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Existence conditions of framed curves for smooth curves

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Abstract

A framed curve is a smooth curve in the Euclidean space with a moving frame. We call the smooth curve in the Euclidean space the framed base curve. In this paper, we give an existence condition of framed curves. Actually, we construct a framed curve such that the image of the framed base curve coincides with the image of a given smooth curve under a condition. As a consequence, polygons in the Euclidean plane can be realised as not only a smooth curve but also a framed base curve.

1 Introduction

A framed curve in the Euclidean space is a curve with a moving frame. It is a generalisation of not only regular curves with the linear independent condition (cf. [7]), but also regular curves with Bishop frame (cf. [2]). Moreover, framed curves may have singular points. It is also a generalisation of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$ (cf. [1, 4]).

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space equipped with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^n a_i b_i$, where $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$. For $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n-1} \in \mathbb{R}^n$, we define the vector product,

$$\boldsymbol{a}_1 \times \cdots \times \boldsymbol{a}_{n-1} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n-11} & \cdots & a_{n-1n} \\ e_1 & \cdots & e_n \end{vmatrix} = \sum_{i=1}^n \det(\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n-1}, e_i) e_i,$$

where $\boldsymbol{a}_i = (a_{i1}, \dots, a_{in})$ for $i = 1, \dots, n-1$ and $e_1, \dots, e_n$ are the canonical basis on $\mathbb{R}^n$. Then we have $(\boldsymbol{a}_1 \times \cdots \times \boldsymbol{a}_{n-1}) \cdot \boldsymbol{a}_i = 0$ for $i = 1, \dots, n-1$. We denote the set Δ_{n-1},

$$\begin{aligned} \Delta_{n-1} &= \{\boldsymbol{\nu} = (\nu_1, \dots, \nu_{n-1}) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \mid \nu_i \cdot \nu_j = \delta_{ij}, i, j = 1, \dots, n-1\} \\ &= \{\boldsymbol{\nu} = (\nu_1, \dots, \nu_{n-1}) \in S^{n-1} \times \cdots \times S^{n-1} \mid \nu_i \cdot \nu_j = 0, i \neq j, i, j = 1, \dots, n-1\}. \end{aligned}$$

Then Δ_{n-1} is an $n(n-1)/2$-dimensional smooth manifold. If $\boldsymbol{\nu} = (\nu_1, \dots, \nu_{n-1}) \in \Delta_{n-1}$, we define the unit vector $\boldsymbol{\mu} = \nu_1 \times \cdots \times \nu_{n-1}$ of $\mathbb{R}^n$. It follows that the pair $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \Delta_n$. By definition, we have $\det(\nu_1, \dots, \nu_{n-1}, \boldsymbol{\mu}) = 1$. Note that $\Delta_2 = S^1$.

Let I be an interval or $\mathbb{R}$.

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Definition 1.1 We say that a smooth map $(\gamma, \boldsymbol{\nu}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ is a *framed curve* if $\dot{\gamma}(t) \cdot \nu_i(t) = 0$ for all $t \in I$ and $i = 1, \ldots, n-1$. We also say that a smooth map $\gamma : I \to \mathbb{R}^n$ is a *framed base curve* if there exists a smooth map $\boldsymbol{\nu} : I \to \Delta_{n-1}$ such that $(\gamma, \boldsymbol{\nu})$ is a framed curve.

For a framed curve $(\gamma, \boldsymbol{\nu}) : I \to \mathbb{R}^n \times \Delta_{n-1}$, the framed base curve γ may have singular points. We denote the set of singular points of γ by $\Sigma(\gamma)$, that is, we set $\Sigma(\gamma) = \{t \in I \mid \dot{\gamma}(t) = \mathbf{0}\}$. The framed curves can be characterised by the moving frame $\{\boldsymbol{\nu}(t), \boldsymbol{\mu}(t)\}$ of the framed base curve $\gamma(t)$ and the curvature of the framed curve, in detail see [6].

In the case of $n = 2$, the framed curve is nothing but a Legendre curve with respect to the canonical contact structure on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$. We have shown that analytic curves are at least locally framed base curves in the cases of plane curves $(n = 2)$ and space curves $(n = 3)$, see [4] and [6], respectively.

For a function f, we denote $f(a-0)$ (respectively, $f(a+0)$) as one sided limit $\lim_{t\to a-0} f(t)$ (respectively, $\lim_{t\to a+0} f(t)$). We denote $\boldsymbol{t}(t)$ as the unit tangent vector of $\gamma(t)$ at regular points, that is, $\boldsymbol{t}(t) = \dot{\gamma}(t)/\|\dot{\gamma}(t)\|$ if $\dot{\gamma}(t) \neq \mathbf{0}$.

The main result in this paper is as follows. We give an existence condition of a framed curve such that the image of the framed base curve coincides with the image of a given smooth curve.

Theorem 1.2 *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a C^∞-curve. Suppose that the singular set $\Sigma(\gamma)$ is finite, and the limit of the derivatives of the tangent vectors $\boldsymbol{t}^{(k)}(s-0)$ and $\boldsymbol{t}^{(k)}(s+0)$ exist for all $s \in \Sigma(\gamma)$ and $k \in \mathbb{N} \cup \{0\}$. Then there exists a framed curve $(\widetilde{\gamma}, \widetilde{\boldsymbol{\nu}}) : [0, 1] \to \mathbb{R}^n \times \Delta_{n-1}$ such that $\widetilde{\gamma}([0, 1]) = \gamma([a, b])$.*

In section 2, we give a proof of the main result by using flat functions. In section 3, we give examples of a polygon and a 3/2-cusp singularity. We also give an example that the smooth curve does not admit as a framed curve.

All maps and manifolds considered here are differential of class C^∞ unless the contrary is explicitly stated.

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2 Proof of the main result

We introduce notations as preparations. Let $\varphi : [0, 1] \to \mathbb{R}$ be a non-analytic smooth function defined by

$$\varphi(t) = \begin{cases} e^{-1/t} & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0. \end{cases}$$

We also define a smooth function $\psi : [0, 1] \to \mathbb{R}$ by

$$\psi(t) = \frac{\varphi(t)}{\varphi(t) + \varphi(1-t)}.$$

The function ψ provides a smooth transition from 0 to 1 on the interval $[0, 1]$ and $\psi^{(n)}(0 + 0) = \psi^{(n)}(1 - 0) = 0$ for all $n \in \mathbb{N}$. Moreover, we define a smooth function $\psi_{a,b} : [0, 1] \to \mathbb{R}$ by $\psi_{a,b}(t) = \psi(t)b + (1 - \psi(t))a$, where $a, b \in \mathbb{R}$ with $a < b$. Note that $\psi_{0,1} = \psi$.

Lemma 2.1 *The function $\psi_{a,b} : [0,1] \to \mathbb{R}$ provides a smooth transition from a to b in the interval $[0,1]$.*

Proof. By definition, $\psi_{a,b}(0) = \psi(0)b + (1-\psi(0))a = a$ and $\psi_{a,b}(1) = \psi(1)b + (1-\psi(1))a = b$. Moreover, we have $\dot{\psi}_{a,b}(t) > 0$ for $0 < t < 1$. Since $\psi_{a,b}^{(n)}(t) = \psi^{(n)}(t)(b-a)$ for all $n \in \mathbb{N}$, we have $\psi_{a,b}^{(n)}(0+0) = \psi^{(n)}(0+0)(b-a) = 0$ and $\psi_{a,b}^{(n)}(1-0) = \psi^{(n)}(1-0)(b-a) = 0$ for all $n \in \mathbb{N}$. □

Let X be a topological space. For two maps on the unit interval $f_1 : [0,1] \to X$ and $f_2 : [0,1] \to X$ with $f_1(1) = f_2(0)$, we define a concatenation map $f_2 * f_1 : [0,1] \to X$ by

$$(f_2 * f_1)(t) = \begin{cases} f_1(2t) & \text{if } 0 \le t \le 1/2, \\ f_2(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Note that the operator $*$ is not associative. The concatenation map of two continuous maps turns out a continuous map again (see [8], for example). On the other hand, in general, the concatenation map of two C^∞-maps does not turn out a C^∞-map. However, we can concatenate two C^∞-maps smoothly by using the smooth transition function.

Lemma 2.2 *Let M be a smooth manifold. Assume $f_1 : [0,1] \to M$ and $f_2 : [0,1] \to M$ are C^∞-maps with $f_1(1) = f_2(0)$. Then the concatenation map $(f_2 \circ \psi) * (f_1 \circ \psi) : [0,1] \to M$ is a C^∞-map.*

Proof. Since the map $(f_2 \circ \psi) * (f_1 \circ \psi)$ is C^∞ on $t \neq 1/2$, it is sufficient to show that $\{(f_2 \circ \psi) * (f_1 \circ \psi)\}^{(n)}(1/2 - 0) = \{(f_2 \circ \psi) * (f_1 \circ \psi)\}^{(n)}(1/2 + 0)$ for all $n \in \mathbb{N}$. By definition of the concatenation map, we have

$$\{(f_2 \circ \psi) * (f_1 \circ \psi)\}^{(n)} \left(\frac{1}{2} - 0 \right) = (f_1 \circ \psi)^{(n)}(1-0)$$

and

$$\{(f_2 \circ \psi) * (f_1 \circ \psi)\}^{(n)} \left(\frac{1}{2} + 0 \right) = (f_2 \circ \psi)^{(n)}(0+0).$$

By the chain rule, we can write each component of $(f_1 \circ \psi)^{(n)}$ (respectively, $(f_2 \circ \psi)^{(n)}$) as a sum of products of each component of $f_1^{(k)}$ (respectively, $f_2^{(k)}$) and $\psi^{(k)}$ for $k \in \{1, \cdots, n\}$. By Lemma 2.1, $\psi^{(k)}(1-0) = 0$ and $\psi^{(k)}(0+0) = 0$ for $k = 1, \cdots, n$. Hence we have $(f_1 \circ \psi)^{(n)}(1-0) = \mathbf{0}$ and $(f_2 \circ \psi)^{(n)}(0+0) = \mathbf{0}$. Therefore, the map $(f_2 \circ \psi) * (f_1 \circ \psi) : [0,1] \to M$ is a C^∞-map. □

Remark 2.3 By Lemma 2.2, piece-wise C^∞-curves can be realised as a C^∞-curve such that the same image. Especially, polygons in the Euclidean plane may be considered as the image of a C^∞-curve.

Proof of the Theorem 1.2. Let $\{s_0, \cdots, s_n\}$ be the set of singular points except a and b.

First step: We define a smooth map $\widetilde{\gamma}_{a,s_0} : [0,1] \to \mathbb{R}^n$ by $\widetilde{\gamma}_{a,s_0}(t) = \gamma(\psi_{a,s_0}(t))$. We show this map has the following properties:

(i) $\widetilde{\gamma}_{a,s_0}(0) = \gamma(a)$ and $\widetilde{\gamma}_{a,s_0}(1) = \gamma(s_0)$,

(ii) $\widetilde{\gamma}_{a,s_0}^{(n)}(0+0) = \widetilde{\gamma}_{a,s_0}^{(n)}(1-0) = \mathbf{0}$ for all $n \in \mathbb{N}$,

(iii) $\widetilde{\gamma}_{a,s_0}([0,1]) = \gamma([a,s_0])$.

By Lemma 2.1, we obtain $\widetilde{\gamma}_{a,s_0}(0) = \gamma(\psi_{a,s_0}(0)) = \gamma(a)$ and $\widetilde{\gamma}_{a,s_0}(1) = \gamma(\psi_{a,s_0}(1)) = \gamma(s_0)$. By the chain rule, we can calculate $\widetilde{\gamma}^{(n)}_{a,s_0}$ as a sum of products of $\gamma^{(k)}$ and $\psi^{(k)}_{a,s_0}$ for $k \in \{1, \cdots, n\}$. By Lemma 2.1, we have $\widetilde{\gamma}^{(n)}_{a,s_0}(0+0) = \widetilde{\gamma}^{(n)}_{a,s_0}(1-0) = \mathbf{0}$ for all $n \in \mathbb{N}$. Since ψ_{a,s_0} is a bijection from $[0,1]$ to $[a,s_0]$, we have $\widetilde{\gamma}_{a,s_0}([0,1]) = \gamma([a,s_0])$. Therefore, (i), (ii) and (iii) hold.

Second step: We construct a map $\widetilde{\boldsymbol{\nu}}_{a,s_0} : [0,1] \to \Delta_{n-1}$ such that $(\widetilde{\gamma}_{a,s_0}, \widetilde{\boldsymbol{\nu}}_{a,s_0}) : [0,1] \to \mathbb{R}^n \times \Delta_{n-1}$ is a framed curve. By the assumption, we have $\boldsymbol{t}(a+0)$. Consider an orthonormal $n-1$ frame $\boldsymbol{\nu}_- = (\nu_{-,1}, \cdots, \nu_{-,n-1})$ with $({}^T\boldsymbol{t}(a+0), {}^T\boldsymbol{\nu}_-) \in SO(n)$, where ${}^T\mathbf{a}$ is the transpose of a vector $\mathbf{a}$ and $SO(n)$ is the $n \times n$ special orthogonal group. Since $\boldsymbol{t}$ is the smooth unit tangent vector field along γ on $[a,s_0)$, there exists a smooth map $A \in C^\infty([a,s_0), SO(n))$ such that $\boldsymbol{t}(t) = \boldsymbol{t}(a+0)A(t)$. By the assumption, the one side derivatives $\boldsymbol{t}^{(k)}(s_0-0)$ exists for all $k \in \mathbb{N} \cup \{0\}$. We can extend A to $t = s_0$, that is, $A \in C^\infty([a,s_0], SO(n))$. Now we define $\boldsymbol{\nu}_{a,s_0} : [a,s_0] \to \Delta_{n-1}$ by $\nu_i(t) = \nu_{-,i}A(t)$ for each component $i = 1, \cdots, n-1$. Then $\widetilde{\boldsymbol{\nu}}_{a,s_0} : [0,1] \to \Delta_{n-1}$ defined by $\widetilde{\boldsymbol{\nu}}_{a,s_0}(t) = \boldsymbol{\nu}_{a,s_0}(\psi_{a,s_0}(t))$ is the required map. In fact, we have $(d/dt)\widetilde{\gamma}_{a,s_0}(t) \in \langle \widetilde{\boldsymbol{\nu}}_{a,s_0}(t) \rangle^\perp$, since

$$\begin{aligned}
\frac{d}{dt}\widetilde{\gamma}_{a,s_0}(t) \cdot \widetilde{\nu}_{a,s_0;i}(t) &= \dot{\gamma}(\psi_{a,s_0}(t))\dot{\psi}_{a,s_0}(t) \cdot \nu_{a,s_0;i}(\psi_{a,s_0}(t)) \\
&= \|\dot{\gamma}(\psi_{a,s_0}(t))\|\boldsymbol{t}(\psi_{a,s_0}(t))\dot{\psi}_{a,s_0}(t) \cdot \nu_{a,s_0;i}(\psi_{a,s_0}(t)) \\
&= \|\dot{\gamma}(\psi_{a,s_0}(t))\|\dot{\psi}_{a,s_0}(t)\boldsymbol{t}(\psi_{a,s_0}(0+0))A(\psi_{a,s_0}(t)) \cdot \nu_{-,i}A(\psi_{a,s_0}(t)) \\
&= \|\dot{\gamma}(\psi_{a,s_0}(t))\|\dot{\psi}_{a,s_0}(t)\boldsymbol{t}(\psi_{a,s_0}(0+0)) \cdot \nu_{-,i} \\
&= 0
\end{aligned}$$

for all $i = 1, \cdots, n-1$, where $\widetilde{\boldsymbol{\nu}}_{a,s_0} = (\widetilde{\nu}_{a,s_0;1}, \cdots, \widetilde{\nu}_{a,s_0;n-1})$, $\boldsymbol{\nu}_{a,s_0} = (\nu_{a,s_0;1}, \cdots, \nu_{a,s_0;n-1})$ and $\langle \widetilde{\boldsymbol{\nu}}_{a,s_0}(t) \rangle^\perp$ is the orthogonal complement of the linear space spanned by $\widetilde{\boldsymbol{\nu}}_{a,s_0}(t)$.

Third step: We define $\widetilde{\gamma}_{s_0} : [0,1] \to \mathbb{R}^n$ by a constant map $\widetilde{\gamma}_{s_0}(t) = \gamma(s_0)$ for all $t \in [0,1]$.

Fourth step: Let $\boldsymbol{\nu}_+$ be an element of Δ_{n-1} with $({}^T\boldsymbol{t}(s_0+0), {}^T\boldsymbol{\nu}_+) \in SO(n)$. We denote $({}^T\boldsymbol{t}(s_0+0), {}^T\boldsymbol{\nu}_+)$ by S_+, and $({}^T\boldsymbol{t}(s_0-0), {}^T\widetilde{\boldsymbol{\nu}}_{a,s_0}(1))$ by S_-. Note that $S_- \in SO(n)$ by the definition of $\widetilde{\boldsymbol{\nu}}_{a,s_0}$ in the second step.

We construct a map $\widetilde{\boldsymbol{\nu}}_{s_0} : [0,1] \to \Delta_{n-1}$ which connects ${}^T\widetilde{\boldsymbol{\nu}}_{a,s_0}(1)$ and ${}^T\boldsymbol{\nu}_+$. By the linear algebra, there is a C^∞-map $P_1 : [0,1] \to SO(n)$, which connects S_- and I_n, where I_n is the unit element of $SO(n)$ (see [5] for example). Further, there is a C^∞-map $P_2 : [0,1] \to SO(n)$, which connects I_n and S_+. We define $\widetilde{P}_i : [0,1] \to SO(n)$ by $\widetilde{P}_i(t) = P_i(\psi(t))$ for $i = 1, 2$. Then we obtain the required map $\widetilde{\boldsymbol{\nu}}_{s_0} : [0,1] \to \Delta_{n-1}$ by $\widetilde{\boldsymbol{\nu}}_{s_0}(t) = ({}^T(\widetilde{P}_2 * \widetilde{P}_1)_2(t), \cdots, {}^T(\widetilde{P}_2 * \widetilde{P}_1)_n(t))$, where $(\widetilde{P}_2 * \widetilde{P}_1)_k$ is the k-th column of the matrix $(\widetilde{P}_2 * \widetilde{P}_1)$. By Lemma 2.2, the map $\widetilde{\boldsymbol{\nu}}_{s_0}$is a C^∞-map. Since $\widetilde{\gamma}_{s_0}$ is a constant map, $(\widetilde{\gamma}_{s_0}, \widetilde{\boldsymbol{\nu}}_{s_0}) : [0,1] \to \mathbb{R}^n \times \Delta_{n-1}$ is also a framed curve.

Fifth step: Similar to the first step to the fourth step, we construct $\widetilde{\gamma}_{s_i,s_{i+1}}$, $\widetilde{\boldsymbol{\nu}}_{s_i,s_{i+1}}$, $\widetilde{\gamma}_{s_{i+1}}$, $\widetilde{\boldsymbol{\nu}}_{s_{i+1}}$, $\widetilde{\gamma}_{s_n,b}$ and $\widetilde{\boldsymbol{\nu}}_{s_n,b}$ for all $i = 1, \cdots, n-1$. Note that we can take $\widetilde{\boldsymbol{\nu}}_{s_i,s_{i+1}}$ (respectively, $\widetilde{\boldsymbol{\nu}}_{s_{i+1},b}$) such that $\widetilde{\boldsymbol{\nu}}_{s_i,s_{i+1}}(0) = \widetilde{\boldsymbol{\nu}}_{s_i}(1)$ for all $i = 1, \ldots, n-1$ (respectively, $\widetilde{\boldsymbol{\nu}}_{s_n,b}(0) = \widetilde{\boldsymbol{\nu}}_{s_n}(1)$).

Sixth step: We concatenate on the all maps, that is, we define a C^∞-map $(\widetilde{\gamma}, \widetilde{\boldsymbol{\nu}}) : [0,1] \to \mathbb{R}^n \times \Delta_{n-1}$ by

$$\widetilde{\gamma}(t) = (\widetilde{\gamma}_{s_n,b} * (\widetilde{\gamma}_{s_n} * (\cdots * (\widetilde{\gamma}_{s_0} * \widetilde{\gamma}_{a,s_0}))))(t), \ \widetilde{\boldsymbol{\nu}}(t) = (\widetilde{\boldsymbol{\nu}}_{s_n,b} * (\widetilde{\boldsymbol{\nu}}_{s_n} * (\cdots * (\widetilde{\boldsymbol{\nu}}_{s_0} * \widetilde{\boldsymbol{\nu}}_{a,s_0}))))(t).$$

By the construction, we have $\langle \dot{\widetilde{\gamma}}(t) \rangle \subset \langle \widetilde{\boldsymbol{\nu}}(t) \rangle^\perp$ for all $t \in [0,1]$. It follows that the map $(\widetilde{\gamma}, \widetilde{\boldsymbol{\nu}}) : [0,1] \to \mathbb{R}^n \times \Delta_{n-1}$ is a framed curve such that $\widetilde{\gamma}([0,1]) = \gamma([a,b])$. □

Remark 2.4 By the above construction, the boundaries 0 and 1 in the unit interval $[0,1]$ are singular points of $\widetilde{\gamma}$ in spite of a and b may be regular points of γ. On the other hand, if we use $\varphi_{s_0,a}(1-t)$ (respectively, $\varphi_{s_n,b}(t)$) instead of $\psi_{a,s_0}(t)$ (respectively $\psi_{s_n,b}(t)$), where $\varphi_{a,b} : [0,1] \to [a,b]$ is defined by $\varphi_{a,b}(t) = (e\varphi(t))b + \{1-(e\varphi(t))\}a$, then 0 (respectively, 1) is a regular point of $\widetilde{\gamma}$ if and only if a (respectively, b) is a regular point of γ.

The assumption that the limit of the derivatives of the tangent vectors $\boldsymbol{t}^{(k)}(s-0)$ and $\boldsymbol{t}^{(k)}(s+0)$ exist for all $s \in \Sigma(\gamma)$ and $k \in \mathbb{N} \cup \{0\}$ is essential. We can construct a C^∞-curve which is not the image of the framed base curves, see Example 3.4.

In the case of the domain of γ is an open interval or $\mathbb{R}$, we also have the following result.

Corollary 2.5 (1) *Let $\gamma : (a,b) \to \mathbb{R}^n$ be a C^∞-curve. Suppose that the singular set $\Sigma(\gamma)$ is finite, and the limit of the derivatives of the tangent vectors $\boldsymbol{t}^{(k)}(s-0)$ and $\boldsymbol{t}^{(k)}(s+0)$ exist for all $s \in \Sigma(\gamma)$ and $k \in \mathbb{N} \cup \{0\}$. Then there exists a framed curve $(\widetilde{\gamma}, \widetilde{\boldsymbol{\nu}}) : (0,1) \to \mathbb{R}^n \times \Delta_{n-1}$ such that $\widetilde{\gamma}((0,1)) = \gamma((a,b))$.*

(2) *Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a C^∞-curve. Suppose that the singular set $\Sigma(\gamma)$ is finite, and the limit of the derivatives of the tangent vectors $\boldsymbol{t}^{(k)}(s-0)$ and $\boldsymbol{t}^{(k)}(s+0)$ exist for all $s \in \Sigma(\gamma)$ and $k \in \mathbb{N} \cup \{0\}$. Then there exists a framed curve $(\widetilde{\gamma}, \widetilde{\boldsymbol{\nu}}) : \mathbb{R} \to \mathbb{R}^n \times \Delta_{n-1}$ such that $\widetilde{\gamma}(\mathbb{R}) = \gamma(\mathbb{R})$.*

Proof. (1) By a similar construction in the proof of Theorem 1.2, we have the result.

(2) Parameter changes preserve the conditions of the framed curves. By using (1) and a diffeomorphism between $\mathbb{R}$ and an open interval, we have the result. □

3 Examples

We give concrete examples of the construction of framed curves in the proof of Theorem 1.2. Furthermore, we give an example of a C^∞-curve which is not the image of the framed base curves.

Example 3.1 Let $\gamma : (-1,1) \to \mathbb{R}^2$ be a C^∞-curve given by

$$\gamma(t) = \begin{cases} (e^{-\frac{1}{t^2}}, 0) & \text{if } -1 < t < 0, \\ (0,0) & \text{if } t = 0, \\ (0, e^{-\frac{1}{t^2}}) & \text{if } 0 < t < 1. \end{cases}$$

Note that this curve is not a frontal (see [4, 6]). However, we can construct a framed curve $(\widetilde{\gamma}, \widetilde{\nu}) : (0,1) \to \mathbb{R}^2 \times S^1$ such that $\widetilde{\gamma}((0,1)) = \gamma((-1,1))$ by using the method in the proof of Theorem 1.2, since the singular set $\Sigma(\gamma) = \{0\}$ and the limit of the derivatives of the tangent vectors exists at the origin.

First, we define $\widetilde{\gamma}_{-1,0} : (0,1] \to \mathbb{R}^2$ by

$$\widetilde{\gamma}_{-1,0}(t) = \gamma(\psi_{-1,0}(t)) = \begin{cases} \left(\exp\left(-\frac{1}{\psi_{-1,0}(t)^2}\right), 0\right) & \text{if } 0 < t < 1, \\ (0,0) & \text{if } t = 1. \end{cases}$$

Second, we define $\widetilde{\nu}_{-1,0} : (0,1] \to S^1$ as follows. By a direct calculation, we have $\boldsymbol{t}(-1+0) = (-1,0)$ and $\nu_- = (0,-1)$. The unit tangent vector is given by $\boldsymbol{t}(t) = (-1,0)$ for all $t \in (-1,0]$.

Hence, we have $\boldsymbol{t}(t) = \boldsymbol{t}(-1+0)I_2$, for all $t \in (-1,0]$, where I_2 is the 2×2 unit matrix. Then we have the constant map $\nu_{-1,0} : (-1,0] \to S^1, \nu_{-1,0}(t) = \nu_- I_2 = \nu_-$. Now we define $\widetilde{\nu}_{-1,0} : (0,1] \to S^1$ by $\widetilde{\nu}_{-1,0}(t) = \nu_{-1,0}(\psi_{-1,0}(t)) = (0,-1)$.

Third, we define a map $\widetilde{\gamma}_0 : [0,1] \to \mathbb{R}^2$ by $\widetilde{\gamma}_0(t) = \gamma(0) = (0,0)$ for all $t \in [0,1]$.

Fourth, we define a map $\widetilde{\nu}_0 : [0,1] \to S^1$ as follows. By a direct calculation, we have $\boldsymbol{t}(0+0) = (0,1)$, $\nu_+ = (-1,0)$, $\boldsymbol{t}(0-0) = (-1,0)$ and $\widetilde{\nu}_{-1,0}(1) = (0,-1)$. Hence,

$$S_+ = ({}^T\boldsymbol{t}(0+0), {}^T\nu_+) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}$$

and

$$S_- = ({}^T\boldsymbol{t}(0-0), {}^T\widetilde{\nu}_{-1,0}(1)) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}.$$

We define maps P_1 (respectively, P_2) from S_- to I_2 (respectively, from I_2 to S_+) by

$$P_1(t) = \begin{pmatrix} \cos(1-t)\pi & -\sin(1-t)\pi \\ \sin(1-t)\pi & \cos(1-t)\pi \end{pmatrix}, P_2(t) = \begin{pmatrix} \cos\frac{t\pi}{2} & -\sin\frac{t\pi}{2} \\ \sin\frac{t\pi}{2} & \cos\frac{t\pi}{2} \end{pmatrix}.$$

Then we have $\widetilde{P}_i(t) = P_i(\psi(t))$, that is,

$$\widetilde{P_1}(t) = \begin{pmatrix} \cos(1-\psi(t))\pi & -\sin(1-\psi(t))\pi \\ \sin(1-\psi(t))\pi & \cos(1-\psi(t))\pi \end{pmatrix}, \widetilde{P_2}(t) = \begin{pmatrix} \cos\frac{\psi(t)\pi}{2} & -\sin\frac{\psi(t)\pi}{2} \\ \sin\frac{\psi(t)\pi}{2} & \cos\frac{\psi(t)\pi}{2} \end{pmatrix}.$$

Now we define

$$\widetilde{\nu}_0(t) = {}^T(\widetilde{P_2} * \widetilde{P_1})_2(t) = \begin{cases} (-\sin(1-\psi(2t))\pi, \cos(1-\psi(2t))\pi) & \text{if } 0 \le t \le 1/2, \\ \left(-\sin\frac{\psi(2t-1)\pi}{2}, \cos\frac{\psi(2t-1)\pi}{2}\right) & \text{if } 1/2 \le t \le 1. \end{cases}$$

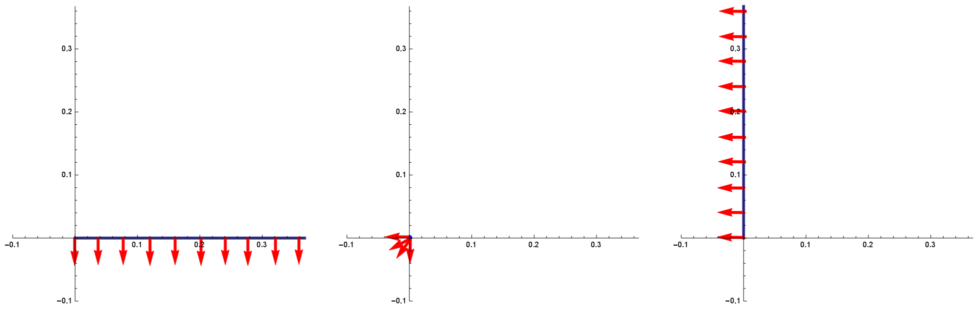
Fifth, we define $\widetilde{\gamma}_{0,1} : [0,1) \to \mathbb{R}^2$ by

$$\widetilde{\gamma}_{0,1}(t) = \gamma(\psi(t)) = \begin{cases} \left(0, \exp\left(-\frac{1}{\psi(t)^2}\right)\right) & \text{if } 0 < t < 1, \\ (0,0) & \text{if } t = 0. \end{cases}$$

Sixth, we define $\widetilde{\nu}_{0,1} : [0,1) \to S^1$ as follows. By a direct calculation, we have $\boldsymbol{t}(0+0) = (0,1)$ and $\nu_- = (-1,0)$. The unit tangent vector is given by $\boldsymbol{t}(t) = (0,1)$ for all $t \in [0,1)$. Hence, we have $\boldsymbol{t}(t) = \boldsymbol{t}(0+0)I_2$, for all $t \in [0,1)$. Then we have the constant map $\nu_{0,1} : [0,1) \to S^1$, $\nu_{0,1}(t) = \nu_- I_2 = \nu_-$. Now we define $\widetilde{\nu}_{0,1} : [0,1) \to S^1$ by $\widetilde{\nu}_{0,1}(t) = \nu_{0,1}(\psi(t)) = (-1,0)$.

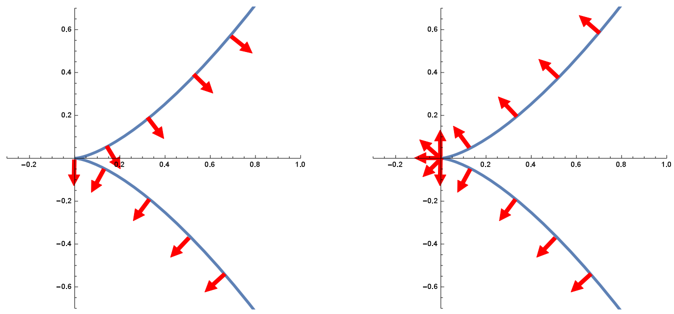
Finally, we concatenate on the all maps, that is, we define $\widetilde{\gamma} : (0,1) \to \mathbb{R}^2$ and $\widetilde{\nu} : (0,1) \to S^1$ by $\widetilde{\gamma}(t) = (\widetilde{\gamma}_{0,1} * (\widetilde{\gamma}_0 * \widetilde{\gamma}_{-1,0}))(t)$ and $\widetilde{\nu}(t) = (\widetilde{\nu}_{0,1} * (\widetilde{\nu}_0 * \widetilde{\nu}_{-1,0}))(t)$. Then we obtain a framed curve $(\widetilde{\gamma}, \widetilde{\nu}) : (0,1) \to \mathbb{R}^2 \times S^1$ such that $\widetilde{\gamma}((0,1)) = \gamma((-1,1))$, see Figure 1.

Remark 3.2 Since piece-wise smooth curves can be realised as a C^∞-curve, see Remark 2.3, it is also realised as a framed base curve by Theorem 1.2 if the conditions satisfy. It follows that polygons in the Euclidean plane can be realised as the image of a framed base curve.



$(\widetilde{\gamma}_{-1,0}(t), \widetilde{\nu}_{-1,0}(t))$ $(\widetilde{\gamma}_{0}(t), \widetilde{\nu}_{0}(t))$ $(\widetilde{\gamma}_{0,1}(t), \widetilde{\nu}_{0,1}(t))$

Figure 1: Legendre curve $(\widetilde{\gamma}, \widetilde{\nu})$. Note that the length of the unit normal vectors is modified.



$(\gamma(t), \nu(t))$ $(\widetilde{\gamma}(t), \widetilde{\nu}(t))$

Figure 2: Images of the 3/2-cusp and unit normal vector fields. Note that the length of the unit normal vectors is modified.

Example 3.3 Let $\gamma : (-1, 1) \to \mathbb{R}^2$ be a 3/2-cusp $\gamma(t) = (t^2/2, t^3/3)$ (cf. [4]). As well known, the 3/2-cusp is a front. In fact, if we take $\nu(t) = (1/\sqrt{t^2+1})(-t, 1)$ (respectively, $-\nu$), then (γ, ν) (respectively, $(\gamma, -\nu)$) is a framed curve and (γ, ν) (respectively, $(\gamma, -\nu)$) is an immersion. Both cases, the unit normal vectors change inner (outer) to outer (inner) of the curve γ around the origin, see Figure 2 left. However, we can construct a framed curve $(\widetilde{\gamma}, \widetilde{\nu}) : (0, 1) \to \mathbb{R}^2 \times S^1$ such that $\widetilde{\gamma}((0, 1)) = \gamma((-1, 1))$ and the unit normal $\widetilde{\nu}$ does not change inner and outer of the curve γ, by using the method of the proof in Theorem 1.2, see Figure 2 right.

By definition of γ, the singular set $\Sigma(\gamma) = \{0\}$ and the limit of the derivatives of the tangent vectors exists at the origin.

First, we define $\widetilde{\gamma}_{-1,0} : (0, 1] \to \mathbb{R}^2$ by

$$\widetilde{\gamma}_{-1,0}(t) = \gamma(\psi_{-1,0}(t)) = \left(\frac{1}{2}\psi_{-1,0}(t)^2, \frac{1}{3}\psi_{-1,0}(t)^3\right).$$

Second, we define $\widetilde{\nu}_{-1,0} : (0, 1] \to S^1$ as follows. By a direct calculation, we have

$$\boldsymbol{t}(-1+0) = \lim_{t \to -1+0} \frac{1}{|t|\sqrt{t^2+1}}(t, t^2) = \frac{1}{\sqrt{2}}(-1, 1)$$

and $\nu_- = (1/\sqrt{2})(-1, -1)$. The unit tangent vector is given by $\boldsymbol{t}(t) = (-1/\sqrt{t^2+1})(1, t)$ for all

$t \in (-1,0]$. Hence, we have $\boldsymbol{t}(t) = \boldsymbol{t}(-1+0)A(t)$, where

$$A(t) = \frac{-\sqrt{2}}{2\sqrt{t^2+1}} \begin{pmatrix} t-1 & -t-1 \\ t+1 & t-1 \end{pmatrix}$$

for all $t \in (-1,0]$. Then we have a map $\nu_{-1,0} : (-1,0] \to S^1$,

$$\nu_{-1,0}(t) = \nu_- A(t) = \frac{1}{\sqrt{2}}(-1,-1)\frac{-\sqrt{2}}{2\sqrt{t^2+1}} \begin{pmatrix} t-1 & -t-1 \\ t+1 & t-1 \end{pmatrix} = \frac{-1}{\sqrt{t^2+1}}(-t,1).$$

Now we define $\widetilde{\nu}_{-1,0} : (0,1] \to S^1$ by

$$\widetilde{\nu}_{-1,0}(t) = \nu_{-1,0}(\psi_{-1,0}(t)) = \frac{-1}{\sqrt{\psi_{-1,0}(t)^2+1}}(-\psi_{-1,0}(t),1).$$

Third, we define a map $\widetilde{\gamma}_0 : [0,1] \to \mathbb{R}^2$ by $\widetilde{\gamma}_0(t) = \gamma(0) = (0,0)$ for all $t \in [0,1]$.

Fourth, we define a map $\widetilde{\nu}_0 : [0,1] \to S^1$ as follows. By a direct calculation, we have $\boldsymbol{t}(0+0) = (1,0)$, $\nu_+ = (0,1)$, $\boldsymbol{t}(0-0) = (-1,0)$ and $\widetilde{\nu}_{-1,0}(1) = (0,-1)$. Hence,

$$S_+ = ({}^T\boldsymbol{t}(0+0), {}^T\nu_+) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix}$$

and

$$S_- = ({}^T\boldsymbol{t}(0-0), {}^T\widetilde{\nu}_{-1,0}(1)) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos \pi & -\sin \pi \\ \sin \pi & \cos \pi \end{pmatrix}.$$

We define a map P_1 from S_- to I_2 by

$$P_1(t) = \begin{pmatrix} \cos(1-t)\pi & -\sin(1-t)\pi \\ \sin(1-t)\pi & \cos(1-t)\pi \end{pmatrix},$$

and we define a map P_2 from I_2 to S_+ by $P_2(t) = I_2$ for all $t \in [0,1]$. Then we have $\widetilde{P}_i(t) = P_i(\psi(t))$, that is,

$$\widetilde{P_1}(t) = \begin{pmatrix} \cos(1-\psi(t))\pi & -\sin(1-\psi(t))\pi \\ \sin(1-\psi(t))\pi & \cos(1-\psi(t))\pi \end{pmatrix}, \ \widetilde{P_2}(t) = I_2$$

for all $t \in [0,1]$. Now we define

$$\widetilde{\nu_0}(t) = {}^T(\widetilde{P_2} * \widetilde{P_1})_2(t) = \begin{cases} (-\sin(1-\psi(2t))\pi, \cos(1-\psi(2t))\pi) & \text{if } 0 \le t \le 1/2, \\ (0,1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Fifth, we define $\widetilde{\gamma}_{0,1} : [0,1) \to \mathbb{R}^2$ by

$$\widetilde{\gamma}_{0,1} = \gamma(\psi(t)) = \left(\frac{1}{2}\psi_{-1,0}(t)^2, \frac{1}{3}\psi_{-1,0}(t)^3\right).$$

Sixth, we define $\widetilde{\nu}_{0,1} : [0,1) \to S^1$ as follows. By a direct calculation, we have $\boldsymbol{t}(0+0) = (1,0)$ and $\nu_- = (0,1)$. The unit tangent vector is given by $\boldsymbol{t}(t) = (1/\sqrt{t^2+1})(1,t)$ for all $t \in [0,1)$. Hence, we have $\boldsymbol{t}(t) = \boldsymbol{t}(0+0)A(t)$, where

$$A(t) = \frac{1}{\sqrt{t^2+1}} \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix}$$

for all $t \in [0,1)$. Then we have a map $\nu_{0,1} : (-1,0] \to S^1$,

$$\nu_{0,1}(t) = \nu_- A(t) = (0,1)\frac{1}{\sqrt{t^2+1}}\begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix} = \frac{1}{\sqrt{t^2+1}}(-t,1).$$

Now we define $\widetilde{\nu}_{0,1} : [0,1) \to S^1$ by

$$\widetilde{\nu}_{0,1}(t) = \nu_{0,1}(\psi(t)) = \frac{1}{\sqrt{\psi(t)^2+1}}(-\psi(t),1).$$

Finally, we concatenate all maps, that is, we define $\widetilde{\gamma} : (0,1) \to \mathbb{R}^2$ and $\widetilde{\nu} : (0,1) \to S^1$ by $\widetilde{\gamma}(t) = (\widetilde{\gamma}_{0,1} * (\widetilde{\gamma}_0 * \widetilde{\gamma}_{-1,0}))(t)$ and $\widetilde{\nu}(t) = (\widetilde{\nu}_{0,1} * (\widetilde{\nu}_0 * \widetilde{\nu}_{-1,0}))(t)$, respectively. Then we obtain a framed curve $(\widetilde{\gamma}, \widetilde{\nu}) : (0,1) \to \mathbb{R}^2 \times S^1$ such that $\gamma((-1,1)) = \widetilde{\gamma}((0,1))$.

Example 3.4 Let $\gamma : [0,1] \to \mathbb{R}^2$ be given by

$$\gamma(t) = \begin{cases} \left(e^{-1/t}\cos\frac{1}{t}, e^{-1/t}\sin\frac{1}{t}\right) & \text{if } 0 < t \leq 1, \\ (0,0) & \text{if } t = 0, \end{cases}$$

see Figure 3. Since $\gamma^{(n)}$ is given by a sum of products of $\varphi^{(k)}$, $\sin^{(k)}$, $\cos^{(k)}$, $(1/t)^{(k)}$ for $k \in \{0,1,\cdots,n\}$ and $\gamma^{(n)}(0+0) = \mathbf{0}$ for all $n \in \mathbb{N}$, γ is a C^∞-curve. The singular set $\Sigma(\gamma) = \{0\}$. However, the unit tangent vector is given by

$$\boldsymbol{t}(t) = \frac{1}{\sqrt{2}}\left(\cos\frac{1}{t} + \sin\frac{1}{t}, \sin\frac{1}{t} - \cos\frac{1}{t}\right)$$

on $(0,1]$. The limit of the tangent vector $\boldsymbol{t}(0+0)$ and hence the limit of a unit normal vector $\boldsymbol{\nu}(0+0)$ oscillate. Therefore, we can not extend the unit normal vector $\boldsymbol{\nu}$ to $[0,1]$. This means that there are no framed curves $(\widetilde{\gamma}, \widetilde{\boldsymbol{\nu}}) : I \to \mathbb{R}^2 \times S^1$ such that $\widetilde{\gamma}(I) = \gamma([0,1])$.

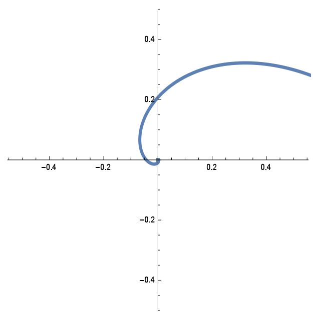


Figure 3: An example of the image of a curve which can not be the image of a framed base curve.

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