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# Lefschetz invariants and Young characters for representations of the hyperoctahedral groups

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## Abstract

The ring  $R(B_n)$  of virtual  $\mathbb{C}$ -characters of the hyperoctahedral group  $B_n$  has two  $\mathbb{Z}$ -bases consisting of permutation characters, and the ring structure associated with each basis of them defines a partial Burnside ring of which  $R(B_n)$  is a homomorphic image. In particular, the concept of Young characters of  $B_n$  arises from a certain set  $\mathcal{U}_n$  of subgroups of  $B_n$ , and the  $\mathbb{Z}$ -basis of  $R(B_n)$  consisting of Young characters, which is presented by L. Geissinger and D. Kinch [7], forces  $R(B_n)$  to be isomorphic to a partial Burnside ring  $\Omega(B_n, \mathcal{U}_n)$ . The linear  $\mathbb{C}$ -characters of  $B_n$  are analyzed with reduced Lefschetz invariants which characterize the unit group of  $\Omega(B_n, \mathcal{U}_n)$ . The parabolic Burnside ring  $\mathcal{PB}(B_n)$  is a subring of  $\Omega(B_n, \mathcal{U}_n)$ , and the unit group of  $\mathcal{PB}(B_n)$  is isomorphic to the four group. The unit group of the parabolic Burnside ring of the even-signed permutation group  $D_n$  is also isomorphic to the four group.

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## 1 Introduction

Let  $G$  be a finite group, and let  $G\text{-set}$  be the category of finite left  $G$ -sets and  $G$ -equivariant maps. The Burnside ring  $\Omega(G)$ , which is the Grothendieck ring of the category  $G\text{-set}$ , is the commutative unital ring consisting of all  $\mathbb{Z}$ -linear combinations of isomorphism classes  $[X]$  of finite left  $G$ -sets  $X$  with disjoint union for addition and cartesian product for multiplication. We denote by  $R(G)$  the ring of virtual  $\mathbb{C}$ -characters of  $G$ . Set  $[n] = \{1, 2, \dots, n\}$ , and let  $S_n$  be the symmetric group on  $[n]$ . We denote by  $\mathcal{Y}_n$  the set of Young subgroups of  $S_n$ , which is closed under intersection and conjugation. By [15, §7],  $\Omega(S_n)$  possesses the partial Burnside ring  $\Omega(S_n, \mathcal{Y}_n)$  relative to the Young subgroups as a subring, and  $\Omega(S_n, \mathcal{Y}_n) \cong R(S_n)$ . This fact means that the characters  $1_Y^{S_n}$  induced from the trivial characters  $1_Y$  of  $Y$  for  $Y \in \mathcal{Y}_n$  form a  $\mathbb{Z}$ -basis of  $R(S_n)$  (see, *e.g.*, [2, Proposition 3]). Let  $C_2$  be a cyclic group of order 2, and let  $V_n$  be the direct product  $C_2^{(n)}$  of  $n$  copies of  $C_2$ . We denote by  $B_n$  the hyperoctahedral group, that is, the wreath product  $C_2 \wr S_n$  defined to be a semidirect product  $V_n \rtimes S_n$  of  $V_n$  with  $S_n$ . Let  $\mathcal{Z}_n$  be the set of all products  $KY$  of  $K \leq V_n$  and  $Y \in \mathcal{Y}_n$  with  $|V_n : K| \leq 2$  and  $Y \leq N_{S_n}(K)$ . We establish in §3 that  $R(B_n)$  is a homomorphic image of the partial Burnside ring  $\Omega(B_n, \tilde{\mathcal{Z}}_n)$  relative to the set  $\tilde{\mathcal{Z}}_n$  of intersections of subgroups contained in  $\mathcal{Z}_n$ .

For a ring  $R$ , we denote by  $R^\times$  the unit group of  $R$ . By [13, Example 2],  $R(S_n)^\times$  is isomorphic to the four group. There exists a unit of  $\Omega(S_n, \mathcal{Y}_n)$  which enables us to describe the sign character  $\text{sgn}_n : S_n \rightarrow \mathbb{C}$  as a  $\mathbb{Z}$ -linear combination of the characters  $1_Y^{S_n}$  for  $Y \in \mathcal{Y}_n$  (see [2, Corollary 2] and [9, §4]); such a description is called Solomon's formula. The ring  $R(B_n)$  includes exactly four linear  $\mathbb{C}$ -characters, and  $R(B_n)^\times$  is generated by the nontrivial linear  $\mathbb{C}$ -characters and  $-1_{B_n}$ . In §4 we identify  $R(B_n)^\times$  with a subgroup of  $\Omega(B_n, \tilde{\mathcal{Z}}_n)^\times$ , and then describe the linear  $\mathbb{C}$ -characters of  $B_n$  as  $\mathbb{Z}$ -linear combinations of the characters  $1_H^{B_n}$  for  $H \in \mathcal{Z}_n$ .

There is a set  $\mathcal{U}_n$  of subgroups of  $B_n$  such that the characters  $1_H^{B_n}$  for  $H \in \mathcal{U}_n$  form a  $\mathbb{Z}$ -basis of  $R(B_n)$  (cf. [7, Corollary II.4]). In §5 we define the partial Burnside ring  $\Omega(B_n, \mathcal{U}_n)$  relative to the Young subgroups of  $B_n$ , which is a subring of  $\Omega(B_n)$  isomorphic to  $R(B_n)$ . The parabolic Burnside ring  $\mathcal{PB}(B_n)$  (cf. [1, §4]) is a subring of  $\Omega(B_n, \mathcal{U}_n)$ . By [4, (66.29) Corollary], the sign character  $\varepsilon_n : B_n \rightarrow \mathbb{C}$  is described as a  $\mathbb{Z}$ -linear combination of the characters  $1_H^{B_n}$  for parabolic subgroups  $H$  of  $B_n$ , whence  $\mathcal{PB}(B_n)$  includes a unit  $\alpha_n$  corresponding to  $\varepsilon_n : B_n \rightarrow \mathbb{C}$ . There also is a unit  $\beta_n$  of  $\Omega(B_n, \mathcal{U}_n)$  corresponding to a natural extension of  $\text{sgn}_n : S_n \rightarrow \mathbb{C}$  to  $B_n$  such that  $\alpha_n \beta_n$  corresponds to the restriction of  $\text{sgn}_{2n} : S_{2n} \rightarrow \mathbb{C}$  to  $B_n$ . By the description of  $\beta_n$  in terms of the characters  $1_H^{B_n}$  for  $H \in \mathcal{Z}_n \cap \mathcal{U}_n$ , we have

$$\beta_n \in \Omega(B_n, \tilde{\mathcal{Z}}_n)^\times \cap (\Omega(B_n, \mathcal{U}_n)^\times - \mathcal{PB}(B_n)^\times),$$

which proves  $\mathcal{PB}(B_n)^\times$  to be isomorphic to the four group.

Let  $X \in G\text{-set}$ . To explore the units of  $\Omega(G)$ , we are mainly concerned with the reduced Lefschetz invariant  $\tilde{\Lambda}_{P(X)}$  of the  $G$ -poset  $P(X)$  consisting of nonempty

and proper subsets of  $X$ . The reduced Euler-Poincaré characteristic  $\tilde{\chi}(P(X)^K)$  of the set of  $K$ -invariants  $P(X)^K$  in  $P(X)$  with  $K \leq G$  is  $(-1)^{|K \setminus X|}$ , so that  $\tilde{\Lambda}_{P(X)}$  is a unit of  $\Omega(G)$  (cf. [11, §5]). As a sequel to this fact, the linear  $\mathbb{C}$ -characters of  $B_n$  are analyzed with reduced Lefschetz invariants which characterize  $\Omega(B_n, \mathcal{U}_n)^\times$ .

Let  $D_n$  be the group of even-signed permutations on  $[n]$ , which is also a Coxeter group of type  $D$ . In §6 we explore the units of the parabolic Burnside ring of  $D_n$ .

## 2 Lefschetz invariant

Following [4, §80], we review the Burnside ring of  $G$  and related facts. Let  $\mathbf{F}(G)$  be the free abelian group on the set of isomorphism classes of finite left  $G$ -sets. Given  $X \in G\text{-set}$ , we denote by  $\overline{X}$  the isomorphism class of left  $G$ -sets including  $X$ . Let  $\mathbf{F}(G)_0$  be the subgroup of  $\mathbf{F}(G)$  generated by the elements  $\overline{X_1 \dot{\cup} X_2} - \overline{X_1} - \overline{X_2}$  for  $X_1, X_2 \in G\text{-set}$ . We define a multiplication on the generators of  $\mathbf{F}(G)$  by

$$\overline{X_1} \cdot \overline{X_2} = \overline{X_1 \times X_2},$$

where  $X_1 \times X_2$  is the cartesian product of  $X_1$  and  $X_2$ , and extend it to  $\mathbf{F}(G)$  by  $\mathbb{Z}$ -linearly. Then  $\mathbf{F}(G)$  is a commutative unital ring, and  $\mathbf{F}(G)_0$  is an ideal of  $\mathbf{F}(G)$ . We define a commutative unital ring  $\Omega(G)$  to be the quotient  $\mathbf{F}(G)/\mathbf{F}(G)_0$ , and call it the Burnside ring of  $G$ . For each  $X \in G\text{-set}$ , let  $[X]$  be the coset  $\overline{X} + \mathbf{F}(G)_0$  of  $\mathbf{F}(G)_0$  in  $\mathbf{F}(G)$  represented by  $\overline{X}$ . Then by [4, (80.4) Lemma],  $[X_1] = [X_2]$  if and only if  $\overline{X_1} = \overline{X_2}$ . Hence we may regard  $[X]$  as the isomorphism class of left  $G$ -sets including  $X \in G\text{-set}$ . Multiplication on the generators of  $\Omega(G)$  is given by

$$[X_1] \cdot [X_2] = [X_1 \times X_2].$$

Let  $\mathcal{C}(G)$  be a full set of non-conjugate subgroups of  $G$ . Given  $H \leq G$ , we denote by  $G/H$  the set of left cosets  $gH$ ,  $g \in G$ , of  $H$  in  $G$ , and make  $G/H$  into a left  $G$ -set by defining  $d(gH) = dgH$  for all  $d, g \in G$ . For  $H, K \leq G$ ,  $G/H \simeq G/K$  if and only if  $H$  is a conjugate of  $K$  (cf. [4, (80.5) Proposition]). The elements  $[G/H]$  for  $H \in \mathcal{C}(G)$  form a free  $\mathbb{Z}$ -basis of  $\Omega(G)$ . We have

$$[G/H] \cdot [G/U] = \sum_{HgU \in H \setminus G/U} [G/(H \cap {}^gU)] \quad (1)$$

for all  $H, U \leq G$ , where  ${}^gU = gUg^{-1}$  (cf. [4, §80 Exercise 2]). The identity of  $\Omega(G)$  is  $[G/G]$ . For shortness' sake, we usually write  $1 = [G/G]$ .

Let  $H \leq G$ . For each  $X \in G\text{-set}$ , we denote by  $\text{inv}_H(X)$  or  $X^H$  the set of  $H$ -invariants in  $X$ . There exists a ring homomorphism  $\phi_H : \Omega(G) \rightarrow \mathbb{Z}$  given by

$$[G/U] \mapsto |\text{inv}_H(G/U)|$$

for all  $U \in \mathcal{C}(G)$ . For each  $X \in G\text{-set}$ , it is obvious that

$$\phi_H([X]) = |X^H|.$$

We set  $\tilde{\Omega}(G) = \prod_{H \in \mathcal{C}(G)} \mathbb{Z}$ , and define a map  $\phi : \Omega(G) \rightarrow \tilde{\Omega}(G)$  by

$$x \mapsto (\phi_H(x))_{H \in \mathcal{C}(G)}$$

for all  $x \in \Omega(G)$ . By [4, (80.12) Proposition], this map is a ring monomorphism. We call  $\tilde{\Omega}(G)$  the ghost ring of  $\Omega(G)$ , and call  $\phi : \Omega(G) \rightarrow \tilde{\Omega}(G)$  the Burnside homomorphism or the mark homomorphism. Obviously,  $\tilde{\Omega}(G)^\times = \prod_{H \in \mathcal{C}(G)} \mathbb{Z}^\times$ . Hence  $\tilde{\Omega}(G)^\times$  is an elementary abelian 2-group, and so is  $\Omega(G)^\times$ .

We turn to the concept of (reduced) Lefschetz invariants for finite  $G$ -sets. A finite (left)  $G$ -set  $P$  equipped with order relation  $\leq$  is called a finite  $G$ -poset if  $\leq$  is invariant under the  $G$ -action. Let  $P$  be a finite  $G$ -poset. For each nonnegative integer  $n$ , we denote by  $Sd_n(P)$  the set of chains  $p_0 < p_1 < \cdots < p_n$  of elements of  $P$  of cardinality  $n + 1$ , and make  $Sd_n(P)$  into a  $G$ -set by defining

$$g(p_0 < p_1 < \cdots < p_n) = gp_0 < gp_1 < \cdots < gp_n$$

for all  $g \in G$  and  $p_0 < p_1 < \cdots < p_n \in Sd_n(P)$ . The Lefschetz invariant  $\Lambda_P$  of  $P$  and the reduced Lefschetz invariant  $\tilde{\Lambda}_P$  of  $P$  are two elements of  $\Omega(G)$  given by

$$\Lambda_P = \sum_{i=0}^{\infty} (-1)^i [Sd_i(P)] \quad \text{and} \quad \tilde{\Lambda}_P = \Lambda_P - 1,$$

respectively, which are introduced by Thévenaz (cf. [3, 11]).

Given  $X \in G\text{-set}$ , we denote by  $P(X)$  the  $G$ -poset consisting of nonempty and proper subsets of  $X$ , and explore  $\tilde{\Lambda}_{P(X)}$  from the point of view of combinatorics.

**Definition 2.1** Let  $X \in G\text{-set}$ . Given  $X_0 \in G\text{-set}$ , we define a finite left  $G$ -set  $\text{Map}(X, X_0)$  to be the set of maps from  $X$  to  $X_0$  with the action given by

$$(gf)(x) = gf(g^{-1}x)$$

for all  $g \in G$ ,  $f \in \text{Map}(X, X_0)$ , and  $x \in X$  (cf. [5, §2]). Given a nonnegative integer  $i$  and  $X_0, X_1, \dots, X_i \in G\text{-set}$ , we denote by  $\text{Map}(X, X_0, X_1, \dots, X_i)$  the set of all  $f \in \text{Map}(X, X_0 \dot{\cup} X_1 \dot{\cup} \cdots \dot{\cup} X_i)$  such that  $\text{Im } f \cap X_j \neq \emptyset$  for any  $j = 1, 2, \dots, i$ , and make it into a left  $G$ -set by defining

$$(gf)(x) = gf(g^{-1}x)$$

for all  $g \in G$ ,  $f \in \text{Map}(X, X_0, X_1, \dots, X_i)$ , and  $x \in X$ .

**Lemma 2.2** Let  $X \in G\text{-set}$ . Set  $n = |X|$  and  $X_1 = \cdots = X_n = G/G$ . Then

$$\tilde{\Lambda}_{P(X)} = \sum_{i=1}^n (-1)^i [\text{Map}(X, \emptyset, X_1, \dots, X_i)].$$

*Proof.* Obviously,  $[\text{Map}(X, \emptyset, X_1)] = [\text{Map}(X, G/G)] = 1$ . We assume that  $2 \leq i \leq n$ , and define a bijection  $\Delta : \text{Map}(X, \emptyset, X_1, \dots, X_i) \rightarrow \text{Sd}_{i-2}(P(X))$  by

$$f \mapsto p_0 < p_1 < \dots < p_{i-2},$$

where

$$p_k = \{x \in X \mid f(x) \in X_j \text{ for some } j \in \{1, 2, \dots, k+1\}\}$$

for each integer  $k$  with  $0 \leq k \leq i-2$ . Let  $g \in G$ , and let  $f \in \text{Map}(X, \emptyset, X_1, \dots, X_i)$ . We have  $(gf)(gx) = f(x)$  for any  $x \in X$ . Hence, if  $\Delta(f) = p_0 < p_1 < \dots < p_{i-2}$ , then  $\Delta(gf) = gp_0 < gp_1 < \dots < gp_{i-2}$ . Consequently, we have

$$[\text{Map}(X, \emptyset, X_1)] = 1 \quad \text{and} \quad [\text{Map}(X, \emptyset, X_1, \dots, X_i)] = [\text{Sd}_{i-2}(P(X))]$$

for all integer  $i$  with  $2 \leq i \leq n$ , which implies that

$$\tilde{\Lambda}_{P(X)} = -1 + \sum_{i=0}^{\infty} (-1)^i [\text{Sd}_i(P(X))] = \sum_{i=1}^n (-1)^i [\text{Map}(X, \emptyset, X_1, \dots, X_i)].$$

This completes the proof.  $\square$

By Eq.(1), the set  $\Omega(G)^+$  consisting of all elements  $\sum_{U \in \mathcal{C}(G)} \ell_U [G/U]$ ,  $\ell_U \geq 0$ , of  $\Omega(G)$  is an additive semigroup closed under multiplication. We fix  $X \in G\text{-set}$ , and define a multiplicative map  $\text{Map}(X, -) : \Omega(G)^+ \rightarrow \Omega(G)$  by

$$[Y] \mapsto [\text{Map}(X, Y)]$$

for all  $Y \in G\text{-set}$ . There exists a unique polynomial map (multiplicative map)  $(-)^{[X]} : \Omega(G) \rightarrow \Omega(G)$ ,  $y \mapsto y^{[X]}$  extending  $\text{Map}(X, -)$  (see [5, §2] and [14, §3]). If  $X = X_1 \dot{\cup} X_2$ , then  $y^{[X]} = y^{[X_1]} \cdot y^{[X_2]}$  for any  $y \in \Omega(G)$ .

By [14, Lemma 3.6],  $\phi((-1)^{[X]}) = ((-1)^{|K \setminus X|})_{K \in \mathcal{C}(G)}$ , where  $K \setminus X$  is the set of  $K$ -orbits in  $X$ , and thus  $(-1)^{[X]} \in \Omega(G)^\times$ . The following proposition is equivalent to [9, Proposition 4.1] and [11, Proposition 5.1].

**Proposition 2.3** *For any  $X \in G\text{-set}$ ,  $\tilde{\Lambda}_{P(X)} = (-1)^{[X]} \in \Omega(G)^\times$ .*

We derive Proposition 2.3 from the combinatorial identity

$$(-1)^n = \sum_{i=1}^n (-1)^i S(n, i) i!, \quad (2)$$

where  $S(n, i)$  is the Stirling number of the second kind (cf. [10, (24d)]). While Eq.(2) is equivalent to [9, Lemma 4.2], the former is nicer than the later for our argument based on entry 3 of the Twelvelfold Way (cf. [10, p. 33]).

*Proof of Proposition 2.3.* Set  $n = |X|$  and  $X_1 = \cdots = X_n = G/G$ . By Lemma 2.2,

$$\tilde{\Lambda}_{P(X)} = \sum_{i=1}^n (-1)^i [\text{Map}(X, \emptyset, X_1, \dots, X_i)].$$

Let  $K \in \mathcal{C}(G)$ , and set  $m_K = |K \setminus X|$ . Then for each integer  $i$  with  $1 \leq i \leq n$ ,

$$|\text{Map}(X, \emptyset, X_1, \dots, X_i)^K| = S(m_K, i)i!,$$

because  $S(m_K, i)$  is the number of partitions of an  $m_K$ -set into  $i$  nonempty subsets. Combining the preceding facts with Eq.(2), we have

$$\phi(\tilde{\Lambda}_{P(X)}) = \left( \sum_{i=1}^{m_K} (-1)^i S(m_K, i)i! \right)_{K \in \mathcal{C}(G)} = ((-1)^{m_K})_{K \in \mathcal{C}(G)},$$

completing the proof.  $\square$

*Remark 2.4* For each  $X \in G\text{-set}$ , the elements  $y^{[X]}$  for  $y \in \Omega(G)$ , which may be called exponentials, were introduced by A. Dress (cf. [5, §2]), including  $(-1)^{[X]}$  (cf. [5, §3]), and the fact that  $\phi(\tilde{\Lambda}_{P(X)}) = ((-1)^{|K \setminus X|})_{K \in \mathcal{C}(G)}$  was generalized in terms of the exponentials (see [12, §6] and [14, §3]).

### 3 The character ring of $B_n$

Set  $C_2 = \mathbb{Z}^\times$ , and let  $V_n$  be the direct product  $C_2^{(n)}$  of  $n$  copies of  $C_2$ . The wreath product  $B_n := C_2 \wr S_n$  of  $C_2$  with  $S_n$  is defined to be the semidirect product

$$V_n \rtimes S_n = \{(x_1, \dots, x_n)\sigma \mid (x_1, \dots, x_n) \in V_n \text{ and } \sigma \in S_n\}$$

in which each permutation on  $[n]$  acts as an inner automorphism on  $V_n$ :

$$\sigma(x_1, \dots, x_n)\sigma^{-1} = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}).$$

If  $L \leq V_n$  or if  $F \leq S_n$ , then we regard  $L$  or  $F$  as a subgroup of  $B_n$ . Given  $K \leq V_n$  and  $F \leq N_{S_n}(K) := N_{B_n}(K) \cap S_n$ ,  $KF$  is the semidirect product  $K \rtimes F$ .

Given  $J \subset [n]$ , we denote by  $S_J$  the symmetric group on  $J$ , and view it as a subgroup of  $S_n$ . For a cycle type  $\lambda = (1^{m_1}, \dots, n^{m_n})$  of a permutation on  $[n]$ , let  $S_\lambda$  denote a Young subgroup of  $S_n$  isomorphic to  $S_1^{(m_1)} \times \cdots \times S_n^{(m_n)}$ , where each  $S_i^{(m_i)}$  is the direct product of  $m_i$  copies of  $S_i$ .

Let  $J \subset [n]$ . There exists a linear  $\mathbb{C}$ -character  $\vartheta_J$  of  $V_n$  given by

$$\vartheta_J((x_1, \dots, x_n)) = \vartheta(x_1) \cdots \vartheta(x_n) \quad \text{with} \quad \vartheta(x_j) = \begin{cases} x_j & \text{if } j \in J, \\ 1 & \text{otherwise} \end{cases}$$

for all  $(x_1, \dots, x_n) \in V_n$ . Set  $\bar{J} = [n] - J$ . The inertia group  $I_{B_n}(\vartheta_J)$  of  $\vartheta_J$ , which is defined to be  $\{a \in B_n \mid \vartheta_J(aba^{-1}) = \vartheta_J(b) \text{ for all } b \in V_n\}$ , is

$$V_n(S_J S_{\bar{J}}) = \{(x_1, \dots, x_n)\sigma \in B_n \mid (x_1, \dots, x_n) \in V_n \text{ and } \sigma \in S_J S_{\bar{J}}\}$$

(cf. [8, Lemma 25.5]). There exists an extension  $\widehat{\vartheta}_J$  of  $\vartheta_J$  to  $I_{B_n}(\vartheta_J)$  given by

$$\widehat{\vartheta}_J((x_1, \dots, x_n)\sigma) = \vartheta_J((x_1, \dots, x_n))$$

for all  $(x_1, \dots, x_n) \in V_n$  and  $\sigma \in S_J S_{\bar{J}}$ . Obviously,  $I_{B_n}(\vartheta_J)/V_n \simeq S_J S_{\bar{J}}$ . For a  $\mathbb{C}$ -character  $\psi$  of  $S_J S_{\bar{J}}$ , we denote by  $\widehat{\psi}$  the  $\mathbb{C}$ -character of  $I_{B_n}(\vartheta_J)$  given by

$$\widehat{\psi}(g\sigma) = \psi(\sigma)$$

for all  $g \in V_n$  and  $\sigma \in S_J S_{\bar{J}}$ . Set  $K_J = \ker \vartheta_J$ . Then  $S_J S_{\bar{J}} \leq I_{B_n}(\vartheta_J) \leq N_{B_n}(K_J)$ .

For each integer  $i$  with  $0 \leq i \leq n$ , we indicate with  $[i] \subset [n]$  that  $[i]$  is the subset  $\{1, 2, \dots, i\}$  of  $[n]$ , where  $[0]$  is the empty set.

Let  $[i] \subset [n]$ . We write  $\vartheta_i = \vartheta_{[i]}$ ,  $K_i = \ker \vartheta_i$ ,  $S_i = S_{[i]}$ , and  $S_{\bar{i}} = S_{[n] \setminus [i]}$  for shortness' sake. Let  $\text{Irr}(S_i S_{\bar{i}})$  be the set of irreducible  $\mathbb{C}$ -characters of  $S_i S_{\bar{i}}$ .

The following proposition is well-known (cf. [7, §II]).

**Proposition 3.1** *The irreducible  $\mathbb{C}$ -characters of  $B_n$  consist of the  $\mathbb{C}$ -characters  $(\widehat{\vartheta}_i \widehat{\psi})^{B_n}$  induced from the product  $\widehat{\vartheta}_i \widehat{\psi}$  of  $\widehat{\vartheta}_i$  and  $\widehat{\psi}$  for  $[i] \subset [n]$  and  $\psi \in \text{Irr}(S_i S_{\bar{i}})$ .*

Let  $J \subset [n]$ , and let  $\mathcal{P}(J)$  be the set of cycle types of permutations on  $J$ . We write  $\mathcal{P}(n) = \mathcal{P}([n])$ . Recall that for each  $\lambda \in \mathcal{P}(J)$  ( $= \mathcal{P}(|J|)$ ),  $S_\lambda$  denotes a Young subgroup of  $S_{|J|}$ . We set  $\mathcal{P}(J, \bar{J}) = \mathcal{P}(J) \times \mathcal{P}(\bar{J})$ . Given  $(\lambda_J, \lambda_{\bar{J}}) \in \mathcal{P}(J, \bar{J})$ , let  $S_{\lambda_J \lambda_{\bar{J}}}$  denote the product  $HK$  of a subgroup  $H$  of  $S_J$  and a subgroup  $K$  of  $S_{\bar{J}}$  such that  $H$  is a conjugate of  $S_{\lambda_J}$  in  $S_n$  and  $K$  is a conjugate of  $S_{\lambda_{\bar{J}}}$  in  $S_n$ .

For each  $X \in G\text{-set}$ , let  $\pi_X$  be the permutation character of  $G$  which assigns each  $g \in G$  the number of fixed elements of  $X$  by  $g$ , that is,  $\pi_X(g) = |X^{(g)}|$ . For each  $H \leq G$ ,  $\pi_{G/H}$  is the character  $1_H^G$  induced from the trivial character  $1_H$  of  $H$ .

**Theorem 3.2** *The characters  $1_{K_i S_{\lambda_i \lambda_{\bar{i}}}}^{B_n}$  induced from the trivial characters  $1_{K_i S_{\lambda_i \lambda_{\bar{i}}}}$  of  $K_i S_{\lambda_i \lambda_{\bar{i}}}$  for  $[i] \subset [n]$  and  $(\lambda_i, \lambda_{\bar{i}}) \in \mathcal{P}([i], \bar{[i]})$  form a  $\mathbb{Z}$ -basis of  $R(B_n)$ . In particular, the number of irreducible  $\mathbb{C}$ -characters of  $B_n$  is  $\sum_{i=0}^n |\mathcal{P}([i], \bar{[i]})|$ .*

*Proof.* The second assertion is well-known, and is also an immediate consequence of the first one. Let  $J \subset [n]$ , and let  $(\lambda_J, \lambda_{\bar{J}}) \in \mathcal{P}(J, \bar{J})$ . If  $g \in V_n$  and  $\sigma \in S_J S_{\bar{J}}$ , then

$$\begin{aligned} g\sigma(h\tau K_J S_{\lambda_J \lambda_{\bar{J}}}) = h\tau K_J S_{\lambda_J \lambda_{\bar{J}}} &\iff \tau^{-1}h^{-1}(g\sigma)h\tau \in K_J S_{\lambda_J \lambda_{\bar{J}}} \\ &\iff \tau^{-1}(h^{-1}g)\tau^{-1}\sigma h\tau^{-1}\sigma\tau \in K_J S_{\lambda_J \lambda_{\bar{J}}} \\ &\iff g^\sigma h \in h^\tau K_J \text{ and } \sigma\tau \in \tau S_{\lambda_J \lambda_{\bar{J}}} \\ &\iff ghK_J = hK_J \text{ and } \sigma\tau S_{\lambda_J \lambda_{\bar{J}}} = \tau S_{\lambda_J \lambda_{\bar{J}}} \end{aligned}$$



for all  $h \in V_n$  and  $\tau \in S_J S_{\bar{J}}$ , because  $\sigma \in N_{S_n}(K_J)$  and  $|V_n : K_J| \leq 2$ , and thus

$$\begin{aligned} 1_{K_J S_{\lambda_J \lambda_{\bar{J}}}}^{I_{B_n}(\vartheta_J)}(g\sigma) &= \pi_{I_{B_n}(\vartheta_J)/(K_J S_{\lambda_J \lambda_{\bar{J}}}}(g\sigma) \\ &= \pi_{V_n/K_J}(g) \cdot \pi_{(S_J S_{\bar{J}})/S_{\lambda_J \lambda_{\bar{J}}}}(\sigma) \\ &= 1_{K_J}^{V_n}(g) \cdot 1_{S_{\lambda_J \lambda_{\bar{J}}}}^{S_J S_{\bar{J}}}(\sigma). \end{aligned}$$

In particular,  $1_{V_n S_{\lambda_{\bar{\emptyset}}}}^{I_{B_n}(\vartheta_{\emptyset})} = \widehat{1_{S_{\lambda_{\bar{\emptyset}}}}}$ . Moreover, if  $J \neq \emptyset$ , then  $\vartheta_J = 1_{K_J}^{V_n} - 1_{V_n}$  and

$$(1_{K_J S_{\lambda_J \lambda_{\bar{J}}}}^{I_{B_n}(\vartheta_J)} - \widehat{1_{S_{\lambda_J \lambda_{\bar{J}}}}^{S_J S_{\bar{J}}}})(g\sigma) = (1_{K_J}^{V_n} - 1_{V_n})(g) \cdot 1_{S_{\lambda_J \lambda_{\bar{J}}}}^{S_J S_{\bar{J}}}(\sigma) = (\widehat{\vartheta_J} \widehat{1_{S_{\lambda_J \lambda_{\bar{J}}}}^{S_J S_{\bar{J}}}})(g\sigma)$$

for all  $g \in V_n$  and  $\sigma \in S_J S_{\bar{J}}$ , and consequently,

$$1_{K_J S_{\lambda_J \lambda_{\bar{J}}}}^{B_n} = \widehat{1_{S_{\lambda_J \lambda_{\bar{J}}}}^{S_J S_{\bar{J}}}}^{B_n} + \left( \widehat{\vartheta_J} \widehat{1_{S_{\lambda_J \lambda_{\bar{J}}}}^{S_J S_{\bar{J}}}} \right)^{B_n} = \widehat{1_{S_{\lambda_J \lambda_{\bar{J}}}}^{S_n}} + \left( \widehat{\vartheta_J} \widehat{1_{S_{\lambda_J \lambda_{\bar{J}}}}^{S_J S_{\bar{J}}}} \right)^{B_n}.$$

Let  $[i] \subset [n]$ . By the above fact with  $J = [i]$  and Proposition 3.1, it suffices to verify that the characters  $1_{S_{\lambda_i \lambda_{\bar{i}}}}^{S_i S_{\bar{i}}}$  for  $(\lambda_i, \lambda_{\bar{i}}) \in \mathcal{P}([i], [\bar{i}])$  form a  $\mathbb{Z}$ -basis of  $R(S_i S_{\bar{i}})$ . We identify  $S_i S_{\bar{i}}$  and the subgroups  $S_{\lambda_i \lambda_{\bar{i}}}$  of  $S_i S_{\bar{i}}$  for  $(\lambda_i, \lambda_{\bar{i}}) \in \mathcal{P}([i], [\bar{i}])$  with  $S_i \times S_{n-i}$  and the subgroups  $S_{\mu} \times S_{\nu}$  of  $S_i \times S_{n-i}$  for  $\mu \in \mathcal{P}(i)$  and  $\nu \in \mathcal{P}(n-i)$ , respectively. By [2, Proposition 3] and [4, §9 Exercise 6], the characters  $1_{S_{\mu} \times S_{n-i}}^{S_i \times S_{n-i}} 1_{S_i \times S_{\nu}}^{S_i \times S_{n-i}}$  for  $\mu \in \mathcal{P}(i)$  and  $\nu \in \mathcal{P}(n-i)$  form a  $\mathbb{Z}$ -basis of  $R(S_i \times S_{n-i})$ . This, combined with [4, (10.19) Corollary], shows that the characters  $1_{S_{\mu} \times S_{\nu}}^{S_i \times S_{n-i}}$  for  $\mu \in \mathcal{P}(i)$  and  $\nu \in \mathcal{P}(n-i)$  form a  $\mathbb{Z}$ -basis of  $R(S_i \times S_{n-i})$ , as desired. This completes the proof.  $\square$

We quote part of [15, §3] and review the concept of generalized Burnside rings.

**Definition 3.3** For a set  $\mathcal{D}$  of subgroups of  $G$ , we define a  $\mathbb{Z}$ -lattice  $\Omega(G, \mathcal{D})$  to be an additive group consisting of all  $\mathbb{Z}$ -linear combinations of the elements  $[G/H]$  of  $\Omega(G)$  for  $H \in \mathcal{D}$ , and define  $\overline{\mathcal{D}} := \{ {}^g H \mid g \in G \text{ and } H \in \mathcal{D} \}$ .

The following theorem is a concise version of [15, 3.11 Theorem].

**Theorem 3.4** *Let  $\mathcal{D}$  be a set of subgroups of  $G$  including  $G$ , and suppose that*

$$\bigcap_{\langle g \rangle U \leq H \in \overline{\mathcal{D}}} H \in \overline{\mathcal{D}}$$

for all  $U \in \overline{\mathcal{D}}$  and  $g \in N_G(U)$ . Then  $\Omega(G, \overline{\mathcal{D}})$  has a unique ring structure such that the group homomorphism  $\Omega(G, \overline{\mathcal{D}}) \rightarrow \prod_{H \in \mathcal{C}(G) \cap \overline{\mathcal{D}}} \mathbb{Z}$  given by

$$x \mapsto (\phi_H(x))_{H \in \mathcal{C}(G) \cap \overline{\mathcal{D}}}$$

for all  $x \in \Omega(G, \overline{\mathcal{D}})$  is a ring homomorphism, and the identity of  $\Omega(G, \overline{\mathcal{D}})$  is 1. If  $\overline{\mathcal{D}}$  is closed under intersection, then  $\Omega(G, \overline{\mathcal{D}})$  is a subring of  $\Omega(G)$ .

We set  $\mathcal{X}_n = \{K_J S_{\lambda_J \lambda_{\bar{J}}} \mid J \subset [n] \text{ and } (\lambda_J, \lambda_{\bar{J}}) \in \mathcal{P}(J, \bar{J})\}$ . Let  $\mathcal{Y}_n$  be the set of Young subgroups of  $S_n$ , and let  $\mathcal{Z}_n$  be the set consisting of all products  $KY$  of  $K \leq V_n$  and  $Y \in \mathcal{Y}_n$  with  $|V_n : K| \leq 2$  and  $Y \leq N_{S_n}(K)$ . We define

$$\tilde{\mathcal{Z}}_n := \left\{ \bigcap_{H \in \mathcal{S}} H \mid \mathcal{S} \in \text{Sub}(\mathcal{Z}_n) \right\},$$

where  $\text{Sub}(\mathcal{Z}_n)$  is the set of nonempty subsets of  $\mathcal{Z}_n$ .

**Lemma 3.5** *The following statements hold.*

- (a) *The set  $\overline{\mathcal{X}}_n$  coincides with  $\mathcal{Z}_n$ . In particular,  $\mathcal{Z}_n$  is closed under conjugation.*
- (b) *The set  $\tilde{\mathcal{Z}}_n$  is closed under intersection and conjugation.*

*Proof.* Suppose that  $J \subset [n]$  and  $(\lambda_J, \lambda_{\bar{J}}) \in \mathcal{P}(J, \bar{J})$ . Let  $\sigma \in S_n$ , and let  $g \in V_n$ . Then we have  ${}^\sigma(K_J S_{\lambda_J \lambda_{\bar{J}}}) = K_{\sigma(J)} {}^\sigma S_{\lambda_J \lambda_{\bar{J}}}$ ,  ${}^\sigma S_{\lambda_J \lambda_{\bar{J}}} \in \mathcal{Y}_n$ , and  ${}^\sigma S_{\lambda_J \lambda_{\bar{J}}} \leq N_{S_n}(K_{\sigma(J)})$ , where  $\sigma(J) = \{\sigma(j) \mid j \in J\}$ . Since  $\vartheta_J(g^\tau g) = 1$  for any  $\tau \in S_J S_{\bar{J}}$ , it follows that

$${}^g(K_J S_{\lambda_J \lambda_{\bar{J}}}) = \{gh^\tau g\tau \mid h \in K_J \text{ and } \tau \in S_{\lambda_J \lambda_{\bar{J}}}\} = K_J S_{\lambda_J \lambda_{\bar{J}}}.$$

In particular,  $\overline{\mathcal{X}}_n \subset \mathcal{Z}_n$ . Suppose that  $K \leq V_n$  and  $Y \in \mathcal{Y}_n$  with  $|V_n : K| \leq 2$  and  $Y \leq N_{S_n}(K)$ . There exists a subset  $J$  of  $[n]$  such that  $K = K_J$ . For each  $\sigma \in Y$ , we have  $K_J = {}^\sigma(K_J) = K_{\sigma(J)}$ , whence  $\sigma(J) = J$  and  $Y = {}^\tau S_{\lambda_J \lambda_{\bar{J}}}$  for some  $\tau \in S_J S_{\bar{J}}$  and  $(\lambda_J, \lambda_{\bar{J}}) \in \mathcal{P}(J, \bar{J})$ . This means that  $KY$  is a conjugate of  $K_J S_{\lambda_J \lambda_{\bar{J}}}$ . Consequently,  $\overline{\mathcal{X}}_n \supset \mathcal{Z}_n$ , and the statement (a) holds. Obviously,  $\tilde{\mathcal{Z}}_n$  is closed under intersection. Hence the statement (b) follows from (a). This completes the proof.  $\square$

By Lemma 3.5,  $\tilde{\mathcal{Z}}_n$  satisfies the hypothesis of Theorem 3.4 with  $\mathcal{D} = \overline{\mathcal{D}} = \tilde{\mathcal{Z}}_n$ , so that  $\Omega(B_n, \tilde{\mathcal{Z}}_n)$  is a subring of  $\Omega(B_n)$  which is called a partial Burnside ring.

We now define a ring homomorphism  $\text{char}_G : \Omega(G) \rightarrow R(G)$  by

$$[X] \mapsto \pi_X$$

for all  $X \in G\text{-set}$  (cf. [14, §6]), and usually write  $\text{char} = \text{char}_G$  by omitting subscript  $G$ . Given  $x \in \Omega(G)$  and  $g \in G$ ,  $\text{char}(x)(g) = \phi_{\langle g \rangle}(x)$ .

We are successful in finding a natural relationship between  $\Omega(B_n, \tilde{\mathcal{Z}}_n)$  and  $R(B_n)$ .

**Theorem 3.6** *The ring homomorphism  $\text{char} : \Omega(B_n) \rightarrow R(B_n)$  induces an epimorphism from the partial Burnside ring  $\Omega(B_n, \tilde{\mathcal{Z}}_n)$  to  $R(B_n)$ .*

*Proof.* The theorem is a consequence of Theorem 3.2.  $\square$

#### 4 Units of the character ring of $B_n$

The set  $[n]$  is viewed as a left  $S_n$ -set. According to [9, Eq.(3)],

$$\tilde{\Lambda}_{\mathcal{P}([n])} = \sum_{\lambda=(1^{m_1}, \dots, n^{m_n}) \in \mathcal{P}(n)} (-1)^{m_1 + \dots + m_n} \frac{(m_1 + \dots + m_n)!}{m_1! \cdots m_n!} [S_n/S_\lambda], \quad (3)$$

so that the sign character  $\text{sgn}_n : S_n \rightarrow \mathbb{C}$  is described as

$$\text{sgn}_n = \sum_{\lambda=(1^{m_1}, \dots, n^{m_n}) \in \mathcal{P}(n)} (-1)^{m_1 + \dots + m_n + n} \frac{(m_1 + \dots + m_n)!}{m_1! \cdots m_n!} 1_{S_\lambda}^{S_n} \quad (4)$$

(see [2, Corollary 2] and [9, Theorem 4.4]). Note that the numbers

$$\frac{(m_1 + \dots + m_n)!}{m_1! \cdots m_n!}$$

for nonnegative integers  $m_1, \dots, m_n$  are multinomial coefficients (cf. [10, 1.2]).

Let  $\kappa_n : B_n \rightarrow \mathbb{C}$  be a linear  $\mathbb{C}$ -character of  $B_n$  given by

$$(x_1, \dots, x_n)\sigma \mapsto \prod_{i=1}^n x_i$$

for all  $(x_1, \dots, x_n) \in V_n$  and  $\sigma \in S_n$ . There also exists an extension  $\rho_n : B_n \rightarrow \mathbb{C}$  of the sign character  $\text{sgn}_n : S_n \rightarrow \mathbb{C}$  to  $B_n$  given by

$$(x_1, \dots, x_n)\sigma \mapsto \text{sgn}_n(\sigma)$$

for all  $(x_1, \dots, x_n) \in V_n$  and  $\sigma \in S_n$ . Let  $\varepsilon_n : B_n \rightarrow \mathbb{C}$  be the product  $\kappa_n \rho_n$  of  $\kappa_n$  and  $\rho_n$ , which coincides with the sign character of  $B_n$ .

We view the set  $\mathbb{Z}^\times = \{1, -1\}$  as a left  $B_n$ -set with the action given by

$$(x_1, \dots, x_n)\sigma.x = x \cdot \prod_{i=1}^n x_i$$

for all  $(x_1, \dots, x_n) \in V_n$ ,  $\sigma \in S_n$ , and  $x \in \mathbb{Z}^\times$ . The set  $[n]$  is naturally viewed as a left  $B_n$ -set on which  $V_n$  acts trivially. Let  $[n]^\diamond$  denote the  $B_n$ -set  $\mathbb{Z}^\times \dot{\cup} [n]$ .

**Lemma 4.1** *There are exactly three nontrivial linear  $\mathbb{C}$ -characters  $\kappa_n : B_n \rightarrow \mathbb{C}$ ,  $\rho_n : B_n \rightarrow \mathbb{C}$ , and  $\varepsilon_n : B_n \rightarrow \mathbb{C}$  defined as above in  $R(B_n)$ , and  $\kappa_n(y) = (-1)^{|\langle y \rangle \setminus \mathbb{Z}^\times|}$ ,  $\rho_n(y) = (-1)^{|\langle y \rangle \setminus [n]| + n}$ , and  $\varepsilon_n(y) = (-1)^{|\langle y \rangle \setminus [n]^\diamond| + n}$  for each  $y \in B_n$ .*

*Proof.* By Proposition 3.1, there are exactly three nontrivial linear  $\mathbb{C}$ -characters of  $B_n$ . Let  $(x_1, \dots, x_n) \in V_n$ , and let  $\sigma \in S_n$ . Set  $y = (x_1, \dots, x_n)\sigma \in B_n$ , and

assume that  $\sigma$  is a product of pairwise disjoint  $n_j$ -cycles  $\sigma_j$  for  $j = 1, 2, \dots, r$  with  $\sum_j n_j = n$ . Obviously,  $\kappa_n(y) = (-1)^{|\langle y \rangle \setminus \mathbb{Z}^\times|}$ . We have  $|\langle y \rangle \setminus [n]| = r$  and

$$|\langle y \rangle \setminus [n]^\circ| = \begin{cases} r + 1 & \text{if } \prod_{i=1}^n x_i = -1, \\ r + 2 & \text{if } \prod_{i=1}^n x_i = 1. \end{cases}$$

Moreover, if  $\ell = \#\{j \mid n_j \text{ is odd}\}$ , then  $\rho_n(y) = \text{sgn}(\sigma) = (-1)^{r-\ell} = (-1)^{r+n}$  and  $\varepsilon_n(y) = (-1)^{r+n} \prod_{i=1}^n x_i$ , because  $\ell \equiv n \pmod{2}$ . This completes the proof.  $\square$

**Lemma 4.2**  $R(B_n)^\times = \langle \kappa_n, \eta_n, -1_{B_n} \rangle$ .

*Proof.* The lemma is a consequence of [6, Theorem 5.5.6] (see also Theorem 3.2), [13, Corollary 1.2 and Lemma 2.1], and Lemma 4.1.  $\square$

We are now in position to establish the following proposition.

**Proposition 4.3** *The nontrivial linear  $\mathbb{C}$ -characters of  $B_n$  are characterized by the reduced Lefschetz invariants. Indeed,  $\kappa_n = \text{char}(\tilde{\Lambda}_{P(\mathbb{Z}^\times)})$ ,  $\rho_n = (-1)^n \text{char}(\tilde{\Lambda}_{P([n])})$ , and  $\varepsilon_n = (-1)^n \text{char}(\tilde{\Lambda}_{P([n]^\circ)})$ . The reduced Lefschetz invariants  $\tilde{\Lambda}_{P(\mathbb{Z}^\times)}$  and  $\tilde{\Lambda}_{P([n])}$ , together with  $-1$ , generate an elementary abelian subgroup of  $\Omega(B_n, \tilde{\mathcal{Z}}_n)^\times$  isomorphic to  $R(B_n)^\times$ , and  $\tilde{\Lambda}_{P([n]^\circ)} = \tilde{\Lambda}_{P([n])} \cdot \tilde{\Lambda}_{P(\mathbb{Z}^\times)}$ . Moreover,*

$$\begin{aligned} \tilde{\Lambda}_{P(\mathbb{Z}^\times)} &= [B_n / (K_n S_n)] - [B_n / B_n], \\ \tilde{\Lambda}_{P([n])} &= \sum_{\lambda=(1^{m_1}, \dots, n^{m_n}) \in \mathcal{P}(n)} (-1)^{m_1 + \dots + m_n} \frac{(m_1 + \dots + m_n)!}{m_1! \cdots m_n!} [B_n / (V_n S_\lambda)], \\ \tilde{\Lambda}_{P([n]^\circ)} &= \sum_{\lambda=(1^{m_1}, \dots, n^{m_n}) \in \mathcal{P}(n)} (-1)^{m_1 + \dots + m_n} \frac{(m_1 + \dots + m_n)!}{m_1! \cdots m_n!} [B_n / (K_n S_\lambda)] \\ &\quad - \sum_{\lambda=(1^{m_1}, \dots, n^{m_n}) \in \mathcal{P}(n)} (-1)^{m_1 + \dots + m_n} \frac{(m_1 + \dots + m_n)!}{m_1! \cdots m_n!} [B_n / (V_n S_\lambda)]. \end{aligned}$$

*Proof.* The first assertion follows from Proposition 2.3 and Lemma 4.1. We prove the last two assertions. By Lemma 2.2 with  $X = \mathbb{Z}^\times$  and  $X_1 = X_2 = B_n / B_n$ ,

$$\tilde{\Lambda}_{P(\mathbb{Z}^\times)} = -[\text{Map}(\mathbb{Z}^\times, \emptyset, X_1)] + [\text{Map}(\mathbb{Z}^\times, \emptyset, X_1, X_2)] = -[B_n / B_n] + [B_n / (K_n S_n)].$$

We obtain the description of  $\tilde{\Lambda}_{P([n])}$  in a similar fashion to the proof of [9, Eq.(3)]. By Proposition 2.3,  $\tilde{\Lambda}_{P([n]^\circ)} = \tilde{\Lambda}_{P([n])} \cdot \tilde{\Lambda}_{P(\mathbb{Z}^\times)}$ , which yields the description of  $\tilde{\Lambda}_{P([n]^\circ)}$ , and the reduced Lefschetz invariants  $\tilde{\Lambda}_{P(\mathbb{Z}^\times)}$ ,  $\tilde{\Lambda}_{P([n])}$ , and  $\tilde{\Lambda}_{P([n]^\circ)}$  are contained

in  $\Omega(B_n, \tilde{\mathcal{Z}}_n)^\times$ . Hence it follows from Lemma 4.2 that  $\tilde{\Lambda}_{P(\mathbb{Z}^\times)}$ ,  $\tilde{\Lambda}_{P([n])}$ , and  $-1$  generate an elementary abelian subgroup of  $\Omega(B_n, \tilde{\mathcal{Z}}_n)^\times$  isomorphic to  $R(B_n)^\times$ . This completes the proof.  $\square$

The following descriptions of nontrivial linear  $\mathbb{C}$ -characters of  $B_n$  are obtained; see Eq.(5) in §5 for Solomon's formula of the sign character  $\varepsilon_n : B_n \rightarrow \mathbb{C}$ .

#### Corollary 4.4

$$\begin{aligned} \kappa_n &= 1_{K_n S_n}^{B_n} - 1_{B_n}, \\ \rho_n &= \sum_{\lambda=(1^{m_1}, \dots, n^{m_n}) \in \mathcal{P}(n)} (-1)^{m_1 + \dots + m_n + n} \frac{(m_1 + \dots + m_n)!}{m_1! \cdots m_n!} 1_{V_n S_\lambda}^{B_n}, \\ \varepsilon_n &= \sum_{\lambda=(1^{m_1}, \dots, n^{m_n}) \in \mathcal{P}(n)} (-1)^{m_1 + \dots + m_n + n} \frac{(m_1 + \dots + m_n)!}{m_1! \cdots m_n!} 1_{K_n S_\lambda}^{B_n} \\ &\quad - \sum_{\lambda=(1^{m_1}, \dots, n^{m_n}) \in \mathcal{P}(n)} (-1)^{m_1 + \dots + m_n + n} \frac{(m_1 + \dots + m_n)!}{m_1! \cdots m_n!} 1_{V_n S_\lambda}^{B_n}. \end{aligned}$$

*Proof.* The corollary is an immediate consequence of Proposition 4.3. (The formulae of  $\kappa_n$  and  $\rho_n$  can also be obtained by a calculation and Eq.(4), respectively.)  $\square$

## 5 The Young subgroups of the hyperoctahedral groups

Given  $J \subset [n]$ , we define a subgroup  $L_J$  of  $V_n$  by

$$L_J = \{(x_1, \dots, x_n) \in V_n \mid x_k = 1 \text{ for all } k \in \bar{J}\}.$$

Let  $\mathcal{U}_n$  denote the set of products  $L_J S_{\lambda_J \lambda_{\bar{J}}}$  of  $L_J$  and  $S_{\lambda_J \lambda_{\bar{J}}}$  for  $J \subset [n]$  and  $(\lambda_J, \lambda_{\bar{J}}) \in \mathcal{P}(J, \bar{J})$ , and let  $\mathcal{E}_n$  denote the set of products  $L_J (S_{\lambda_J \lambda_{\bar{J}}} S_J)$  of  $L_J$  and  $S_{\lambda_J \lambda_{\bar{J}}} S_J$  for  $J \subset [n]$  and  $(\lambda_J, \lambda_{\bar{J}}) \in \mathcal{P}(J, \bar{J})$ . Obviously,  $\mathcal{E}_n \subset \mathcal{U}_n$ .

We call the subgroups  $L_J S_{\lambda_J \lambda_{\bar{J}}}$  of  $B_n$  and the characters  $1_{L_J S_{\lambda_J \lambda_{\bar{J}}}}^{B_n}$  for  $J \subset [n]$  and  $(\lambda_J, \lambda_{\bar{J}}) \in \mathcal{P}(J, \bar{J})$  the Young subgroups and the Young characters, respectively.

The sets  $\mathcal{U}_n$  and  $\mathcal{E}_n$  are closed under intersection; they are not closed under conjugation, however. Recall that  $\bar{\mathcal{D}} = \{yH \mid y \in B_n \text{ and } H \in \mathcal{D}\}$  where  $\mathcal{D}$  is  $\mathcal{U}_n$  or  $\mathcal{E}_n$ . Given  $[i] \subset [n]$  and  $\lambda \in \mathcal{P}(i)$ , we write  $L_{\bar{i}} = L_{[i]}$  and  $S_\lambda B_{n-i} = L_{\bar{i}}(S_\lambda S_{\bar{i}})$ . The set  $\bar{\mathcal{E}}_n$  consists of the conjugates of the parabolic subgroups  $S_\lambda B_{n-i}$  for  $[i] \subset [n]$  and  $\lambda \in \mathcal{P}(i)$ , and is closed under intersection (cf. [6, Exercise 2.2]). To explore  $\bar{\mathcal{U}}_n$ , we make  $\mathbb{Z}^\times \times [n]$  into a left  $B_n$ -set by defining

$$(x_1, x_2, \dots, x_n) \sigma.(x, i) = (x_{\sigma(i)} x, \sigma(i))$$

for all  $(x_1, x_2, \dots, x_n) \in V_n$ ,  $\sigma \in S_n$ , and  $(x, i) \in \mathbb{Z}^\times \times [n]$ .

**Lemma 5.1** *The set  $\overline{\mathcal{U}}_n$  is closed under intersection.*

*Proof.* Suppose that  $J_1, J_2 \subset [n]$ ,  $(\lambda_{J_1}, \lambda_{\overline{J_1}}) \in \mathcal{P}(J_1, \overline{J_1})$ ,  $(\lambda_{J_2}, \lambda_{\overline{J_2}}) \in \mathcal{P}(J_2, \overline{J_2})$ ,  $g \in V_n$ , and  $\sigma \in S_n$ . Then  ${}^g(L_{\sigma(J_1)} {}^\sigma S_{\lambda_{J_1} \lambda_{\overline{J_1}}}) \cap L_{J_2} S_{\lambda_{J_2} \lambda_{\overline{J_2}}}$  is considered to be the intersection of the stabilizers of disjoint subsets

$$N_1^+, \dots, N_k^+, N_1^-, \dots, N_k^-, N_{k+1}, \dots, N_r$$

obtained by a certain partition of  $\mathbb{Z}^\times \times [n]$  into nonempty subsets such that

$$N_i^+ = \{g_i \cdot (1, q) \mid q \in Q_i\} \quad \text{and} \quad N_i^- = \{g_i \cdot (-1, q) \mid q \in Q_i\}$$

with  $Q_i \subset [n]$  and  $g_i \in L_{Q_i}$  for  $i = 1, 2, \dots, k$  and

$$N_i = \{(1, q), (-1, q) \mid q \in Q_i\}$$

with  $Q_i \subset [n]$  for  $i = k+1, \dots, r$ . Set  $g' = g_1 \cdots g_k$  and  $J = Q_{k+1} \dot{\cup} \cdots \dot{\cup} Q_r$ . Then

$$\begin{aligned} {}^{g\sigma}(L_{J_1} S_{\lambda_{J_1} \lambda_{\overline{J_1}}}) \cap L_{J_2} S_{\lambda_{J_2} \lambda_{\overline{J_2}}} &= {}^g(L_{\sigma(J_1)} {}^\sigma S_{\lambda_{J_1} \lambda_{\overline{J_1}}}) \cap L_{J_2} S_{\lambda_{J_2} \lambda_{\overline{J_2}}} \\ &= {}^{g'}(L_J {}^\tau S_{\lambda_J \lambda_{\overline{J}}}) \\ &= {}^{g'\tau}(L_J S_{\lambda_J \lambda_{\overline{J}}}) \end{aligned}$$

for some  $\tau \in S_J S_{\overline{J}}$  and  $(\lambda_J, \lambda_{\overline{J}}) \in \mathcal{P}(J, \overline{J})$ . Consequently,  $\overline{\mathcal{U}}_n$  is closed under intersection. This completes the proof.  $\square$

By Lemma 5.1 and [6, Exercise 2.2],  $\Omega(B_n, \mathcal{U}_n)$  and  $\Omega(B_n, \mathcal{E}_n)$  are subrings of  $\Omega(B_n)$  (cf. Theorem 3.4) called partial Burnside rings. The partial Burnside ring  $\Omega(B_n, \mathcal{E}_n)$  is known as the parabolic Burnside ring. As for the partial Burnside ring  $\Omega(B_n, \mathcal{U}_n)$  relative to the Young subgroups of  $B_n$ , we quote [7, Corollary II.4]:

**Theorem 5.2** *The characters  $1_{L_{\overline{i}} S_{\lambda_i \lambda_{\overline{i}}}}^{B_n}$  induced from the trivial characters  $1_{L_{\overline{i}} S_{\lambda_i \lambda_{\overline{i}}}}$  of  $L_{\overline{i}} S_{\lambda_i \lambda_{\overline{i}}}$  for  $[i] \subset [n]$  and  $(\lambda_i, \lambda_{\overline{i}}) \in \mathcal{P}([i], \overline{[i]})$  form a  $\mathbb{Z}$ -basis of  $R(B_n)$ .*

**Corollary 5.3** *The ring homomorphism  $\text{char} : \Omega(B_n) \rightarrow R(B_n)$  induces a ring isomorphism  $\overline{\text{char}} : \Omega(B_n, \mathcal{U}_n) \rightarrow R(B_n)$ . In particular,  $\Omega(B_n, \mathcal{U}_n)^\times \simeq R(B_n)^\times$ .*

*Proof.* The corollary is a consequence of Theorem 5.2, because  $\mathcal{U}_n$  is a set of conjugates of the subgroups  $L_{\overline{i}} S_{\lambda_i \lambda_{\overline{i}}}$  for  $[i] \subset [n]$  and  $(\lambda_i, \lambda_{\overline{i}}) \in \mathcal{P}([i], \overline{[i]})$ .  $\square$

The rest of this section is devoted to quite a new view of the units of  $\Omega(B_n, \mathcal{U}_n)$ .

**Proposition 5.4**  $|\Omega(B_n, \mathcal{E}_n)^\times| = 4$ .

*Proof.* By [4, (66.29) Corollary] and Corollary 5.3, there is a unique unit  $\alpha_n$  of  $\Omega(B_n, \mathcal{E}_n)$  such that  $\text{char}(\alpha_n) = \varepsilon_n$ . Obviously,  $-1 \in \Omega(B_n, \mathcal{E}_n)^\times$ . Hence we have  $|\Omega(B_n, \mathcal{E}_n)^\times| \geq 4$ . By Proposition 4.3 and Theorem 5.2,  $\Lambda_{P([n])} \in \Omega(B_n, \mathcal{U}_n)^\times$  and  $\tilde{\Lambda}_{P([n])} \notin \Omega(B_n, \mathcal{E}_n)^\times$ . Thus  $|\Omega(B_n, \mathcal{U}_n)^\times : \Omega(B_n, \mathcal{E}_n)^\times| \geq 2$ . By Lemma 4.1 and Corollary 5.3, we have  $|\Omega(B_n, \mathcal{U}_n)^\times| = |R(B_n)^\times| = 8$ , whence  $|\Omega(B_n, \mathcal{E}_n)^\times| = 4$ . This completes the proof.  $\square$

We present a technical lemma by which [4, (66.29) Corollary] deduces Eq.(4) and a description of  $\varepsilon_n : B_n \rightarrow \mathbb{C}$  (see also [6, Propositions 2.3.8 and 2.3.10]):

$$\varepsilon_n = \sum_{i=0}^n \sum_{\lambda=(1^{m_1}, \dots, i^{m_i}) \in \mathcal{P}(i)} (-1)^{m_1 + \dots + m_i + n} \frac{(m_1 + \dots + m_i)!}{m_1! \dots m_i!} 1_{S_\lambda B_{n-i}}. \quad (5)$$

**Lemma 5.5** *Let  $(S_n, X)$  be the Coxeter system of type  $A_{n-1}$ . Given  $\lambda \in \mathcal{P}(n)$ , let  $W(\lambda)$  be the set of parabolic subgroups  $W_I$  of  $S_n$  for  $I \subset X$  which are conjugates of  $S_\lambda$ . Suppose that  $I \subset X$  and  $W_I \in W(\lambda)$  with  $\lambda = (1^{m_1}, \dots, n^{m_n}) \in \mathcal{P}(n)$ . Then  $|I| \equiv m_1 + \dots + m_n + n \pmod{2}$ , so that  $(-1)^{|I|} = (-1)^{m_1 + \dots + m_n + n}$ .*

*Proof.* We use induction with respect to the partially order  $\leq$  on  $\mathcal{P}(n)$  given by

$$\mu \leq \nu \quad :\iff \quad S_\mu \text{ is a conjugate of a subgroup of } S_\nu.$$

If  $\lambda = (1^n)$ , then  $I = \emptyset$ , and hence  $|I| \equiv 2n \pmod{2}$ . Assume that  $(1^n) < \lambda$ . Then  $m_k \neq 0$  and  $m_{k+1} = \dots = m_n = 0$  for some  $k \in [n]$ . We set

$$\mu = \begin{cases} (1^{m_1+2}, 2^{m_2-1}) & \text{if } k = 2, \\ (1^{m_1+1}, 2^{m_2}, \dots, (k-1)^{m_{k-1}+1}, k^{m_k-1}, 0, \dots, 0) & \text{if } k > 2. \end{cases}$$

Suppose that  $I' \subset X$  and  $W_{I'} \in W(\mu)$ . Then  $\mu < \lambda$  and  $|I'| = |I| - 1$ . By the inductive assumption,  $|I'| \equiv m_1 + \dots + m_n + 1 + n \pmod{2}$ . Since  $|I| = |I'| + 1$ , it follows that  $|I| \equiv m_1 + \dots + m_n + n \pmod{2}$ . This completes the proof.  $\square$

What about a unique unit  $\gamma_n$  of  $\Omega(B_n, \mathcal{U}_n)$  satisfying  $\text{char}(\gamma_n) = \kappa_n$ ? We are interested in the reduced Lefschetz invariant  $\tilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])}$ .

**Lemma 5.6**  $\kappa_n = \text{char}(\tilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])})$ .

*Proof.* By Proposition 2.3,  $\text{char}(\tilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])})(y) = (-1)^{|(y) \setminus (\mathbb{Z}^\times \times [n])|}$  for all  $y \in B_n$ . Let  $\sigma \in S_n$ , and assume that  $\sigma$  is the product of pairwise disjoint  $n_j$ -cycles  $\sigma_j$  for  $j = 1, 2, \dots, r$  with  $\sum_j n_j = n$ . Let  $(x_1, \dots, x_n) \in V_n$ , and set  $y = (x_1, \dots, x_n)\sigma$ . For each  $j \in \{1, 2, \dots, r\}$ , let  $I_j$  be the minimal subset of  $[n]$  with  $\sigma_j \in S_{I_j}$ , and set

$$y_j = (x_1^{(j)}, x_2^{(j)}, \dots, x_n^{(j)})\sigma_j \quad \text{with} \quad x_i^{(j)} = \begin{cases} x_i & \text{if } i \in I_j, \\ 1 & \text{otherwise.} \end{cases}$$

Obviously,  $y = \prod_{j=1}^r y_j$ . We now set  $s = \#\{j \in \{1, 2, \dots, r\} \mid \prod_{i=1}^n x_i^{(j)} = 1\}$ , so that  $|\langle y \rangle \backslash (\mathbb{Z}^\times \times [n])| = r + s$ . Hence it turns out that

$$\kappa_n(y) = \prod_{i=1}^n x_i = \prod_{j=1}^r \prod_{i=1}^n x_i^{(j)} = (-1)^{r-s} = (-1)^{|\langle y \rangle \backslash (\mathbb{Z}^\times \times [n])|}.$$

Consequently, we obtain  $\kappa_n = \text{char}(\tilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])})$ , completing the proof.  $\square$

The following lemma, which is a basic fact for the left  $B_n$ -set  $\mathbb{Z}^\times \times [n]$ , is crucial.

**Lemma 5.7** *Let  $\{M_1, \dots, M_i\}$ ,  $i$  a positive integer, be a partition of  $\mathbb{Z}^\times \times [n]$  into nonempty subsets, and view them as elements of the  $B_n$ -poset  $P(\mathbb{Z}^\times \times [n])$ . If each  $M_j$  for  $j = 1, 2, \dots, i$  does not include both  $(1, q)$  and  $(-1, q)$  for any  $q \in [n]$ , then there exists an element  $\lambda$  of  $\mathcal{P}(n)$  such that the intersection of stabilizers of  $M_j$  in  $B_n$  for  $j = 1, 2, \dots, i$  is a conjugate of  $S_\lambda$ .*

*Proof.* There is a partition  $\{N_1, \dots, N_k\}$ ,  $k$  a positive integer, of  $[n]$  into nonempty subsets such that each  $M_j$  for  $j = 1, 2, \dots, i$  consists of either  $(1, q)$  or  $(-1, q)$ , but not both, for each  $q \in N_{\ell_1} \dot{\cup} \dots \dot{\cup} N_{\ell_r}$  with  $\{N_{\ell_1}, \dots, N_{\ell_r}\} \subset \{N_1, \dots, N_k\}$ . Let  $\widehat{\mathcal{P}}(n)$  be the set of all cycle types to which such partitions  $\{N_1, \dots, N_k\}$  of  $[n]$  into nonempty subsets correspond, and take the maximal element  $\mu$  of  $\widehat{\mathcal{P}}(n)$  with respect to the partially order  $\leq$  on  $\mathcal{P}(n)$  given in the proof of Lemma 5.5. Let  $\{N_1, \dots, N_k\}$  be a partition of  $[n]$  into nonempty subsets corresponding to  $\mu$  which satisfy the above condition. We set  $J = N_\ell$ , where  $\ell$  is an arbitrary integer with  $1 \leq \ell \leq k$ . There exists a unique subset  $Q$  of  $J$  such that

$$J^+ := \{(1, q) \mid q \in Q\} \dot{\cup} \{(-1, q) \mid q \in J - Q\} \subset M_{j_1}$$

and

$$J^- := \{(1, q) \mid q \in J - Q\} \dot{\cup} \{(-1, q) \mid q \in Q\} \subset M_{j_2}$$

for some integers  $j_1$  and  $j_2$  with  $1 \leq j_1 \neq j_2 \leq i$ . Let  $g = (x_1, \dots, x_n) \in L_Q$ , and suppose that  $x_q = -1$  for all  $q \in Q$ . Then the stabilizer of  $J^+$  in  $B_n$  is  ${}^g(L_{J^+} S_{J^+})$ , and so is that of  $J^-$  in  $B_n$ . Observe now that the intersection of stabilizers of  $M_j$  for  $j = 1, 2, \dots, i$  in  $B_n$  coincides with the intersection of such subgroups of  $B_n$ . Hence the assertion is a consequence of Lemma 5.1. This completes the proof.  $\square$

Identifying  $(-1, q)$  with  $n + q \in [2n]$  for all  $q \in [n]$ , we may consider  $S_{2n}$  to be the symmetric group on  $\mathbb{Z}^\times \times [n]$ . In particular,  $B_n$  is viewed as a subgroup of  $S_{2n}$ .

**Lemma 5.8** *Let  $\lambda \in \mathcal{P}(2n)$ . Then  $B_n \cap {}^\sigma S_\lambda \in \overline{\mathcal{U}}_n$  for all  $\sigma \in S_{2n}$ , and*

$$[\text{res}_{B_n}^{S_{2n}}(S_{2n}/S_\lambda)] = \sum_{\sigma \in \overline{B_n \backslash S_{2n}/S_\lambda}} [B_n / (B_n \cap {}^\sigma S_\lambda)] \in \Omega(B_n, \mathcal{U}_n),$$

where  $\text{res}_{B_n}^{S_{2n}}$  indicates restriction of operators from  $S_{2n}$  to  $B_n$  and  $\overline{B_n \backslash S_{2n}/S_\lambda}$  is a complete set of representatives of double cosets  $B_n \sigma S_\lambda$ ,  $\sigma \in S_{2n}$ , in  $S_{2n}$ .



*Proof.* Let  $\sigma \in S_{2n}$ . By Lemma 5.7,  $B_n \cap {}^\sigma S_\lambda = {}^{g\tau}(L_J S_{\mu_J \mu_{\bar{J}}})$  for some  $J \subset [n]$ ,  $g \in L_{\bar{J}}$ ,  $\tau \in S_J S_{\bar{J}}$ , and  $(\mu_J, \mu_{\bar{J}}) \in \mathcal{P}(J, \bar{J})$ . Hence  $B_n \cap {}^\sigma S_\lambda \in \bar{\mathcal{U}}_n$ . The second assertion follows from [4, (80.27) Subgroup Theorem]. This completes the proof.  $\square$

There is a formula of the reduced Lefschetz invariant  $\tilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])}$  (cf. Eq.(6)) which is implicit in the proof of a conclusion from the proceeding facts:

**Theorem 5.9** *Define three elements  $\alpha_n$ ,  $\beta_n$ , and  $\gamma_n$  of  $\Omega(B_n, \mathcal{U}_n)$  by*

$$\alpha_n = \sum_{i=0}^n \sum_{\lambda=(1^{m_1}, \dots, i^{m_i}) \in \mathcal{P}(i)} (-1)^{m_1 + \dots + m_i + n} \frac{(m_1 + \dots + m_i)!}{m_1! \dots m_i!} [B_n / (S_\lambda B_{n-i})],$$

$$\beta_n = (-1)^n \tilde{\Lambda}_{P([n])}, \quad \text{and} \quad \gamma_n = \tilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])}.$$

Then  $\varepsilon_n = \text{char}(\alpha_n)$ ,  $\rho_n = \text{char}(\beta_n)$ ,  $\kappa_n = \text{char}(\gamma_n)$ , and  $\alpha_n = (-1)^n \tilde{\Lambda}_{P([n] \dot{\cup} (\mathbb{Z}^\times \times [n]))}$ . Moreover,  $\Omega(B_n, \mathcal{E}_n)^\times = \langle \alpha_n, -1 \rangle$ ,  $\Omega(B_n, \mathcal{U}_n)^\times = \langle \beta_n, \gamma_n, -1 \rangle$ , and  $\alpha_n = \beta_n \gamma_n$ .

*Proof.* By Eq.(5),  $\varepsilon_n = \text{char}(\alpha_n)$ . Obviously,  $\alpha_n \in \Omega(B_n, \mathcal{E}_n)$ . Since  $\alpha_n \neq 1, -1$ , it follows from Proposition 5.4 that  $\Omega(B_n, \mathcal{E}_n)^\times$  is generated by  $\alpha_n$  and  $-1$ . By Proposition 4.3 and Lemma 5.6, we have  $\rho_n = \text{char}(\beta_n)$ ,  $\beta_n \in \Omega(B_n, \mathcal{U}_n)$ , and  $\kappa_n = \text{char}(\gamma_n)$ . The reduced Lefschetz invariant  $\tilde{\Lambda}_{P([2n])}$  of the left  $S_{2n}$ -set  $[2n]$  is an element of  $\Omega(S_{2n}, \mathcal{Y}_{2n})$  (cf. [9, §4]); for its description, see Eq.(3). We may identify  $\tilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])}$  with  $\text{res}_{B_n}^{S_{2n}}(\tilde{\Lambda}_{P([2n])})$  which is the element of  $\Omega(B_n)$  obtained by restriction of operators on  $S_{2n}$ -sets appearing in the components of  $\tilde{\Lambda}_{P([2n])}$  from  $S_{2n}$  to  $B_n$ . By Lemma 5.8,  $\text{res}_{B_n}^{S_{2n}}(\tilde{\Lambda}_{P([2n])}) \in \Omega(B_n, \mathcal{U}_n)$ , and thus  $\tilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])} \in \Omega(B_n, \mathcal{U}_n)$ . Moreover, it follows from Lemma 4.2 and Corollary 5.3 that  $\Omega(B_n, \mathcal{U}_n)^\times$  is generated by  $\beta_n$ ,  $\gamma_n$ , and  $-1$ . Also,  $\alpha_n = \beta_n \gamma_n$ , because  $\varepsilon_n = \rho_n \kappa_n$ . By Proposition 2.3, it turns out that  $\alpha_n = (-1)^n \tilde{\Lambda}_{P([n] \dot{\cup} (\mathbb{Z}^\times \times [n]))}$ . This completes the proof.  $\square$

Since  $\tilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])} = \text{res}_{B_n}^{S_{2n}}(\tilde{\Lambda}_{P([2n])})$ , it follows from Eq.(3) and Lemma 5.8 that

$$\tilde{\Lambda}_{P(\mathbb{Z}^\times \times [n])} = \sum_{\lambda=(1^{m_1}, \dots, (2n)^{m_{2n}}) \in \mathcal{P}(2n)} \sum_{\sigma \in \overline{B_n \setminus S_{2n} / S_\lambda}} (-1)^{m_1 + \dots + m_{2n}} \times \frac{(m_1 + \dots + m_{2n})!}{m_1! \dots m_{2n}!} [B_n / (B_n \cap {}^\sigma S_\lambda)]. \quad (6)$$

We close this section with a character theoretical explanation of the formula of  $\kappa_n$  obtained by Eq.(6). For each  $\mathbb{C}$ -character  $\chi$  of  $G$ , let  $\chi|_H$  with  $H \leq G$  denote the  $\mathbb{C}$ -character obtained by restriction of  $\chi$  from  $G$  to  $H$ .

**Lemma 5.10** *Let  $\mathbf{M} : G \rightarrow GL_n(\mathbb{C})$  be a  $\mathbb{C}$ -representation of  $G$  affording a real valued character  $\chi$  of  $G$ . Then for any  $g \in G$ ,*

$$\det \mathbf{M}(g) = (-1)^{n - \langle \chi|_{\langle g \rangle}, 1_{\langle g \rangle} \rangle},$$

where  $\langle \chi|_{\langle g \rangle}, 1_{\langle g \rangle} \rangle$  is the inner product of  $\chi|_{\langle g \rangle}$  and  $1_{\langle g \rangle}$ .

*Proof.* See the later part of the proof of [14, Theorem A].  $\square$

There is a representation  $\mathbf{M}_n : S_n \rightarrow GL_n(\mathbb{C})$  given by

$$\sigma \mapsto (\delta_{\sigma^{-1}(i)j})_{1 \leq i, j \leq n}, \quad \delta \text{ the Kronecker delta,}$$

which affords the permutation character  $\pi_{[n]} : S_n \rightarrow \mathbb{C}$ . Obviously, the sign character  $\text{sgn}_n : S_n \rightarrow \mathbb{C}$  coincides with the linear  $\mathbb{C}$ -character  $\det \mathbf{M}_n : S_n \rightarrow \mathbb{C}$  given by

$$\sigma \mapsto \det \mathbf{M}_n(\sigma)$$

for all  $\sigma \in S_n$ . Recall that  $B_n$  is viewed as a subgroup of  $S_{2n}$ . By Lemma 5.10,

$$\det \mathbf{M}_{2n}(\sigma) = (-1)^{\langle \pi_{[2n]} |_{\langle \sigma \rangle}, 1_{\langle \sigma \rangle} \rangle} = (-1)^{|\langle \sigma \rangle \setminus [2n]|}$$

for all  $\sigma \in S_{2n}$  (see also [9, Lemma 3.3]). This, combined with Proposition 2.3 and Lemma 5.6, shows that the linear  $\mathbb{C}$ -character  $\det \mathbf{M}_{2n}|_{B_n} : B_n \rightarrow \mathbb{C}$  coincides with  $\kappa_n : B_n \rightarrow \mathbb{C}$ . Consequently, we have  $\kappa_n = \text{sgn}_{2n}|_{B_n}$ . Hence it follows from Eq.(4) and Lemma 5.8 (see also [4, (10.13) Subgroup Theorem]) that

$$\kappa_n = \sum_{\lambda=(1^{m_1}, \dots, (2n)^{m_{2n}}) \in \mathcal{P}(2n)} \sum_{\sigma \in B_n \setminus S_{2n}/S_\lambda} (-1)^{m_1 + \dots + m_{2n}} \frac{(m_1 + \dots + m_{2n})!}{m_1! \cdots m_{2n}!} 1_{B_n \cap \sigma S_\lambda}$$

and  $B_n \cap \sigma S_\lambda \in \bar{U}_n$  for all  $\lambda \in \mathcal{P}(2n)$  and  $\sigma \in S_{2n}$ .

## 6 The parabolic Burnside rings of even-signed permutation groups

We set  $D_n = \ker \kappa_n$  and call it the even-signed permutation group on  $[n]$ . Obviously,  $D_n = K_n S_n$ , where  $K_n = \ker \vartheta_n$ . Suppose that  $[i] \subset [n]$  and  $\lambda \in \mathcal{P}(i)$ . We set  $S_\lambda D_{n-i} = (K_n \cap L_{\bar{i}}) S_\lambda S_{\bar{i}}$  and set  $t = (0, 0, \dots, 1) \in V_n$ . Observe that

$$[\text{res}_{D_n}^{B_n}(B_n/(S_\lambda B_{n-i}))] = \begin{cases} [D_n/(S_\lambda D_{n-i})] & \text{if } 0 \leq i \leq n-1, \\ [D_n/S_\lambda] + [D_n/{}^t S_\lambda] & \text{if } i = n \end{cases}$$

by [4, (80.27) Subgroup Theorem], which are contained in the parabolic Burnside ring  $\mathcal{PB}(D_n)$  (cf. [6, 2.3.11]). We define a map  $\text{res}_{D_n}^{B_n} : \mathcal{PB}(B_n) \rightarrow \mathcal{PB}(D_n)$  by

$$[B_n/(S_\lambda B_{n-i})] \mapsto [\text{res}_{D_n}^{B_n}(B_n/(S_\lambda B_{n-i}))]$$

for all  $[i] \subset [n]$  and  $\lambda \in \mathcal{P}(i)$ . Set  $\alpha'_n = \text{res}_{D_n}^{B_n}(\alpha_n)$  (see Theorem 5.9). Then

$$\begin{aligned} \alpha'_n &= \sum_{i=0}^{n-1} \sum_{\lambda=(1^{m_1}, \dots, i^{m_i}) \in \mathcal{P}(i)} (-1)^{m_1 + \dots + m_i + n} \frac{(m_1 + \dots + m_i)!}{m_1! \cdots m_i!} [D_n/(S_\lambda D_{n-i})] \\ &+ \sum_{\lambda=(1^{m_1}, \dots, n^{m_n}) \in \mathcal{P}(n)} (-1)^{m_1 + \dots + m_n + n} \frac{(m_1 + \dots + m_n)!}{m_1! \cdots m_n!} ([D_n/S_\lambda] + [D_n/{}^t S_\lambda]). \end{aligned}$$

**Proposition 6.1**  $\mathcal{PB}(D_n)^\times = \langle \alpha'_n, -1 \rangle$ .

*Proof.* By the proof of [1, Theorem 4.5], there is an injection from  $\mathcal{PB}(D_n)^\times$  to  $R(D_n)^\times$  inherited from the ring homomorphism  $\text{char} : \Omega(D_n) \rightarrow R(D_n)$ . The sign character  $\varepsilon_n|_{D_n} : D_n \rightarrow \mathbb{C}$  is the only nontrivial  $\mathbb{C}$ -character of  $D_n$  and  $\mathbb{Q}$  is a splitting field for  $D_n$  (cf. [6, §5.6]). This, combined with [13, Corollary 1.2 and Lemma 2.1], shows that  $R(D_n)^\times$  is isomorphic to the four group. Moreover, by [4, (10.13) Subgroup Theorem] and Eq.(5), we have  $\varepsilon_n|_{D_n} = \text{char}(\alpha'_n)$ . Consequently,  $\mathcal{PB}(D_n)^\times$  is generated by  $\alpha'_n$  and  $-1$ . This completes the proof.  $\square$

*Remark 6.2* Let  $(W, S)$  be a Coxeter system of type  $E_6$ ,  $E_7$ , or  $E_8$ . Then every character of  $W$  is rational-valued (cf. [6, 5.3.6]). Moreover, there are exactly two linear  $\mathbb{C}$ -characters of  $W$  (cf. [6, pp. 413–416]). Hence  $R(W)^\times$  is isomorphic to the four group and  $\mathcal{PB}(W)^\times$  is isomorphic to a subgroup of  $R(W)^\times$  (see the proof of Proposition 6.1). Thus it follows from [4, (66.29) Corollary] that  $\mathcal{PB}(W)^\times$  is of order 4 and is generated by  $\sum_{J \subset S} (-1)^{|J|} [W/W_J]$  and  $-1$ , where  $W_J = \langle s \mid s \in J \rangle$ .

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