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Orbital angular momentum of Liénard–Wiechert fields

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We derive a general expression for the electromagnetic field radiated by a relativistic charged particle with arbitrary periodic orbit, in the form of multi-pole expansion of the Liénard–Wiechert potential, which explicitly includes the charged particle motion. Using this expression, we discuss the orbital angular momentum radiated from a relativistic charged particle. It has recently been indicated that the radiation emitted by circularly orbiting charged particles carries well-defined orbital angular momentum. We show that, even for the general cases of arbitrary periodic orbits, the radiation field possesses well-defined orbital angular momentum.

Subject Index A00, A01

1. Introduction

Electromagnetic waves carry not only energy and linear momentum but also angular momentum. In particular, it is well known that electromagnetic waves carry spin angular momentum, which is associated with circular polarization. However, it is less well known that they also carry orbital angular momentum. In 1992, Allen et al. showed that an electromagnetic wave in Laguerre–Gaussian modes carries well-defined orbital angular momentum, distinct from spin angular momentum [1]. Such light waves have spiral wavefronts and are called optical vortices, photons carrying orbital angular momentum, or twisted photons. Since this pioneering work, optical vortices have been investigated not only from the scientific but also the technological viewpoint [2–4]. Some methods for artificially modifying the wavefront structure from a plane to a spiral by using spiral phase plates or numerically designed holograms have been established [2], and various applications of their unique features have been discussed. For example, optical vortices can be used to manipulate micrometer-sized objects by using their torque [5], and can be made useful for future high-capacity communications owing to the orthogonality between their different spiral modes [4]. Another attractive application of optical vortices may be for astronomical observations. It has been indicated that much more information can be obtained from electromagnetic wave signals from the universe by considering their spiral modes rather than conventional plane-wave spectrum channels [4,6–8]. However, there have only been a few discussions on the generation of optical vortices in nature [6,9].

It has recently been indicated that the radiation emitted by circularly orbiting charged particles exhibits spiral wavefronts and carries well-defined orbital angular momentum [10]. This process forms the basis for various important radiation processes in plasma physics and astrophysics, such as cyclotron radiation, synchrotron radiation, and Compton scattering of circularly polarized light. Therefore, this radiation has been well investigated in the literature [11,12]; however, its angular momentum has not been discussed explicitly until recently [13–15].

In this paper, we discuss the angular momentum of the radiation field emitted by a relativistic charged particle in more general cases, using a multi-pole expansion of the Liénard–Wiechert fields [16,17]. The multi-pole expansion of electromagnetic fields has been discussed in textbooks in the context of angular momentum [18–21]. However, in these textbooks, the authors consider electromagnetic radiation fields that are solutions of the homogeneous Helmholtz equation. This means that they examine electromagnetic waves propagating in free space without considering their source. In contrast, here, we treat the radiation field by explicitly including the motion of the charged particle.

2. Multipole expansion of Liénard–Wiechert fields

In this section, we briefly review the multi-pole expansion of Liénard–Wiechert fields [16,17]. The Liénard–Wiechert potentials, which are general expressions for the electromagnetic field produced by a relativistic charged particle, are given as follows:

$$\varphi(t,\boldsymbol{x}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{R(\tau) - \boldsymbol{R}(\tau)\cdot\frac{\boldsymbol{v}(\tau)}{c}}, \tag{1}$$

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{q}{4\pi\varepsilon_0 c^2}\frac{\boldsymbol{v}(\tau)}{R(\tau) - \boldsymbol{R}(\tau)\cdot\frac{\boldsymbol{v}(\tau)}{c}}, \tag{2}$$

where q is the electric charge of the particle, ε_0 is the dielectric constant, c is the velocity of light in vacuum, $\boldsymbol{R}(\tau) = \boldsymbol{x} - \boldsymbol{s}(\tau), R(\tau) = |\boldsymbol{R}(\tau)|$, $\boldsymbol{s}(t)$ is the charged particle trajectory, and $\boldsymbol{v}(t) = \dot{\boldsymbol{s}}(t)$ is the particle velocity. The right-hand sides of Eqs. (1) and (2) should be evaluated at the retarded time τ, which is defined by the following recursive causality relation:

$$\tau = t - \frac{|\boldsymbol{x} - \boldsymbol{s}(\tau)|}{c}. \tag{3}$$

The Fourier expansion of the potentials are expressed as follows:

$$\varphi(t,\boldsymbol{x}) = \frac{\omega q}{8\pi^2\varepsilon_0 c}\sum_{n=-\infty}^{\infty}\int_a^b \frac{\exp\left\{in\left(\omega t - \sigma - \frac{\omega}{c}|\boldsymbol{x} - \boldsymbol{s}(\sigma)|\right)\right\}}{|\boldsymbol{x} - \boldsymbol{s}(\sigma)|}\frac{c}{\omega}d\sigma, \tag{4}$$

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{\omega q}{8\pi^2\varepsilon_0 c^2}\sum_{n=-\infty}^{\infty}\int_a^b \frac{\exp\left\{in\left(\omega t - \sigma - \frac{\omega}{c}|\boldsymbol{x} - \boldsymbol{s}(\sigma)|\right)\right\}}{|\boldsymbol{x} - \boldsymbol{s}(\sigma)|}d\boldsymbol{s}(\sigma), \tag{5}$$

where ω is the angular frequency of the original electron motion and $\sigma = \omega t$ is the phase of the periodic motion. Then, by employing the well-known formula [20]

$$\frac{e^{-ik|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|} = -ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(2l+1)(-1)^m h_l^{(2)}(kr)j_l(kr')P_l^m(\cos\theta)P_l^{-m}(\cos\theta')e^{-im(\varphi-\varphi')}, \tag{6}$$

and replacing x' by $\boldsymbol{s}(\sigma)$, Eqs. (4) and (5) can be rewritten as

$$\varphi(t,\boldsymbol{x}) = \frac{\omega q}{8\pi^2\varepsilon_0 c}\sum_{n=-\infty}^{\infty}(-in)\int_0^{2\pi} d\sigma\, e^{in(\omega t-\sigma)}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(2l+1)(-1)^m$$
$$\times h_l^{(2)}(n\frac{\omega}{c}r)j_l(n\frac{\omega}{c}r(\sigma))P_l^m(\cos\theta)P_l^{-m}(\cos\theta(\sigma))e^{-im(\phi-\phi(\sigma))}, \tag{7}$$

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{\omega q}{8\pi^2\varepsilon_0 c^2}\sum_{n=-\infty}^{\infty}(-in)\int_0^{2\pi} d\sigma\frac{\boldsymbol{v}(\sigma)}{c}e^{in(\omega t-\sigma)}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(2l+1)(-1)^m$$
$$\times h_l^{(2)}(n\frac{\omega}{c}r)j_l(n\frac{\omega}{c}r(\sigma))P_l^m(\cos\theta)P_l^{-m}(\cos\theta(\sigma))e^{-im(\phi-\phi(\sigma))}, \tag{8}$$

where r,θ, and φ are the spherical coordinates, and $h_l^{(2)}(x), j_l(x)$, and $P_l^m(x)$ are the spherical Hankel function of the second type, the spherical Bessel function, and the associated Legendre function, respectively. If we introduce the spherical harmonic functions [20]

$$Y_{l,m}(\theta,\phi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}, \tag{9}$$

$$Y_{l,-m}(\theta,\phi) \equiv (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{-im\phi}, \tag{10}$$

Eqs. (7) and (8) can be expressed as

$$\varphi(t,\boldsymbol{x}) = \sum_{n=-\infty}^{\infty}(-in\omega)e^{in\omega t}\sum_{l=0}^{\infty}h_l^{(2)}(n\frac{\omega}{c}r)\sum_{m=-l}^{l}c\widetilde{M}^0_{n,l,m}Y_{l,-m}(\theta,\phi), \tag{11}$$

$$\boldsymbol{A}(t,\boldsymbol{x}) = \sum_{n=-\infty}^{\infty}(-in\omega)e^{in\omega t}\sum_{l=0}^{\infty}h_l^{(2)}(n\frac{\omega}{c}r)\sum_{m=-l}^{l}\widetilde{\boldsymbol{M}}_{n,l,m}Y_{l,-m}(\theta,\phi), \tag{12}$$

where

$$\widetilde{M}^0_{n,l,m} = \frac{q}{8\pi^2\varepsilon_0 c^2}\int_0^{2\pi}\sqrt{4\pi(2l+1)\frac{(l+m)!}{(l-m)!}}j_l(n\frac{\omega}{c}r(\sigma))P_l^{-m}(\cos\theta(\sigma))e^{-in\sigma}e^{im\varphi(\sigma)}d\sigma, \tag{13}$$

$$\widetilde{\boldsymbol{M}}_{n,l,m} = \frac{q}{8\pi^2\varepsilon_0 c^2}\int_0^{2\pi}\sqrt{4\pi(2l+1)\frac{(l+m)!}{(l-m)!}}\frac{\boldsymbol{v}(\sigma)}{c}j_l(n\frac{\omega}{c}r(\sigma))P_l^{-m}(\cos\theta(\sigma))e^{-in\sigma}e^{im\phi(\sigma)}d\sigma. \tag{14}$$

are definite integrals with respect to $\sigma = \omega t$ over the period of the particle motion; these have constant values in Cartesian coordinates $\left(\hat{\boldsymbol{x}},\ \hat{\boldsymbol{y}},\ \hat{\boldsymbol{z}}\right)$.

In accordance with convention [19,20], we interpret Eqs. (11) and (12) as the $(l-m)$ multi-pole expansions of the Liénard–Wiechert potential fields and Eqs. (13) and (14) as their components.

3. Linear and angular momenta of Liénard–Wiechert multipole fields

In this section, we evaluate the linear and angular momenta of the Liénard–Wiechert fields. For this purpose, it is convenient to express the multi-pole expansion of the Liénard–Wiechert potentials in spherical coordinates $\left(\hat{\boldsymbol{r}},\quad \hat{\boldsymbol{\theta}},\quad \hat{\phi}\right)$ [10]. While Eq. (12) is already written in spherical coordinates, Eq. (14) needs to be transformed from Cartesian to spherical coordinates, as follows:

$$\begin{aligned}\boldsymbol{M}_{n,l,m} &= (-1)^m\sqrt{\tfrac{2l+1}{4\pi}\tfrac{(l-m)!}{(l+m)!}}\widetilde{\boldsymbol{M}}_{n,l,m} = (-1)^m\sqrt{\tfrac{2l+1}{4\pi}\tfrac{(l-m)!}{(l+m)!}}\left(\widetilde{M}^x_{n,l,m}\hat{\boldsymbol{x}} + \widetilde{M}^y_{n,l,m}\hat{\boldsymbol{y}} + \widetilde{M}^z_{n,l,m}\hat{\boldsymbol{z}}\right)\\ &\equiv M^x_{n,l,m}\hat{\boldsymbol{x}} + M^y_{n,l,m}\hat{\boldsymbol{y}} + M^z_{n,l,m}\hat{\boldsymbol{z}} \equiv M^r_{n,l,m}(\theta,\phi)\hat{\boldsymbol{r}} + M^\theta_{n,l,m}(\theta,\phi)\hat{\boldsymbol{\theta}} + M^\phi_{n,l,m}(\theta,\phi)\hat{\phi}\end{aligned}, \tag{15}$$

$$\begin{cases} M^x_{n,l,m} = \frac{q}{8\pi^2\varepsilon_0 c^2}(2l+1)(-1)^m\int_0^{2\pi}\frac{v_x(\sigma)}{c}j_l\left(n\frac{\omega}{c}r(\sigma)\right)P_l^{-m}(\cos\theta(\sigma))e^{-in\sigma}e^{im\phi(\sigma)}d\sigma\\ M^y_{n,l,m} = \frac{q}{8\pi^2\varepsilon_0 c^2}(2l+1)(-1)^m\int_0^{2\pi}\frac{v_y(\sigma)}{c}j_l\left(n\frac{\omega}{c}r(\sigma)\right)P_l^{-m}(\cos\theta(\sigma))e^{-in\sigma}e^{im\phi(\sigma)}d\sigma\\ M^z_{n,l,m} = \frac{q}{8\pi^2\varepsilon_0 c^2}(2l+1)(-1)^m\int_0^{2\pi}\frac{v_z(\sigma)}{c}j_l\left(n\frac{\omega}{c}r(\sigma)\right)P_l^{-m}(\cos\theta(\sigma))e^{-in\sigma}e^{im\phi(\sigma)}d\sigma\end{cases}, \tag{16}$$

$$\begin{cases} M^r_{n,l,m}(\theta,\phi) = \sin\theta\cos\phi M^x_{n,l,m} + \sin\theta\sin\phi M^y_{n,l,m} + \cos\theta M^z_{n,l,m}\\ M^\theta_{n,l,m}(\theta,\phi) = \cos\theta\cos\phi M^x_{n,l,m} + \cos\theta\sin\phi M^y_{n,l,m} - \sin\theta M^z_{n,l,m}\\ M^\phi_{n,l,m}(\phi) = -\sin\phi M^x_{n,l,m} + \cos\phi M^y_{n,l,m}\end{cases}. \tag{17}$$

The vector $\boldsymbol{M}_{n,l,m}$ in Eq. (12) has a constant value in Cartesian coordinates, but depends on θ,ϕ in spherical coordinates. From Eq. (17), we find the following relations between the components of $\boldsymbol{M}_{n,l,m}$:

$$\frac{\partial M^\theta_{n,l,m}(\theta,\phi)}{\partial\theta} = -M^r_{n,l,m}(\theta,\phi), \tag{18}$$

$$\frac{\partial^2 M^\phi_{n,l,m}(\phi)}{\partial\phi^2} = -M^\phi_{n,l,m}(\phi), \tag{19}$$

$$\frac{\partial M^\theta_{n,l,m}(\theta,\phi)}{\partial\phi} = \cos\theta M^\phi_{n,l,m}(\phi), \tag{20}$$

$$\frac{\partial M^r_{n,l,m}(\theta,\phi)}{\partial\phi} = \sin\theta M^\phi_{n,l,m}(\phi). \tag{21}$$

In spherical coordinates, the multi-pole expansion of the Liénard–Wiechert potentials (11) and (12) is

$$\varphi(t,\boldsymbol{x}) = \sum_{n=-\infty}^{\infty}(-in\omega)e^{in\omega t}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}cM^0_{n,l,m}h_l^{(2)}\left(n\frac{\omega}{c}r\right)P_l^m(\cos\theta)e^{-im\phi}, \tag{22}$$

$$A_r(t,\boldsymbol{x}) = \sum_{n=-\infty}^{\infty}(-in\omega)e^{in\omega t}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}M^r_{n,l,m}(\theta,\varphi)h_l^{(2)}\left(n\frac{\omega}{c}r\right)P_l^m(\cos\theta)e^{-im\varphi}, \tag{23}$$

$$A_\theta(t,\boldsymbol{x}) = \sum_{n=-\infty}^{\infty}(-in\omega)e^{in\omega t}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}M^\theta_{n,l,m}(\theta,\varphi)h_l^{(2)}\left(n\frac{\omega}{c}r\right)P_l^m(\cos\theta)e^{-im\varphi}, \tag{24}$$

$$A_\varphi(t,\boldsymbol{x}) = \sum_{n=-\infty}^{\infty}(-in\omega)e^{in\omega t}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}M^\phi_{n,l,m}(\phi)h_l^{(2)}\left(n\frac{\omega}{c}r\right)P_l^m(\cos\theta)e^{-im\phi}. \tag{25}$$

Accordingly, the electromagnetic field components in the far field corresponding to Eqs. (22)–(25) can be expressed approximately as follows (Appendix A):

$$E_r^{rad}(t,\boldsymbol{x}) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} ci^{l+1}\frac{e^{-in\frac{\omega}{c}r}}{r^2}e^{i(n\omega t-m\phi)}\left\{-\left(\frac{c}{n\omega}\right)^2\frac{i}{r}\left[\left(P_l^m(\cos\theta)M_{n,l,m}^r(\theta,\phi)\right)\right.\right.$$
$$+\frac{1}{\sin\theta}\frac{\partial\left(\sin\theta P_l^m(\cos\theta)M_{n,l,m}^\theta(\theta,\phi)\right)}{\partial\theta}+\frac{1}{\sin\theta}P_l^m(\cos\theta)\left.\left(\frac{\partial M_{n,l,m}^\phi(\phi)}{\partial\phi}-imM_{n,l,m}^\phi(\phi)\right)\right]$$
$$\left.+\frac{c}{n\omega}\frac{1}{\sin\theta}\left[\frac{\partial\left(\sin\theta P_l^m(\cos\theta)M_{n,l,m}^\theta(\theta,\phi)\right)}{\partial\theta}+P_l^m(\cos\theta)\left(\frac{\partial M_{n,l,m}^\phi(\phi)}{\partial\phi}-imM_{n,l,m}^\phi(\phi)\right)\right]\right\}, \quad (26)$$

$$E_\theta^{rad}(t,\boldsymbol{x}) = \sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} ci^{l+1}\frac{e^{-in\frac{\omega}{c}r}}{r}e^{i(n\omega t-m\phi)}$$
$$\times\left(\frac{c}{n\omega}\frac{1}{r}\frac{dP_l^m(\cos\theta)}{d\theta}M_{n,l,m}^0+iP_l^m(\cos\theta)M_{n,l,m}^\theta(\theta,\phi)\right), \quad (27)$$

$$E_\phi^{rad}(t,\boldsymbol{x}) = \sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} ci^{l+1}\frac{e^{-in\frac{\omega}{c}r}}{r}e^{i(n\omega t-m\phi)}\left(\frac{c}{n\omega}\frac{-im}{r\sin\theta}M_{n,l,m}^0+iM_{n,l,m}^\phi(\phi)\right)P_l^m(\cos\theta), \quad (28)$$

$$B_r^{rad}(t,\boldsymbol{x}) = \sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^{l+1}\frac{e^{-in\frac{\omega}{c}r}}{r^2}e^{i(n\omega t-m\phi)}$$
$$\times\frac{c}{n\omega}\frac{1}{\sin\theta}\left\{\frac{-\partial\left(\sin\theta P_l^m(\cos\theta)\right)}{\partial\theta}M_{n,l,m}^\phi(\phi)\right.$$
$$\left.+\left(\frac{\partial M_{n,l,m}^\theta(\theta,\phi)}{\partial\phi}-imM_{n,l,m}^\theta(\theta,\phi)\right)P_l^m(\cos\theta)\right\}, \quad (29)$$

$$B_\theta^{rad}(t,\boldsymbol{x}) = \sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^{l+1}\frac{e^{-in\frac{\omega}{c}r}}{r}e^{i(n\omega t-m\phi)}$$
$$\times\left\{\frac{-1}{r\sin\theta}\frac{c}{n\omega}\left(\frac{\partial M_{n,l,m}^r(\theta,\phi)}{\partial\phi}-imM_{n,l,m}^r(\theta,\phi)\right)-iM_{n,l,m}^\phi(\phi)\right\}P_l^m(\cos\theta), \quad (30)$$

$$B_\phi^{rad}(t,\boldsymbol{x}) = \sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^{l+1}\frac{e^{-in\frac{\omega}{c}r}}{r}e^{i(n\omega t-m\phi)}$$
$$\times\left\{iP_l^m(\cos\theta)M_{n,l,m}^\theta(\theta,\phi)+\frac{1}{r}\frac{c}{n\omega}\frac{d\left(P_l^m(\cos\theta)M_{n,l,m}^r(\theta,\phi)\right)}{d\theta}\right\}. \quad (31)$$

Now, we shall calculate the radiation power emitted by the ultra-relativistic charged particle:

$$\frac{dU}{dt} = \frac{1}{T}\int_0^T dt \int_S \left(\boldsymbol{E}^{\mathrm{rad}*} \times \boldsymbol{H}^{\mathrm{rad}}\right) \cdot \boldsymbol{n} dS = \frac{1}{T}\int_0^T dt \int_S \left(\boldsymbol{E}^{\mathrm{rad}*} \times \frac{1}{\mu_0}\boldsymbol{B}^{\mathrm{rad}}\right) \cdot \boldsymbol{n} dS, \tag{32}$$

where $T = \frac{2\pi}{\omega}$ is the period of the particle motion, n is a unit vector normal to the surface S, and the surface integral $dS = r^2 \sin\theta d\theta d\phi$ is calculated for a spherical surface with a sufficiently large radius compared to the charged particle motion region. Considering that only the terms of order $\frac{1}{r}$ in Eqs. (26)–(31) contribute to Eq. (32), we only need to consider the second terms of Eqs. (27), (28), and (30) and the first term of Eq. (31). Substituting these terms into Eq. (32), we obtain

$$\begin{aligned}
\frac{dU}{dt} &= \frac{1}{\mu_0}\frac{1}{T}\int_0^T dt \int_0^{2\pi} d\phi \int_0^{\pi} r^2 \sin\theta d\theta \left(E_\theta^{\mathrm{rad}*} B_\phi^{\mathrm{rad}} - E_\varphi^{\mathrm{rad}*} B_\theta^{\mathrm{rad}}\right) \\
&= \frac{c}{\mu_0}\frac{1}{T}\int_0^T dt \int_0^{2\pi} d\phi \int_0^{\pi} r^2 \sin\theta d\theta \\
&\times \left\{ \left(\sum_{n=-\infty} (-in\omega) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i) M_{n,l,m}^{\theta}(\theta,\Phi)^* P_l^m(\cos\theta) (i^{l+1})^* \frac{e^{in\frac{\omega}{c}r}}{r} e^{-i(n\omega t - m\phi)} \right) \right. \\
&\times \left(\sum_{n'=-\infty} (-in'\omega) \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} i M_{n',l',m'}^{\theta}(\theta,\Phi) P_{l'}^{m'}(\cos\theta) i^{l'+1} \frac{e^{-in'\frac{\omega}{c}r}}{r} e^{i(n'\omega t - m'\phi)} \right) \\
&- \left(\sum_{n=-\infty} (-in\omega) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i) M_{n,l,m}^{\theta}(\Phi)^* P_l^m(\cos\theta) (i^{l+1})^* \frac{e^{in\frac{\omega}{c}r}}{r} e^{-i(n\omega t - m\phi)} \right) \\
&\times \left(\sum_{n'=-\infty} (-in'\omega) \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} (-i) M_{n',l',m'}^{\theta}(\Phi) P_{l'}^{m'}(\cos\theta) i^{l'+1} \frac{e^{-in'\frac{\omega}{c}r}}{r} e^{i(n'\omega t - m'\phi)} \right)
\end{aligned} \tag{33}$$

and carrying out the integral with respect to t over the period T, we arrive at

$$\begin{aligned}
\frac{dU}{dt} &= \frac{c}{\mu_0}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \sum_{n=-\infty}^{\infty} (n\omega)^2 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} \\
&\times i^{l'-l} \left(M_{n,l,m}^{\theta}(\theta,\phi)^* M_{n,l',m'}^{\theta}(\theta,\phi) + M_{n,l,m}^{\phi}(\phi)^* M_{n,l',m'}^{\phi}(\phi) \right) P_l^m(\cos\theta) P_{l'}^{m'}(\cos\theta) e^{i(m-m')\phi}
\end{aligned} \tag{34}$$

Note that the orthogonal formulas relating to spherical harmonics,

$$\int_0^{\pi} P_l^m(\cos\theta) P_{l'}^m(\cos\theta) \sin\theta d\theta = \frac{2}{2l+1}\frac{(l+|m|)!}{(l-|m|)!}\delta_{l,l'}, \tag{35}$$

$$\int_0^{2\pi} e^{-i(m-m')\phi} d\phi = 2\pi\,\delta_{m,m'}, \tag{36}$$

cannot be used for Eq. (34) since $M_{n,l,m}^{\theta}(\theta,\phi), M_{n,l,m}^{\phi}(\phi)$ depend on θ, ϕ. In Eqs. (35) and (36), $\delta_{n,m}$ denotes the Kronecker delta.

Next we shall calculate the angular momentum of the radiated fields. We consider its projection onto an arbitrary axis. As a matter of convenience, we shall call it the z-axis. The z-component of the radiated angular momentum,

$$\frac{dL_z}{dt} = \frac{1}{T}\int_0^T cdt \int_S \left\{ \boldsymbol{r} \times \frac{1}{c^2}\left(\boldsymbol{E}^{\mathrm{rad}*} \times \boldsymbol{H}^{\mathrm{rad}}\right)\right\} \cdot \hat{z} dS$$
$$= \frac{1}{\mu_0}\frac{1}{cT}\int_0^T cdt \int_S \left\{ \left(\boldsymbol{r}\cdot\boldsymbol{B}^{\mathrm{rad}}\right)\boldsymbol{E}^{\mathrm{rad}*}\cdot\hat{z} - \left(\boldsymbol{r}\cdot\boldsymbol{E}^{\mathrm{rad}*}\right)\boldsymbol{B}^{\mathrm{rad}}\cdot\hat{z}\right\} dS, \tag{37}$$

can be expressed in spherical coordinate as follows:

$$\frac{dL_z}{dt} = \frac{1}{\mu_0 cT}\int_0^T dt \int_S \left\{ rB_r^{\mathrm{rad}}\left(\cos\theta E_r^{\mathrm{rad}*} - \sin\theta E_\theta^{\mathrm{rad}*}\right) - rE_r^{\mathrm{rad}*}\left(\cos\theta B_r^{\mathrm{rad}} - \sin\theta B_\theta^{\mathrm{rad}}\right)\right\} dS$$
$$= \frac{1}{\mu_0 cT}\int_0^T dt \int_0^{2\pi} d\phi \int_0^\pi r\sin\theta\left(E_r^{\mathrm{rad}*}B_\theta^{\mathrm{rad}} - E_\theta^{\mathrm{rad}*}B_r^{\mathrm{rad}}\right) r^2 \sin\theta d\theta. \tag{38}$$

This means that we only need to consider terms of order $\frac{1}{r^3}$ in the integrands of Eq. (38). Since the r-components of the electric (26) and magnetic (29) fields are at least of order $\frac{1}{r^2}$, we only need to consider the second terms of Eqs. (27) and (30). Similarly to Eq. (33), substituting these terms into Eq. (38) and carrying out the integral with respect to t over the period T, we obtain the following:

$$\frac{dL_z}{dt} = \frac{1}{\mu_0 cT}\int_0^T dt \int_0^{2\pi} d\phi \int_0^\pi r^3 \sin^2\theta \left(E_r^{\mathrm{rad}*}B_\theta^{\mathrm{rad}} - E_\theta^{\mathrm{rad}*}B_r^{\mathrm{rad}}\right) d\theta$$
$$= \frac{1}{\mu_0}\int_0^{2\pi} d\phi \int_0^\pi d\theta \sum_{n=-\infty}^{\infty} (n\omega)^2 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} \frac{c}{n\omega} i^{l'-l} e^{i(m-m')\phi} \sin\theta(-i)$$
$$\times \left[\left\{ \frac{\partial\left(\sin\theta P_l^m(\cos\theta)M_{n,l,m}^\theta(\theta,\phi)^*\right)}{\partial\theta}\right.\right.$$
$$\left. + iP_l^m(\cos\theta)\left(mM_{n,l,m}^\phi(\phi)^* - i\frac{\partial M_{n,m,l}^\phi(\phi)^*}{\partial\phi}\right)\right\} M_{n,l',m'}^\phi(\phi)P_{l'}^{m'}(\cos\theta)$$
$$+ P_l^m(\cos\theta)M_{n,l,m}^\theta(\theta,\phi)^* \left\{ \frac{\partial\left(\sin\theta P_{l'}^{m'}(\cos\theta)\right)}{\partial\theta} M_{n,l',m'}^\phi(\phi)\right.$$
$$\left.\left. + i\left(m'M_{n,l',m'}^\theta(\theta,\phi) + i\frac{\partial M_{n,l',m'}^\theta(\theta,\phi)}{\partial\phi}\right)P_{l'}^{m'}(\cos\theta)\right\}\right]. \tag{39}$$

The first, second, and third terms of Eq. (26) have been neglected in Eq. (39) due to the order $\frac{1}{r^3}$ requirement. The summations of the first and third terms of Eq. (39) disappear in the integral with respect to θ and we finally obtain the following equation:

$$\frac{dL_z}{dt} = \frac{\alpha^2}{\mu_0}\int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \sum_{n=-\infty}^{\infty} (n\omega)^2 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} \frac{c}{n\omega}$$
$$\times i^{l'-l}\left\{ mM_{n,l,m}^\phi(\phi)^* M_{n,l',m'}^\phi(\phi) + m'M_{n,l,m}^\theta(\theta,\phi)^* M_{n,l',m'}^\theta(\theta,\phi)\right. \tag{40}$$
$$\left. -i\left(\frac{\partial M_{n,l,m}^\phi(\phi)^*}{\partial\phi}M_{n,l',m'}^\phi(\phi) - M_{n,l,m}^\theta(\theta,\phi)^*\frac{\partial M_{n,l',m'}^\theta(\theta,\phi)}{\partial\phi}\right)\right\} P_l^m(\cos\theta)P_{l'}^{m'}(\cos\theta)e^{i(m-m')\phi}$$

If we assume, e.g., that the charged particle motion is axis-symmetric with respect to the z-axis on average over the period of the particle motion, meaning that $M^r_{n,l,m}(\theta,\phi), M^\theta_{n,l,m}(\theta,\phi), M^\phi_{n,l,m}(\phi)$ are independent of ϕ, then

$$\frac{\partial M^r_{n,l',m'}(\theta,\varphi)}{\partial\phi}=0$$
$$\frac{\partial M^\theta_{n,l',m'}(\theta,\varphi)}{\partial\phi}=0,$$
$$\frac{\partial M^\varphi_{n,l,m}(\varphi)}{\partial\varphi}=0 \tag{41}$$

the third and fourth terms of Eq. (40) become zero, and we can carry out the integral with respect to ϕ by simply using the orthogonal relation (36), yielding

$$\begin{aligned}\frac{dL_z}{dt}&=\frac{1}{\mu_0}2\pi\int_0^\pi\sin\theta d\theta\sum_{n=-\infty}^{\infty}(n\omega)^2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\frac{c}{n\omega}m\\&\times i^{l'-l}\left(M^\phi_{n,l,m}{}^*M^\phi_{n,l',m}+M^\theta_{n,l,m}(\theta)^*M^\theta_{n,l',m}(\theta)\right)P^m_l(\cos\theta)P^m_{l'}(\cos\theta)\end{aligned}. \tag{42}$$

In addition, Eq. (34) becomes

$$\begin{aligned}\frac{dU}{dt}&=\frac{1}{\mu_0}2\pi\int_0^\pi\sin\theta d\theta\sum_{n=-\infty}^{\infty}(n\omega)^2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}c\\&\times i^{l'-l}\left(M^\phi_{n,l,m}{}^*M^\phi_{n,l',m}+M^\theta_{n,l,m}(\theta)^*M^\theta_{n,l',m}(\theta)\right)P^m_l(\cos\theta)P^m_{l'}(\cos\theta)\end{aligned}. \tag{43}$$

From Eqs. (42) and (43), we obtain

$$\begin{aligned}\left(\frac{\frac{dL_z}{dt}}{\frac{dU}{dt}}\right)_{n,l,m}&=\frac{\begin{array}{c}\int_0^\pi\sin\theta d\theta\sum_{l'=0}^{\infty}(n\omega)^2\frac{cm}{n\omega}i^{l'-l}\\ \left(M^\phi_{n,l,m}{}^*M^\phi_{n,l',m}+M^\theta_{n,l,m}(\theta)^*M^\theta_{n,l',m}(\theta)\right)P^m_l(\cos\theta)P^m_{l'}(\cos\theta)\end{array}}{\begin{array}{c}\int_0^\pi\sin\theta d\theta\sum_{l'=0}^{\infty}(n\omega)^2ci^{l'-l}\left(M^\phi_{n,l,m}{}^*M^\phi_{n,l',m}\right.\\ \left.+M^\theta_{n,l,m}(\theta)^*M^\theta_{n,l',m}(\theta)\right)P^m_l(\cos\theta)P^m_{l'}(\cos\theta)\end{array}}\\&=\frac{m}{n\omega}=\frac{m\hbar}{\hbar(n\omega)}\end{aligned} \tag{44}$$

for the (n-l-m) multi-pole components. We can infer from Eq. (44) that the radiation fields emitted by the ultra-relativistic charged particle carry angular momenta of $m\hbar$ per $\hbar(n\omega)$ photons for each nth time harmonic and (l-m)th multi-pole if the charged particle motion is axis-symmetric with respect to the z-axis on average over the period of the particle motion, as in Eq. (41).

Another interesting case is when the charged particle motion has the following rotational symmetry:

$$M^r_{n,l,m}(\theta,\phi)=M^r_{n,l,m}(\theta)e^{-i\alpha\phi}$$
$$M^\theta_{n,l,m}(\theta,\phi)=M^\theta_{n,l,m}(\theta)e^{-i\alpha\phi}.$$
$$M^\phi_{n,l,m}(\phi)=M^\phi_{n,l,m}e^{-i\alpha\phi} \tag{45}$$

In this case, the integrand of Eq. (40) becomes

$$\begin{aligned}&\left(mM^{\phi}_{n,l,m}(\phi)^{*}M^{\phi}_{n,l',m'}(\phi)+m'M^{\theta}_{n,l,m}(\theta,\phi)^{*}M^{\theta}_{n,l',m'}(\theta,\phi)\right)\\&\quad-i\left(\frac{\partial M^{\phi}_{n,l,m}(\phi)^{*}}{\partial\phi}M^{\phi}_{n,l',m'}(\phi)-M^{\theta}_{n,l,m}(\theta,\phi)^{*}\frac{\partial M^{\theta}_{n,l',m'}(\theta,\phi)}{\partial\phi}\right).\\&=(m+\alpha)M^{\phi}_{n,l,m}{}^{*}M^{\phi}_{n,l',m'}+(m'+\alpha)M^{\theta}_{n,l,m}(\theta)^{*}M^{\theta}_{n,l',m'}(\theta)\end{aligned}\tag{46}$$

In a similar way, using the orthogonal relation (36), the relation (44) can be extended to

$$\left(\frac{\frac{dL_z}{dt}}{\frac{dU}{dt}}\right)_{n,l,m}=\frac{m+\alpha}{n\omega}=\frac{(m+\alpha)\hbar}{\hbar(n\omega)}\tag{47}$$

for the (n-l-m) multi-pole components.

For the general case, after tedious calculations (Appendix B), the relation between the momentum and angular momentum of the multi-pole components of the radiation fields can be expressed in the following form, similar to Eq. (47):

$$\left(\frac{\frac{dL_z}{dt}}{\frac{dU}{dt}}\right)_{n,l,m}=\frac{m+\alpha_{n,l,m}}{n\omega},\tag{48}$$

where $\alpha_{n,l,m}$ is a constant value, which must be determined for each (n-l-m) multi-pole component (see Eq. (B7)).

The relation (44) between the momentum and angular momentum of the multi-pole components of the radiation fields emitted by ultra-relativistic charged particles is in agreement with discussions of general multi-pole fields in Refs. [18–21]. These discussions assume that the radiation fields are homogeneous and governed by the homogeneous Helmholtz equation, corresponding to the case where $M^{r}_{n,l,m}(\theta,\phi), M^{\theta}_{n,l,m}(\theta,\phi), M^{\phi}_{n,l,m}(\phi)$ in Eq. (17) are either unity or constant, and the relation (44) was obtained. The above discussion tells us that Eq. (44) is extended to Eq. (48) for the Liénard–Wiechert fields.

4. Synchrotron radiation

As a typical example of radiating charged particle motion, we apply here the above discussion to circular motion at constant velocity, as shown in Fig. 1. The charged particle's motion can be described as

$$\boldsymbol{s}(t)=a\cos\omega t\hat{\boldsymbol{x}}+a\sin\omega t\hat{\boldsymbol{y}},\tag{49}$$

$$\boldsymbol{v}(t)=\frac{d\mathbf{s}(t)}{dt}=-a\omega\sin\omega t\hat{\boldsymbol{x}}+a\omega\cos\omega t\hat{\boldsymbol{y}}=-v\sin\omega t\hat{\boldsymbol{x}}+v\cos\omega t\hat{\boldsymbol{y}},\tag{50}$$

where $v=a\omega$ is the velocity of the particle. This motion can be expressed in spherical coordinates as

$$\boldsymbol{s}(t)=\Big(r(t),\quad\theta(t),\quad\phi(t)\Big)=\Big(a,\quad\tfrac{\pi}{2},\quad\omega t\Big),\tag{51}$$

$$\boldsymbol{v}(t)=\Big(-v\sin\theta\sin(\omega t-\varphi),\quad-v\cos\theta\sin(\omega t-\varphi),\quad v\cos(\omega t-\varphi)\Big).\tag{52}$$

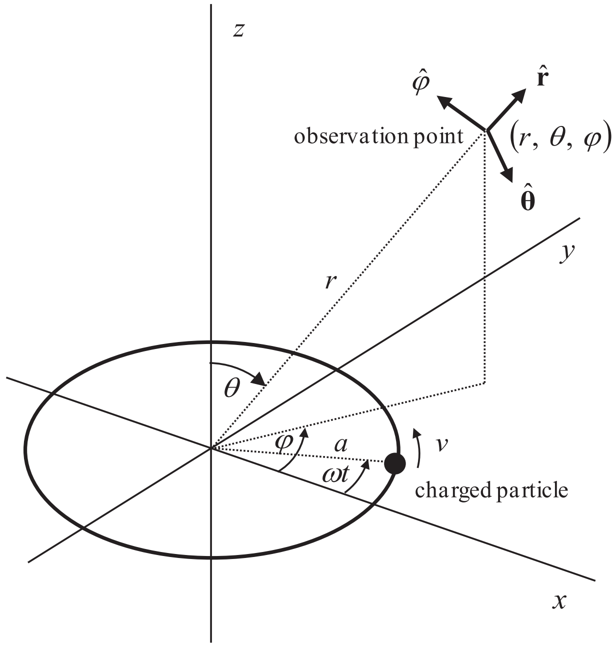

Fig. 1. Circular motion of a charged particle

Substituting Eq. (51) and (52) into Eqs. (15) and (16) yields

$$M^0_{n,l,m} = \frac{q}{8\pi^2\varepsilon_0 c^2}(2l+1)(-1)^m \int_0^{2\pi} j_l\left(n\frac{a\omega}{c}\right) P_l^{-m}(0)e^{-in\sigma}e^{im\sigma}\,d\sigma$$
$$= \frac{q}{4\pi\varepsilon_0 c^2}(2l+1)(-1)^m j_l\left(n\frac{v}{c}\right) P_l^{-m}(0)\delta_{n,m}, \tag{53}$$

$$M^x_{n,l,m} = \frac{q}{8\pi^2\varepsilon_0 c^2}(2l+1)(-1)^m \int_0^{2\pi} \frac{-v}{c}\sin(\sigma) j_l\left(n\frac{\omega}{c}a\right) P_l^{-m}(0)e^{-in\sigma}e^{im\sigma}\,d\sigma$$
$$= \frac{q}{4\pi\varepsilon_0 c^2}(2l+1)(-1)^m \frac{i}{2}\frac{v}{c} j_l\left(n\frac{v}{c}\right) P_l^{-m}(0)\left(\delta_{n,m+1}-\delta_{n,m-1}\right), \tag{54}$$

$$M^y_{n,l,m} = \frac{q}{8\pi^2\varepsilon_0 c^2}(2l+1)(-1)^m \int_0^{2\pi} \frac{v}{c}\cos(\sigma) j_l\left(n\frac{\omega}{c}a\right) P_l^{-m}(0)e^{-in\sigma}e^{im\sigma}\,d\sigma$$
$$= \frac{q}{4\pi\varepsilon_0 c^2}(2l+1)(-1)^m \frac{1}{2}\frac{v}{c} j_l\left(n\frac{v}{c}\right) P_l^{-m}(0)\left(\delta_{n,m+1}+\delta_{n,m-1}\right), \tag{55}$$

$$M^z_{n,l,m} = 0. \tag{56}$$

In this case, only two harmonics $n = m \pm 1$ exist in the summation $\sum_{n=-\infty}^{\infty}$ of Eq. (12) for each l-m multi-pole due to the Kronecker delta:

$$\boldsymbol{M}^z_{n=m+1,l,m} = \begin{pmatrix} iM^+_{l,m} & M^+_{l,m} & 0 \end{pmatrix}, \tag{57}$$

$$\boldsymbol{M}^z_{n=m-1,l,m} = \begin{pmatrix} -iM^-_{l,m} & M^-_{l,m} & 0 \end{pmatrix}, \tag{58}$$

where $M_{l,m}^{\pm} = \frac{q}{4\pi\varepsilon_0 c^2}(2l+1)(-1)^m \frac{1}{2}\frac{v}{c} j_l\left((m\pm 1)\frac{v}{c}\right) P_l^{-m}(0)$. For Eqs. (57) and (58), Eq. (17) becomes

$$\begin{cases} M^r_{n=m+1,l,m}(\theta,\phi) = \sin\theta\left(\cos\phi M^x_{n=m+1,l,m} + \sin\varphi M^y_{n=m+1,l,m}\right) + \cos\theta M^z_{n,l,m} = i\sin\theta M^+_{l,m}e^{-i\phi} \\ M^\theta_{n=m+1,l,m}(\theta,\phi) = \cos\theta\left(\cos\phi M^x_{n=m+1,l,m} + \sin\phi M^y_{n=m+1,l,m}\right) - \sin\theta M^z_{n,l,m} = i\cos\theta M^+_{l,m}e^{-i\phi} \\ M^\phi_{n=m+1,l,m}(\phi) = -\sin\phi M^x_{n=m+1,l,m} + \cos\phi M^y_{n=m+1,l,m} = M^+_{l,m}e^{-i\phi} \end{cases}, \tag{59}$$

$$\begin{cases} M^r_{n=m-1,l,m}(\theta,\phi) = \sin\theta\left(\cos\phi M^x_{n=m-1,l,m} + \sin\phi M^y_{n=m-1,l,m}\right) + \cos\theta M^z_{n,l,m} = -i\sin\theta M^-_{l,m}e^{i\phi} \\ M^\theta_{n=m-1,l,m}(\theta,\phi) = \cos\theta\left(\cos\phi M^x_{n=m-1,l,m} + \sin\phi M^y_{n=m-1,l,m}\right) - \sin\theta M^z_{n,l,m} = -i\cos\theta M^-_{l,m}e^{i\phi} \\ M^\phi_{n=m-1,l,m}(\phi) = -\sin\phi M^x_{n=m-1,l,m} + \cos\phi M^y_{n=m-1,l,m} = M^-_{l,m}e^{i\phi} \end{cases}. \tag{60}$$

Applying Eqs. (45), (46), and (47) to Eqs. (59) and (60), we can obtain

$$\left(\frac{\frac{dL_z}{dt}}{\frac{dU}{dt}}\right)_{n=m+1,l,m} = \frac{m+1}{n\omega} = \frac{m+1}{(m+1)\omega} = \frac{1}{\omega}, \tag{61}$$

$$\left(\frac{\frac{dL_z}{dt}}{\frac{dU}{dt}}\right)_{n=m-1,l,m} = \frac{m-1}{n\omega} = \frac{m-1}{(m-1)\omega} = \frac{1}{\omega}, \tag{62}$$

which agree with the result in Ref. [10].

5. Numerical examples

Finally, numerically calculated examples of Eq. (48) are given. Two numerical examples, circular and helical motion of the charged particle (see Fig. 2), are considered. The values of $\alpha_{n=20,l,m}$ for the circular motion (Fig. 2(a), radius $a = 1.0 \times 10^{-4}$ m, charged particle energy $E = 1.2$ GeV, harmonics $n = 20$) are depicted in Fig. 3(a). We can make sure that the results of Eqs. (61) and (62), i.e., $\alpha_{n,l,m}$, have non-zero values of ± 1 only for $m = n \mp 1$. In Fig. 3(b), values of $\alpha_{n=20,l,m}$ for the helical motion (Fig. 2(b), radius $a = 1.0 \times 10^{-4}$ m, pitch length $L = 3.14 \times 10^{-4}$ m, charged particle energy $E = 1.2$ GeV, harmonics $n = 20$) are depicted. There are no more simple relations as in the circular motion (53)–(56), and $\alpha_{n=20,l,m}$ has non-zero values for various sets of l and m. Most of $\alpha_{n=20,l,m}$ has an approximate value of -1, and some much bigger values appear depending on the original charged particle motion.

6. Conclusions

In this paper, we have generalized the discussion on the orbital angular momentum carried by the radiation field from a charged particle in circular motion [10] to arbitrary trajectories, by using a multipole expansion of the Liénard–Wiechert fields. The expression that we have derived is applicable to arbitrary charged particle motion with periodic orbit. We have shown that when the particle motion has an axis of symmetry, the field carries a well-defined angular momentum along the symmetry axis and that this expression for the angular momentum can be extended to the general case.

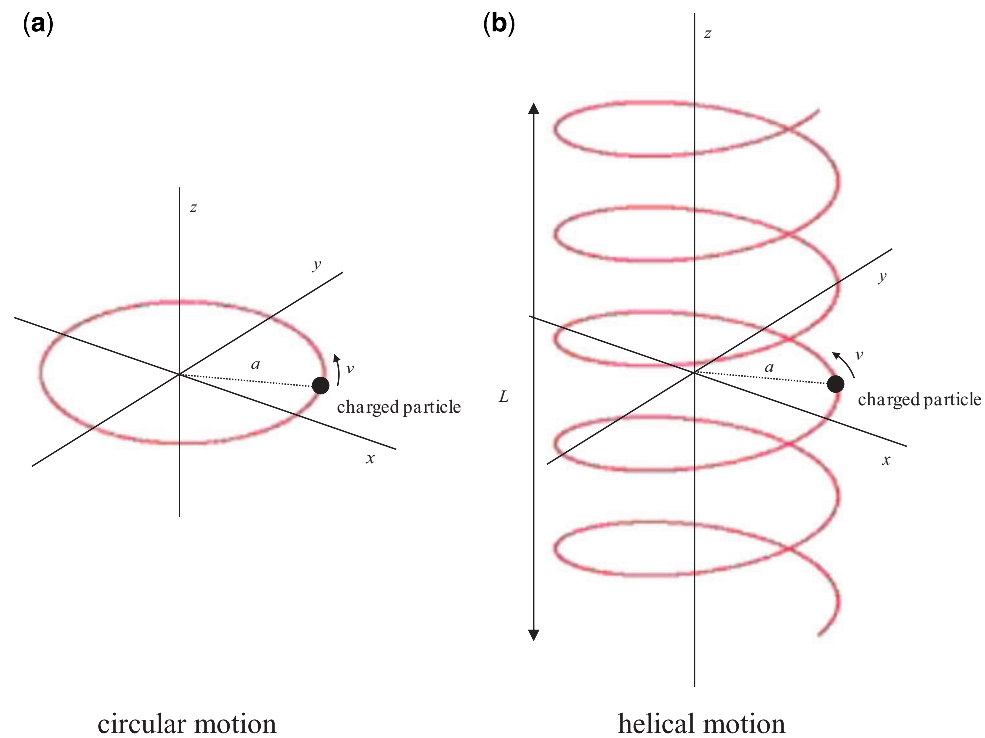

Fig. 2. Circular and helical motion of a charged particle

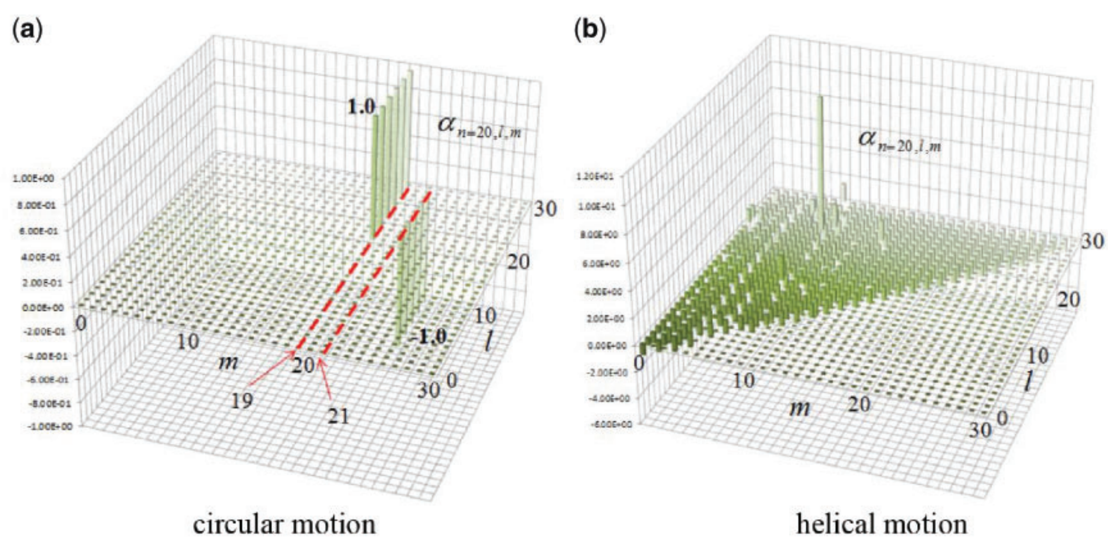

Fig. 3. Values of $\alpha_{n=20,l,m}$ for circular and helical motion of a charged particle

A. Appendix A

First, substituting Eqs. (22)–(25) into the Lorentz condition $\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \cdot \boldsymbol{A} = 0$ in spherical coordinates,

$$\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} = 0, \quad (A1)$$

yields the following relation for the multi-pole components:

$$\frac{in\omega}{c}h_l^{(2)}\left(n\frac{\omega}{c}r\right)P_l^m(\cos\theta)M_{n,l,m}^0+\left(\frac{2}{r}h_l^{(2)}\left(n\frac{\omega}{c}r\right)+\frac{d}{dr}h_l^{(2)}\left(n\frac{\omega}{c}r\right)\right)P_l^m(\cos\theta)M_{n,l,m}^r$$
$$+\frac{1}{r\sin\theta}h_l^{(2)}\left(n\frac{\omega}{c}r\right)\frac{\partial\left(\sin\theta P_l^m(\cos\theta)M_{n,l,m}^\theta\right)}{\partial\theta}$$
$$+\frac{1}{r\sin\theta}\left(\frac{\partial M_{n,l,m}^\varphi}{\partial\varphi}-imM_{\phi_m}^{n^l}\right)h_l^{(2)}\left(n\frac{\omega}{c}r\right)P_l^m(\cos\theta)=0. \tag{A2}$$

The electromagnetic fields corresponding to Eqs. (22)–(25) then become the following:

$$\begin{aligned}E_r(t,x)&=\frac{-\partial\varphi}{\partial r}-\frac{\partial A_r}{\partial t}\\&=\sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}e^{i(n\omega t-m\phi)}P_l^m(\cos\theta)\\&\quad\times\left(\frac{d}{dr}h_l^{(2)}\left(n\frac{\omega}{c}r\right)cM_{n,l,m}^0+in\omega h_l^{(2)}\left(n\frac{\omega}{c}r\right)M_{n,l,m}^r(\theta,\varphi)\right),\end{aligned} \tag{A3}$$

$$\begin{aligned}E_\theta(t,x)&=\frac{-1}{r}\frac{\partial\varphi}{\partial\theta}-\frac{\partial A_\theta}{\partial t}\\&=\sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}e^{i(n\omega t-m\phi)}h_l^{(2)}\left(n\frac{\omega}{c}r\right)\\&\quad\times\left(\frac{1}{r}\frac{dP_l^m(\cos\theta)}{d\theta}cM_{n,l,m}^0+in\omega P_l^m(\cos\theta)M_{n,l,m}^\theta(\theta,\varphi)\right),\end{aligned} \tag{A4}$$

$$\begin{aligned}E_\varphi(t,x)&=\frac{-1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}-\frac{\partial A_\varphi}{\partial t}\\&=\sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}e^{i(n\omega t-m\phi)}h_l^{(2)}\left(n\frac{\omega}{c}r\right)P_l^m(\cos\theta)\left(\frac{-im}{r\sin\theta}cM_{n,l,m}^0+in\omega M_{n,l,m}^\phi(\phi)\right),\end{aligned} \tag{A5}$$

$$\begin{aligned}B_r(t,x)&=\frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\\&=\sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}e^{i(n\omega t-m\phi)}h_l^{(2)}\left(n\frac{\omega}{c}r\right)\frac{1}{r}\left\{\frac{-1}{\sin\theta}\frac{\partial\left(\sin\theta P_l^m(\cos\theta)\right)}{\partial\theta}M_{n,l,m}^\phi(\phi)\right.\\&\quad\left.+\frac{1}{\sin\theta}\left(-imM_{n,l,m}^\theta(\theta,\phi)+\frac{\partial M_{n,l,m}^\theta(\theta,\phi)}{\partial\phi}\right)P_l^m(\cos\theta)\right\},\end{aligned} \tag{A6}$$

$$\begin{aligned}B_\theta(t,x)&=\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\\&=\sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}e^{i(n\omega t-m\phi)}\left\{\frac{-1}{r\sin\theta}h_l^{(2)}\left(n\frac{\omega}{c}r\right)\left(-imM_{n,l,m}^r(\theta,\phi)+\frac{\partial M_{n,l,m}^r(\theta,\phi)}{\partial\phi}\right)\right.\end{aligned}$$

$$+\left(\frac{1}{r}h_l^{(2)}\left(n\frac{\omega}{c}r\right)+\frac{d}{dr}h_l^{(2)}\left(n\frac{\omega}{c}r\right)\right)M^{\phi}_{n,l,m}(\phi)\Bigg\}P_l^m(\cos\theta), \tag{A7}$$

$$\begin{aligned}B_\varphi(t,x) &= \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r}-\frac{1}{r}\frac{\partial A_r}{\partial\theta}\\ &= \sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}e^{i(n\omega t-m\phi)}\Bigg\{\left(\frac{-1}{r}h_l^{(2)}\left(n\frac{\omega}{c}r\right)-\frac{d}{dr}h_l^{(2)}\left(n\frac{\omega}{c}r\right)\right)P_l^m(\cos\theta)M^{\theta}_{n,l,m}(\theta,\phi)\\ &\quad+\frac{1}{r}h_l^{(2)}\left(n\frac{\omega}{c}r\right)\frac{d\left(P_l^m(\cos\theta)M^{r}_{n,l,m}(\theta,\phi)\right)}{d\theta}\Bigg\}.\end{aligned} \tag{A8}$$

Now, we focus our discussion on the far field, where the spherical Bessel function of the second kind can be approximated by the asymptotic forms

$$h_l^{(2)}\left(n\frac{\omega}{c}r\right)\cong i^{l+1}\frac{c}{n\omega}\frac{e^{-in\frac{\omega}{c}r}}{r}, \tag{A9}$$

and

$$\frac{d}{dr}h_l^{(2)}\left(n\frac{\omega}{c}r\right)\cong i^{l+1}\left(\frac{-i}{r}-\frac{c}{n\omega}\frac{1}{r^2}\right)e^{-in\frac{\omega}{c}r}. \tag{A10}$$

Substituting Eqs. (A9) and (A10) into the Lorentz condition (A2)' yields

$$\begin{aligned}&-P_l^m(\cos\theta)M^0_{n,l,m}+\left(i\frac{c}{n\omega}\frac{1}{r}+1\right)P_l^m(\cos\theta)M^r_{n,l,m}(\theta,\phi)\\ &+\frac{c}{n\omega}\frac{i}{r}\frac{1}{\sin\theta}\frac{\partial\left(\sin\theta P_l^m(\cos\theta)M^\theta_{n,l,m}(\theta,\phi)\right)}{\partial\theta}+\frac{P_l^m(\cos\theta)}{r\sin\theta}\frac{c}{n\omega}\left(mM^\phi_{n,l,m}(\phi)+i\frac{\partial M^\phi_{n,l,m}(\phi)}{\partial\phi}\right)=0,\end{aligned} \tag{A11}$$

and the r-component of the far electric field can be approximated as

$$\begin{aligned}E_r^{\mathrm{rad}}(t,x) &= \sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}i^{l+1}\frac{e^{-in\frac{\omega}{c}r}}{r}e^{i(n\omega t-m\varphi)}\\ &\quad\times\left\{-\left(i+\frac{c}{n\omega}\frac{1}{r}\right)cM^0_{n,l,m}+icM^r_{n,l,m}(\theta,\varphi)\right\}P_l^m(\cos\theta).\end{aligned} \tag{A12}$$

Substituting the first and third terms of Eq. (A11) into the first and third terms of Eq. (A12), we obtain the following:

$$\begin{aligned}E_r^{\mathrm{rad}}(t,\boldsymbol{x}) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c\frac{c}{n\omega}\Bigg\{\left(-P_l^m(\cos\theta)M^0_{n,l,m}+P_l^m(\cos\theta)M^r_{n,l,m}(\theta,\phi)\right)\\ &\quad\frac{+\partial\left(\sin\theta P_l^m(\cos\theta)M^\theta_{n,l,m}(\theta,\varphi)\right)}{\sin\theta\partial\theta}\\ &\quad+\frac{P_l^m(\cos\theta)}{\sin\theta}\left(\frac{\partial M^\phi_{n,l,m}(\phi)}{\partial\phi}-imM^\phi_{n,l,m}(\phi)\right)\Bigg\}i^{l+1}\frac{e^{-in\frac{\omega}{c}r}}{r^2}e^{i(n\omega t-m\phi)}.\end{aligned} \tag{A13}$$

Substituting the first and third terms of Eq. (A11) into the first and second terms of Eq. (A13) once more,

$$E_r^{\rm rad}(t,x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(in\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} ci^{l+1}\frac{e^{-in\frac{\omega}{c}r}}{r^2}e^{i(n\omega t-m\varphi)}\left\{-\left(\frac{c}{n\omega}\right)^2\frac{i}{r}\left[\left(P_l^m(\cos\theta)M_{n,l,m}^r(\theta,\phi)\right)\right.\right.$$

$$+\frac{1}{\sin\theta}\frac{\partial\left(\sin\theta P_l^m(\cos\theta)M_{n,l,m}^{\theta}(\theta,\varphi)\right)}{\partial\theta}$$

$$\left.+\frac{1}{\sin\theta}P_l^m(\cos\theta)\left(\frac{\partial M_{n,l,m}^{\phi}(\phi)}{\partial\phi}-imM_{n,l,m}^{\phi}(\phi)\right)\right]$$

$$+\frac{c}{n\omega}\frac{1}{\sin\theta}\left[\frac{\partial\left(\sin\theta P_l^m(\cos\theta)M_{n,l,m}^{\theta}(\theta,\varphi)\right)}{\partial\theta}\right.$$

$$\left.\left.+P_l^m(\cos\theta)\left(\frac{\partial M_{n,l,m}^{\phi}(\phi)}{\partial\phi}-imM_{n,l,m}^{\phi}(\phi)\right)\right]\right\}. \tag{A14}$$

All other electromagnetic field components can now be calculated straightforwardly.

B. Appendix B

Substituting Eq. (17) into the integrand of Eq. (34) yields

$$M_{n,l,m}^{\phi}(\phi)^*M_{n,l',m'}^{\phi}(\phi)+M_{n,l,m}^{\theta}{}^*(\theta,\phi)M_{n,l',m'}^{\theta}(\theta,\phi)$$

$$=\left(-\sin\phi M_{n,l,m}^{x}{}^*+\cos\phi M_{n,l,m}^{y}{}^*\right)\left(-\sin\phi M_{n,l',m'}^{x}+\cos\phi M_{n,l',m'}^{y}\right)$$

$$+\left(\cos\theta\cos\phi M_{n,l,m}^{x}{}^*+\cos\theta\sin\phi M_{n,l,m}^{y}{}^*-\sin\theta M_{n,l,m}^{z}{}^*\right)$$

$$\left(\cos\theta\cos\phi M_{n,l',m'}^{x}+\cos\theta\sin\phi M_{n,l',m'}^{y}-\sin\theta M_{n,l',m'}^{z}\right)$$

$$=\left(\sin^2\phi M_{n,l,m}^{x}{}^*M_{n,l',m'}^{x}-\sin\phi\cos\phi M_{n,l,m}^{x}{}^*M_{n,l',m'}^{y}-\sin\phi\cos\phi M_{n,l,m}^{y}{}^*M_{n,l',m'}^{x}\right.$$

$$\left.+\cos^2\phi M_{n,l,m}^{y}{}^*M_{n,l',m'}^{y}\right)+\cos^2\theta\left(\cos^2\phi M_{n,l,m}^{x}{}^*M_{n,l',m'}^{x}+\sin\phi\cos\phi M_{n,l,m}^{x}{}^*M_{n,l',m'}^{y}\right.$$

$$\left.+\sin\phi\cos\phi M_{n,l,m}^{y}{}^*M_{n,l',m'}^{x}+\sin^2\phi M_{n,l,m}^{y}{}^*M_{n,l',m'}^{y}\right)$$

$$-\cos\theta\sin\theta\left(\cos\phi M_{n,l,m}^{x}{}^*M_{n,l',m'}^{z}+\sin\phi M_{n,l,m}^{y}{}^*M_{n,l',m'}^{z}\right.$$

$$\left.+\cos\phi M_{n,l,m}^{z}{}^*M_{n,l',m'}^{x}+\sin\phi M_{n,l,m}^{z}{}^*M_{n,l',m'}^{y}\right)+\sin^2\theta M_{n,l,m}^{z}{}^*M_{n,l',m'}^{z}. \tag{B1}$$

Multiplying Eq. (B1) by $e^{i(m-m')\phi}$ gives

$$\left(M_{n,l,m}^{\phi}(\phi)^*M_{n,l',m'}^{\phi}(\phi)+M_{n,l,m}^{\theta}{}^*(\theta,\phi)M_{n,l',m'}^{\theta}(\theta,\phi)\right)e^{i(m-m')\phi}$$

$$=\frac{1}{4}\left(\left(2e^{i(m-m')\phi}-e^{i(m+2-m')\phi}-e^{i(m-2-m')\phi}\right)M_{n,l,m}^{x}{}^*M_{n,l',m'}^{x}\right.$$

$$+ i\left(e^{i(m+2-m')\phi} - e^{i(m-2-m')\phi}\right) M^x_{n,l,m}{}^* M^y_{n,l',m'} + i\left(e^{i(m+2-m')\phi} - e^{i(m-2-m')\phi}\right) M^y_{n,l,m}{}^* M^x_{n,l',m'}$$

$$+ \left(2e^{i(m-m')\phi} + e^{i(m+2-m')\phi} + e^{i(m-2-m')\phi}\right) M^y_{n,l,m}{}^* M^y_{n,l',m'}\Big)$$

$$+ \frac{1}{4}\Big\{\cos^2\theta\left(\left(2e^{i(m-m')\phi} + e^{i(m+2-m')\phi} + e^{i(m-2-m')\phi}\right) M^x_{n,l,m}{}^* M^x_{n,l',m'}\right.$$

$$- i\left(e^{i(m+2-m')\phi} - e^{i(m-2-m')\phi}\right) M^x_{n,l,m}{}^* M^y_{n,l',m'} - i\left(e^{i(m+2-m')\phi} - e^{i(m-2-m')\phi}\right) M^y_{n,l,m}{}^* M^x_{n,l',m'}$$

$$+ \left(2e^{i(m-m')\phi} - e^{i(m+2-m')\phi} - e^{i(m-2-m')\phi}\right) M^y_{n,l,m}{}^* M^y_{n,l',m'}\Big)$$

$$- 2\cos\theta\sin\theta\left(\left(e^{i(m+1-m')\phi} + e^{i(m-1-m')\phi}\right) M^x_{n,l,m}{}^* M^z_{n,l',m'}\right.$$

$$- i\left(e^{i(m+1-m')\phi} - e^{i(m-1-m')\phi}\right) M^y_{n,l,m}{}^* M^z_{n,l',m'}$$

$$+ \left(e^{i(m+1-m')\phi} + e^{i(m-1-m')\phi}\right) M^z_{n,l,m}{}^* M^x_{n,l',m'} - i\left(e^{i(m+1-m')\phi} - e^{i(m-1-m')\phi}\right) M^z_{n,l,m}{}^* M^y_{n,l',m'}\Big)$$

$$+ 4\sin^2\theta e^{i(m-m')\phi} M^z_{n,l,m}{}^* M^z_{n,l',m'}\Big\}, \tag{B2}$$

and carrying out the integral with respect to ϕ and the summation with respect to m' in Eq. (34) in order to use Eq. (B2), we obtain

$$L_{n,l,m,l'}(\theta) \equiv \sum_{m'=-l'}^{l'} \frac{1}{2\pi}\int_0^{2\pi} \left(M^\phi_{n,l,m}(\phi)^* M^\phi_{n,l',m'}(\phi) + M^\theta_{n,l,m}{}^*(\theta,\phi) M^\theta_{n,l',m'}(\theta,\phi)\right) e^{i(m-m')\phi} d\phi$$

$$= \frac{1}{4}\Big(2M^x_{n,l,m}{}^* M^x_{n,l',m} - M^x_{n,l,m}{}^* M^x_{n,l',m+2} - M^x_{n,l,m}{}^* M^x_{n,l',m-2}$$

$$+ iM^x_{n,l,m}{}^* M^y_{n,l',m+2} - iM^x_{n,l,m}{}^* M^y_{n,l',m-2}$$

$$+ iM^y_{n,l,m}{}^* M^x_{n,l',m+2} - iM^y_{n,l,m}{}^* M^x_{n,l',m-2} + 2M^y_{n,l,m}{}^* M^y_{n,l',m}$$

$$+ M^y_{n,l,m}{}^* M^y_{n,l',m+2} + M^y_{n,l,m}{}^* M^y_{n,l',m-2}\Big)$$

$$+ \frac{1}{4}\Big\{\cos^2\theta\Big(2M^{x*}_{n,l,m} M^x_{n,l',m} + M^x_{n,l,m}{}^* M^x_{n,l',m+2} + M^x_{n,l,m}{}^* M^x_{n,l',m-2}$$

$$- iM^x_{n,l,m}{}^* M^y_{n,l',m+2} + iM^x_{n,l,m}{}^* M_{n,l',m-2}.$$

$$- iM^y_{n,l,m}{}^* M^x_{n,l',m+2} + iM^y_{n,l,m}{}^* M^x_{n,l',m-2} + 2M^y_{n,l,m}{}^* M^y_{n,l',m}$$

$$- M^y_{n,l,m}{}^* M^y_{n,l',m+2} - M^y_{n,l,m}{}^* M^y_{n,l',m-2}\Big)$$

$$- 2\cos\theta\sin\theta\Big(M^x_{n,l,m}{}^* M^z_{n,l',m+1} + M^x_{n,l,m}{}^* M^z_{n,l',m-1}$$

$$- iM^y_{n,l,m}{}^* M^z_{n,l',m+1} + iM^y_{n,l,m}{}^* M^z_{n,l',m-1}$$

$$+ M^{z}_{n,l,m}{}^{*}M^{x}_{n,l',m+1} + M^{z}_{n,l,m}{}^{*}M^{x}_{n,l',m-1} - iM^{z}_{n,l,m}{}^{*}M^{y}_{n,l',m+1} + iM^{z}_{n,l,m}{}^{*}M^{y}_{n,l',m-1}\Big)$$

$$+4\sin^2\theta m M^{z}_{n,l,m}{}^{*}M^{z}_{n,l',m}\Big\}. \tag{B3}$$

In a similar way, the integrand of Eq. (40) is

$$\sum_{m'=-l'}^{l'} \frac{1}{2\pi}\int_0^{2\pi}\Bigg\{\Big(mM^{\phi}_{n,l,m}(\phi)^{*}M^{\phi}_{n,l',m'}(\phi) + m'M^{\theta}_{n,l,m}{}^{*}(\theta,\phi)M^{\theta}_{n,l',m'}(\theta,\phi)\Big)$$
$$-i\left(\frac{\partial M^{\phi}_{n,l,m}(\phi)^{*}}{\partial\phi}M^{\phi}_{n,l',m'}(\phi) - M^{\theta}_{n,l,m}{}^{*}(\theta,\phi)\frac{\partial M^{\theta}_{n,l',m'}(\theta,\phi)}{\partial\phi}\right)\Bigg\} e^{i(m-m')\phi}d\phi$$
$$= mL_{n,l,m,l'}(\theta) + N_{n,l,m,l'}(\theta) \tag{B4}$$

where $N_{n,l,m,l'}(\theta)$ is defined as

$$N_{n,l,m,l'}(\theta)\equiv$$
$$\frac{1}{4}\Big\{\cos^2\theta\Big(M^{x}_{n,l,m}{}^{*}M^{x}_{n,l',m+2} - M^{x}_{n,l,m}{}^{*}M^{x}_{n,l',m-2} - iM^{x}_{n,l,m}{}^{*}M^{y}_{n,l',m+2} - iM^{x}_{n,l,m}{}^{*}M^{y}_{n,l',m-2}$$
$$-iM^{y}_{n,l,m}{}^{*}M^{x}_{n,l',m+2} - iM^{y}_{n,l,m}{}^{*}M^{x}_{n,l',m-2} - M^{y}_{n,l,m}{}^{*}M^{y}_{n,l',m+2} + M^{y}_{n,l,m}{}^{*}M^{y}_{n,l',m-2}\Big) - 2\cos\theta$$
$$\sin\theta\Big(M^{x}_{n,l,m}{}^{*}M^{z}_{n,l',m+1} - M^{x}_{n,l,m}{}^{*}M^{z}_{n,l',m-1} - iM^{y}_{n,l,m}{}^{*}M^{z}_{n,l',m+1} - iM^{y}_{n,l,m}{}^{*}M^{z}_{n,l',m-1}\Big)\Big\}$$
$$+\frac{1}{4}\Big\{\Big(-M^{x}_{n,l,m}{}^{*}M^{x}_{n,l',m+2} + M^{x}_{n,l,m}{}^{*}M^{x}_{n,l',m-2} + 2iM^{x}_{n,l,m}{}^{*}M^{y}_{n,l',m} + iM^{x}_{n,l,m}{}^{*}M^{y}_{n,l',m+2}$$
$$+ iM^{x}_{n,l,m}{}^{*}M^{y}_{n,l',m-2} - 2iM^{y}_{n,l,m}{}^{*}M^{x}_{n,l',m} + iM^{y}_{n,l,m}{}^{*}M^{x}_{n,l',m+2} + iM^{y}_{n,l,m}{}^{*}M^{x}_{n,l',m-2}$$
$$+M^{y}_{n,l,m}{}^{*}M^{y}_{n,l',m+2} - M^{y}_{n,l,m}{}^{*}M^{y}_{n,l',m-2}\Big) + \cos^2\theta\Big(2iM^{x}_{n,l,m}{}^{*}M^{y}_{n,l',m} - 2iM^{y}_{n,l,m}{}^{*}M^{x}_{n,l',m}\Big)\Big\}. \tag{B5}$$

From this, we finally obtain the following for the relation between the momentum and angular momentum as follows:

$$\left(\frac{\frac{dL_z}{dt}}{\frac{dU}{dt}}\right)_{n,l,m} = \frac{\int_0^{\pi}\sin\theta d\theta\sum\limits_{l'=0}^{\infty}(n\omega)^2\frac{c}{n\omega}i^{l'-l}\left(mL_{n,l,m,l'}(\theta) + N_{n,l,m,l'}(\theta)\right)P_l^m(\cos\theta)P_{l'}^m(\cos\theta)}{\int_0^{\pi}\sin\theta d\theta\sum\limits_{l'=0}^{\infty}(n\omega)^2 c i^{l'-l}L_{n,l,m,l'}(\theta)P_l^m(\cos\theta)P_{l'}^m(\cos\theta)}$$
$$= \frac{m+\alpha_{n,l,m}}{n\omega}, \tag{B6}$$

where $\alpha_{n,l,m}$ is defined by

$$\alpha_{n,l,m} = \frac{\int_0^{\pi}\sum\limits_{l'=0}^{\infty}i^{l'-l}N_{n,l,m,l'}(\theta)P_l^m(\cos\theta)P_{l'}^m(\cos\theta)\sin\theta d\theta}{\int_0^{\pi}\sum\limits_{l'=0}^{\infty}i^{l'-l}L_{n,l,m,l'}(\theta)P_l^m(\cos\theta)P_{l'}^m(\cos\theta)\sin\theta d\theta}, \tag{B7}$$

which is a constant for each multi-pole.

References

[1] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A **45**, 8185 (1992).
[2] M. Padgett, J. Courtial, and L. Allen, Phys. Today **57**, 35 (2004).
[3] G. Molina-Terriza, J. P. Torres, and L. Torner, Nat. Phys. **3**, 305 (2007).
[4] A. E. Willner, IEEE Spectrum **53**, 34 (2016).
[5] H. He, M. E. J. Friese, N. R. Heckenberg, and H. Rubinsztein-Dunlop, Phys. Rev. Lett. **75**, 826 (1995).
[6] M. Harwit, Astrophys. J. **597**, 1266 (2003).
[7] M. Harwit, Phys. Today **56**, 38 (2003).
[8] N. M. Elias II, Astron. Astrophys. **492**, 883 (2008).
[9] F. Tamburini, B. Thidé, G. Molina-Terriza, and G. Anzolin, Nat. Phys. **7**, 195 (2011).
[10] M. Katoh, M. Fujimoto, H. Kawaguchi, K. Tsuchiya, K. Ohmi, T. Kaneyasu, Y. Taira, M. Hosaka, A. Mochihashi, and Y. Takashima, Phys. Rev. Lett. **118**, 094801 (2017).
[11] J. Schwinger, Phys. Rev. **75**, 1912 (1949).
[12] A. A. Sololov and I. M. Ternov, Synchrotron Radiation (Akademic Verlag, Berlin, 1968).
[13] Y. Taira, T. Hayakawa, and M. Katoh, Sci. Rep. **7**, 5018 (2017).
[14] Y. Taira and M. Katoh, Astrophys. J. **860**, 45 (2018).
[15] Y. Taira and M. Katoh, Phys. Rev. A **98**, 052130 (2018).
[16] H. Kawaguchi and S. Murata, J. Phys. Soc. Jpn. **58**, 848 (1989).
[17] H. Kawaguchi and T. Honma, J. Phys. A: Math. Gen. **28**, 469 (1995).
[18] L. D. Landau and E. M. Lifshitz, Classical Theory of Fields (Pergamon, Oxford, UK, 1971).
[19] W. K. H Panofsky and M. Phillips, Classical Electricity and Magnetism (Addison-Wesley, Boston, MA, 1962), 2nd ed.
[20] J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1999), 3rd ed.
[21] W. Heitler, The Quantum Theory of Radiation (Clarendon Press, Oxford, UK, 1960), 3rd ed.