(t, \lambda) dt \right), \\ \nu(t, \lambda) &= (\cos \theta(t, \lambda), \sin \theta(t, \lambda)). \end{aligned}$$

By a direct calculation,  $(\gamma, \nu)$  is a one-parameter family of Legendre curves with the curvature  $(\ell, m, \beta)$ .  $\square$

**Theorem 3.7** (The Uniqueness Theorem for one-parameter families of Legendre curves.) *Let  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu}) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be one-parameter families of Legendre curves with the curvatures  $(\ell, m, \beta)$  and  $(\tilde{\ell}, \tilde{m}, \tilde{\beta})$  respectively. Then  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu})$  are congruent as one-parameter family of Legendre curves if and only if  $(\ell, m, \beta)$  and  $(\tilde{\ell}, \tilde{m}, \tilde{\beta})$  coincides.*

*Proof.* Suppose that  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu})$  are congruent as one-parameter families of Legendre curves. By a direct calculation, we have

$$\begin{aligned}\tilde{\gamma}_t(t, \lambda) &= \frac{\partial}{\partial t}(A(\gamma(t, \lambda)) + \mathbf{a}(\lambda)) = A(\gamma_t(t, \lambda)) = \beta(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = \beta(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda), \\ \tilde{\nu}_t(t, \lambda) &= \frac{\partial}{\partial t}(A(\nu(t, \lambda))) = A(\nu_t(t, \lambda)) = \ell(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = \ell(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda), \\ \tilde{\nu}_\lambda(t, \lambda) &= \frac{\partial}{\partial \lambda}(A(\nu(t, \lambda))) = A(\nu_\lambda(t, \lambda)) = m(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = m(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda).\end{aligned}$$

Therefore the curvatures  $(\ell, m, \beta)$  and  $(\tilde{\ell}, \tilde{m}, \tilde{\beta})$  coincides.

Conversely, suppose that  $(\ell, m, \beta)$  and  $(\tilde{\ell}, \tilde{m}, \tilde{\beta})$  coincides. Let  $(t_0, \lambda_0) \in I \times \Lambda$  be fixed. By using a congruence as one-parameter family of Legendre curves, we may assume  $\gamma(t_0, \lambda_0) = \tilde{\gamma}(t_0, \lambda_0)$  and  $\nu(t_0, \lambda_0) = \tilde{\nu}(t_0, \lambda_0)$ . Moreover, we have  $\theta(t, \lambda) = \tilde{\theta}(t, \lambda)$  for all  $(t, \lambda) \in I \times \Lambda$  in the proof of Theorem 3.6. It follows from the construction that we have  $\nu(t, \lambda) = \tilde{\nu}(t, \lambda)$ , and  $\gamma(t, \lambda) = \tilde{\gamma}(t, \lambda)$  up to a smooth translation mapping  $\mathbf{a}(\lambda)$  for all  $(t, \lambda) \in I \times \Lambda$ .  $\square$

## 4 Envelopes of Legendre curves

Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a one-parameter family of Legendre curves with the curvature  $(\ell, m, \beta)$ , and let  $e_L : U \rightarrow I \times \Lambda, e_L(u) = (t(u), \lambda(u))$  be a smooth curve. We denote  $\Gamma_\lambda(t) = \gamma(t, \lambda)$  and  $E_L = \gamma \circ e_L(u)$ . Note that we don't assume  $e_L$  is a regular curve, see section 2.

**Definition 4.1** We call  $E_L$  an *envelope* (and  $e_L$  a *pre-envelope*) for the family of Legendre curves  $(\gamma, \nu)$ , when the following conditions are satisfied.

- (i) The function  $\lambda$  is non-constant on any non-trivial subinterval of  $U$ . (The Variability Condition.)
- (ii) For all  $u$  the curve  $E_L$  is tangent at  $u$  to the curve  $\Gamma_{\lambda(u)}$  at the parameter  $t(u)$ , meaning that  $E'_L(u)$  and  $\boldsymbol{\mu}(t(u), \lambda(u))$  are linearly dependent. (The Tangency Condition.)

Note that the tangency condition is equivalent to the condition  $E'_L(u) \cdot \nu(e_L(u)) = 0$  for all  $u \in U$ .

**Example 4.2** (Example 3.3) Let  $(\gamma, \nu) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2 \times S^1, \gamma(t, \lambda) = (t^2 + \lambda, t^3), \nu(t, \lambda) = (4 + 9t^2)^{-1/2}(-3t, 2)$ . Let  $e_L : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, e_L(u) = (t(u), \lambda(u)) = (0, u)$ . Then  $E_L(u) = \gamma \circ e_L(u) = (u, 0)$ . Since  $\lambda'(u) = 1$  and  $E'_L(u) \cdot \nu(0, u) = 0$ ,  $E_L$  is an envelope of  $(\gamma, \nu)$ .

**Example 4.3** (Example 3.4) Let  $(\gamma, \nu) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2 \times S^1, \gamma(t, \lambda) = (t^2, t^3 + \lambda), \nu(t, \lambda) = (4 + 9t^2)^{-1/2}(-3t, 2)$ . Let  $e_L : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, e_L(u) = (t(u), \lambda(u)) = (0, u)$ . Then  $E_L(u) = \gamma \circ e_L(u) = (0, u)$  and  $\lambda'(u) = 1$ . Since  $E'_L(u) \cdot \nu(0, u) = 1 \neq 0$ ,  $E_L$  is not an envelope of  $(\gamma, \nu)$ .

**Proposition 4.4** Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a one-parameter family of Legendre curves with the curvature  $(\ell, m, \beta)$ . Suppose that  $e_L : U \rightarrow I \times \Lambda$  is a pre-envelope and  $E_L = \gamma \circ e_L : U \rightarrow \mathbb{R}^2$  is an envelope of  $(\gamma, \nu)$ . Then  $E_L$  is a frontal. More precisely,  $(E_L, \nu \circ e_L) : U \rightarrow \mathbb{R}^2 \times S^1$  is a Legendre curve with the curvature

$$\begin{aligned}\ell_E(u) &= t'(u)\ell(e_L(u)) + \lambda'(u)m(e_L(u)), \\ \beta_E(u) &= t'(u)\beta(e_L(u)) + \lambda'(u)\gamma_\lambda(e_L(u)) \cdot \boldsymbol{\mu}(e_L(u)).\end{aligned}$$

*Proof.* We denote  $e_L(u) = (t(u), \lambda(u))$ . Since  $E_L$  is an envelope,  $E'_L(u) \cdot \nu(e_L(u)) = 0$  for all  $u \in U$ . It follows that  $(E_L, \nu \circ e_L) : U \rightarrow \mathbb{R}^2 \times S^1$  is a Legendre curve. Then  $\ell_E(u) = (d/du)(\nu(e_L(u))) \cdot \boldsymbol{\mu}(e_L(u)) = (t'(u)\nu_t(e_L(u)) + \lambda'(u)\nu_\lambda(e_L(u))) \cdot \boldsymbol{\mu}(e_L(u)) = t'(u)\ell(e_L(u)) + \lambda'(u)m(e_L(u))$  and  $\beta_E(u) = (d/du)(\gamma(e_L(u))) \cdot \boldsymbol{\mu}(e_L(u)) = (t'(u)\gamma_t(e_L(u)) + \lambda'(u)\gamma_\lambda(e_L(u))) \cdot \boldsymbol{\mu}(e_L(u)) = t'(u)\beta(e_L(u)) + \lambda'(u)\gamma_\lambda(e_L(u)) \cdot \boldsymbol{\mu}(e_L(u))$ .  $\square$

We give the envelope theorem for one-parameter family of Legendre curves.

**Theorem 4.5** Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a one-parameter family of Legendre curves, and let  $e_L : U \rightarrow I \times \Lambda$  be a smooth curve satisfying the variability condition. Then  $e_L$  is a pre-envelope of  $(\gamma, \nu)$  (and  $E_L$  is an envelope) if and only if  $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$  for all  $u \in U$ .

*Proof.* Suppose that  $e_L$  is a pre-envelope of  $(\gamma, \nu)$ . By the tangency condition, there exists a function  $c(u) \in \mathbb{R}$  such that  $E'_L(u) = c(u)\boldsymbol{\mu}(e_L(u))$ . By differentiate  $E_L(u) = \gamma \circ e_L(u)$ , we have  $E'_L(u) = t'(u)\gamma_t(e_L(u)) + \lambda'(u)\gamma_\lambda(e_L(u))$ . It follows from  $\gamma_t(t, \lambda) = \beta(t, \lambda)\boldsymbol{\mu}(t, \lambda)$  that  $(t'(u)\beta(e_L(u)) - c(u))\boldsymbol{\mu}(e_L(u)) + \lambda'(u)\gamma_\lambda(e_L(u)) = 0$ . Then we have  $\lambda'(u)\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ . By the variability condition, we have  $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$  for all  $u \in U$ .

Conversely, suppose that  $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$  for all  $u \in U$ . Since  $E'_L(u) \cdot \nu(e_L(u)) = (t'(u)\gamma_t(e_L(u)) + \lambda'(u)\gamma_\lambda(e_L(u))) \cdot \nu(e_L(u)) = 0$ ,  $e_L$  is a pre-envelope of  $(\gamma, \nu)$ .  $\square$

**Example 4.6** Let  $i, j, m, n, j, k$  be natural numbers with  $j = i + h, n = m + k$ . Moreover, we take  $h = 1$  or  $k = 1$ , or  $h, k$  are relatively prime numbers. Let  $(\gamma, \nu) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2 \times S^1$ ,

$$\gamma(t, \lambda) = \left( \frac{t^m}{m} + \frac{\lambda^i}{i}, \frac{t^n}{n} + \frac{\lambda^j}{j} \right), \quad \nu(t, \lambda) = \frac{1}{\sqrt{t^{2k} + 1}}(-t^k, 1).$$

Since  $\gamma_t(t, \lambda) = (t^{m-1}, t^{n-1})$ , we have  $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$  for all  $(t, \lambda) \in \mathbb{R} \times \mathbb{R}$ . Moreover, since  $\gamma_\lambda(t, \lambda) = (\lambda^{i-1}, \lambda^{j-1})$ , we have  $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = (\lambda^{i-1}/\sqrt{t^{2k} + 1})(-t^k + \lambda^h)$ . If we take  $e_L : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, e_L(u) = (u^h, u^k)$ , then the variability condition holds. Furthermore, since

$$\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = \frac{u^{k(i-1)}}{\sqrt{u^{2kh} + 1}}(-u^{hk} + u^{hk}) = 0,$$

$e_L$  is a pre-envelope of  $(\gamma, \nu)$  by Theorem 4.5. Hence, the envelope  $(E_L, \nu_L) : \mathbb{R} \rightarrow \mathbb{R}^2 \times S^1$  is given by

$$E_L(u) = \left( \frac{u^{mh}}{m} + \frac{u^{ik}}{i}, \frac{u^{nh}}{n} + \frac{u^{jk}}{j} \right), \quad \nu_L(u) = \frac{1}{\sqrt{u^{2kh} + 1}}(-u^{kh}, 1).$$

**Proposition 4.7** Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a one-parameter family of Legendre curves. Suppose that  $e_L : U \rightarrow I \times \Lambda$  is a pre-envelope and  $E_L = \gamma \circ e_L$  is an envelope of  $(\gamma, \nu)$ . Then  $e_L : U \rightarrow I \times \Lambda$  is also a pre-envelope and  $E_L = \gamma \circ e_L$  is also an envelope of  $(\gamma, -\nu)$ .

*Proof.* By Remark 3.2,  $(\gamma, -\nu)$  is also a one-parameter family of Legendre curves. It follows from Theorem 4.5 that we have the same pre-envelopes and the envelopes of  $(\gamma, \nu)$  and  $(\gamma, -\nu)$ .  $\square$

**Definition 4.8** We say that a map  $\Phi : \tilde{I} \times \tilde{\Lambda} \rightarrow I \times \Lambda$  is a one-parameter family of parameter change if  $\Phi$  is a diffeomorphism and given by the form  $\Phi(s, k) = (\phi(s, k), \varphi(k))$ .

**Proposition 4.9** Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a one-parameter family of Legendre curves. Suppose that  $e_L : U \rightarrow I \times \Lambda$  is a pre-envelope,  $E_L = \gamma \circ e_L$  is an envelope and  $\Phi : \tilde{I} \times \tilde{\Lambda} \rightarrow I \times \Lambda$  is a one-parameter family of parameter change. Then  $(\tilde{\gamma}, \tilde{\nu}) = (\gamma \circ \Phi, \nu \circ \Phi) : \tilde{I} \times \tilde{\Lambda} \rightarrow \mathbb{R}^2 \times S^1$  is also a one-parameter family of Legendre curves. Moreover,  $\Phi^{-1} \circ e_L : U \rightarrow \tilde{I} \times \tilde{\Lambda}$  is a pre-envelope and  $E_L$  is also an envelope of  $(\tilde{\gamma}, \tilde{\nu})$ .

*Proof.* Since  $\tilde{\gamma}_s(s, k) = \phi_s(s, k)\gamma_t(\Phi(s, k))$  and  $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$  for all  $(t, \lambda) \in I \times \Lambda$ , we have  $\tilde{\gamma}_s(s, k) \cdot \tilde{\nu}(s, k) = 0$  for all  $(s, k) \in \tilde{I} \times \tilde{\Lambda}$ . Therefore,  $(\tilde{\gamma}, \tilde{\nu})$  is a one-parameter family of Legendre curves. By the form of the diffeomorphism  $\Phi(s, k) = (\phi(s, k), \varphi(k))$ ,  $\Phi^{-1} : I \times \Lambda \rightarrow \tilde{I} \times \tilde{\Lambda}$  is given by the form  $\Phi^{-1}(t, \lambda) = (\psi(t, \lambda), \varphi^{-1}(\lambda))$ . It follows that  $\Phi^{-1} \circ e_L(u) = (\phi(t(u), \lambda(u)), \varphi^{-1}(\lambda(u)))$ . Since  $(d/du)\varphi^{-1}(\lambda(u)) = \varphi_\lambda^{-1}(\lambda(u))\lambda'(u)$ , the variability condition holds. Moreover, we have  $\tilde{\gamma}_k(s, k) \cdot \tilde{\nu}(s, k) = (\gamma_t(\Phi(s, k))\phi_k(s, k) + \gamma_\lambda(\Phi(s, k))\varphi'(k)) \cdot \nu(\Phi(s, k)) = \varphi'(k)\gamma_\lambda(\Phi(s, k)) \cdot \nu(\Phi(s, k))$ . It follows that  $\tilde{\gamma}_k(\Phi^{-1} \circ e_L(u)) \cdot \tilde{\nu}(\Phi^{-1} \circ e_L(u)) = \varphi'(\varphi^{-1}(\lambda(u)))\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ . By Theorem 4.5,  $\Phi^{-1} \circ e_L$  is a pre-envelope of  $(\tilde{\gamma}, \tilde{\nu})$ . Therefore,  $\tilde{\gamma} \circ \Phi^{-1} \circ e_L = \gamma \circ \Phi \circ \Phi^{-1} \circ e_L = \gamma \circ e_L = E_L$  is also an envelope of  $(\tilde{\gamma}, \tilde{\nu})$ .  $\square$

We give a relationship between envelopes which are given by implicit functions (Definition 2.1) and one-parameter families of Legendre curves.

**Proposition 4.10** Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a one-parameter family of Legendre curves, and let  $F(x, y, \lambda) = 0$  be an implicit function of the one-parameter family of frontals, that is,  $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$ , where  $\gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda))$ . If  $e_L : U \rightarrow I \times \Lambda$  is a pre-envelope and  $E_L : U \rightarrow \mathbb{R}^2$  is an envelope of  $(\gamma, \nu)$ , then  $E_L(U) \subset E_I$ .

*Proof.* By differentiate  $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$ , we have

$$x_t(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_t(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) = 0$$

and

$$x_\lambda(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_\lambda(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) + F_\lambda(x(t, \lambda), y(t, \lambda), \lambda) = 0.$$

Equivalently,  $\gamma_t(t, \lambda) \cdot (F_x, F_y)(x(t, \lambda), y(t, \lambda), \lambda) = 0$  and  $\gamma_\lambda(t, \lambda) \cdot (F_x, F_y)(x(t, \lambda), y(t, \lambda), \lambda) + F_\lambda(x(t, \lambda), y(t, \lambda), \lambda) = 0$ . Since  $(\gamma, \nu)$  is a one-parameter family of Legendre curves, there exists a function  $c(t, \lambda)$  such that  $(F_x, F_y)(x(t, \lambda), y(t, \lambda), \lambda) = c(t, \lambda)\nu(t, \lambda)$ . Moreover,  $e_L(u) = (t(u), \lambda(u))$  is a pre-envelope of  $(\gamma, \nu)$ , we have  $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$  for all  $u \in U$ . It follows that  $F_\lambda(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) = 0$ . Therefore, we have  $E(u) = \gamma \circ e_L(u) \in E_I$  with respect to  $\lambda(u)$  for all  $u \in U$ .  $\square$

In order to consider the converse result, we need the following lemma and proposition.

**Lemma 4.11** Let  $\mathbf{a}, \mathbf{b} : U \rightarrow \mathbb{R}^2$  be smooth maps. Suppose that the set of non-zero points of smooth function  $k : U \rightarrow \mathbb{R}$  is dense in  $U$ . If  $k(u)\mathbf{a}(u)$  and  $\mathbf{b}(u)$  are linearly dependent, then  $\mathbf{a}(u)$  and  $\mathbf{b}(u)$  are linearly dependent for all  $u \in U$ .



*Proof.* Since  $\det(k(u)\mathbf{a}(u), \mathbf{b}(u)) = 0$ , we have  $k(u)\det(\mathbf{a}(u), \mathbf{b}(u)) = 0$ . By the condition and continuous property, we have  $\det(\mathbf{a}(u), \mathbf{b}(u)) = 0$  for all  $u \in U$ .  $\square$

**Proposition 4.12** *Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2$  be a one-parameter family of Legendre curves, and let  $e_L : U \rightarrow I \times \Lambda$  be a smooth curve satisfying the variability condition. If the set of regular points of  $\gamma$  on  $e_L(U)$  is dense in  $U$  and the trace of  $e_L$  lies in the singular set of  $\gamma$ , then  $e_L$  is a pre-envelope of  $(\gamma, \nu)$  (and  $E_L$  is an envelope).*

*Proof.* Since  $e_L(u)$  belong to the singular set of  $\gamma$ , we have  $\det(\gamma_t(e_L(u)), \gamma_\lambda(e_L(u))) = 0$  for all  $u \in U$ . Therefore  $\gamma_t(e_L(u)) = \beta(e_L(u))\boldsymbol{\mu}(e_L(u))$  and  $\gamma_\lambda(e_L(u))$  are linearly dependent. By the assumption, the set of non-zero points of  $\beta \circ e_L$  is dense in  $U$ . It follows from Lemma 4.11 that  $\boldsymbol{\mu}(e_L(u))$  and  $\gamma_\lambda(e_L(u))$  are linearly dependent. Therefore  $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$  for all  $u \in U$ . By Theorem 4.5,  $e_L$  is a pre-envelope of  $(\gamma, \nu)$ .  $\square$

**Proposition 4.13** *Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a one-parameter family of Legendre curves, and let  $F(x, y, \lambda) = 0$  be an implicit function of the one-parameter family of frontals, that is,  $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$ , where  $\gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda))$ . Let  $e_L : U \rightarrow I \times \Lambda, e(u) = (t(u), \lambda(u))$  be a smooth curve satisfying the variability condition. If the set of regular points of  $\gamma$  on  $e_L(U)$  is dense in  $U$ ,  $E_L(u) = \gamma \circ e_L(u) \in E_I$  with respect to  $\lambda(u)$  and*

$$(F_x, F_y)(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) \neq (0, 0)$$

for all  $u \in U$ , then  $e_L$  is a pre-envelope of  $(\gamma, \nu)$  (and  $E_L$  is an envelope).

*Proof.* By differentiate  $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$ , we have

$$x_t(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_t(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) = 0$$

and

$$x_\lambda(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_\lambda(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) + F_\lambda(x(t, \lambda), y(t, \lambda), \lambda) = 0.$$

Since  $E_L(u) = \gamma \circ e_L(u) \in E_I$  with respect to  $\lambda(u)$ , we have  $F_\lambda(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) = 0$ . It follows that

$$\begin{pmatrix} x_t(t(u), \lambda(u)) & y_t(t(u), \lambda(u)) \\ x_\lambda(t(u), \lambda(u)) & y_\lambda(t(u), \lambda(u)) \end{pmatrix} \begin{pmatrix} F_x(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) \\ F_y(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then the trace of  $e_L$  lies in the singular set of  $\gamma$ . By Proposition 4.12,  $e_L$  is a pre-envelope of  $(\gamma, \nu)$ .  $\square$

We give interesting examples of envelopes of one-parameter families of Legendre curves by using two Legendre curves. Also see [6, 9, 10].

Let  $(\mathbf{p}, \nu_p) : I \rightarrow \mathbb{R}^2 \times S^1$  and  $(\mathbf{q}, \nu_q) : \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be Legendre curves with the curvature  $(\ell_p, \beta_p)$  and  $(\ell_q, \beta_q)$  respectively, see Appendix A. We denote

$$\mathbf{p}(t) = (p_1(t), p_2(t)), \nu_p(t) = (\nu_{p1}(t), \nu_{p2}(t)), \boldsymbol{\mu}_p(t) = (-\nu_{p2}(t), \nu_{p1}(t)),$$

$$\mathbf{q}(\lambda) = (q_1(\lambda), q_2(\lambda)), \nu_q(\lambda) = (\nu_{q1}(\lambda), \nu_{q2}(\lambda)), \boldsymbol{\mu}_q(\lambda) = (-\nu_{q2}(\lambda), \nu_{q1}(\lambda)),$$

respectively. Suppose that  $\mathbf{p}(0) = (0, 0)$  and  $\nu_p(0) = (0, 1)$ .

We define  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  by

$$\gamma(t, \lambda) = \mathbf{q}(\lambda) + A(\theta(\lambda))\mathbf{p}(t), \nu(t, \lambda) = A(\theta(\lambda))\nu_p(t), \quad (3)$$

where  $\theta : \Lambda \rightarrow \mathbb{R}$  and

$$A(\theta(\lambda)) = \begin{pmatrix} \cos \theta(\lambda) & -\sin \theta(\lambda) \\ \sin \theta(\lambda) & \cos \theta(\lambda) \end{pmatrix}.$$

By a direct calculation,  $(\gamma, \nu)$  is a one-parameter family of Legendre curves.

First, we consider a Legendre curve  $\mathbf{p}$  along a Legendre curve  $\mathbf{q}$  which satisfying the both unit normal vectors coincide. Suppose that  $\nu(0, \lambda) = \nu_q(\lambda)$ . This means that the unit normal vector of  $\gamma$  at  $(0, \lambda)$  coincide with the unit normal vector of  $\mathbf{q}$  at  $\lambda$ . It follows that  $\cos \theta(\lambda) = \nu_{q2}(\lambda)$  and  $\sin \theta(\lambda) = -\nu_{q1}(\lambda)$ . By a direct calculation, we have  $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = -\beta_q(\lambda)\nu_{p1}(t) - \ell_q(\lambda)\mathbf{p}(t) \cdot \boldsymbol{\mu}_p(t)$ . By a corollary of Theorem 4.5, we have the following.

**Corollary 4.14** *Under the above notations, let  $(\gamma, \nu)$  be given by (3) with the conditions  $\mathbf{p}(0) = (0, 0)$ ,  $\nu_p(0) = (0, 1)$  and  $\nu(0, \lambda) = \nu_q(\lambda)$ . Let  $e_L : U \rightarrow I \times \Lambda$  be a smooth curve satisfying the variability condition. If  $\beta_q(\lambda(u))\nu_{p1}(t(u)) + \ell_q(\lambda(u))\mathbf{p}(t(u)) \cdot \boldsymbol{\mu}_p(t(u)) = 0$ , then  $e_L$  is a pre-envelope of  $(\gamma, \nu)$  (and  $E_L$  is an envelope).*

Note that  $e_L(u) = (0, u)$  is a pre-envelope of  $(\gamma, \nu)$ . Thus,  $E_L(u) = \mathbf{q}(\lambda)$  is always an envelope of  $(\gamma, \nu)$ .

Second, we consider a Legendre curve  $\mathbf{p}$  along a Legendre curve  $\mathbf{q}$  which satisfying the unit normal vector of  $\mathbf{p}$  coincide with the tangent vector of  $\mathbf{q}$ . Suppose that  $\nu(0, \lambda) = \boldsymbol{\mu}_q(\lambda)$ . This means that the unit normal vector of  $\gamma$  at  $(0, \lambda)$  coincide with the unit tangent vector of  $\mathbf{q}$  at  $\lambda$ . It follows that  $\cos \theta(\lambda) = \nu_{q1}(\lambda)$  and  $\sin \theta(\lambda) = \nu_{q2}(\lambda)$ . By a direct calculation, we have  $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = \beta_q(\lambda)\nu_{p2}(t) - \ell_q(\lambda)\mathbf{p}(t) \cdot \boldsymbol{\mu}_p(t)$ . By a corollary of Theorem 4.5, we have the following.

**Corollary 4.15** *Under the above notations, let  $(\gamma, \nu)$  be given by (3) with the conditions  $\mathbf{p}(0) = (0, 0)$ ,  $\nu_p(0) = (0, 1)$  and  $\nu(0, \lambda) = \boldsymbol{\mu}_q(\lambda)$ . Let  $e_L : U \rightarrow I \times \Lambda$  be a smooth curve satisfying the variability condition. If  $\beta_q(\lambda(u))\nu_{p2}(t(u)) - \ell_q(\lambda(u))\mathbf{p}(t(u)) \cdot \boldsymbol{\mu}_p(t(u)) = 0$ , then  $e_L$  is a pre-envelope of  $(\gamma, \nu)$  (and  $E_L$  is an envelope).*

**Example 4.16** Let  $(\mathbf{p}, \nu_p) : [0, 2\pi) \rightarrow \mathbb{R}^2 \times S^1$  be an astroid  $\mathbf{p}(t) = (\cos^3 t - 1, \sin^3 t)$ ,  $\nu_p(t) = (\sin t, \cos t)$  and  $(\mathbf{q}, \nu_q) : [0, 2\pi) \rightarrow \mathbb{R}^2 \times S^1$  be the unit circle  $\mathbf{q}(\lambda) = (\cos \lambda, \sin \lambda)$ ,  $\nu_q(\lambda) = (\cos \lambda, \sin \lambda)$ , see Figure 3. Then we have  $\beta_p(t) = 3 \cos t \sin t$ ,  $\ell_p(t) = -1$ ,  $\beta_q(\lambda) = 1$  and  $\ell_q(\lambda) = 1$ . Moreover, the conditions  $\mathbf{p}(0) = (0, 0)$  and  $\nu_p(0) = (0, 1)$  are satisfied.

First, we consider a Legendre curve  $\mathbf{p}$  along a Legendre curve  $\mathbf{q}$  which satisfying the both unit normal vectors coincide. By (3) and the condition  $\nu(0, \lambda) = \nu_q(\lambda)$ , the one-parameter family of Legendre curves  $(\gamma, \nu) : [0, 2\pi) \times [0, 2\pi) \rightarrow \mathbb{R}^2 \times S^1$  is given by

$$\begin{aligned} \gamma(t, \lambda) &= \begin{pmatrix} \cos \lambda \\ \sin \lambda \end{pmatrix} + \begin{pmatrix} \sin \lambda & \cos \lambda \\ -\cos \lambda & \sin \lambda \end{pmatrix} \begin{pmatrix} \cos^3 t - 1 \\ \sin^3 t \end{pmatrix}, \\ \nu(t, \lambda) &= \begin{pmatrix} \sin \lambda & \cos \lambda \\ -\cos \lambda & \sin \lambda \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}. \end{aligned}$$

By a direct calculation, we have  $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = -4 \cos(t - (\pi/4)) \cos((t/2) - (\pi/4)) \sin t/2$ . It follows that  $e_L : [0, 2\pi) \rightarrow [0, 2\pi) \times [0, 2\pi)$ ,  $e_L(u) = (0, u), (3\pi/4, u), (3\pi/2, u), (7\pi/4, u)$  are pre-envelopes of  $(\gamma, \nu)$  respectively, by Corollary 4.14. Therefore, the envelopes  $E_L : [0, 2\pi) \rightarrow \mathbb{R}^2$  of  $(\gamma, \nu)$  are given by  $E_L(u) = (\cos u, \sin u), (\sqrt{2} + (1/2))(\cos(u + \pi/4), \sin(u + \pi/4)), (\cos(u + \pi/2), \sin(u + \pi/2)), (\sqrt{2} - (1/2))(\cos(u + \pi/4), \sin(u + \pi/4))$ , respectively see Figure 4 left.

Second, we consider a Legendre curve  $\mathbf{p}$  along a Legendre curve  $\mathbf{q}$  which satisfying the unit normal vector of  $\mathbf{p}$  coincide with the tangent vector of  $\mathbf{q}$ . By (3) and the condition  $\nu(0, \lambda) = \mu_q(\lambda) = (-\sin \lambda, \cos \lambda)$ , the one-parameter family of Legendre curves  $(\gamma, \nu) : [0, 2\pi) \times [0, 2\pi) \rightarrow \mathbb{R}^2 \times S^1$  is given by

$$\begin{aligned}\gamma(t, \lambda) &= \begin{pmatrix} \cos \lambda \\ \sin \lambda \end{pmatrix} + \begin{pmatrix} \cos \lambda & -\sin \lambda \\ \sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} \cos^3 t - 1 \\ \sin^3 t \end{pmatrix}, \\ \nu(t, \lambda) &= \begin{pmatrix} \cos \lambda & -\sin \lambda \\ \sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.\end{aligned}$$

By a direct calculation, we have  $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = \cos 2t$ . It follows that  $e_L : [0, 2\pi) \rightarrow [0, 2\pi) \times [0, 2\pi)$ ,  $e_L(u) = (\pi/4, u), (3\pi/4, u), (5\pi/4, u), (7\pi/4, u)$  are pre-envelopes of  $(\gamma, \nu)$  respectively, by Corollary 4.15. Therefore the envelopes  $E_L : [0, 2\pi) \rightarrow \mathbb{R}^2$  of  $(\gamma, \nu)$  are given by  $E_L(u) = (1/2)(\cos(u + \pi/4), \sin(u + \pi/4)), (1/2)(\cos(u + 3\pi/4), \sin(u + 3\pi/4)), (1/2)(\cos(u + 5\pi/4), \sin(u + 5\pi/4)), (1/2)(\cos(u + 7\pi/4), \sin(u + 7\pi/4))$ , respectively see Figure 4 right.

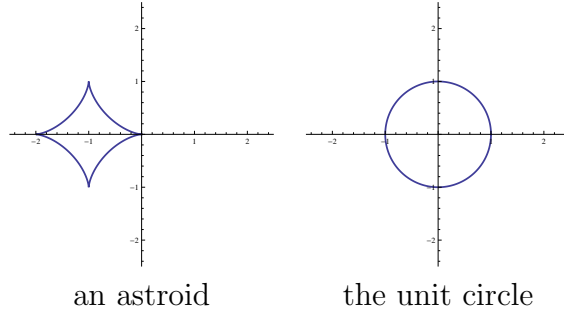


Figure 3.

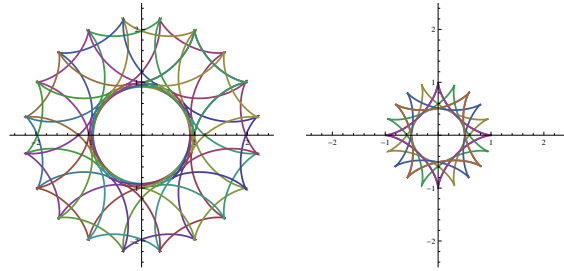


Figure 4.

## 5 Bi-Legendre curves and envelopes

We consider a special class of one-parameter families of Legendre curves. Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a smooth mapping.

**Definition 5.1** We say that  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  is a *bi-Legendre curve* if  $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$  and  $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = 0$  for all  $(t, \lambda) \in I \times \Lambda$ .

Then  $(\gamma, \nu)$  is a one-parameter family of Legendre curves with respect to both parameters  $t$  and  $\lambda$ . We define  $\boldsymbol{\mu}(t, \lambda) = J(\nu(t, \lambda))$ . Since  $\{\nu(t, \lambda), \boldsymbol{\mu}(t, \lambda)\}$  is a moving frame along  $\gamma(t, \lambda)$ , we have the Frenet type formula.

$$\begin{aligned} \begin{pmatrix} \nu_t(t, \lambda) \\ \boldsymbol{\mu}_t(t, \lambda) \end{pmatrix} &= \begin{pmatrix} 0 & \ell(t, \lambda) \\ -\ell(t, \lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t, \lambda) \\ \boldsymbol{\mu}(t, \lambda) \end{pmatrix}, \\ \begin{pmatrix} \nu_\lambda(t, \lambda) \\ \boldsymbol{\mu}_\lambda(t, \lambda) \end{pmatrix} &= \begin{pmatrix} 0 & m(t, \lambda) \\ -m(t, \lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t, \lambda) \\ \boldsymbol{\mu}(t, \lambda) \end{pmatrix}, \\ \gamma_t(t, \lambda) &= \beta(t, \lambda)\boldsymbol{\mu}(t, \lambda), \\ \gamma_\lambda(t, \lambda) &= \alpha(t, \lambda)\boldsymbol{\mu}(t, \lambda), \end{aligned}$$

where

$$\begin{aligned} \ell(t, \lambda) &= \nu_t(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda), \quad m(t, \lambda) = \nu_\lambda(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda), \\ \beta(t, \lambda) &= \gamma_t(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda), \quad \alpha(t, \lambda) = \gamma_\lambda(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda). \end{aligned}$$

By the integrability conditions  $\nu_{t\lambda}(t, \lambda) = \nu_{\lambda t}(t, \lambda)$ ,  $\gamma_{t\lambda}(t, \lambda) = \gamma_{\lambda t}(t, \lambda)$ ,  $\ell, m, \beta, \alpha$  satisfies the conditions

$$\ell_\lambda(t, \lambda) = m_t(t, \lambda), \beta_\lambda(t, \lambda) = \alpha_t(t, \lambda), \ell(t, \lambda)\alpha(t, \lambda) = m(t, \lambda)\beta(t, \lambda) \quad (4)$$

for all  $(t, \lambda) \in I \times \Lambda$ . We call the pair  $(\ell, m, \beta, \alpha)$  with the integrability conditions (4) a *curvature of the bi-Legendre curve*  $(\gamma, \nu)$ .

**Definition 5.2** Let  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu}) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be bi-Legendre curves. We say that  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu})$  are *congruent as bi-Legendre curves* if there exist a constant rotation  $A \in SO(2)$  and a translation  $\mathbf{a}$  on  $\mathbb{R}^2$  such that  $\tilde{\gamma}(t, \lambda) = A(\gamma(t, \lambda)) + \mathbf{a}$  and  $\tilde{\nu}(t, \lambda) = A(\nu(t, \lambda))$  for all  $(t, \lambda) \in I \times \Lambda$ .

**Theorem 5.3** (The Existence Theorem for bi-Legendre curves.) *Let  $(\ell, m, \beta, \alpha) : I \times \Lambda \rightarrow \mathbb{R}^4$  be a smooth mapping with the integrability conditions. There exists a bi-Legendre curve  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  whose associated curvature is  $(\ell, m, \beta, \alpha)$ .*

*Proof.* Let  $(t_0, \lambda_0) \in I \times \Lambda$  be fixed. We define a smooth mapping  $\theta : I \times \Lambda \rightarrow \mathbb{R}$  by

$$\theta(t, \lambda) = \int_{t_0}^t \ell(t, \lambda) dt + \int_{\lambda_0}^\lambda m(t_0, \lambda) d\lambda.$$

Then  $\theta$  satisfy the conditions  $\theta_t(t, \lambda) = \ell(t, \lambda)$  and  $\theta_\lambda(t, \lambda) = m(t, \lambda)$ . We also define a smooth mapping  $(x, y) : I \times \Lambda \rightarrow \mathbb{R}^2$  by

$$\begin{aligned} x(t, \lambda) &= -\int_{t_0}^t \beta(t, \lambda) \sin \theta(t, \lambda) dt - \int_{\lambda_0}^\lambda \alpha(t_0, \lambda) \sin \theta(t_0, \lambda) d\lambda \\ y(t, \lambda) &= \int_{t_0}^t \beta(t, \lambda) \cos \theta(t, \lambda) dt + \int_{\lambda_0}^\lambda \alpha(t_0, \lambda) \cos \theta(t_0, \lambda) d\lambda. \end{aligned}$$

By the integrability condition (4), we have

$$x_t(t, \lambda) = -\beta(t, \lambda) \sin \theta(t, \lambda), \quad x_\lambda(t, \lambda) = -\alpha(t, \lambda) \sin \theta(t, \lambda),$$

$$y_t(t, \lambda) = \beta(t, \lambda) \cos \theta(t, \lambda), y_\lambda(t, \lambda) = \alpha(t, \lambda) \cos \theta(t, \lambda).$$

We define a smooth mapping  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  by

$$\gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda)), \nu(t, \lambda) = (\cos \theta(t, \lambda), \sin \theta(t, \lambda)).$$

By a direct calculation,  $(\gamma, \nu)$  is a bi-Legendre curve with the curvature  $(\ell, m, \beta, \alpha)$ .  $\square$

**Theorem 5.4** (The Uniqueness Theorem for bi-Legendre curves.) *Let  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu}) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be bi-Legendre curves with the curvatures  $(\ell, m, \beta, \alpha)$  and  $(\tilde{\ell}, \tilde{m}, \tilde{\beta}, \tilde{\alpha})$  respectively. Then  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu})$  are congruent as bi-Legendre curves if and only if  $(\ell, m, \beta, \alpha)$  and  $(\tilde{\ell}, \tilde{m}, \tilde{\beta}, \tilde{\alpha})$  coincides.*

*Proof.* Suppose that  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu})$  are congruent as bi-Legendre curves. By a direct calculation, we have

$$\begin{aligned} \tilde{\gamma}_t(t, \lambda) &= \frac{\partial}{\partial t}(A(\gamma(t, \lambda)) + \mathbf{a}) = A(\gamma_t(t, \lambda)) = \beta(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = \beta(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda), \\ \tilde{\gamma}_\lambda(t, \lambda) &= \frac{\partial}{\partial \lambda}(A(\gamma(t, \lambda)) + \mathbf{a}) = A(\gamma_\lambda(t, \lambda)) = \alpha(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = \alpha(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda), \\ \tilde{\nu}_t(t, \lambda) &= \frac{\partial}{\partial t}A(\nu(t, \lambda)) = A(\nu_t(t, \lambda)) = \ell(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = \ell(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda), \\ \tilde{\nu}_\lambda(t, \lambda) &= \frac{\partial}{\partial \lambda}A(\nu(t, \lambda)) = A(\nu_\lambda(t, \lambda)) = m(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = m(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda). \end{aligned}$$

Therefore the curvatures  $(\ell, m, \beta, \alpha)$  and  $(\tilde{\ell}, \tilde{m}, \tilde{\beta}, \tilde{\alpha})$  coincides.

Conversely, suppose that  $(\ell, m, \beta, \alpha)$  and  $(\tilde{\ell}, \tilde{m}, \tilde{\beta}, \tilde{\alpha})$  coincides. Let  $(t_0, \lambda_0) \in I \times \Lambda$  be fixed. By using a congruence as bi-Legendre curves,  $\gamma(t_0, \lambda_0) = \tilde{\gamma}(t_0, \lambda_0)$  and  $\nu(t_0, \lambda_0) = \tilde{\nu}(t_0, \lambda_0)$ . Moreover, we have  $\theta(t, \lambda) = \tilde{\theta}(t, \lambda)$  in the proof of Theorem 5.3. It follows from the construction that  $\nu(t, \lambda) = \tilde{\nu}(t, \lambda)$  and  $\gamma(t, \lambda) = \tilde{\gamma}(t, \lambda)$  for all  $(t, \lambda) \in I \times \Lambda$ .  $\square$

Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a bi-Legendre curve. Then  $(\gamma, \nu)$  is a one-parameter family of Legendre curves with respect to the parameter  $\lambda$ . We denote a smooth map  $e_L : U \rightarrow I \times \Lambda$ ,  $e_L(u) = (t(u), \lambda(u))$ . Since  $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = 0$  for all  $(t, \lambda) \in I \times \Lambda$ , we have  $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$  for all  $u \in U$ . If the function  $\lambda$  is non-constant on any non-trivial subinterval of  $U$ , then  $E_L = \gamma \circ e_L$  is an envelope of  $(\gamma, \nu)$  with respect to the parameter  $\lambda$  by Theorem 4.5. Moreover,  $(\gamma, \nu)$  is also a one-parameter family of Legendre curves with respect to the parameter  $t$ . Since  $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$  for all  $(t, \lambda) \in I \times \Lambda$ , we have  $\gamma_t(e_L(u)) \cdot \nu(e_L(u)) = 0$  for all  $u \in U$ . If the function  $t$  is non-constant on any non-trivial subinterval of  $U$ , then  $E_L = \gamma \circ e_L$  is an envelope of  $(\gamma, \nu)$  with respect to the parameter  $t$  by Theorem 4.5. Summary we have the following result.

**Proposition 5.5** *Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a bi-Legendre curve. If  $e_L : U \rightarrow I \times \Lambda$ ,  $e_L(u) = (t(u), \lambda(u))$  satisfy the conditions that the functions  $t$  and  $\lambda$  are non-constant on any non-trivial subinterval of  $U$ , then  $E_L = \gamma \circ e_L$  is an envelope of  $(\gamma, \nu)$  with respect to the both parameter  $t$  and  $\lambda$  respectively.*

Let  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$  be a bi-Legendre curve. Since  $\gamma_t(t, \lambda) = \beta(t, \lambda)\boldsymbol{\mu}(t, \lambda)$  and  $\gamma_\lambda(t, \lambda) = \alpha(t, \lambda)\boldsymbol{\mu}(t, \lambda)$ , we have  $\det(\gamma_t(t, \lambda), \gamma_\lambda(t, \lambda)) = 0$  for all  $(t, \lambda) \in I \times \Lambda$ . It follows that for any  $(t, \lambda) \in I \times \Lambda$  are singular points of  $\gamma : I \times \Lambda \rightarrow \mathbb{R}^2$ . Hence, at a rank 1 point, the image of  $\gamma$  is a curve at least locally. We give a concrete example of bi-Legendre curves.

**Example 5.6** Let  $k, n$  be natural numbers. We define  $(\ell, m, \beta, \alpha) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^4$  by  $\ell(t, \lambda) = \lambda^{k+1}t^k, m(t, \lambda) = \lambda^k t^{k+1}, \beta(t, \lambda) = \lambda^{n+1}t^n, \alpha(t, \lambda) = \lambda^n t^{n+1}$ . Then the integrability conditions  $\ell_\lambda(t, \lambda) = m_t(t, \lambda), \beta_\lambda(t, \lambda) = \alpha_t(t, \lambda), \alpha(t, \lambda)\ell(t, \lambda) = \beta(t, \lambda)m(t, \lambda)$  hold for all  $(t, \lambda) \in \mathbb{R} \times \mathbb{R}$ . It follows that  $\theta(t, \lambda) = \lambda^{k+1}t^{k+1}/(k+1)$ . By the construction in the proof of Theorem 5.3, we give a bi-Legendre curve  $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ ,

$$\begin{aligned}\gamma(t, \lambda) &= (x(t, \lambda), y(t, \lambda)) = \left( -\int_0^t \lambda^{n+1}t^n \sin\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right) dt, \int_0^\lambda \lambda^{n+1}t^n \cos\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right) dt \right), \\ \nu(t, \lambda) &= (\cos \theta(t, \lambda), \sin \theta(t, \lambda)) = \left( \cos\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right), \sin\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right) \right).\end{aligned}$$

## A Legendre curves in the unit tangent bundle

We quickly review on the theory of Legendre curves in the unit tangent bundle over  $\mathbb{R}^2$ , see detail [8]. We say that  $(\gamma, \nu) : I \rightarrow \mathbb{R}^2 \times S^1$  is a *Legendre curve* if  $(\gamma, \nu)^*\theta = 0$  for all  $t \in I$ , where  $\theta$  is a canonical contact form on the unit tangent bundle  $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$  over  $\mathbb{R}^2$  (cf. [1, 2]). This condition is equivalent to  $\dot{\gamma}(t) \cdot \nu(t) = 0$  for all  $t \in I$ . We say that  $\gamma : I \rightarrow \mathbb{R}^2$  is a *frontal* if there exists  $\nu : I \rightarrow S^1$  such that  $(\gamma, \nu)$  is a Legendre curve. Examples of Legendre curves see [13, 14]. We have the Frenet formula of a frontal  $\gamma$  as follows. We put on  $\boldsymbol{\mu}(t) = J(\nu(t))$ . Then we call the pair  $\{\nu(t), \boldsymbol{\mu}(t)\}$  a *moving frame of a frontal*  $\gamma(t)$  in  $\mathbb{R}^2$  and we have the Frenet formula of a frontal (or, Legendre curve),

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \quad \dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t),$$

where  $\ell(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$  and  $\beta(t) = \dot{\gamma}(t) \cdot \boldsymbol{\mu}(t)$ . We call the pair  $(\ell, \beta)$  the *curvature of the Legendre curve*.

**Definition A.1** Let  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu}) : I \rightarrow \mathbb{R}^2 \times S^1$  be Legendre curves. We say that  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu})$  are *congruent as Legendre curves* if there exist a constant rotation  $A \in SO(2)$  and a translation  $\mathbf{a}$  on  $\mathbb{R}^2$  such that  $\tilde{\gamma}(t) = A(\gamma(t)) + \mathbf{a}$  and  $\tilde{\nu}(t) = A(\nu(t))$  for all  $t \in I$ .

**Theorem A.2** (The Existence Theorem for Legendre curves.) *Let  $(\ell, \beta) : I \rightarrow \mathbb{R}^2$  be a smooth mapping. There exists a Legendre curve  $(\gamma, \nu) : I \rightarrow \mathbb{R}^2 \times S^1$  whose associated curvature of the Legendre curve is  $(\ell, \beta)$ .*

**Theorem A.3** (The Uniqueness Theorem for Legendre curves.) *Let  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu}) : I \rightarrow \mathbb{R}^2 \times S^1$  be Legendre curves with the curvatures of Legendre curves  $(\ell, \beta)$  and  $(\tilde{\ell}, \tilde{\beta})$ . Then  $(\gamma, \nu)$  and  $(\tilde{\gamma}, \tilde{\nu})$  are congruent as Legendre curves if and only if  $(\ell, \beta)$  and  $(\tilde{\ell}, \tilde{\beta})$  coincides.*

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