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著者	TAKAHASHI Masatomo
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Envelopes of families of Legendre mappings in the unit tangent bundle over the Euclidean space

Masatomo Takahashi

Dedicated to Professor Takashi Nishimura on the occasion of his 60th birthday

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Abstract

For families of hypersurfaces with singular points, a classical definition of an envelope is vague. In order to define an envelope for a family of hypersurfaces with singular points, we consider r-parameter families of frontals and of Legendre mappings in the unit tangent bundle over the Euclidean space. We define an envelope for the r-parameter family of Legendre mappings. Then the envelope is also a frontal. We investigate properties of the envelopes. As an application, we give a condition that the projection of a singular solution of a first order partial differential equation is an envelope.

1 Introduction

Envelopes are classical object in the differential geometry. There are a lot of applications of envelopes to differential geometry, differential equations and physics, for instance [4, 5, 7, 8, 9, 12, 15, 16, 18, 21, 23]. An envelope of a family of surfaces is a surface that is "tangent" to each member of the family at some point. If the surfaces are regular, then the tangent is well-defined. However, a definition of an envelope is vague for singular surfaces (surfaces with singular points). In [22], a definition and properties of an envelope for a one-parameter family of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$ were given. In this paper, we clarify a definition of an envelope for a family of singular surfaces. As singular surfaces, we consider frontals and Legendre mappings in the unit tangent bundle over $\mathbb{R}^{n+1}$. The basic results on the singularity theory see [1, 2, 4, 13, 14, 17].

We consider r-parameter families of Legendre mappings and define an envelope in §3. Then the envelope of an r-parameter family of Legendre mappings is also a frontal. We give a necessary and sufficient condition that the r-parameter family of Legendre mappings has an envelope, see Theorem 3.6 as the envelope theorem. Moreover, we give relations between envelopes of a classical definition and of a family of Legendre mappings. As an application,

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we give a condition that the projection of a singular solution of a first order partial differential equation is an envelope by using the envelope theorem in §4.

All maps and manifolds considered here are differentiable of class C^∞.

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2 Preliminaries

Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional Euclidean space with the inner product $x \cdot y = x_1y_1 + \cdots + x_{n+1}y_{n+1}$, where $x = (x_1, \dots, x_{n+1}), y = (y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1}$. The norm of $x \in \mathbb{R}^{n+1}$ is given by $|x| = \sqrt{x \cdot x}$.

Let $F : W \times \Lambda \to \mathbb{R}$ be an r-parameter family of smooth functions, where W and Λ are domains in $\mathbb{R}^{n+1}$ and in $\mathbb{R}^r$, respectively. Then one of the classical definition of an envelope E_I is as follows, see for instance [3, 4, 11]:

Definition 2.1 The *envelope* of the family F is the discriminant set of F, that is, the set of points given by

$$E_I = \{x \in \mathbb{R}^{n+1} | \text{ for some } \lambda \in \Lambda, F(x,\lambda) = F_{\lambda_j}(x,\lambda) = 0, j = 1, \dots, r\}.$$

If $F(x,\lambda) = F_{\lambda_j}(x,\lambda) = 0, j = 1, \dots, r$, we say that $x \in E_I$ *with respect to* $\lambda = (\lambda_1, \dots, \lambda_r)$. Here $F_{\lambda_j}(x,\lambda) = (\partial F/\partial \lambda_j)(x,\lambda)$.

Example 2.2 Let $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, F(x,y,z,\lambda) = (x-\lambda)^3 - y^2$. Then $F = 0$ is the image of the cuspidal edge for each fixed $\lambda \in \mathbb{R}$, see Figure 1 and Example 3.8. The definition and properties of cuspidal edges see [10, 20]. Since $F_\lambda(x,y,z,\lambda) = -3(x-\lambda)^2$, the envelope of the family F is given by $E_I = \{(\lambda, 0, z)\} = xz$-plane.

Example 2.3 Let $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, F(x,y,z,\lambda) = x^3 - (y-\lambda)^2$. Then $F = 0$ is the image of the cuspidal edge for each fixed $\lambda \in \mathbb{R}$, see Figure 2 and Example 3.9. Since $F_\lambda(x,y,z,\lambda) = 2(y-\lambda)$, the envelope of the family F is given by $E_I = \{(0, \lambda, z)\} = yz$-plane.

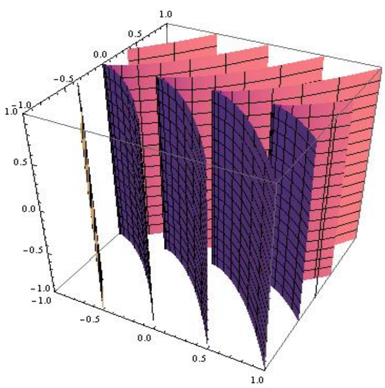

Figure 1.

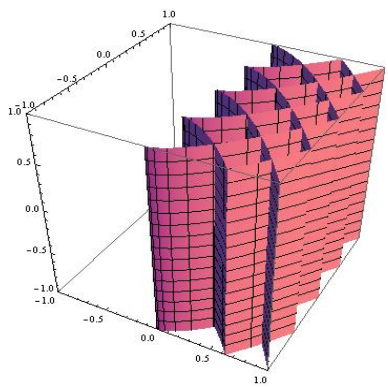

Figure 2.

However, in the sense of the limit tangent plane of the cuspidal edge, yz-plane is not tangent to the cuspidal edge. Therefore, we would like to distinguish as envelopes, see Examples 3.8 and 3.9.

Let $U \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. We say that $(f,\nu): U \to \mathbb{R}^{n+1} \times S^n$ is a *Legendre mapping* if $(f,\nu)^*\theta = 0$, where θ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \times S^n$ over $\mathbb{R}^{n+1}$ (cf. [1, 2]). Moreover, $f: U \to \mathbb{R}^{n+1}$ is a *frontal* (respectively, a *front*) if there exists a smooth mapping $\nu: U \to S^n$ such that (f,ν) is a Legendre mapping (respectively, a Legendre immersion). The condition $(f,\nu)^*\theta = 0$ is equivalent to $df(u) \cdot \nu(u) = 0$ for all $u \in U$. If we denote $f(u) = (f_1(u), \dots, f_{n+1}(u)), \nu(u) = (\nu_1(u), \dots, \nu_{n+1}(u))$ and $u = (u_1, \dots, u_n)$, then the condition $df(u) \cdot \nu(u) = 0$ for all $u \in U$ is equivalent to

$$f_{u_i}(u) \cdot \nu(u) = f_{1u_i}(u)\nu_1(u) + \cdots + f_{n+1u_i}(u)\nu_{n+1}(u) = 0,$$

for all $u \in U$ and $i = 1, \dots, n$.

The *parallel* of a Legendre mapping $(f,\nu): U \to \mathbb{R}^{n+1} \times S^n$ is defined by $f^k: U \to \mathbb{R}^{n+1}, f^k(u) = f(u) + k\nu(u)$, where $k \in \mathbb{R}$. Then it is easy to see that $(f^k,\nu): U \to \mathbb{R}^{n+1} \times S^n$ is also a Legendre mapping for each fixed $k \in \mathbb{R}$.

3 Envelopes of families of Legendre mappings

We say that $(f,\nu): U \times \Lambda \to \mathbb{R}^{n+1} \times S^n$ is an *r-parameter family of Legendre mapping* if $(f(\cdot,\lambda), \nu(\cdot,\lambda)): U \to \mathbb{R}^{n+1} \times S^n$ is a Legendre mapping for each $\lambda \in \Lambda \subset \mathbb{R}^r$.

Let $(f,\nu): U \times \Lambda \to \mathbb{R}^{n+1} \times S^n$ be an r-parameter family of Legendre mappings. Let $V \subset \mathbb{R}^n$ be an open subset and $e: V \to U \times \Lambda, e(p) = (u(p), \lambda(p))$ be a smooth mapping. We denote $E = f \circ e: V \to \mathbb{R}^{n+1}$.

Definition 3.1 We call E an *envelope* (and e a *pre-envelope*) for the r-parameter family of Legendre mappings (f,ν), when the following conditions are satisfied.

(i) The set of regular points of $\lambda: V^n \to \Lambda^r$ is dense in V. (The Variability Condition.)

(ii) For all $p \in V$ and $i = 1, \dots, n$, $E_{p_i}(p) \cdot \nu(e(p)) = 0$. (The Tangency Condition.)

The definition of the envelope is a generalisation of the definition of the envelope of a one-parameter family of Legendre curves in [22]. By definition, we have the following.

Proposition 3.2 *Let $(f,\nu): U \times \Lambda \to \mathbb{R}^{n+1} \times S^n$ be an r-parameter family of Legendre mappings. Suppose that $e: V \to U \times \Lambda$ is a pre-envelope and $E = f \circ e: V \to \mathbb{R}^{n+1}$ is an envelope of (f,ν). Then E is a frontal. More precisely, $(E, \nu \circ e): V \to \mathbb{R}^{n+1} \times S^n$ is a Legendre mapping.*

Proof. Since the tangency condition, we have $E_{p_i}(p) \cdot \nu(e(p)) = 0$ for all $p \in V$. It follows that $dE(p) \cdot (\nu \circ e)(p) = 0$ for all $p \in V$. That is, $(E, \nu \circ e): V \to \mathbb{R}^{n+1} \times S^n$ is a Legendre mapping. □

Proposition 3.3 *Let $(f,\nu): U \times \Lambda \to \mathbb{R}^{n+1} \times S^n$ be an r-parameter family of Legendre mappings. Suppose that $e: V \to U \times \Lambda$ is a pre-envelope and $E = f \circ e$ is an envelope of (f,ν). Then we have the following.*

(1) $e : V \to U \times \Lambda$ *is also a pre-envelope of* $(f, -\nu)$ *and* $E = f \circ e$ *is also an envelope of* $(f, -\nu)$.

(2) $e : V \to U \times \Lambda$ *is also a pre-envelope of* $(-f, \nu)$ *and* $-E = -f \circ e$ *is an envelope of* $(-f, \nu)$.

Proof. (1) By definition, $(f, -\nu)$ is also an r-parameter family of Legendre mappings. Since e is a pre-envelope of (f, ν), $E_{p_i}(p) \cdot (-\nu(e(p)) = -E_{p_i}(p) \cdot \nu(e(p)) = 0$ for all $p \in V$. Hence, e is also a pre-envelope and $E = f \circ e$ is also an envelope of $(f, -\nu)$.

(2) By similarly, we have the result. $\square$

Remark 3.4 By Proposition 3.3 (1), we may define an envelope for an r-parameter family of Legendre mapping in $PT^*\mathbb{R}^{n+1}$.

Remark 3.5 As the same definition, we can define an envelope of a family of Legendre mappings in the unit tangent bundle over a smooth manifold. Especially, we can define envelopes not only of families of Legendre mappings in the unit spherical bundle (cf. [19]), but also of families of frontals in the hyperbolic or de-Sitter space (cf. [6]).

We give a necessary and sufficient condition that the r-parameter family of Legendre mappings has an envelope. We call this result *the envelope theorem* (cf. [11, 22]).

Theorem 3.6 (The Envelope Theorem) *Let* $(f, \nu) : U \times \Lambda \to \mathbb{R}^{n+1} \times S^n$ *be an* r*-parameter family of Legendre mappings, and let* $e : V \to U \times \Lambda$ *be a smooth mapping satisfying the variability condition. Suppose that* $n \geq r$*. Then* e *is a pre-envelope of* (f, ν) *(and* E *is an envelope) if and only if* $f_{\lambda_j}(e(p)) \cdot \nu(e(p)) = 0$ *for all* $p \in V$ *and* $j = 1, \dots, r$*.*

Proof. Suppose that e is a pre-envelope of (f, ν). We denote $f = (f_1, \dots, f_{n+1}), \nu = (\nu_1, \dots, \nu_{n+1})$. By a direct calculation,

$$\begin{aligned} E_{p_i}(p) &= \frac{\partial}{\partial p_i}(f \circ e(p)) \\ &= \Big(\sum_{k=1}^{n} f_{1u_k}(e(p)) u_{kp_i}(p) + \sum_{j=1}^{r} f_{1\lambda_j}(e(p)) \lambda_{jp_i}(p), \dots, \\ &\qquad \sum_{k=1}^{n} f_{n+1u_k}(e(p)) u_{kp_i}(p) + \sum_{j=1}^{r} f_{n+1\lambda_j}(e(p)) \lambda_{jp_i}(p) \Big). \end{aligned}$$

Since $E_{p_i}(p) \cdot \nu(e(p)) = 0$ for all $p \in V$ and $i = 1, \dots, n$, and (f, ν) is an r-parameter family of Legendre mappings, we have

$$(f_{\lambda_1}(e(p)) \cdot \nu(e(p))) \lambda_{1p_i}(p) + \cdots + (f_{\lambda_r}(e(p)) \cdot \nu(e(p))) \lambda_{rp_i}(p) = 0,$$

for all $p \in V$ and $i = 1, \dots, n$. It follows that

$$\begin{pmatrix} \lambda_{1p_1}(p) & \cdots & \lambda_{rp_1}(p) \\ \vdots & \cdots & \vdots \\ \lambda_{1p_n}(p) & \cdots & \lambda_{rp_n}(p) \end{pmatrix} \begin{pmatrix} f_{\lambda_1}(e(p)) \cdot \nu(e(p)) \\ \vdots \\ f_{\lambda_r}(e(p)) \cdot \nu(e(p)) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By the assumption $n \geq r$ and the variability condition, we have $f_{\lambda_j}(e(p)) \cdot \nu(e(p)) = 0$ for all $p \in V$ and $j = 1, \dots, r$.

Conversely, suppose that $f_{\lambda_j}(e(p)) \cdot \nu(e(p)) = 0$ for all $p \in V$ and $j = 1, \dots, r$. By a direct calculation, we have

$$
\begin{aligned}
E_{p_i}(p) \cdot \nu(e(p)) &= \left(\sum_{k=1}^{n} f_{1u_k}(e(p)) u_{kp_i}(p) + \sum_{j=1}^{r} f_{1\lambda_j}(e(p)) \lambda_{jp_i}(p) \right) \cdot \nu_1(e(p)) \\
&\quad + \cdots + \left(\sum_{k=1}^{n} f_{n+1u_k}(e(p)) u_{kp_i}(p) + \sum_{j=1}^{r} f_{n+1\lambda_j}(e(p)) \lambda_{jp_i}(p) \right) \cdot \nu_{n+1}(e(p)) \\
&= \sum_{k=1}^{n} u_{kp_i}(p) f_{u_k}(e(p)) \cdot \nu(e(p)) + \sum_{j=1}^{r} \lambda_{jp_i}(p) f_{\lambda_j}(e(p)) \cdot \nu(e(p)) \\
&= 0
\end{aligned}
$$

for all $p \in V$ and $i = 1, \dots, n$. It follows that e is a pre-envelope of (f, ν). $\square$

Remark 3.7 In Theorem 3.6, the assumption $n \geq r$ does not need to prove the converse.

Example 3.8 Let $(f, \nu) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3 \times S^2$ be

$$
f(u, v, \lambda) = (u^2 + \lambda, u^3, v), \nu(u, v, \lambda) = \frac{1}{\sqrt{9u^2 + 4}}(3u, -2, 0).
$$

Then (f, ν) is a one-parameter family of Legendre mappings (immersions) and f is the cuspidal edge for each fixed $\lambda \in \mathbb{R}$. Since $f_\lambda(u, v, \lambda) \cdot \nu(u, v, \lambda) = 3u/\sqrt{9u^2 + 4}$, if we take $e : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}, e(p, q) = (0, p, q)$, then the variability condition holds and $f_\lambda(e(p, q)) \cdot \nu(e(p, q)) = 0$ for all $(p, q) \in \mathbb{R}^2$. By Theorem 3.6, e is a pre-envelope and $E(p, q) = f \circ e(p, q) = (q, 0, p)$ is an envelope. Hence xz-plane is an envelope of (f, ν), see Example 2.2.

Example 3.9 Let $(f, \nu) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3 \times S^2$ be

$$
f(u, v, \lambda) = (u^2, u^3 + \lambda, v), \nu(u, v, \lambda) = \frac{1}{\sqrt{9u^2 + 4}}(3u, -2, 0).
$$

Then (f, ν) is a one-parameter family of Legendre mappings (immersions) and f is the cuspidal edge for each fixed $\lambda \in \mathbb{R}$. Since $f_\lambda(u, v, \lambda) \cdot \nu(u, v, \lambda) = -2/\sqrt{9u^2 + 4} \neq 0$, (f, ν) does not have an envelope by Theorem 3.6. Hence yz-plane is not an envelope of (f, ν), see Example 2.3.

Definition 3.10 We say that a map $\Phi : \widetilde{U} \times \widetilde{\Lambda} \to U \times \Lambda$ is an *r-parameter family of parameter change* if Φ is a diffeomorphism and given by the form $\Phi(q, k) = (\phi(q, k), \varphi(k))$.

Proposition 3.11 *Let $(f, \nu) : U \times \Lambda \to \mathbb{R}^{n+1} \times S^n$ be an r-parameter family of Legendre mappings. Suppose that $n \geq r$, $e : V \to U \times \Lambda$ is a pre-envelope, $E = f \circ e$ is an envelope and $\Phi : \widetilde{U} \times \widetilde{\Lambda} \to U \times \Lambda$ is an r-parameter family of parameter change. Then $(\widetilde{f}, \widetilde{\nu}) = (f \circ \Phi, \nu \circ \Phi) : \widetilde{U} \times \widetilde{\Lambda} \to \mathbb{R}^{n+1} \times S^n$ is also an r-parameter family of Legendre mappings. Moreover, $\Phi^{-1} \circ e : V \to \widetilde{U} \times \widetilde{\Lambda}$ is a pre-envelope and E is also an envelope of $(\widetilde{f}, \widetilde{\nu})$.*

Proof. By the chain rule, we have $d(f \circ \Phi) \cdot \nu \circ \Phi = df(\Phi) d\Phi \cdot \nu(\Phi) = 0$ for fixed $k \in \widetilde{\Lambda}$. Therefore, $(\widetilde{f}, \widetilde{\nu})$ is also an r-parameter family of Legendre mappings. Since the form of Φ, there exists a smooth map $\psi : U \times \Lambda \to \widetilde{U}$ such that $\Phi^{-1}(u,\lambda) = (\psi(u,\lambda), \varphi^{-1}(\lambda))$. It follows that $\Phi^{-1} \circ e$ satisfies the variability condition. By a direct calculation, we have

$$\begin{aligned}
\widetilde{f}_{k_j}(q,k) &= \frac{\partial}{\partial k_j} f \circ \Phi(q,k) \\
&= \Big(\sum_{i=1}^{n} f_{1u_i}(\Phi(q,k))\phi_{ik_j}(q,k) + \sum_{\ell=1}^{r} f_{1\lambda_\ell}(\Phi(q,k))\varphi_{\ell k_j}(k), \cdots, \\
&\qquad \sum_{i=1}^{n} f_{n+1u_i}(\Phi(q,k))\phi_{ik_j}(q,k) + \sum_{\ell=1}^{r} f_{n+1\lambda_\ell}(\Phi(q,k))\varphi_{\ell k_j}(k)\Big)
\end{aligned}$$

for all $(q,k) \in \widetilde{U} \times \widetilde{\Lambda}$ and $j = 1, \dots, r$. Then

$$\widetilde{f}_{k_j}(\Phi^{-1}\circ e(p))\cdot\widetilde{\nu}(\Phi^{-1}\circ e(p)) = \varphi_{1k_j}(\lambda(p))f_{\lambda_1}(e(p))\cdot\nu(e(p)) + \cdots + \varphi_{rk_j}(\lambda(p))f_{\lambda_r}(e(p))\cdot\nu(e(p)) = 0$$

for all $p \in V$ and $j = 1, \dots, r$. It follows that $\Phi^{-1} \circ e$ is a pre-envelope of $(\widetilde{f}, \widetilde{\nu})$, and hence $\widetilde{f} \circ \Phi^{-1} \circ e = f \circ \Phi \circ \Phi^{-1} \circ e = f \circ e = E$ is also an envelope of $(\widetilde{f}, \widetilde{\nu})$. $\square$

Let $(f,\nu) : U \times \Lambda \to \mathbb{R}^{n+1} \times S^n$ be an r-parameter family of Legendre mappings. We define the parallel of the r-parameter family of Legendre mappings by $f^k : U \times \Lambda \to \mathbb{R}^{n+1}$, $f^k(u,\lambda) = f(u,\lambda) + k\nu(u,\lambda)$, where $k \in \mathbb{R}$. It is easy to see that (f^k, ν) is also an r-parameter family of Legendre mappings for each fixed $k \in \mathbb{R}$.

Proposition 3.12 *Suppose that $e : V \to U \times \Lambda$ is a pre-envelope of (f,ν) (and E is an envelope) and $n \geq r$. Then the envelope of the parallel of the r-parameter family of Legendre mappings is given by the parallel of the envelope.*

Proof. Since ν is a unit vector, $\nu_{\lambda_j}(u,\lambda)\cdot\nu(u,\lambda) = 0$. Therefore, $f^k_{\lambda j}(u,\lambda)\cdot\nu(u,\lambda) = (f_{\lambda j}(u,\lambda) + k\nu_{\lambda j}(u,\lambda))\cdot\nu(u,\lambda) = f_{\lambda j}(u,\lambda)\cdot\nu(u,\lambda)$. If e is a pre-envelope of (f,ν), then $f^k_{\lambda_j}(e(p))\cdot\nu(e(p)) = f_{\lambda_j}(e(p))\cdot\nu(e(p)) = 0$ for all $p \in V$ and $j = 1, \dots, r$. It follows that e is also a pre-envelope of (f^k, ν) by Theorem 3.6. By definition, the envelope of the parallel of the r-parameter family of Legendre mappings is given by $E^k = f^k \circ e = f \circ e + k\nu \circ e = E + k\nu \circ e$. It follows that E^k is the parallel of the Legendre mapping $(E, \nu \circ e)$. $\square$

We give a relation between the envelope E_I of the classical definition by using an implicit function (Definition 2.1) and the envelope E of an r-parameter family of Legendre mappings (Definition 3.1).

Proposition 3.13 *Let $(f,\nu) : U \times \Lambda \to \mathbb{R}^{n+1} \times S^n$ be an r-parameter family of Legendre mappings, and let $F(x,\lambda) = 0$ be an implicit function of the r-parameter family of frontals, that is, assume $F(f(u,\lambda),\lambda) = 0$ and $(F_{x_1}, \dots, F_{x_{n+1}})(f(u,\lambda),\lambda)$ is parallel to $\nu(u,\lambda)$ for all $(u,\lambda) \in U \times \Lambda$. Suppose that $n \geq r$. If $e : V \to U \times \Lambda$ is a pre-envelope and $E : V \to \mathbb{R}^{n+1}$ is an envelope of (f,ν), then $E(V) \subset E_I$.*

Proof. By differentiating $F(f(u,\lambda),\lambda) = 0$ with respect to λ_j, we have

$$F_{x_1}(f(u,\lambda),\lambda)f_{1\lambda_j}(u,\lambda) + \cdots + F_{x_{n+1}}(f(u,\lambda),\lambda)f_{n+1\lambda_j}(u,\lambda) + F_{\lambda_j}(f(u,\lambda),\lambda) = 0,$$

where $j = 1, \dots, r$. By the assumption, there exists a smooth function $a : U \times \Lambda \to \mathbb{R}$ such that $(F_{x_1}, \dots, F_{x_{n+1}})(f(u,\lambda),\lambda) = a(u,\lambda)(\nu_1, \dots, \nu_{n+1})(u,\lambda)$ for all $(u,\lambda) \in U \times \Lambda$. By

Theorem 3.6, we have $f_{\lambda_j}(e(p)) \cdot \nu(e(p)) = 0$ for all $p \in V$ and $j = 1, \ldots, r$. It follows that $F_{\lambda_j}(f(e(p)), \lambda(p)) = 0$ for all $p \in V$ and $j = 1, \ldots, r$. Therefore $E(p) \in E_I$ with respect to $\lambda(p)$ for all $p \in V$. $\square$

Proposition 3.14 *Let $(f, \nu) : U \times \Lambda \to \mathbb{R}^{n+1} \times S^n$ be an r-parameter family of Legendre mappings, and let $e : V \to U \times \Lambda$ be a smooth map satisfying the variability condition. If* $\mathrm{rank}(f_{u_1}, \ldots, f_{u_n})(e(p)) = n$ *and the trace of e lies in the singular set of f, then e is a pre-envelope of (f, ν) (and E is an envelope).*

Proof. We denote the set of singular points (singular set) of f by $\Sigma(f)$. Since $e(p) \in \Sigma(f)$, we have the condition

$$\mathrm{rank}\begin{pmatrix} f_{1u_1} & \cdots & f_{1u_n} & f_{1\lambda_1} & \cdots & f_{1\lambda_r} \\ \vdots & & \vdots & \vdots & & \vdots \\ f_{n+1u_1} & \cdots & f_{n+1u_n} & f_{n+1\lambda_1} & \cdots & f_{n+1\lambda_r} \end{pmatrix}(e(p)) < n+1.$$

By the assumption $\mathrm{rank}(f_{u_1}, \ldots, f_{u_n})(e(p)) = n$, there exist smooth functions $a_{ij} : V \to \mathbb{R}$, $i = 1, \ldots, n, j = 1 \ldots, r$ such that $f_{\lambda_j}(e(p)) = a_{1j}(p) f_{u_1}(e(p)) + \cdots + a_{nj}(p) f_{u_n}(e(p))$. It follows that $f_{\lambda_j}(e(p)) \cdot \nu(e(p)) = 0$ for all $p \in V$ and $j = 1, \ldots, r$. Hence e is a pre-envelope of (f, ν). $\square$

Proposition 3.15 *Let $(f, \nu) : U \times \Lambda \to \mathbb{R}^{n+1} \times S^n$ be an r-parameter family of Legendre mappings, and let $F(x, \lambda) = 0$ be an implicit function of the r-parameter family of frontals, that is, assume $F(f(x, \lambda), \lambda) = 0$ and $(F_{x_1}, \ldots, F_{x_{n+1}})(f(u, \lambda), \lambda)$ is parallel to $\nu(u, \lambda)$ for all $(u, \lambda) \in U \times \Lambda$. Suppose that $e : V \to U \times \Lambda, e(p) = (u(p), \lambda(p))$ is a smooth mapping satisfying the variability condition. If $E(p) = f \circ e(p) \in E_I$ with respect to $\lambda(p)$,* $\mathrm{rank}(f_{u_1}, \ldots, f_{u_n})(e(p)) =$ *n and*

$$(F_{x_1}, \ldots, F_{x_{n+1}})(f(e(p)), \lambda(p)) \neq (0, \ldots, 0)$$

for all $p \in V$, then e is a pre-envelope of (f, ν) (and E is an envelope).

Proof. By differentiating $F(f(u, \lambda), \lambda) = 0$ with respect to u_i and λ_j, we have

$$\begin{aligned} &F_{x_1}(f(u, \lambda), \lambda) f_{1u_i}(u, \lambda) + \cdots + F_{x_{n+1}}(f(u, \lambda), \lambda) f_{n+1u_i}(u, \lambda) = 0, \\ &F_{x_1}(f(u, \lambda), \lambda) f_{1\lambda_j}(u, \lambda) + \cdots + F_{x_{n+1}}(f(u, \lambda), \lambda) f_{n+1\lambda_j}(u, \lambda) + F_{\lambda_j}(f(u, \lambda), \lambda) = 0, \end{aligned}$$

where $i = 1, \ldots, n, j = 1, \ldots, r$. Since $E(p) \in E_I$ with respect to $\lambda(p)$, we have $F_{\lambda_j}(f(e(p)), \lambda(p)) =$ 0 for all $p \in V$ and $j = 1, \ldots, r$. It follows that

$$\begin{pmatrix} f_{1u_1}(e(p)) & \cdots & f_{n+1u_1}(e(p)) \\ \vdots & & \vdots \\ f_{1u_n}(e(p)) & \cdots & f_{n+1u_n}(e(p)) \\ f_{1\lambda_1}(e(p)) & \cdots & f_{n+1\lambda_1}(e(p)) \\ \vdots & & \vdots \\ f_{1\lambda_r}(e(p)) & \cdots & f_{n+1\lambda_r}(e(p)) \end{pmatrix} \begin{pmatrix} F_{x_1}(f(e(p)), \lambda(p)) \\ \vdots \\ F_{x_{n+1}}(f(e(p)), \lambda(p)) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If the rank $df(e(p)) = n+1$, then $(F_{x_1}, \ldots, F_{x_{n+1}})(f(e(p)), \lambda(p)) = (0, \ldots, 0)$. By the assumption $(F_{x_1}, \ldots, F_{x_{n+1}})(f(e(p)), \lambda(p)) \neq (0, \ldots, 0)$, we have rank $df(e(p)) < n+1$. It follows that $e(p) \in \Sigma(f)$. By Proposition 3.14, e is a pre-envelope of (f, ν). $\square$

4 Singular solutions of first order partial differential equations

As an application of the theory of envelopes of families of Legendre mappings, we give a condition that the projection of a singular solution of a first order partial differential equation is an envelope.

We quickly review the theory of singular solutions and of Clairaut type of first order partial differential equations, in detail see [15, 16].

An equation is a submersion germ $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}, 0)$ on the 1-jet space of functions of n-variables, where $z_0 = (x_0, y_0, p_0)$. Let θ be a canonical contact 1-form on $J^1(\mathbb{R}^n, \mathbb{R})$ which is given by $\theta = dy - \sum_{i=1}^n p_i dx_i$, where $(x, y, p) = (x_1, \dots, x_n, y, p_1, \dots, p_n)$ is the canonical coordinate on $J^1(\mathbb{R}^n, \mathbb{R})$. We define a *geometric solution* of $F = 0$ to be an immersion germ $i : (L, q_0) \to (F^{-1}(0), z_0)$ of an n-dimensional manifold such that $i^*\theta = 0$, that is, a Legendre submanifold which is contained in $F^{-1}(0)$. We say that z_0 is a *contact singular point* if $\theta(T_{z_0} F^{-1}(0)) = 0$. It is easy to see that z_0 is a contact singular point if and only if $F = F_{p_i} = F_{x_i} + p_i F_y = 0$ for $i = 1, \dots, n$ at z_0. We also say that z_0 is a *π-singular point* if $F = F_{p_i} = 0$ for $i = 1, \dots, n$ at z_0. We denote the set of contact singular points by $\Sigma_c(F)$, the set of π-singular points by $\Sigma_\pi(F)$ and $\pi(\Sigma_\pi(F)) = D_F$, where $\pi : J^n(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^{n+1}$ is the canonical projection $\pi(x, y, p) = (x, y)$. We call the set D_F a *discriminant set* of the equation $F = 0$.

An equation $F = 0$ is said to be *completely integrable* at z_0 if there exists a foliation by geometric solution on $F^{-1}(0)$ around z_0, that is, there exists an immersion germ $\Gamma : (\mathbb{R}^n \times \mathbb{R}^n, (u_0, c_0)) \to (F^{-1}(0), z_0)$ such that $\Gamma(\cdot, c)$ is a geometric solution of $F = 0$ for each $c \in (\mathbb{R}^n, c_0)$. In this case, such a foliation is called a *complete solution* of $F = 0$ at z_0. We say that an n-parameter family of function germs $f : (\mathbb{R}^n \times \mathbb{R}^n, (x_0, c_0)) \to (\mathbb{R}, y_0)$ is a *classical complete solution* of $F = 0$ at z_0 if a complete solution is a form of $j^1 f : (\mathbb{R}^n \times \mathbb{R}^n, (x_0, c_0)) \to (F^{-1}(0), z_0)$, that is, $F(x, f(x, c), f_x(x, c)) = 0$ and $j^1 f(x, c) = (x, f(x, c), f_x(x, c))$ is an immersion germ. An equation $F = 0$ is said to be *classical completely integrable* at z_0 if there exists a classical complete solution of $F = 0$ at z_0.

A geometric solution $i : (L, q_0) \to (F^{-1}(0), z_0)$ of $F = 0$ is called a *singular solution* of $F = 0$ at z_0 if for any representative $\widetilde{i} : U \to F^{-1}(0)$ of i and any open subset $V \subset U$, $\widetilde{i}(V)$ is not contained in a leaf of any complete solutions of $F = 0$.

An equation $F = 0$ is called of *Clairaut type* at z_0 if there exist smooth function germs $B_{ji}, A_i : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to \mathbb{R}$ for $i, j = 1, \dots, n$ such that

$$F_{x_i} + p_i F_y = \sum_{j=1}^n B_{ji} F_{p_j} + A_i F, \ \ B_{ji} = B_{ij}$$

and

$$\frac{\partial B_{jk}}{\partial x_i} + p_i \frac{\partial B_{jk}}{\partial y} + \sum_{\ell=1}^n B_{\ell i} \frac{\partial B_{jk}}{\partial p_\ell} = \frac{\partial B_{ji}}{\partial x_k} + p_k \frac{\partial B_{ji}}{\partial y} + \sum_{\ell=1}^n B_{\ell k} \frac{\partial B_{ji}}{\partial p_\ell}$$

at any $(x, y, p) \in (F^{-1}(0), z_0)$ for $i, j, k = 1, \dots, n$. Then we have the following result.

Theorem 4.1 ([15, 16]) *Let $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}, 0)$ be a first order partial differential equation germs.*

(1) $F=0$ *is completely integrable at* z_0 *if and only if* $\Sigma_c(F)=\emptyset$ *or* $\Sigma_c(F)$ *is an* n*-dimensional submanifold around* z_0. *Moreover, if* $\Sigma_c(F)\neq\emptyset$, *then* $\Sigma_c(F)$ *is a singular solution of* $F=0$ *at* z_0.

(2) $F=0$ *is smooth completely integrable at* z_0 *if and only if* $F=0$ *is of Clairaut type at* z_0. *In this case, if* $\Sigma_\pi(F)\neq\emptyset$, *then* $\Sigma_\pi(F)$ *is a singular solution of* $F=0$ *at* z_0 *and the discriminant set* D_F *is the envelope of the family of graphs of the smooth complete solution.*

By using the envelope theorem (Theorem 3.6), we have the following result.

Theorem 4.2 *Let* $F:(J^1(\mathbb{R}^n,\mathbb{R}),z_0)\to(\mathbb{R},0)$ *be a first order partial differential equation germs and not of Clairaut type at* z_0. *Suppose that* $\Gamma=(x,y,p):(\mathbb{R}^n\times\mathbb{R}^n,(u_0,c_0))\to(F^{-1}(0),z_0)$ *is a complete solution of* $F=0$ *at* z_0, $\Sigma_c(F)=\Sigma_\pi(F)\neq\emptyset$ *and* $e:(\mathbb{R}^n,\widetilde{q}_0)\to(\mathbb{R}^n\times\mathbb{R}^n,(u_0,c_0))$ *is a smooth mapping satisfying the variability condition. Then* e *is a pre-envelope and* $E=\pi\circ\Gamma\circ e$ *is an envelope of* $(\pi\circ\Gamma,\nu)$ *if and only if* $E(q)\in\pi(\Sigma_c(F))$ *for all* $q\in(\mathbb{R}^n,\widetilde{q}_0)$, *where* $\nu(u,c)=(-p(u,c),1)/\sqrt{1+|p(u,c)|^2}$.

Proof. By the assumption and Theorem 4.1 (1), $\Sigma_c(F)=\Sigma_\pi(F)$ is an n-dimensional manifold around z_0 and a singular solution of $F=0$ at z_0. Since $z_0\in\Sigma_c(F)$ and $F=0$ is submersion, we may consider $F(x,y,p)=-y+g(x,p)$, where g is a smooth function, $x=(x_1,\dots,x_n)$ and $p=(p_1,\dots,p_n)$. We denote the complete solution of $F=0$ at z_0 by

$$\Gamma(u,c)=(x(u,c),y(u,c),p(u,c))=(x_1(u,c),\dots,x_n(u,c),y(u,c),p_1(u,c),\dots,p_n(u,c)),$$

where $u=(u_1,\dots,u_n), c=(c_1,\dots,c_n)$. Since $y(u,c)=g(x(u,c),p(u,c))$ and $\Gamma^*\theta=0$, we have $y_{u_i}(u,c)=p_1(u,c)x_{1u_i}(u,c)+\cdots+p_n(u,c)x_{nu_i}(u,c)$ for $i=1,\dots,n$. Since $F=0$ is not of Clairaut type at z_0, we have

$$\operatorname{rank}\begin{pmatrix} x_{1u_1} & \cdots & x_{nu_1}\\ \vdots & & \vdots\\ x_{1u_n} & \cdots & x_{nu_n}\end{pmatrix}(u_0,c_0)<n$$

and

$$\begin{aligned}&\operatorname{rank}\begin{pmatrix} x_{1u_1} & \cdots & x_{nu_1} & y_{u_1} & p_{1u_1} & \cdots & p_{nu_1}\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ x_{1u_n} & \cdots & x_{nu_n} & y_{u_n} & p_{1u_n} & \cdots & p_{nu_n}\end{pmatrix}(u_0,c_0)\\ &=\operatorname{rank}\begin{pmatrix} x_{1u_1} & \cdots & x_{nu_1} & p_{1u_1} & \cdots & p_{nu_1}\\ \vdots & & \vdots & \vdots & & \vdots\\ x_{1u_n} & \cdots & x_{nu_n} & p_{1u_n} & \cdots & p_{nu_n}\end{pmatrix}(u_0,c_0)=n. \qquad (1)\end{aligned}$$

Set $f(u,c)=\pi\circ\Gamma(u,c)$ and $\nu(u,c)=(-p(u,c),1)/\sqrt{1+|p(u,c)|^2}$. By a direct calculation, we have $f_{u_i}(u,c)\cdot\nu(u,c)=0$ for all $(u,c)\in(\mathbb{R}^n\times\mathbb{R}^n,(u_0,c_0))$ and $i=1,\dots,n$. It follows that (f,ν) is an n-parameter family of Legendre mappings (immersions). Moreover, we have

$$\begin{aligned} f_{c_i}(u,c) &= (x_{1c_i}(u,c),\cdots,x_{nc_i}(u,c),y_{c_i}(u,c))\\ &= \Big(x_{1c_i}(u,c),\cdots,x_{nc_i}(u,c),\\ &\qquad \sum_{j=1}^n x_{jc_i}(u,c)g_{x_j}(x(u,c),p(u,c))+\sum_{j=1}^n p_{jc_i}(u,c)g_{p_j}(x(u,c),p(u,c))\Big).\end{aligned}$$

It follows that $f_{c_i}(u,c)\cdot\nu(u,c) =$

$$\frac{1}{\sqrt{1+|p(u,c)|^2}}\left(\sum_{j=1}^{n}(-p_j(u,c)+g_{x_j}(x(u,c),p(u,c)))x_{jc_i}(u,c)+\sum_{j=1}^{n}p_{jc_i}(u,c)g_{p_j}(x(u,c),p(u,c))\right).$$

We now consider the following case. Suppose that

$$\operatorname{rank}\begin{pmatrix} p_{1u_1} & \cdots & p_{nu_1} \\ \vdots & & \vdots \\ p_{1u_n} & \cdots & p_{nu_n}\end{pmatrix}(u_0,c_0)=n.$$

By using an n-parameter family of parameter change, we may assume that $p_i(u,c)=u_i$ for $i=1,\dots,n$ by Proposition 3.11. If $e:(\mathbb{R}^n,\widetilde{q_0})\to(\mathbb{R}^n\times\mathbb{R}^n,(u_0,c_0))$ is a pre-envelope of (f,ν), then

$$\begin{pmatrix} x_{1c_1}(e(q)) & \cdots & x_{1c_n}(e(q)) \\ \vdots & & \vdots \\ x_{nc_n}(e(q)) & \cdots & x_{nc_n}(e(q))\end{pmatrix}\begin{pmatrix}(-p_1+g_{x_1})(e(q)) \\ \vdots \\ (-p_n+g_{x_n})(e(q))\end{pmatrix}=\begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}.$$

Here we denote a local coordinate $(\mathbb{R}^n,\widetilde{q_0})$ by q instead of p in §3. Since Γ is an immersion germ and by Theorem 3.6, we have $(-p_i+g_{x_i})(e(q))=0$ for $i=1,\dots,n$. It follows that $g_{p_i}(e(q))=0$ for $i=1,\dots,n$ and hence $E(q)\in\pi(\Sigma_\pi(F))=\pi(\Sigma_c(F))$. Conversely, if $E(q)\in\pi(\Sigma_c(F))$, then $f_{c_i}(e(q))\cdot\nu(e(q))=0$ for all $i=1,\dots,n$. By Theorem 3.6, e is a pre-envelope of (f,ν).

Moreover, suppose that

$$\operatorname{rank}\begin{pmatrix} x_{1u_1} & \cdots & x_{nu_k} & p_{k+1u_1} & \cdots & p_{nu_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{1u_n} & \cdots & x_{nu_k} & p_{k+1u_n} & \cdots & p_{nu_n}\end{pmatrix}(u_0,c_0)=n.$$

By using an n-parameter family of parameter change, we may assume that $x_i(u,c)=u_i$ for $i=1,\dots,k$ and $p_j(u,c)=u_j$ for $j=k+1,\dots,n$ by Proposition 3.11. Then we also have $(-p_i+g_{x_i})(e(q))=0$ for $i=1,\dots,k$ and $g_{p_j}(e(q))=0$ for $j=k+1,\dots,n$. It follows that $g_{p_i}(e(q))=0$ for $i=1,\dots,k$ and hence $E(q)\in\pi(\Sigma_\pi(F))=\pi(\Sigma_c(F))$. Conversely, if $E(q)\in\pi(\Sigma_c(F))$, then $f_{c_i}(e(q))\cdot\nu(e(q))=0$ for all $i=1,\dots,n$. By Theorem 3.6, e is a pre-envelope of (f,ν).

The other cases, we can also prove by similarly. This completes the proof of Theorem. □

By Theorems 4.1 and 4.2, if $\Sigma_c(F)=\Sigma_\pi(F)$ is an n-dimensional submanifold around z_0, then $\Sigma_c(F)$ is a singular solution of $F=0$ at z_0 and the projection $\pi(\Sigma_c(F))$ is an envelope when the variability condition holds.

We give concrete examples for completely integrable partial differential equations with a singular solution.

Example 4.3 Let $F: J^1(\mathbb{R}^2,\mathbb{R})\to\mathbb{R}$ be given by $F(x_1,x_2,y,p_1,p_2)=-y+p_1^{n_1}+p_2^{n_2}=0$, where $n_1,n_2\geq 2$. That is, we consider the partial differential equation

$$y=\left(\frac{\partial y}{\partial x_1}\right)^{n_1}+\left(\frac{\partial y}{\partial x_2}\right)^{n_2}.$$

Then $\Sigma_c(F) = \Sigma_\pi(F) = \{(x_1, x_2, 0, 0, 0)\}$ is a 2-dimensional submanifold. The complete solution $\Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to F^{-1}(0)$ is given by

$$\Gamma(u_1, u_2, c_1, c_2) = \left(\frac{n_1}{n_1 - 1}u_1^{n_1-1} + c_1, \frac{n_2}{n_2 - 1}u_2^{n_2-1} + c_2, u_1^{n_1} + u_2^{n_2}, u_1, u_2\right).$$

Then $(f, \nu) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^3 \times S^2$ is a 2-parameter family of Legendre mappings, where

$$\begin{aligned} f(u_1, u_2, c_1, c_2) &= \pi \circ \Gamma(u_1, u_2, c_1, c_2) = \left(\frac{n_1}{n_1 - 1}u_1^{n_1-1} + c_1, \frac{n_2}{n_2 - 1}u_2^{n_2-1} + c_2, u_1^{n_1} + u_2^{n_2}\right), \\ \nu(u_1, u_2, c_1, c_2) &= \frac{1}{\sqrt{1 + u_1^2 + u_2^2}}(-u_1, -u_2, 1). \end{aligned}$$

Since

$$f_{c_i}(u_1, u_2, c_1, c_2) \cdot \nu(u_1, u_2, c_1, c_2) = -u_i/\sqrt{1 + u_1^2 + u_2^2},\ i = 1, 2,$$

$e : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2, e(q_1, q_2) = (0, 0, q_1, q_2)$ is a pre-envelope and hence $E(q) = f \circ e(q) = (q_1, q_2, 0) \in \pi(\Sigma_c(F))$ is an envelope of (f, ν).

Example 4.4 Let $F : J^1(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}$ be given by $F(x_1, x_2, y, p_1, p_2) = -y + p_1^{n_1} + x_2 p_2 + x_2^{n_2} = 0$, where $n_1, n_2 \geq 2$. That is, we consider the partial differential equation

$$y = \left(\frac{\partial y}{\partial x_1}\right)^{n_1} + x_2 \frac{\partial y}{\partial x_2} + x_2^{n_2}.$$

Then $\Sigma_c(F) = \Sigma_\pi(F) = \{(x_1, 0, 0, 0, p_2)\}$ is a 2-dimensional submanifold. The complete solution $\Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to F^{-1}(0)$ is given by

$$\Gamma(u_1, u_2, c_1, c_2) = \left(\frac{n_1}{n_1 - 1}u_1^{n_1-1} + c_1, u_2, u_1^{n_1} + \frac{2n_2 - 1}{n_2 - 1}u_2^{n_2} + c_2 u_2, u_1, \frac{n_2}{n_2 - 1}u_2^{n_2-1} + c_2\right).$$

Then $(f, \nu) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^3 \times S^2$ is a 2-parameter family of Legendre mappings, where

$$\begin{aligned} f(u_1, u_2, c_1, c_2) &= \pi \circ \Gamma(u_1, u_2, c_1, c_2) = \left(\frac{n_1}{n_1 - 1}u_1^{n_1-1} + c_1, u_2, u_1^{n_1} + \frac{2n_2 - 1}{n_2 - 1}u_2^{n_2} + c_2 u_2\right), \\ \nu(u_1, u_2, c_1, c_2) &= \frac{1}{\sqrt{1 + u_1^2 + (\frac{n_2}{n_2-1}u_2^{n_2-1} + c_2)^2}}\left(-u_1, -\frac{n_2}{n_2 - 1}u_2^{n_2-1} - c_2, 1\right). \end{aligned}$$

Since

$$\begin{aligned} f_{c_1}(u_1, u_2, c_1, c_2) \cdot \nu(u_1, u_2, c_1, c_2) &= -\frac{u_1}{\sqrt{1 + u_1^2 + (\frac{n_2}{n_2-1}u_2^{n_2-1} + c_2)^2}}, \\ f_{c_2}(u_1, u_2, c_1, c_2) \cdot \nu(u_1, u_2, c_1, c_2) &= \frac{u_2}{\sqrt{1 + u_1^2 + (\frac{n_2}{n_2-1}u_2^{n_2-1} + c_2)^2}}, \end{aligned}$$

$e : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2, e(q_1, q_2) = (0, 0, q_1, q_2)$ is a pre-envelope and hence $E(q) = f \circ e(q) = (q_1, 0, 0) \in \pi(\Sigma_c(F))$ is an envelope of (f, ν).

Example 4.5 Let $F : J^1(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}$ be given by $F(x_1, x_2, y, p_1, p_2) = -y + p_1^{n_1} + x_2 p_2 + g(p_2) = 0$, where $n_1 \geq 2$ and g is a smooth function. That is, we consider the partial differential equation

$$y = \left(\frac{\partial y}{\partial x_1}\right)^{n_1} + x_2 \frac{\partial y}{\partial x_2} + g\left(\frac{\partial y}{\partial x_2}\right).$$

Then $\Sigma_c(F) = \Sigma_\pi(F) = \{(x_1, -g'(p_2), -g'(p_2)p_2 + g(p_2), 0, p_2)\}$ is a 2-dimensional submanifold. The complete solution $\Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to F^{-1}(0)$ is given by

$$\Gamma(u_1, u_2, c_1, c_2) = \left(\frac{n_1}{n_1 - 1} u_1^{n_1 - 1} + c_1, u_2, u_1^{n_1} + c_2 u_2 + g(c_2), u_1, c_2\right).$$

Then $(f, \nu) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^3 \times S^2$ is a 2-parameter family of Legendre mappings, where

$$\begin{aligned} f(u_1, u_2, c_1, c_2) &= \pi \circ \Gamma(u_1, u_2, c_1, c_2) = \left(\frac{n_1}{n_1 - 1} u_1^{n_1 - 1} + c_1, u_2, u_1^{n_1} + c_2 u_2 + g(c_2)\right), \\ \nu(u_1, u_2, c_1, c_2) &= \frac{1}{\sqrt{1 + u_1^2 + c_2^2}}(-u_1, -c_2, 1). \end{aligned}$$

Since

$$\begin{aligned} f_{c_1}(u_1, u_2, c_1, c_2) \cdot \nu(u_1, u_2, c_1, c_2) &= -u_1 / \sqrt{1 + u_1^2 + u_2^2}, \\ f_{c_2}(u_1, u_2, c_1, c_2) \cdot \nu(u_1, u_2, c_1, c_2) &= (u_2 + g'(c_2)) / \sqrt{1 + u_1^2 + u_2^2}, \end{aligned}$$

$e : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2, e(q_1, q_2) = (0, -g'(q_2), q_1, q_2)$ is a pre-envelope and hence $E(q) = f \circ e(q) = (q_1, -g'(q_2), -g'(q_2)q_2 + g(q_2)) \in \pi(\Sigma_c(F))$ is an envelope of (f, ν).

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Masatomo Takahashi,
Muroran Institute of Technology, Muroran 050-8585, Japan,
E-mail address: masatomo@mmm.muroran-it.ac.jp