### Multiplicative induction and units for the ring of monomial representations

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Multiplicative induction and units for the ring of monomial representations

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Abstract

Let $G$ be a finite group, and let $A$ be a finite abelian $G$-group. For each subgroup $H$ of $G$, $\Omega(H, A)$ denotes the ring of monomial representations of $H$ with coefficients in $A$, which is a generalization of the Burnside ring $\Omega(H)$ of $H$. We research the multiplicative induction map $\Omega(H, A) \to \Omega(G, A)$ derived from the tensor induction map $\Omega(H) \to \Omega(G)$, and also research the unit group of $\Omega(G, A)$. The results are explained in terms of the first cohomology groups $H^1(K, A)$ for $K \leq G$. We see that tensor induction for 1-cocycles plays a crucial role in a description of multiplicative induction. The unit group of $\Omega(G, A)$ is identified as a finitely generated abelian group. We especially study the group of torsion units of $\Omega(G, A)$, and study the unit group of $\Omega(G)$ as well.

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1 Introduction

Let $G$ be a finite group, and let $A$ be a finite abelian group on which $G$ acts via a homomorphism from $G$ to the group of automorphisms of $A$. We are concerned with the ring $\Omega(G, A)$ of monomial representations of $G$ with coefficients in $A$, which was introduced by Dress [12] and is called the monomial Burnside ring for short. This ring contains the ordinary Burnside ring $\Omega(G)$ as a subring, and is applicable to the representation theory of finite groups. There are some well-known facts about $\Omega(G, A)$ (see, e.g., [2, 3, 12, 13, 22, 23]). Many properties of Burnside rings seem to be extended to monomial Burnside rings; for instance, the prime ideal spectrum of $\Omega(G, A)$ was studied in [12] (see also [10]). In this paper, among others, we focus our mind on the concept of multiplicative induction for monomial Burnside rings and the unit group of $\Omega(G, A)$. There are some specific characterizations of them which mean the algebraic peculiarities of $\Omega(G, A)$.

Following [12], we give the concept of $(G, A)$-sets and define simple $(G, A)$-sets $(G/K)_\nu$ for $K \leq G$ and 1-cocycles $\nu : K \to A$ in Section 2. The monomial Burnside ring $\Omega(G, A)$, which is defined to be the Grothendieck ring of the category of $(G, A)$-sets (see Definition 2.13), is the commutative unital ring consisting of all formal $\mathbb{Z}$-linear combinations of the symbols $[(G/K)_\nu]$ corresponding to the isomorphism classes of $(G, A)$-sets containing simple $(G, A)$-sets $(G/K)_\nu$ (see Proposition 2.14).

The concept of multiplicative induction for Burnside rings was introduced by tom Dieck [9] and Dress [11], and was developed by Yoshida [32]. In an attempt to introduce multiplicative induction for monomial Burnside rings, Barker [2] successfully defined the tenduction map $\widetilde{T}_Cten_H^G : B(C, H) \to B(C, G)$ for each $H \leq G$, where $C$ is a supercyclic group and $B(C, H)$ is the monomial Burnside ring for $H$ with fibre group $C$, as a generalization of multiplicative induction for Burnside rings. (If $C$ is a finite cyclic group on which $G$ acts trivially, then $\Omega(G, C) \simeq B(C, G).$)

In Section 3, we introduce the multiplicative induction map

$$\overline{\text{Map}}_H(G, -) : \Omega(H, A) \to \Omega(G, A), \quad x \mapsto \overline{\text{Map}}_H(G, x)$$
for each \( H \leq G \). When \( A \) is a cyclic group on which \( G \) acts trivially, this map is associated with tensor induction for linear characters of \( G \) (cf. [8, §13A]). We have \( \text{Map}_H(G, [(H/H)_{\sigma}]) = [\text{Map}_H(G, (H/H)_{\sigma})] = [(G/G)_{\sigma^G}] \) for all 1-cocycles \( \sigma : H \to A \) (see Example 3.13), where 1-cocycles \( \sigma^G : G \to A \) are obtained from \( \sigma : H \to A \) by tensor induction. There is a nice formula of multiplicative (tensor) induction for Burnside rings (cf. [8, (80.49) Corollary]). The methods used in [8, §80C] enable us to establish that for any \((H, A)\)-sets \( T_0 \) and \( T \),

\[
\overline{\text{Map}}_H(G, [T_0] - [T]) = \sum_{i=0}^{n} (-1)^i[\overline{\text{Map}}_H(G, T_0, T_1, \ldots, T_i)],
\]

where \( n = |G : H| \) and \( T = T_1 = \cdots = T_n \) (see Proposition 3.22).

The mark homomorphism \( \rho_G \), which was introduced by Dress [12], is a ring monomorphism from \( \Omega(G, A) \) to the set \( \mathcal{U}(G, A) := (\prod_{K \leq G} \mathbb{Z}H^1(K, A))^G \) of \( G \)-invariants in the direct product of integral group rings of the first cohomology groups \( H^1(K, A) \) for \( K \leq G \), where the action of \( G \) on \( \prod_{K \leq G} \mathbb{Z}H^1(K, A) \) is given by the conjugation maps \( \text{con}_K^g : \mathbb{Z}H^1(K, A) \to \mathbb{Z}H^1(gK, A) \) for \( K \leq G \) and \( g \in G \). For each \( U \leq G \), there is a ring homomorphism \( - \otimes^G : \mathbb{Z}H^1(U, A) \to \mathbb{Z}H^1(G, A) \) derived from tensor induction which assigns to a 1-cocycle \( \tau : U \to A \) the 1-cocycle \( \tau \otimes^G : G \to A \). In Section 4, we describe \( \overline{\text{Map}}_H(G, x) \in \Omega(G, A) \) for each \( x \in \Omega(H, A) \) via \( \rho_G \) as

\[
\rho_G(\overline{\text{Map}}_H(G, x)) = \left( \prod_{K \leq G} \text{con}_K^G(x_{K \cap H \leq G}) \right) \in \mathcal{U}(G, A)
\]

under the assumption that \( \rho_H(x) = (x_L)_{L \leq H} \), where \( \rho_H : \Omega(H, A) \to \mathcal{U}(H, A) \) is the mark homomorphism (see Theorem 4.16). This fact is a generalization of [32, §3(h.3)]. We make use of Eq.(1.1) to prove Eq.(1.2).

The fundamental theorem of the Burnside ring \( \Omega(G) \) (cf. [32, Lemma 2.1]) is a useful instrument for finding the idempotents of \( \Omega(G) \) (cf. [33, 4.12 Theorem]), and is also essential to the Yoshida criterion (see Theorem 6.4) for the units of \( \Omega(G) \). In Section 5, we insist on the existence of a short exact sequence

\[
0 \to \Omega(G, A) \xrightarrow{\varphi} \tilde{\Omega}(G, A) \xrightarrow{\psi} \text{Obs}(G, A) \to 0
\]

of additive groups (see Theorem 5.9) derived from the Cauchy-Frobenius lemma (see, e.g., [33, 2.7 Lemma]), which generalizes the fundamental theorem of \( \Omega(G) \).

Information of the primitive idempotents of the Burnside algebra \( \mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G) \) can help us to realize the units of \( \Omega(G) \). Following [33, §4], we review the primitive idempotents of \( \mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G) \) and those of \( \Omega(G) \); the latter are precisely the primitive idempotents of \( \Omega(G, A) \) (see Theorem 5.18).

The unit group \( \Omega(G)^\times \) of the Burnside ring \( \Omega(G) \) is studied in many papers (see, e.g., [6, 9, 11, 15, 18, 19, 20, 24, 30, 32]). Section 6 is devoted to a review of some
well-known facts about $\Omega(G)^\times$. We also study a certain specific type of units (see Proposition 6.11), and present an additional fact about the structure of $\Omega(G)^\times$ for which the Yoshida criterion plays a crucial role (see Corollary 6.18).

The unit group $\Omega(G,A)^\times$ of the monomial Burnside ring $\Omega(G,A)$ was studied in [2, 22]. In Section 7, we show that $\Omega(G,A)^\times$ is a finitely generated abelian group (see Proposition 7.2). Consequently, the group $\Omega(G,A)^\omega$ of torsion units of $\Omega(G,A)$ is a finite abelian group. The basic structure of $\Omega(G,A)^\omega$ is analyzed on the basis of a generalization of the Yoshida criterion (see Theorem 7.3). We adapt the methods presented in [2, 8] for an analysis of $\Omega(G,A)^\omega$, and successfully elucidate the structure of $\Omega(G,A)^\omega$ in the sequel (see Corollary 7.4). Specifically, if $G$ is nilpotent, then the universal result deduces that $\Omega(G,A)^\omega \cong \Omega(G)^\times \times H^1(G,A)$ (see Example 7.6). This fact is a generalization of [22, Proposition 5.1].

Notation Let $G$ be a finite group. We denote by $\epsilon$ the identity of $G$, and denote by $S(G)$ the set of subgroups of $G$. The subgroup generated by $g_1, \ldots, g_k \in G$ is denoted by $\langle g_1, \ldots, g_k \rangle$. We write $H \leq G$ if $H$ is a subgroup of $G$, and write $H < G$ if $H$ is a proper subgroup of $G$. The Möbius function on the poset $(S(G), \subseteq)$ of all subgroups of $G$ is denoted by $\mu$ (see, e.g., [1]). We denote by $C(G)$ a full set of non-conjugate subgroups of $G$. Let $H \subseteq G$. We set $gH = gHg^{-1}$ and $H^g = g^{-1}Hg$ for $g \in G$, and denote by $\langle H \rangle$ the set of conjugates of $H$ in $G$. The normalizer of $H$ in $G$ is denoted by $N_G(H)$. We denote by $|G : H|$ the index of $H$ in $G$, and denote by $G/H$ the set of left cosets $gH$, $g \in G$, of $H$ in $G$. Given $K, U \leq G$, $K \setminus G/U$ denotes the set of $(K, U)$-double cosets $KgU$, $g \in G$, in $G$. The category of finite left $G$-sets and $G$-equivariant maps is denoted by $G$-set. For each finite set $X$, we denote by $|X|$ the cardinality of $X$. The natural numbers, the rational numbers, and the complex numbers are denoted by $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{C}$, respectively. We set $[n] = \{1, 2, \ldots, n\}$ for each $n \in \mathbb{N}$. The identity map on a set $\Sigma$ is denoted by $\text{id}_\Sigma$. For each group $V$, we denote by $\text{Hom}(V, (-1))$ the group consisting of all group homomorphisms from $V$ to the unit group $\langle -1 \rangle$ of $\mathbb{Z}$ with pointwise product.

2 Monomial Burnside rings

2A 1-cocycles

Throughout the paper, let $G$ be a finite group, and let $A$ be a finite $G$-group, that is, $A$ is a finite group on which $G$ acts via a homomorphism from $G$ to the group of automorphisms of $A$ (cf. [26, Chapter 1, Definition 8.1]). We start with the definition of $(G, A)$-sets introduced by Dress [12] (see also [27]). Given $g \in G$
and \( a \in A \), the effect of \( g \) on \( a \) is denoted by \( ga \). A finite free right \( A \)-set \( Y \) is called a \((G, A)\)-set if it is also a left \( G \)-set and if

\[
g(ya) = (gy)a
\]

for all \( g \in G \), \( a \in A \), and \( y \in Y \). A map between \((G, A)\)-sets is called a \((G, A)\)-equivariant map if it is a morphism of both left \( G \)-sets and right \( A \)-sets. We now obtain the category of \((G, A)\)-sets such that the empty set is an initial object, which is denoted by \((G, A)\)-set. Under the assumption that \( A \) is abelian, the set of isomorphism classes of \((G, A)\)-sets forms a commutative unital semiring, and the monomial Burnside ring \( \Omega(G, A) \) is defined to be the associated Grothendieck ring (cf. [12]).

For a \((G, A)\)-set \( Y \), we denote by \( Y/A \) the set of \( A \)-orbits \( yA := \{ya \mid a \in A\} \), \( y \in Y \), on \( Y \), which is considered as a left \( G \)-set with the action of \( G \) given by

\[
g(yA) = gyA
\]

for all \( g \in G \) and \( y \in Y \). A \((G, A)\)-set \( Y \) is said to be simple if \( Y/A \) is a transitive left \( G \)-set. Given a pair of \((G, A)\)-sets \( Y_1 \) and \( Y_2 \), their disjoint union \( Y_1 \cup Y_2 \) is also a \((G, A)\)-set. Every \((G, A)\)-set is a disjoint union of simple \((G, A)\)-sets. A subset of a \((G, A)\)-set is said to be a \((G, A)\)-subset if it is closed under the actions of \( G \) and \( A \).

Let \( A^\circ \) be the opposite group of \( A \). For each \( a \in A \), let \( a^\circ \) denote the element of \( A^\circ \) corresponding to \( a \). By definition, \( a^\circ b^\circ = (ba)^\circ \) for all \( a, b \in A \). We view \( A^\circ \) as a \( G \)-group with the action given by that of \( G \) on \( A \), and denote by \( F \) the semidirect product \( A^\circ \rtimes G \) of \( A^\circ \) and \( G \). Each \((G, A)\)-set \( Y \) is viewed as a left \( F \)-set with the action of \( F \) given by

\[
(a^\circ, g)y = (gy)a
\]

for all \((a^\circ, g) \in F \) and \( y \in Y \). A \((G, A)\)-set is simple if and only if it is a transitive left \( F \)-set. A bijection between \((G, A)\)-sets is an isomorphism of \((G, A)\)-sets if and only if it is an isomorphism of left \( F \)-sets.

Let \( H \leq G \). By restriction of operators from \( G \) to \( H \), we view \( A \) as an \( H \)-group. A map \( \sigma : H \to A \) is called a 1-cocycle or a crossed homomorphism if

\[
\sigma(h_1 h_2) = \sigma(h_1)^{h_2} \sigma(h_2)
\]

for all \( h_1, h_2 \in H \) (cf. [26, I, p. 243]). We define a 1-cocycle \( 1_H : H \to A \) by \( 1_H(h) = \epsilon_A \) for all \( h \in H \), where \( \epsilon_A \) is the identity of \( A \).

**Definition 2.1.** For each \( H \leq G \), we denote by \( Z^1(H, A) \) the set of 1-cocycles from \( H \) to \( A \). Let \( S(G, A) \) be the set of pairs \((H, \sigma)\) of \( H \leq G \) and \( \sigma \in Z^1(H, A) \). Given \((H, \sigma) \in S(G, A)\), we fix a complete set \( \{g_1, g_2, \ldots, g_n\} \) with \( g_1 = \epsilon \) of representatives of \( G/H \), and define a \((G, A)\)-set \((G/H)_\sigma\) to be the cartesian product \( A \times (G/H) \) with the left action of \( G \) and the right action of \( A \) given by

\[
g(a, g_j H) = (g_j \sigma(g_j^{-1} g g_j)^\circ a, g_j H) \quad \text{and} \quad (a, g_j H)b = (ab, g_j H),
\]
where \( gg_j H = g_j' H \), for all \( g \in G \), \( a, b \in A \), and \( j \in [n] \), respectively.

Let \( (H, \sigma) \in \mathcal{S}(G, A) \). Then \((G/H)_\sigma\) is a transitive left \( F\)-set. We define
\[
F_{(H, \sigma)} := \{(\sigma(h)^{o-1}, h) \in F \mid h \in H\},
\]
so that \( F_{(H, \sigma)} \) is the stabilizer of \((\epsilon_A, H) \in (G/H)_\sigma\) in \( F \) (see [27, §2]), and make the set \( F/F_{(H, \sigma)} \) of left cosets of \( F_{(H, \sigma)} \) in \( F \) into a \((G, A)\)-set by defining
\[
g((a^o, r)F_{(H, \sigma)}) = (g a^o, gr)F_{(H, \sigma)} \quad \text{and} \quad ((a^o, r)F_{(H, \sigma)})b = ((ab)^o, r)F_{(H, \sigma)} \tag{2.2}
\]
for all \( g \in G \), \( b \in A \), and \((a^o, r) \in F\).

**Lemma 2.2** Let \( (H, \sigma) \in \mathcal{S}(G, A) \). Then \((G/H)_\sigma \simeq F/F_{(H, \sigma)}\) as \((G, A)\)-sets. In particular, the isomorphism class of \((G, A)\)-sets containing \((G/H)_\sigma\) is independent of the choice of \( g_2, \ldots, g_n \) in Definition 2.1.

**Proof.** There exists an isomorphism \( F/F_{(H, \sigma)} \overset{\sim}{\to} (G/H)_\sigma \) of \( F\)-sets given by
\[
(a^o, g)F_{(H, \sigma)} \mapsto (g(\epsilon_A, H))a
\]
for all \((a^o, g) \in F\), because \( F_{(H, \sigma)} \) is the stabilizer of \((\epsilon_A, H) \in (G/H)_\sigma\) in \( F \). Thus we have \((G/H)_\sigma \simeq F/F_{(H, \sigma)}\) as \((G, A)\)-sets, completing the proof. □

**Remark 2.3** Given a simple \((G, A)\)-set \( Y \) and \( y \in Y \), the stabilizer \( F_y \) of \( y \) in \( F \) coincides with \( F_{(H, \sigma)} \) for some \((H, \sigma) \in \mathcal{S}(G, A)\) (see the proof of [27, Lemma 2.1]), and hence \( Y \simeq F/F_{(H, \sigma)}\) as \((G, A)\)-sets. Under the notation of Definition 2.1, we may define \((G/H)_\sigma\) without assuming that \( g_1 = \epsilon \). In such a case, \( F_{(H, \sigma)} \) is the stabilizer of \((\sigma(g_1)^{-1}, H) \in (G/H)_\sigma\) in \( F \), which yields \((G/H)_\sigma \simeq F/F_{(H, \sigma)}\).

**2B Isomorphism classes**

We give a complete set of representatives of isomorphism classes of \((G, A)\)-sets.

**Definition 2.4** Let \((H, \sigma) \in \mathcal{S}(G, A)\). Suppose that \( g \in G \) and \( a \in A \). We define two 1-cocycles \( g\sigma : g^o H \to A \) and \( \sigma^o : H \to A \) by
\[
(g\sigma)(ghg^{-1}) = g\sigma(h) \quad \text{and} \quad \sigma^o(h) = a^{-1}\sigma(h)h^oa
\]
for all \( h \in H \), respectively.

Let \( H \leq G \), and let \( \sigma, \tau \in Z^1(H, A) \). We write \( \sigma =_A \tau \) if \( \tau = \sigma^a \) for some \( a \in A \).

**Lemma 2.5** Let \((H, \sigma) \in \mathcal{S}(G, A)\). Then \( g\sigma =_A g(\sigma^a) \) for any \( g \in G \) and \( a \in A \).
Proof. We have \( g(\sigma^a) = (g\sigma)^{ha} \) for any \( g \in G \) and \( a \in A \), completing the proof. \( \square \)

The argument of the proof of Lemma 2.5 ensures that \( S(G, A) \) is a left \( F \)-set with the action of \( F \) given by

\[
(a^\circ, g)(H, \sigma) = (gH, (g\sigma)^a)
\]

for all \((a^\circ, g) \in F\) and \((H, \sigma) \in S(G, A)\).

By [27, Lemma 2.3], \((H, \sigma)\) and \((U, \tau)\) are contained in the same \( F \)-orbit on \( S(G, A) \) if and only if \((G/H)_\sigma \simeq (G/U)_\tau\) as \((G, A)\)-sets.

Lemma 2.6 Let \( H \leq G \), and let \( \sigma \in Z^1(H, A) \). Then \( h\sigma = \sigma^{(h)} \) for any \( h \in H \). Moreover, given \( \sigma_0 \in Z^1(H, A) \), \( \sigma_0 =_A \sigma \) if and only if \((H/H)_{\sigma_0} \simeq (H/H)_\sigma\).

Proof. The first assertion is shown in the proof of [27, Lemma 3.2]. Suppose that \( \sigma_0 \in Z^1(H, A) \). By [27, Lemma 2.3], \((H/H)_{\sigma_0} \simeq (H/H)_\sigma\) if and only if there exist some \( h \in H \) and \( a \in A \) such that \( \sigma_0 = (h\sigma)^a \). Hence the second assertion follows from the first one. This completes the proof. \( \square \)

Definition 2.7 We define a subset \( R(G, A) \) of \( S(G, A) \) to be a complete set of representatives of \( F \)-orbits on \( S(G, A) \) such that \( H \in C(G) \) for any \((H, \sigma) \in R(G, A)\).

The following proposition is [27, Proposition 2.4].

Proposition 2.8 Let \( Y \) be a simple \((G, A)\)-set. There exists a unique element \((H, \sigma) \) of \( R(G, A) \) such that \( Y \simeq (G/H)_\sigma \) as \((G, A)\)-sets.

Let \( H \leq G \), and let \( X \in H\text{-}set \). We define a left action of \( H \) on the cartesian product \( G \times X \) of \( G \) and \( X \) by

\[
h(g, x) = (gh^{-1}, hx)
\]

for all \( h \in H \) and \( (g, x) \in G \times X \). Given \( (g, x) \in G \times X \), let \( g \otimes x \) denote the \( H \)-orbit containing \( (g, x) \). The left \( G \)-set \( \text{ind}^G_H(X) \) induced from \( X \) is the set of \( H \)-orbits on \( G \times X \) with the action of \( G \) given by

\[
g(r \otimes x) = gr \otimes x
\]

for all \( g, r \in G \) and \( x \in X \) (cf. [11, §4]). Let \( g \in G \), and set \( g \otimes X = \{ g \otimes x \mid x \in X \} \), which is a subset of \( \text{ind}^G_H(X) \). The left \( gH \)-set \( \text{con}_H^g(X) \) conjugate to \( X \) is the set \( g \otimes X \) with the action of \( gH \) given by

\[
ghg^{-1}(g \otimes x) = g \otimes hx
\]

for all \( h \in H \) and \( x \in X \), and is denoted simply by \( gX \).
Definition 2.9 Let $H \leq G$, and let $T$ be an $(H, A)$-set. The $(G, A)$-set $\text{ind}_H^G(T)$ induced from $T$ is the left $G$-set $\text{ind}_H^G(T)$ with the right action of $A$ given by

$$(r \otimes t)a = r \otimes t^{r^{-1}}a$$

for all $r \in G$, $t \in T$, and $a \in A$ (cf. [27, Remark 6.2]). Let $g \in G$. The $(gH, A)$-set $\text{con}_H^g(T)$ conjugate to $T$ is the left $gH$-set $gT$ with the right action of $A$ given by

$$(g \otimes t)a = g \otimes t^{g \cdot a}$$

for all $t \in T$ and $a \in A$ (cf. [27, Remark 6.4]), and is denoted simply by $gT$.

Lemma 2.10 If $U \leq H \leq G$ and $\tau \in Z^1(U, A)$, then $\text{ind}_H^G((H/U)_\tau) \simeq (G/U)_\tau$, $g((H/U)_\tau) \simeq (gH/gU)_\tau$ for each $g \in G$, and $h((H/U)_\tau) \simeq (H/U)_\tau$ for all $h \in H$.

Proof. The proof is straightforward. Note that the last assertion follows from the second one and [27, Lemma 2.3].

Let $(H, \sigma) \in S(G, A)$, and let $T$ be an $(H, A)$-set. For each $K \leq H$, we define a 1-cocycle $\sigma|_K : K \to A$, the restriction of $\sigma$, to be the map obtained by restriction of $\sigma : H \to A$ from $H$ to $K$, and define a $(K, A)$-set $\text{res}_K^H(T)$, the restriction of $T$, to be the $(K, A)$-set $T$ obtained by restriction of operators from $H$ to $K$.

We show a Mackey decomposition formula for $(G, A)$-sets (cf. [27, Lemma 6.5]).

Lemma 2.11 Let $H \leq G$, and let $(U, \tau) \in S(G, A)$. Then

$$\text{res}_H^G((G/U)_\tau) \simeq \bigcup_{HgU \in H \backslash G/U} (H/(H \cap gU))_{(g\tau)|_{H \cap gU}},$$

where the disjoint union is taken over all $(H, U)$-double cosets $HgU$, $g \in G$, in $G$.

Proof. Let $\{g_1, g_2, \ldots, g_m\}$ be a complete set of representatives of $H \backslash G/U$. For each $i \in [m]$, let $\{h_{i1}, h_{i2}, \ldots, h_{i[\ell_i]}\}$ be a complete set of representatives of $H/(H \cap g_iU)$. Then $\{h_{ij}g_i \mid i \in [m] \text{ and } j \in [\ell_i]\}$ is a complete set of representatives of $G/U$. We define a map $\Gamma : \text{res}_H^G((G/U)_\tau) \to \bigcup_{i \in [m]}(H/(H \cap g_iU))_{(g_i\tau)|_{H \cap g_iU}}$ by

$$(a, h_{ij}g_iU) \mapsto (a, h_{ij}(H \cap g_iU))$$

for all $i \in [m]$, $j \in [\ell_i]$, and $a \in A$. Obviously, this map is bijective and $A$-equivariant. Given $h \in H$, $i \in [m]$, and $j \in [\ell_i]$, if $hh_{ij} = h_{ij}h'$, then we have $h(h_{ij}g_i) = h_{ij}g_i(h^{-1}h')g_i = h_{ij}g_iU$ and

$$h_{ij}g_i\tau((h_{ij}g_i)^{-1}h(h_{ij}g_i)) = h_{ij}g_i\tau(g_i^{-1}h'h') = h_{ij}\tau(g_i\tau)(h_{ij}^{-1}h)h_{ij}.$$
From now on, we assume that $A$ is abelian. Hence $A = A^\circ$. Following [12], we define the tensor product $Y_1 \otimes Y_2$ of $(G, A)$-sets $Y_1$ and $Y_2$. The cartesian product $Y_1 \times Y_2$ is viewed as a free right $A$-set with the action of $A$ given by

$$(y_1, y_2)a = (y_1a^{-1}, y_2a)$$

for all $a \in A$ and $(y_1, y_2) \in Y_1 \times Y_2$. For each $(y_1, y_2) \in Y_1 \times Y_2$, let $y_1 \otimes y_2$ be the $A$-orbit containing $(y_1, y_2)$. We set

$$Y_1 \otimes Y_2 = \{ y_1 \otimes y_2 \mid (y_1, y_2) \in Y_1 \times Y_2 \},$$

and make it into a $(G, A)$-set by defining

$$g(y_1 \otimes y_2) = gy_1 \otimes gy_2 \quad \text{and} \quad (y_1 \otimes y_2)a = y_1 \otimes y_2a$$

for all $g \in G$, $a \in A$, and $(y_1, y_2) \in Y_1 \times Y_2$. These actions are well-defined, because

$$g((y_1b^{-1} \otimes y_2b)a) = g(y_1b^{-1}) \otimes g(y_2ba) = (gy_1)^{gb^{-1}} \otimes g(y_2a)^{gb} = g((y_1 \otimes y_2)a)$$

for all $g \in G$, $a, b \in A$, and $(y_1, y_2) \in Y_1 \times Y_2$. Obviously, $Y_1 \otimes Y_2 \simeq Y_2 \otimes Y_1$.

**Lemma 2.12** Let $K \leq H \leq G$, and let $g \in G$. For any $(H, A)$-sets $T_1$ and $T_2$,

$$\text{res}_K^H(T_1 \otimes T_2) \simeq \text{res}_K^H(T_1) \otimes \text{res}_K^H(T_2) \quad \text{and} \quad g(T_1 \otimes T_2) \simeq gT_1 \otimes gT_2.$$ 

**Proof.** The proof is straightforward. □

Let $F(G, A)$ be the free abelian group on the set of isomorphism classes of $(G, A)$-sets. For each $(G, A)$-set $Y$, we denote by $\overline{Y}$ the isomorphism class of $(G, A)$-sets containing $Y$. Let $F(G, A)_0$ be the subgroup of $F(G, A)$ generated by the elements $\overline{Y_i}$ for $(G, A)$-sets $Y_1$ and $Y_2$. We define multiplication on the generators of $F(G, A)$ by

$$\overline{Y_1} \cdot \overline{Y_2} = \overline{Y_1 \otimes Y_2}$$

for all $(G, A)$-sets $Y_1$ and $Y_2$, and extend it to $F(G, A)$ by linearity. Then $F(G, A)$ is a commutative unital ring; moreover, $F(G, A)_0$ is an ideal of $F(G, A)$.

**Definition 2.13** We define a commutative unital ring $\Omega(G, A)$ to be the quotient $F(G, A)/F(G, A)_0$, which is the ring of monomial representations of $G$ with coefficients in $A$ introduced by Dress [12] (see also [2]).
When $A = \{\epsilon_A\}$, which is the group consisting of only the identity, $\Omega(G, A)$ is isomorphic to the Burnside ring $\Omega(G)$ (see §5C).

For each $(G, A)$-set $Y$, we denote by $[Y]$ the coset $\overline{Y} + F(G, A)_0$ of $F(G, A)_0$ in $F(G, A)$. By [12, Proposition 1(b)] (or [27, Lemma 2.6]), $[Y_1] = [Y_2]$ if and only if $\overline{Y_1} = \overline{Y_2}$. Multiplication on the generators of $\Omega(G, A)$ is given by

$$[Y_1] \cdot [Y_2] = [Y_1 \otimes Y_2]$$

equation of all $(G, A)$-sets $Y_1$ and $Y_2$. The identity of $\Omega(G, A)$ is $[(G/G)_1c]$.

A $Z$-lattice is a finitely generated $Z$-free $Z$-module. Obviously, $\Omega(G, A)$ is a $Z$-lattice. The statement of the following proposition is given in [12, Proposition 1(a)] (see also [2, Remark 2.2] and [27, Proposition 2.7]).

**Proposition 2.14** The elements $[(G/H)\sigma]$ for $(H, \sigma) \in \mathcal{R}(G, A)$ form a free $Z$-basis of the $Z$-lattice $\Omega(G, A)$.

**Proof.** The assertion follows from Proposition 2.8. $\square$

We obtain a product formula of simple $(G, A)$-sets (see also [2, Remark 2.3]).

**Lemma 2.15** Let $(H, \sigma), (U, \tau) \in S(G, A)$. Then

$$(G/H)\sigma \otimes (G/U)_{\tau} \simeq \bigcup_{HgU \in H\cap G/U} (G/(H \cap gU))_{\sigma \cdot (g\tau)},$$

where $\sigma \cdot (g\tau) : H \cap gU \to A$ is the pointwise product of $\sigma|_{H \cap gU}$ and $(g\tau)|_{H \cap gU}$.

**Proof.** We view the tensor product $(F/F_{(H, \sigma)}) \otimes (F/F_{(U, \tau)})$ of $(G, A)$-sets $F/F_{(H, \sigma)}$ and $F/F_{(U, \tau)}$ as a left $F$-set. The left $F$-set $(F/F_{(H, \sigma)}) \otimes (F/F_{(U, \tau)})$ is expressed as a disjoint union of $F$-orbits. We identify each $g \in G$ with $(\epsilon_A, g) \in F$ for shortness' sake. For any $(a, g), (b, r) \in F$,

$$(a, g)^{-1}((a, g)_{F(H, \sigma)} \otimes (b, r)_{F(U, \tau)}) = (g^{-1}b, \epsilon)(F_{(H, \sigma)} \otimes \epsilon^{-1}r_{F(U, \tau)})$$

(see Eqs. (2.1) and (2.2)), which means that there exists an $F$-orbit containing both $(a, g)_{F(H, \sigma)} \otimes (b, r)_{F(U, \tau)}$ and $F_{(H, \sigma)} \otimes \epsilon^{-1}r_{F(U, \tau)}$. Let $g, r \in G$. Suppose that

$$F_{(H, \sigma)} \otimes r_{F(U, \tau)} = (a, h)(F_{(H, \sigma)} \otimes g_{F(U, \tau)}) = h_{F_{(H, \sigma)} \otimes (a, hg)}_{F(U, \tau)}$$

with $(a, h) \in F$. Then $h \in H$ and $r^{-1}hg \in U$, which yields $g \in HrU$. Conversely, if $g \in HrU$ and $r^{-1}hg \in U$ with $h \in H$, then we have

$$F_{(H, \sigma)} \otimes r_{F(U, \tau)} = (hg\tau(g^{-1}h^{-1}r)_{\sigma(h)^{-1}, h})(F_{(H, \sigma)} \otimes g_{F(U, \tau)}).$$

Consequently, both $F_{(H, \sigma)} \otimes r_{F(U, \tau)}$ and $F_{(H, \sigma)} \otimes g_{F(U, \tau)}$ are contained in the same $F$-orbit if and only if $g \in HrU$. Suppose that

$$F_{(H, \sigma)} \otimes g_{F(U, \tau)} = (a, h)(F_{(H, \sigma)} \otimes g_{F(U, \tau)}) = (a, h)_{F_{(H, \sigma)} \otimes hg_{F(U, \tau)}}.$$
with \((a, h) \in F\). Then there exists some \(b \in A\) such that
\[
(F_{\mathcal{H}, \sigma}, gF_{\mathcal{U}, \tau}) = (((a, h)F_{\mathcal{H}, \sigma})b^{-1}, (hgF_{\mathcal{U}, \tau})b) = ((b^{-1}a, h)F_{\mathcal{H}, \sigma}, (b, hg)F_{\mathcal{U}, \tau}),
\]
which yields \(h \in H \cap gU\) and
\[
(\sigma \cdot (g\tau))(h) = \sigma(h)g\tau(g^{-1}hg) = (a^{-1}b)g(g^{-1}b^{-1}) = a^{-1}.
\]
Hence \((a, h) \in F_{\mathcal{H} \cap gU, \sigma \cdot (g\tau)}\). Moreover, it is easily verified that \(F_{\mathcal{H} \cap gU, \sigma \cdot (g\tau)}\) is the stabilizer of \(F_{\mathcal{H}, \sigma} \otimes gF_{\mathcal{U}, \tau}\). Thus it turns out that
\[
(F/F_{\mathcal{H}, \sigma}) \otimes (F/F_{\mathcal{U}, \tau}) \simeq \bigcup_{HgU \in H \cap G/U} F/F_{\mathcal{H} \cap gU, \sigma \cdot (g\tau)}
\]
as left \(F\)-sets. The lemma now follows from Lemma 2.2. This completes the proof. □

For each \(K \leq H \leq G\) and \(g \in G\), there are additive maps
\[
\begin{align*}
\text{con}_H^g : \Omega(H, A) &\to \Omega(^gH, A), \quad \sum_T \ell_T[T] \mapsto \sum_T \ell_T[\text{con}_H^g(T)], \\
\text{res}_K^H : \Omega(H, A) &\to \Omega(K, A), \quad \sum_T \ell_T[T] \mapsto \sum_T \ell_T[\text{res}_K^H(T)], \quad \text{and} \\
\text{ind}_K^H : \Omega(K, A) &\to \Omega(H, A), \quad \sum_S k_S[S] \mapsto \sum_S k_S[\text{ind}_K^H(S)],
\end{align*}
\]
where \(S \in (K, A)\)-set, \(T \in (H, A)\)-set, and \(k_S, \ell_T \in \mathbb{Z}\); these maps are called the conjugation map, the restriction map, and the induction map, respectively. By Lemma 2.12, conjugation maps and restriction maps are ring homomorphisms.

**Proposition 2.16** The family of \(\mathbb{Z}\)-algebras \(\Omega(H, A)\) for \(H \leq G\), together with conjugation, restriction, and induction maps, defines a Green functor on \(G\).

**Proof.** The axioms of Green functor follow from Lemmas 2.10, 2.11, and 2.15 (cf. [4, 1.1. Definition]). As for the Frobenius axiom, we have
\[
\text{res}_K^G((G/H)_{\sigma} \otimes (K/U)_{\tau}) \simeq \bigcup_{KgH \in K \cap G/H} \bigcup_{Lg \in L \setminus K \cap H / U} (K/((^gH \cap ^cU))_{(\sigma)}|_{Lg \cap (e\tau)});
\]
where \(L_g = K \cap ^gH\), and
\[
\text{ind}_K^G(\text{res}_K^G((G/H)_{\sigma} \otimes (K/U)_{\tau})) \simeq (G/H)_{\sigma} \otimes \text{ind}_K^G((K/U)_{\tau})
\]
for all \(K \leq G\), \((H, \sigma) \in \mathcal{S}(G, A)\), and \((U, \tau) \in \mathcal{S}(K, A)\), completing the proof. □
3 Multiplicative induction

3A Tensor induction

To begin with, we review the multiplicative induction $\text{Ind}_H^G : X \mapsto \text{Map}_H(G, X)$, where $H \leq G$ and $X \in H\text{-set}$, given in [32, §3(a.3)] (see also [11, §4]).

**Definition 3.1** Let $H \leq G$, and let $T$ be an $(H, A)$-set. We define a left $G$-set $\text{Map}_H(G, T)$ to be the set of maps $f : G \rightarrow T$ such that $f(hg) = hf(g)$ for all $h \in H$ and $g \in G$ with the action of $G$ given by

$$(gf)(r) = f(rg)$$

for all $g, r \in G$ and $f \in \text{Map}_H(G, T)$.

**Remark 3.2** Under the notation of Definition 3.1, the left $G$-set $\text{Map}_H(G, T)$ is viewed as a $(G, A)$-set with the right action of $A$ given by

$$(fa)(r) = f(r) \cdot a$$

for all $r \in G$, $a \in A$, and $f \in \text{Map}_H(G, T)$. However, we need hardly recall such a right action of $A$ on $\text{Map}_H(G, T)$ (see Definitions 3.3 and 3.5) in relation to multiplicative induction for monomial Burnside rings (see Proposition 3.20).

Let $H \leq G$, and let $T$ be an $(H, A)$-set. The tensor induced $G$-set $T \otimes^G G$ obtained from $T$ (see [8, §80C]) is isomorphic to $\text{Map}_H(G, T)$ and is related to tensor induction of modules. By modifying $\text{Map}_H(G, T)$, we define tensor induction for $(H, A)$-sets, and then define multiplicative induction for monomial Burnside rings in §3C.

Let $Hg, g \in G$, be the right coset of $H$ in $G$ containing $g$. Given $g, r \in G$ with $Hg \neq Hr$ and $a \in A$, we define a relation $\sim_{(g, r, a)}$ on $\text{Map}_H(G, T)$ by

$$f \sim_{(g, r, a)} f' : \iff f(hg)^{hr}a = f'(hg)$

and $f(hr) = f'(hr)^{hr}a$ for all $h \in H$, and $f(g') = f'(g')$ for all $g' \in G - Hg \cup Hr$.

Let $\sim_A$ be the equivalence relation on $\text{Map}_H(G, T)$ generated by the relations $\sim_{(g, r, a)}$ for $g, r \in G$ and $a \in A$. For each $f \in \text{Map}_H(G, T)$, we denote by $\hat{f}$ the equivalence class containing $f$ with respect to the equivalence relation $\sim_A$.

**Definition 3.3** Let $H \leq G$, and let $T$ be an $(H, A)$-set. We define

$$\widehat{\text{Map}}_H(G, T) := \{ \hat{f} \mid f \in \text{Map}_H(G, T) \},$$

and make it into a free right $A$-set by defining

$$\hat{fa} = \hat{f}_a \quad \text{with} \quad f_a : G \rightarrow T, \quad r \mapsto f_a(r) = \begin{cases} f(r)^r & \text{if } r \in H, \\ f(r) & \text{if } r \in G - H \end{cases} \quad (3.1)$$

for all $a \in A$ and $f \in \text{Map}_H(G, T)$. 
The following lemma tells us of a suitable left action of $G$ on the free right $A$-set $\text{Map}_H(G, T)$ defined above for an extension of the multiplicative induction $\text{Jnd}_H^G : X \mapsto \text{Map}_H(G, X)$ where $X \in H$-set.

**Lemma 3.4** Let $H \leq G$, and let $T$ be an $(H, A)$-set. Then $\widehat{gf_a} = \widehat{gf} \cdot a$, where $f_a$ is given in Eq. (3.1), for all $g \in G, a \in A$, and $f \in \text{Map}_H(G, T)$.

**Proof.** Suppose that $g \in G, a \in A$, and $f \in \text{Map}_H(G, T)$. By definition,

$$(gf_a)(r) = \begin{cases} f(rg)^\cdot a & \text{if } rg \in H, \\ f(rg) & \text{if } rg \in G - H, \end{cases}$$

and

$$(gf)_a(r) = \begin{cases} f(rg)^\cdot a & \text{if } r \in H, \\ f(rg) & \text{if } r \in G - H. \end{cases}$$

Hence we may assume that $g \notin H$. Observe that for any $h \in H$,

$$(gf_a)(h)^\cdot a = (gf)_a(h) \quad \text{and} \quad (gf_a)(h^{-1}) = (gf)_a(h^{-1})^\cdot a.$$

Moreover, $(gf_a)(r) = (gf)_a(r)$ for all $r \in G - H \cup H^{-1}$. Thus $gf_a \sim_{(r, g^{-1}, a)} (gf)_a$.

We now obtain $\widehat{gf_a} = \widehat{gf} \cdot a$, completing the proof. □

**Definition 3.5 (Tensor induction)** Let $H \leq G$, and let $T$ be an $(H, A)$-set. We make the free right $A$-set $\text{Map}_H(G, T)$ into a left $G$-set by defining

$$\widehat{g} = \widehat{gf}$$

for all $g \in G$ and $f \in \text{Map}_H(G, T)$, so that $\widehat{\text{Map}}_H(G, T)$ is a $(G, A)$-set. The operation which assigns to $T$ the $(G, A)$-set $\widehat{\text{Map}}_H(G, T)$ is called tensor induction (cf. [8, §80C]), and is related to tensor induction for 1-cocycles (see §3B).

**Remark 3.6** Keep the notation of Definition 3.5, and assume further that $G$ acts trivially on $A$. Then the $(G, A)$-sets are considered as the $A$-fibred $G$-sets defined by Barker [2, §2], and the $(G, A)$-set $\widehat{\text{Map}}_H(G, T)$ obtained from $T$ by tensor induction is identified with the $A$-fibred $G$-set $\text{Ten}_H^G(T)$ defined by Barker [2, §9].

We present a fundamental lemma which is essential to the investigation of multiplicative induction for monomial Burnside rings.

**Lemma 3.7** Let $H \leq G$, and let $T$ be an $(H, A)$-set. Suppose that $\{g_1, g_2, \ldots, g_n\}$ with $g_1 = e$ is a complete set of representatives of $G/H$. Let $f \in \text{Map}_H(G, T)$, and define $f^{(0)} \in \text{Map}_H(G, T)$ by $f^{(0)}(h^{-1}g_j) = f((h^{-1}g_j)a_j$ with $a_j \in A$ for all $h \in H$ and $j \in [n]$. Then $f^{(0)} \sim_A f_a$, where $a = g_1^{a_1}g_2^{a_2} \cdots g_n^{a_n}$, and hence $\widehat{f^{(0)}} = \widehat{f_a}$. 
Proof. For each integer \( k \) with \( 1 \leq k \leq n \), we define \( f^{(k)} \in \text{Map}_H(G, T) \) by

\[
f^{(k)}(hg_j^{-1}) = \begin{cases} 
  f(hg_j^{-1}) & \text{if } j \in [k], \\
  f^{(0)}(hg_j^{-1}) & \text{if } j = k + 1, k + 2, \ldots, n
\end{cases}
\]

for all \( h \in H \). In particular, \( f^{(n)} = f \). Obviously, \( f^{(1)} = f_{\text{res}_{\text{H}}} \). Let \( k \) be an integer with \( 2 \leq k \leq n \). Then

\[
f^{(k)}(hg_k^{-1}) = f^{(k-1)}(hg_k^{-1}h_k^{-1}) \quad \text{and} \quad f^{(k)}(hg_j^{-1}) = f^{(k-1)}(hg_j^{-1})
\]

for all \( h \in H \) and \( j \in [n] \) with \( j \neq k \), and

\[
f^{(k-1)}_{\text{res}_{\text{H}}}(hg_1^{-1}) = f^{(k-1)}(hg_1^{-1}h_1^{-1}a_k^{-1}) \quad \text{and} \quad f^{(k-1)}_{\text{res}_{\text{H}}}(hg_j^{-1}) = f^{(k-1)}(hg_j^{-1})
\]

for all \( h \in H \) and \( j = 2, 3, \ldots, n \). This shows that \( f^{(k-1)}_{\text{res}_{\text{H}}} = f^{(k-1)}(g_k^{-1}, g_1^{-1}, a_k^{-1}) f^{(k)} \). Hence we have \( f^{(0)} \sim_A f_a \), completing the proof. \( \square \)

Remark 3.8 Let \( H \leq G \), and let \( T \) be an \((H, A)\)-set. By Lemma 3.7, we have

\[
|\overline{\text{Map}_H(G, T)/A}| = |\text{Map}_H(G, T/A)|,
\]

whence

\[
|\overline{\text{Map}_H(G, T)}| = |T/A|^{G/H} \cdot |A|.
\]

The following proposition, which is a generalization of [32, §3(a.13)], describes a Mackey decomposition formula (see also [2, Lemma 9.1]).

Proposition 3.9 Let \( H, K \leq G \). For each \((H, A)\)-set \( T \),

\[
\text{res}^G_K(\overline{\text{Map}_H(G, T)}) \simeq \bigotimes_{K \not\subset H \in K \setminus G/H} \overline{\text{Map}}_{K \cap \not\subset H}(K, \text{res}^{gH}_{K \cap \not\subset H}(gT)).
\]

Proof. Let \( \{g_1, g_2, \ldots, g_m\} \) with \( g_1 = \epsilon \) be a complete set of representatives of \( K \setminus G/H \). For each \( i \in [m] \), let \( \{r_{i1}, r_{i2}, \ldots, r_{i\ell_i}\} \) be a complete set of representatives of \( K/(K \cap g_i H) \). Then \( \{r_{ij}g_i \mid i \in [m] \text{ and } j \in [\ell_i]\} \) is a complete set of representatives of \( G/H \). Let \( i \in [m] \). There is a map

\[
\Phi_i: \text{res}^G_K(\overline{\text{Map}_H(G, T)}) \rightarrow \overline{\text{Map}}_{K \cap \not\subset H}(K, \text{res}^{gH}_{K \cap \not\subset H}(gT))
\]

given by

\[
\Phi_i(f)(g_{ij}r_{ij}^{-1}) = g_i \otimes f((r_{ij}g_i)^{-1})(= g_i \otimes f((r_{ij}g_i)^{-1})) \in g_i T
\]

for all \( h \in g_i^{-1}K \cap H \), \( j \in [\ell_i] \), and \( f \in \text{Map}_H(G, T) \). Given \( j \in [\ell_i] \), we have

\[
\Phi_i(rf)(r_{ij}^{-1}) = g_i \otimes f((r_{ij}g_i)^{-1}r) = g_i \otimes f(g_i^{-1}(r_{ij}r_{ii'})g_i(r_{ij'}g_i)^{-1})
\]
and
\[(r\Phi_i(f))(r_{ij}^{-1}) = \Phi_i(f)(r_{ij}^{-1}) = \Phi_i(f)((r_{ij}^{-1}rr_{ij})r_{ij}^{-1}),\]
where \(r_{ij}(K \cap g_H) = rr_{ij}(K \cap g_H),\) for all \(r \in K\) and \(f \in \text{Map}_H(G, T).\) Thus \(\Phi_i\) is a \(K\)-equivariant map. We now define a \(K\)-equivariant map
\[
\hat{\Phi} : \text{res}^G_K(\text{Map}_H(G, T)) \to \bigotimes_{i=1}^m \text{Map}_{K \cap g_H}(K, \text{res}^{g_H}_{K \cap g_H}(g_H T))
\]
by
\[
\hat{f} \mapsto \Phi_1(f) \otimes \Phi_2(f) \otimes \cdots \otimes \Phi_m(f)
\]
for all \(f \in \text{Map}_H(G, T).\) (Of course this map is well-defined; see Remark 3.10.) The map \(\hat{\Phi}\) is also a \((K, A)\)-equivariant map, because
\[
(\Phi_1(f) \otimes \Phi_2(f) \otimes \cdots \otimes \Phi_m(f))a = \Phi_1(f)_a \otimes \Phi_2(f)_a \otimes \cdots \otimes \Phi_m(f)_a
\]
and
\[
\Phi_i(f_a)(r_{ij}^{-1}) = \begin{cases} 
\epsilon \otimes f(r_{ij}^{-1})^{-1} & \text{if } i = 1 \text{ and } r_{ij} \in H, \\
\epsilon \otimes f(r_{ij}^{-1})^{-1} & \text{if } i = 1 \text{ and } r_{ij} \notin H, \\
g_i \otimes f((r_{ij}g_i)^{-1})^{-1} & \text{if } i \neq 1
\end{cases}
\]
for all \(i \in [m], j \in [\ell_i], a \in A,\) and \(f \in \text{Map}_H(G, T).\) Thus it only remains for us to show that \(\hat{\Phi}\) is bijective. For each \(i \in [m],\) choose \(f_i \in \text{Map}_{K \cap g_H}(K, \text{res}^{g_H}_{K \cap g_H}(g_H T)).\) Given \(i \in [m] \text{ and } j \in [\ell_i],\) we suppose that \(f_i(r_{ij}^{-1}) = g_i \otimes t_{ij} \in g_H T\) with \(t_{ij} \in T.\) Now define \(f \in \text{Map}_H(G, T)\) by
\[
f(h(r_{ij}g_i)^{-1}) = ht_{ij} \in T
\]
for all \(h \in H, i \in [m], \text{ and } j \in [\ell_i].\) Then \(\hat{\Phi}(f) = \hat{f}_1 \otimes \hat{f}_2 \otimes \cdots \otimes \hat{f}_m.\) Thus \(\hat{\Phi}\) is surjective, which means that it is also injective, because
\[
|\text{Map}_H(G, T)| = |T/A|^\sum_{i=1}^m \ell_i \cdot |A| = \left| \bigotimes_{i=1}^m \text{Map}_{K \cap g_H}(K, \text{res}^{g_H}_{K \cap g_H}(g_H T)) \right|
\]
by Remark 3.8. We now conclude that \(\hat{\Phi}\) is bijective, completing the proof. □

**Remark 3.10** In the proof of Proposition 3.9, assume that \(f \sim ((r_{ij}g_i)^{-1}, (r_{ij}'g_{ij})^{-1}, a) f'\) with \(f, f' \in \text{Map}_H(G, T)\) and \(a \in A.\) Let \(u \in [m]\) and \(v \in [\ell_u].\) Then we have
\[
\Phi_u(f)(g_H r_{uv}^{-1}) = \begin{cases} 
\Phi_i(f')(g_H r_{ij}^{-1}) g_H r_{ij}^{-1} a^{-1} & \text{if } (u, v) = (i, j), \\
\Phi_{i'}(f')(g_H r_{ij}^{-1}) g_H r_{ij}^{-1} a^{-1} & \text{if } (u, v) = (i', j'), \\
\Phi_{i'}(f')(g_H r_{ij}^{-1}) & \text{otherwise}
\end{cases}
\]
for all $h \in g^{-1}K \cap H$. This, combined with Lemma 3.7, shows that
\[ \Phi_u(f) = \Phi_u(f') \quad \text{if} \quad u \neq i, i', \quad \Phi_i(f) = \Phi_i(f)a^{-1}, \quad \text{and} \quad \Phi_{i'}(f) = \Phi_{i'}(f'a). \]
Hence we have $\tilde{\Phi}(f) = \tilde{\Phi}(f')$. Consequently, the map $\tilde{\Phi}$ is well-defined.

3B Tensor induction for 1-cocycles

We introduce tensor induction for 1-cocycles, and see that it is closely allied to tensor induction for $(H;A)$-sets with $H \subseteq G$.

**Definition 3.11** Let $(H, \sigma) \in S(G, A)$. We fix a complete set $\{g_1, g_2, \ldots, g_n\}$ with $g_1 = \epsilon$ of representatives of $G/H$, and define a 1-cocycle $\sigma^G : G \to A$ by
\[ \sigma^G(g) = \prod_{j=1}^{n} g_j^a(g_{j'}^{-1}gg_j), \]
where $gg_jH = g_{j'}H$, for all $g \in G$. The operation which assigns to $\sigma$ the 1-cocycle $\sigma^G : G \to A$ is called tensor induction (cf. [8, §13A]).

**Remark 3.12** Keep the notation of Definition 3.11, and let $h_1, h_2, \ldots, h_n \in H$. Then
\[ (\sigma^G)^a(g) = \prod_{j=1}^{n} g_j^a(h_j^{-1}g_j^{-1}gg_jh_j) \quad \text{with} \quad a = \prod_{j=1}^{n} g_j^a(h_j) \]
for all $g \in G$ (see Definition 2.4), because
\[ g_j^a(h_j^{-1}g_j^{-1}gg_jh_j) = g_j^a(h_jg_j^{-1})g_j^a(g_j^{-1}gg_jh_j) = g_j^a(h_j^{-1}g_j^{-1}gg_jh_j) \]
for all $j \in [n]$. Hence the subset $\{(\sigma^G)^a \mid a \in A\}$ of $Z^1(G, A)$ is independent of the choice of a complete set of representatives of $G/H$. Likewise, if $b \in A$, then
\[ (\sigma^G)^c = (a^b)^G \quad \text{with} \quad c = \prod_{j=1}^{n} g_j^b. \]

**Example 3.13** Let $(H, \sigma) \in S(G, A)$. Obviously, $A$ is a free right $A$-set with the action given by the product operation on $A$. We make it into an $(H, A)$-set $A^{(\sigma)}$ isomorphic to $(H/H)_{\sigma}$ by defining
\[ ha = \sigma(h)^h_a \]
for all $h \in H$ and $a \in A(\sigma)$. For any $K \leq H$, $\text{res}_K^H(A(\sigma)) = A(\sigma|_K)$ (see Lemma 2.11). Keep the notation of Definition 3.11, and identify $(H/H)_\sigma$ with $A(\sigma)$. We define an element $\tilde{\sigma}$ of $\text{Map}_H(G, A(\sigma))$ by

$$\tilde{\sigma}(h_{g^{-1}}) = \sigma(h)$$

for all $h \in H$ and $i \in [n]$. Let $f \in \text{Map}_H(G, A(\sigma))$. For each $j \in [n]$, we set $a_j = f(g_j^{-1}) \in A(\sigma)$. Since $f(h_{g_j^{-1}}) = \tilde{\sigma}(h_{g_j^{-1}})h_{a_j}$ for all $h \in H$ and $j \in [n]$, it follows from Lemma 3.7 with $f(0) = f$ that $\tilde{f} = \tilde{\sigma}a$ where $a = g_{a_1}g_{a_2} \cdots g_{a_n}$. Hence $\text{Map}_H(G, A(\sigma)) = \{\tilde{\sigma}a \mid a \in A\}$. Let $g \in G$. We have

$$(g\tilde{\sigma})(h_{g_j^{-1}}) = \tilde{\sigma}(h_{g_j^{-1}}g) = \sigma(h_{g_j^{-1}}g_{g_j}) = \tilde{\sigma}(h_{g_j^{-1}}h_{g_{g_j}g_j}) = \tilde{\sigma}(h_{g_j^{-1}}h_{g_{g_j}g_j}g_j),$$

where $g_jH = gg_jH$, for all $h \in H$ and $j \in [n]$. Thus it follows from Lemma 3.7 that $g\tilde{\sigma} = \tilde{\sigma}a \otimes G(g)$. Moreover, there exists an isomorphism $\text{Map}_H(G, A(\sigma)) \simeq A(\sigma \otimes G)$ of $(G, A)$-sets given by

$$\tilde{\sigma}a \mapsto a$$

for all $a \in A$. Thus $\text{Map}_H(G, (H/H)_\sigma) \simeq (G/G)_\sigma \otimes \sigma$.

The following proposition describes a Mackey decomposition formula.

**Proposition 3.14** Let $H, K \leq G$. For each $\sigma \in Z^1(H, A)$,

$$\sigma \otimes G|_K = A \prod_{KgH \in K \cap G/H} (g\sigma)|_{K \cap gH} \otimes K.$$ 

**Proof.** By Lemma 2.10, Proposition 3.9, and Example 3.13, we have

$$(K/K)_{\sigma \otimes G}|_K \simeq \bigotimes_{KgH \in K \cap G/H} (K/K)(g\sigma)|_{K \cap gH} \otimes K,$$

which, combined with Lemma 2.15, implies that

$$(K/K)_{\sigma \otimes G}|_K \simeq (K/K) \prod_{KgH \in K \cap G/H} (g\sigma)|_{K \cap gH} \otimes K.$$ 

The assertion follows from this fact and Lemma 2.6. This completes the proof. □

The following lemma states basic properties of tensor induction for 1-cocycles.

**Lemma 3.15** Let $U \leq K \leq H$, and let $g \in G$. Then

$$g\nu \otimes H = A (g\nu) \otimes^g H \quad \text{and} \quad (\tau \otimes K) \otimes H = A \tau \otimes H$$

for all $\nu \in Z^1(K, A)$ and $\tau \in Z^1(U, A)$. 

Proof. Fix a complete set \( \{ h_1, h_2, \ldots, h_m \} \) with \( h_1 = \epsilon \) of representatives of \( H/K \) and a complete set \( \{ r_1, r_2, \ldots, r_k \} \) with \( r_1 = \epsilon \) of representatives of \( K/U \). Given \( \nu \in Z^1(K,A) \) and \( \tau \in Z^1(U,A) \), we have

\[
\begin{align*}
(g\nu \otimes H)(gh_j) &= \prod_{j=1}^{m} q h_j \nu(h_j^{-1}hh_j) \\
&= \prod_{j=1}^{m} q h_j \nu((h_j^{-1}g^{-1}) q h_j(gh_j)) \\
&= \prod_{j=1}^{m} q h_j \nu(g^{-1}(q h_j^{-1} q h_j)g) \\
&= \prod_{j=1}^{m} q h_j (g\nu)(q h_j^{-1} q h_j),
\end{align*}
\]

where \( hh_j K = h_j' K \), and

\[
(\tau \otimes K) \otimes H(h) = \prod_{j=1}^{m} h_j' \otimes K(h_j^{-1}hh_j) \\
= \prod_{j=1}^{m} \prod_{i=1}^{k} h_j' r_i \tau((h_j'^{-1} h_j^{-1})^{-1} h(h_j r_i)),
\]

where \( (h_j^{-1}hh_j)r_i U = r_i' U \), for all \( h \in H \). Consequently, the assertions follow from Remark 3.12. This completes the proof. \( \Box \)

3C Algebraic maps

We define a subset \( \Omega(G,A)^+ \) of \( \Omega(G,A) \) to be the set consisting of all elements \( \sum_{(U,\tau) \in \mathcal{R}(G,A)} \ell(U,\tau) [(G/U)_{\tau}] \) with \( \ell(U,\tau) \geq 0 \), which is an additive semigroup. By Lemma 2.15, \( \Omega(G,A)^+ \) is closed under multiplication. For each \( H \leq G \), there is a map (tensor induction) \( \text{Map}_H(G,-) : \Omega(H,A)^+ \to \Omega(G,A) \) given by

\[ [T] \mapsto [\text{Map}_H(G,T)] \]

for all \( (H,A) \)-sets \( T \) (cf. [8, (80.42)]). This map is multiplicative (see Lemma 3.19).

We review the concept of algebraic maps which is due to Dress [11]. Let \( B \) be an additive semigroup with zero element, and let \( E \) be an additive group. Given \( c \in B \) and a map \( f : B \to E \), we define a map \( D_c f : B \to E \) by

\[ d \mapsto f(c + d) - f(d) \]

for all \( d \in B \). A map \( f : B \to E \) is said to be algebraic of degree \( n \) if \( n \) is the least integer such that

\[ D_{c_1} D_{c_2} \cdots D_{c_n+1} f = 0 \]
for all $c_1, c_2, \ldots, c_{n+1} \in B$ (cf. [8, §80C]). Let $f : B \to E$ be an algebraic map of degree $n$, and let $\overline{B}$ be the additive group generated by the elements of $B$. According to Dress [11, Proposition 1.1], there is a unique map $\overline{f} : \overline{B} \to E$ extending $f$, and $\overline{f}$ is also algebraic of degree $n$ (see also [8, (80.44) Theorem (Dress)]). Assume further that $\overline{B}$ and $E$ are commutative rings and $B$ is closed under multiplication. If $f : B \to E$ is multiplicative, then the unique extension $\overline{f} : \overline{B} \to E$ of $f$ to $\overline{B}$ is also multiplicative (cf. [8, (80.47) Theorem]).

**Definition 3.16** Let $H \leq G$, and let $T_0, T_1, \ldots, T_i$ be $(H,A)$-sets, where $i$ is an integer with $0 \leq i \leq |G : H| + 1$. We define a $(G,A)$-set $\overline{\text{Map}}_H(G, T_0, T_1, \ldots, T_i)$ to be the set consisting of all elements $\widehat{f}$ of $\overline{\text{Map}}_H(G, T_0 \cup T_1 \cup \cdots \cup T_i)$ containing $f \in \text{Map}_H(G, T_0 \cup T_1 \cup \cdots \cup T_i)$ such that $|\text{Im} f \cap T_\ell| \neq 0$ whenever $\ell \neq 0$ with the left action of $G$ and the right action of $A$ given by

$$g \widehat{f} = \overline{g} \widehat{f} \quad \text{and} \quad \widehat{f} a = \widehat{f_a}$$

for all $g \in G$, $a \in A$, and $\widehat{f} \in \overline{\text{Map}}_H(G, T_0, T_1, \ldots, T_i)$.

Under the notation of Definition 3.16, we have

$$\overline{\text{Map}}_H(G, T_0, T_1, \ldots, T_i) = \begin{cases} \text{Map}_H(G, T_0) & \text{if } i = 0, \\ \text{Map}_H(G, T_1) & \text{if } T_0 = \emptyset \text{ and } i = 1, \\ \emptyset & \text{if } i = |G : H| + 1. \end{cases}$$

**Proposition 3.17** For each $H \leq G$, the map $\overline{\text{Map}}_H(G, -) : \Omega(H, A)^+ \to \Omega(G, A)$ is algebraic of degree $|G : H|$.

This proposition is analogous to [8, (80.43) Proposition (Dress)], and is an immediate consequence of the following lemma.

**Lemma 3.18** Keep the notation of Definition 3.16, and assume further that $i \geq 1$. Set $\Theta_i = D_{[T_1]} \cdots D_{[T_i]} \bar{\text{Map}}_H(G, -)$. Then

$$\Theta_i([T_0]) = [\overline{\text{Map}}_H(G, T_0, T_1, \ldots, T_i)].$$

**Proof.** The assertion is proved by an argument analogous to that in the proof of [8, (80.43) Proposition (Dress)]. □

Tensor induction is multiplicative.

**Lemma 3.19** For each $H \leq G$,

$$\overline{\text{Map}}_H(G, T_1 \otimes T_2) \simeq \overline{\text{Map}}_H(G, T_1) \otimes \overline{\text{Map}}_H(G, T_2)$$

for all $(H,A)$-sets $T_1$ and $T_2$. 

Proof. If \( f \in \text{Map}_H(G, T_1 \otimes T_2) \), then by Lemma 3.7, there exists a unique element \( \Psi_i(f) \otimes \Psi_2(f) \) of \( \text{Map}_H(G, T_1) \otimes \text{Map}_H(G, T_2) \), where \( \Psi_i(f) \in \text{Map}_H(G, T_i) \) with \( i = 1, 2 \), such that

\[
f(g) = \Psi_1(f)(g) \otimes \Psi_2(f)(g)
\]

for all \( g \in G \). Obviously, \( \Psi_1(f) \otimes \Psi_2(f) = \Psi_1(f') \otimes \Psi_2(f') \) whenever \( f \sim_A f' \). We now define a map \( \hat{\Psi} : \text{Map}_H(G, T_1 \otimes T_2) \to \text{Map}_H(G, T_1) \otimes \text{Map}_H(G, T_2) \) by

\[
\hat{\Psi}(f) = \Psi_1(f) \otimes \Psi_2(f)
\]

for all \( f \in \text{Map}_H(G, T_1 \otimes T_2) \). Observe that this map is \((G, A)\)-equivariant and surjective. Moreover, by Remark 3.8,

\[
|\text{Map}_H(G, T_1 \otimes T_2)| = (|T_1/A| \cdot |T_2/A|)^{|G/H|} \cdot |A| = |\text{Map}_H(G, T_1) \otimes \text{Map}_H(G, T_2)|.
\]

Hence \( \hat{\Psi} \) is an isomorphism of \((G, A)\)-sets. This completes the proof. \( \square \)

Combining Proposition 3.17 and Lemma 3.19 with [8, (80.47) Theorem], we obtain a result analogous to [8, (80.48) Theorem (Dress)].

Proposition 3.20 For any \( H \leq G \), there is a unique multiplicative map

\[
\text{Map}_H(G, -) : \Omega(H, A) \to \Omega(G, A), \quad x \mapsto \text{Map}_H(G, x)
\]

extending \( \text{Map}_H(G, -) \), called multiplicative induction or tensor induction, and this map is algebraic of degree \(|G : H|\).

Remark 3.21 The multiplicative induction map \( \text{Map}_H(G, -) : \Omega(H, A) \to \Omega(G, A) \) with \( A = \{\epsilon_A\} \) is introduced by Dress [11, §4].

Our concern is an explicit description of each element of \( \text{Im Map}_H(G, -) \) with \( H \leq G \), and is to prove Eq.(1.1) (see also [8, (80.49) Corollary]).

Proposition 3.22 Let \( H \leq G \). For any \((H, A)\)-sets \( T_0 \) and \( T \),

\[
\text{Map}_H(G, [T_0] - [T]) = \sum_{i=0}^{n} (-1)^i [\text{Map}_H(G, T_0, T_1, \ldots, T_i)],
\]

where \( n = |G : H| \) and \( T = T_1 = \cdots = T_n \).

Proof. We set \( D^{0}_{[T]} \Theta = \Theta = \text{Map}_H(G, -) : \Omega(H, A) \to \Omega(G, A) \), and define inductively \( D^{i}_{[T]} \Theta : \Omega(H, A) \to \Omega(G, A) \), \( i = 1, 2, \ldots \), by \( D^{i}_{[T]} \Theta = D^{i}_{[T]}(D^{i-1}_{[T]} \Theta) \). From [8, (80.45)], we know that \( \Theta([T_0] - [T]) = \sum_{i=0}^{\infty} (-1)^i D^{i}_{[T]} \Theta([T_0]) \). Hence the assertion follows from Proposition 3.17 and Lemma 3.18. This completes the proof. \( \square \)
Remark 3.23 Let $(H, \sigma) \in \mathcal{S}(G, A)$. By Lemma 3.7 and Proposition 3.22, we can describe the structure of $\overline{\text{Map}}_H(G, -[(H/H)_\sigma])$. For each $X \in G$-set, let $\tilde{\Lambda}_P(X)$ be the reduced Lefschetz invariant of the poset $P(X)$ consisting of non-empty and proper subsets of $X$, which is an element of the Burnside ring $\Omega(G)$ (cf. [5, 29]). When $A = \{\epsilon_A\}$, $\overline{\text{Map}}_H(G, -[(H/H)_1])$ is identified with $\tilde{\Lambda}_P(G/H)$.

There is a Mackey decomposition formula which generalizes [32, §3(G.5)] (see also [2, Proposition 9.5]).

Proposition 3.24 Let $H, K \leq G$. For each $x \in \Omega(H, A)$,

$$\text{res}_K^G(\overline{\text{Map}}_H(G, x)) = \prod_{KgH \in K \setminus G/H} \overline{\text{Map}}_{K \cap gH}(K, \text{res}_K^{gH} \circ \text{con}_{gH}^H(x)).$$

Proof. By [8, (80.44) Theorem (Dress)], $\text{res}_K^G \circ \overline{\text{Map}}_H(G, -) : \Omega(H, A) \to \Omega(K, A)$ is the unique map extending the algebraic map

$$\text{res}_K^G \circ \overline{\text{Map}}_H(G, -) = \prod_{KgH \in K \setminus G/H} \overline{\text{Map}}_{K \cap gH}(K, -) \circ \text{res}_K^{gH} \circ \text{con}_{gH}^H : \Omega(H, A)^+ \to \Omega(K, A),$$

and so is $\prod_{KgH \in K \setminus G/H} \overline{\text{Map}}_{K \cap gH}(K, -) \circ \text{res}_K^{gH} \circ \text{con}_{gH}^H : \Omega(H, A) \to \Omega(K, A)$ (see [11, Proposition 1.2] and Propositions 3.9, 3.17, and 3.20). Thus the assertion holds. \(\square\)

4 The mark homomorphism

4A The first cohomology group

Following [12, §2], we provide preliminaries of the mark homomorphism for $\Omega(G, A)$ which is given in §4B.

Let $H \leq G$. The set $Z^1(H, A)$ is a right $A$-set with the action of $A$ given in Definition 2.4, and is an abelian group with the product operation given by

$$\sigma \cdot \tau(h) = \sigma(h)\tau(h)$$

for all $\sigma, \tau \in Z^1(H, A)$ and $h \in H$. Obviously, the identity of $Z^1(H, A)$ is $1_H$.

For each $\sigma \in Z^1(H, A)$, we denote by $\overline{\sigma}$ the $A$-orbit $\{\sigma^a \mid a \in A\}$ containing $\sigma$. Given $\sigma, \tau \in Z^1(H, A)$ and $a, b \in A$, it is easily seen that $\overline{\sigma^a \cdot \tau^b} = (\overline{\sigma \cdot \tau})^{ab} = \overline{\sigma \cdot \tau}$.

Definition 4.1 For each $H \leq G$, we define

$$H^1(H, A) := \{\overline{\sigma} \mid \sigma \in Z^1(H, A)\},$$

$$\text{res}_K^G(\overline{\text{Map}}_H(G, x)) = \prod_{KgH \in K \setminus G/H} \overline{\text{Map}}_{K \cap gH}(K, \text{res}_K^{gH} \circ \text{con}_{gH}^H(x)).$$

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$$\text{res}_K^G \circ \overline{\text{Map}}_H(G, -) = \prod_{KgH \in K \setminus G/H} \overline{\text{Map}}_{K \cap gH}(K, -) \circ \text{res}_K^{gH} \circ \text{con}_{gH}^H : \Omega(H, A)^+ \to \Omega(K, A),$$

and so is $\prod_{KgH \in K \setminus G/H} \overline{\text{Map}}_{K \cap gH}(K, -) \circ \text{res}_K^{gH} \circ \text{con}_{gH}^H : \Omega(H, A) \to \Omega(K, A)$ (see [11, Proposition 1.2] and Propositions 3.9, 3.17, and 3.20). Thus the assertion holds. \(\square\)

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For each $\sigma \in Z^1(H, A)$, we denote by $\overline{\sigma}$ the $A$-orbit $\{\sigma^a \mid a \in A\}$ containing $\sigma$. Given $\sigma, \tau \in Z^1(H, A)$ and $a, b \in A$, it is easily seen that $\overline{\sigma^a \cdot \tau^b} = (\overline{\sigma \cdot \tau})^{ab} = \overline{\sigma \cdot \tau}$.

Definition 4.1 For each $H \leq G$, we define

$$H^1(H, A) := \{\overline{\sigma} \mid \sigma \in Z^1(H, A)\},$$
the set of $A$-orbits on $Z^1(H, A)$, and make it into an abelian group by defining
\[ \sigma \cdot \tau = \sigma \cdot \tau \]
for all $\sigma, \tau \in Z^1(H, A)$. (This product operation is well-defined.)

Let $H \leq G$. We denote by $ZH^1(H, A)$ the group ring of $H^1(H, A)$ over $\mathbb{Z}$. Given $K \leq H$ and $g \in G$, there are ring homomorphisms
\[
\text{con}^g_H : ZH^1(H, A) \rightarrow ZH^1(ghA), \quad \sum_{\sigma \in H^1(H, A)} \ell_\sigma \sigma \mapsto \sum_{\sigma \in H^1(H, A)} \ell_\sigma g^{-1}\sigma \quad \text{and}
\text{res}_K^g : ZH^1(H, A) \rightarrow ZH^1(K, A), \quad \sum_{\sigma \in H^1(H, A)} \ell_\sigma \sigma \mapsto \sum_{\sigma \in H^1(H, A)} \ell_\sigma \sigma |_K, \]
where $\ell_\sigma \in \mathbb{Z}$ with $\sigma \in Z^1(H, A)$ (see §2B), which are called the conjugation map and the restriction map, respectively (cf. [12, §2.2]). Obviously, the restriction map is well-defined. Let $\sigma \in Z^1(H, A)$. By Lemma 2.5, we have $g(\sigma^a) = g\sigma$ for any $a \in A$. Thus the conjugation map is well-defined.

Let $Y$ be a $(G, A)$-set. The set of $A$-orbits $yA$, $y \in Y$, on $Y$ is a left $G$-set. For each $y \in Y$, we denote by $G_{yA}$ the stabilizer of the $A$-orbit $yA$ in $G$, that is,
\[ G_{yA} = \{ g \in G \mid gy = ya \text{ for some } a \in A \}, \]
and define a 1-cocycle $\sigma_y : G_{yA} \rightarrow A$ by
\[ gy = y\sigma_y(g) \]
for all $g \in G_{yA}$. Obviously, $G_{(ya)A} = G_{yA}$ and $\sigma_{ya} = \sigma_y^a$ for any $y \in Y$ and $a \in A$.

**Definition 4.2** Let $Y$ be a $(G, A)$-set, and let $H \leq G$. We define
\[ \text{inv}_H(Y) := \{ y \in Y \mid H \leq G_{yA} \}, \]
which is viewed as an $(H, A)$-subset of $\text{res}_H^Y(Y)$, and define
\[ [Y]_H := \frac{1}{|A|} \sum_{y \in \text{inv}_H^Y(Y)} \text{res}_{H}^{G_{yA}}(\sigma_y) = \sum_{yA \in \text{inv}_H^Y(Y)/A} \text{res}_{H}^{G_{yA}}(\sigma_y) \in ZH^1(H, A). \]

Let $Y_1$ and $Y_2$ be $(G, A)$-sets, and let $H \leq G$. Obviously,
\[ [Y_1 \cup Y_2]_H = [Y_1]_H + [Y_2]_H. \]
Let $(y_1, y_2) \in Y_1 \times Y_2$. Given $g \in G$ and $a \in A$, $g(y_1 \otimes y_2) = (y_1 \otimes y_2)a$ if and only if $(gy_1b^{-1}, gy_2b) = (y_1, y_2a)$ for some $b \in A$. Hence we have
\[ G_{(y_1 \otimes y_2)A} = G_{y_1A} \cap G_{y_2A} \quad \text{and} \quad \sigma_{y_1 \otimes y_2} = \sigma_{y_1}|_{G_{(y_1 \otimes y_2)A}} \cdot \sigma_{y_2}|_{G_{(y_1 \otimes y_2)A}}. \]
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Moreover, $y_1 \otimes y_2 \in \text{inv}^A_H(Y_1 \otimes Y_2)$ if and only if $y_1 \in \text{inv}^A_H(Y_1)$ and $y_2 \in \text{inv}^A_H(Y_2)$. This means that

$$\left[ Y_1 \right]_H \cdot \left[ Y_2 \right]_H = \frac{1}{|A|} \left( \sum_{y \in \text{inv}^A_H(Y_1)} \text{res}^G_H(\overline{y}) \right) \left( \sum_{yA \in \text{inv}^A_H(Y_2)/A} \text{res}^G_H(\overline{yA}) \right)$$

$$= \frac{1}{|A|} \sum_{y_1 \otimes y_2 \in \text{inv}^A_H(Y_1 \otimes Y_2)} \text{res}^G_H(y_1 \otimes y_2)$$

$$= \left[ Y_1 \otimes Y_2 \right]_H.$$

Given $H \leq G$, we define a ring homomorphism $\rho^H_G : \Omega(G, A) \to ZH^1(H, A)$ by

$$[Y] \mapsto [Y]_H$$

for all $(G, A)$-sets $Y$ (cf. [12, §2.4]).

The ring homomorphisms $\rho^H_G : \Omega(G, A) \to ZH^1(H, A)$ for $H \leq G$ form the map

$$\prod_{H \leq G} \rho^H_G : \Omega(G, A) \to \prod_{H \leq G} ZH^1(H, A), \quad x \mapsto (\rho^H_G(x))_{H \leq G}$$

(cf. [12, §2.5]), which is injective (cf. [12, Theorem 1]).

4B The ghost ring

We continue reviewing part of [12, §2.4, §2.5], and define a ring monomorphism $\rho_G : \Omega(G, A) \to \overline{\Omega}(G, A), \ x \mapsto \prod_{H \leq G} \rho^H_G(x)$ (see Eq.(4.2)).

**Definition 4.3** Let $Y$ be a $(G, A)$-set, and let $(H, \sigma) \in S(G, A)$. We define a subset $\text{inv}((H, \sigma))(Y)$ of $Y$ to be the set of $F_{(H, \sigma)}$-invariants in $Y$, so that

$$\text{inv}((H, \sigma))(Y) = \{ y \in Y \mid hy = y\sigma(h) \text{ for all } h \in H \} = \{ y \in \text{inv}^A_H(Y) \mid \sigma|_H = \sigma \},$$

and denote by $A_{\sigma}$ the stabilizer $\{ a \in A \mid \sigma = \sigma^a \}$ of $\sigma \in Z^1(H, A)$ in $A$.

Under the notation of Definition 4.3, the set $\text{inv}((H, \sigma))(Y)$ is a free right $A_{\sigma}$-set with the action inherited from that of $A$ on $Y$. For each $(H, \sigma) \in S(G, A)$, we denote by $\text{inv}((H, \sigma))(Y)/A_{\sigma}$ the set of $A_{\sigma}$-orbits on $\text{inv}((H, \sigma))(Y)$.

**Lemma 4.4** Let $Y$ be a $(G, A)$-set, and let $H \leq G$. Then

$$\left[ Y \right]_H = \sum_{\sigma \in H^1(H, A)} \left| \text{inv}((H, \sigma))(Y)/A_{\sigma} \right| \cdot \sigma.$$

Moreover, $|\text{inv}((gH, g\sigma))(Y)/A_{g\sigma}| = |\text{inv}((H, \sigma))(Y)/A_{\sigma}|$ for any $\sigma \in Z^1(H, A)$ and $g \in G$. 
Proof. The second statement is clear. To prove the first statement, we set
\[(Y/A)_{(H, \sigma)} = \{yA \in \text{inv}^A_H(Y)/A \mid \text{res}^G\text{inv}^A_H(\sigma y) = \sigma\}\]
for each \(\sigma \in Z^1(H, A)\), so that
\[|Y|_H = \sum_{\sigma \in H^1(H, A)} |(Y/A)_{(H, \sigma)}| \cdot \sigma.\]

Hence it suffices to verify that \(|(Y/A)_{(H, \sigma)}| = |\text{inv}^A_{(H, \sigma)}(Y)/A_{\sigma}|\) for any \(\sigma \in Z^1(H, A)\).

Let \(\sigma \in Z^1(H, A)\). We make the set \(\text{inv}^A_H(Y)\) into a free right \(A_{\sigma}\)-set by restriction of operators from \(A\) to \(A_{\sigma}\). By definition,
\[\text{inv}^A_{(H, \sigma)}(Y) = \{yA_{\sigma} \in \text{inv}^A_H(Y)/A_{\sigma} \mid \sigma y|_H = \sigma\},\]
where \(\text{inv}^A_H(Y)/A_{\sigma}\) is the set of \(A_{\sigma}\)-orbits \(yA_{\sigma} := \{ya \mid a \in A_{\sigma}\}, y \in \text{inv}^A_H(Y)\), on \(\text{inv}^A_H(Y)\). Let \(y \in \text{inv}^A_H(Y)\), and suppose that \(\sigma y|_H = \sigma^a = \sigma^b\) for some \(a, b \in A\). Then \(ab^{-1} \in A_{\sigma}\) and \(ya^{-1}A_{\sigma} = yb^{-1}A_{\sigma} \in \text{inv}^A_{(H, \sigma)}(Y)/A_{\sigma}\). (Note that \(\sigma y|_H = \sigma^{ac}\) for any \(c \in A\).) Hence there is a bijection \((Y/A)_{(H, \sigma)} \rightarrow \text{inv}^A_{(H, \sigma)}(Y)/A_{\sigma}\) given by
\[yA \mapsto ya^{-1}A_{\sigma},\]
where \(\sigma y|_H = \sigma^a\) with \(a \in A\), for all \(yA \in (Y/A)_{(H, \sigma)}\). This completes the proof. \(\Box\)

The following lemma is [27, Lemma 3.3].

**Lemma 4.5** Let \((H, \sigma), (U, \tau) \in S(G, A)\). Then
\[|\text{inv}^A_{(H, \sigma)}((G/U)\tau)/A_{\sigma}| = |\{gU \in G/U \mid H \leq gU \text{ and } (g\tau)|_H = A \sigma\}|.\]

Let \(H, U \leq G\), and consider \(G/U\) to be a left \(G\)-set with the action of \(G\) given by the product operation on \(G\). Following [8, (80.5) Proposition], we define
\[\text{inv}^A_H(G/U) := \{gU \in G/U \mid H \leq gU\}.\] (4.1)

**Lemma 4.6**

(a) Let \(H \leq G\), and let \((U, \tau) \in S(G, A)\). Then
\[|(G/U)\tau|_H = \sum_{gU \in \text{inv}^A_H(G/U)} \text{res}^G_H \circ \text{con}^G_U(\tau).\]

(b) Let \(K \leq H \leq G\), and let \((U, \tau) \in S(H, A)\). Then for any \(r \in G\),
\[\left[\tau\left((H/U)\tau\right)\right]_K = \text{con}^G_K\left(\left[(H/U)\tau\right]_K\right).\]

If \(H = G\), then for any \(r \in G\),
\[\left[(G/U)\tau\right]_K = \text{con}^G_K\left(\left[(G/U)\tau\right]_K\right).\]
Proof. (a) Although the assertion follows from Lemmas 4.4 and 4.5, we directly prove it. In the proof of Lemma 4.4, if \( Y = (G/U)_\tau \), then by Lemmas 2.5 and 2.6, 

\[
(Y/A)_{(H,\sigma)} = \{(\epsilon_A, gU)A \in \text{inv}_H^A(Y)/A \mid \res_H^g U \circ \con_U^g(\tau) = \sigma\},
\]

whence

\[
[(G/U)_\tau]_H = \sum_{\sigma \in H^1(H,A)} [(Y/A)_{(H,\sigma)}] \cdot \sigma = \sum_{gU \in \text{inv}_H(G/U)} \res_H^g U \circ \con_U^g(\tau).
\]

(b) By Lemma 2.10, it suffices to prove the first assertion. We have

\[
[(H/rU)_\tau]_K = \sum_{h' \in \text{inv}_K(H/rU)} \res_{h'K}^{h'rU} \circ \con_{h'U}(\tau)
\]

\[
= \sum_{h' \in \text{inv}_K(H/rU)} \con_{h'K} \circ \res_K^{r^{-1} h'U} \circ \con_U^{r^{-1} h'U}(\tau)
\]

\[
= \con_{h'K}([H/U)_\tau]_K).
\]

Hence the first assertion follows from Lemma 2.10. This completes the proof. \(\square\)

**Definition 4.7** We define 

\[
\mathcal{U}(G, A) := \left\{ (x_H)_{H \leq G} \in \prod_{H \leq G} \mathbb{Z}H^1(H, A) \mid \con_H^g(x_H) = x_\sigma for all g \in G \right\},
\]

the ghost ring of \( \Omega(G, A) \), which is a subring of \( \prod_{H \leq G} \mathbb{Z}H^1(H, A) \).

**Remark 4.8** The family of \( \mathbb{Z} \)-algebras \( \mathbb{Z}H^1(H, A) \) for \( H \leq G \), together with conjugation maps and restriction maps, defines a \( \mathbb{Z} \)-algebra restriction functor \( \mathbb{Z}H^1(-, A) \) defined in [4, 1.1. Definition]. The rings \( \Omega(G, A) \) and \( \mathcal{U}(G, A) \) are identified with \( \mathbb{Z}H^1(G, A)_+ \) and \( \mathbb{Z}H^1(G, A)_+^+ \), respectively, which are obtained by the plus constructions \( \mathbb{Z}H^1(-, A) \mapsto \mathbb{Z}H^1(-, A)_+ \) and \( \mathbb{Z}H^1(-, A) \mapsto \mathbb{Z}H^1(-, A)_+^+ \); moreover, the Green functor given in Proposition 2.16 is identified with \( \mathbb{Z}H^1(-, A)_+ \) (see [4]).

From Proposition 2.14 and Lemma 4.6, we know that there is an additive map \( \rho_G : \Omega(G, A) \to \mathcal{U}(G, A) \) given by

\[
[(G/U)_\tau] \mapsto \left( \sum_{gU \in \text{inv}_H(G/U)} \res_H^g U \circ \con_U^g(\tau) \right)_{H \leq G}
\]

for all \( (U, \tau) \in R(G, A) \) (cf. [4, 2.3.]), which is called the mark homomorphism. Since

\[
\rho_G([Y]) = ([Y]_H)_{H \leq G}
\]

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for all \((G, A)\)-sets \(Y\), the mark homomorphism is a ring homomorphism defined by

\[
\rho_G(x) = (\rho_H^G(x))_{H \leq G}
\]

(4.2)

for all \(x \in \Omega(G, A)\) (cf. [12, §2.5]). We write \(\rho = \rho_G\) for shortness’ sake.

According to [4, (2.3a)], there is a map \(\mathcal{M}^* : \Omega(G, A) \rightarrow \Omega(G, A)\) given by

\[
\left(\sum_{\sigma \in Z^1(H, A)} \ell_{(H, \sigma)}(\phi) \right)_{H \leq G} \mapsto \sum_{H \leq U \leq H} \sum_{\sigma \in Z^1(H, A)} |U| \mu(U, H) \sum_{\sigma \in Z^1(H, A)} \ell_{(H, \sigma)}((G/U)_{\sigma|U})
\]

for all \(\ell_{(H, \sigma)} \in \mathbb{Z}\) with \(H \leq G\) and \(\sigma \in Z^1(H, A)\).

We quote concise versions of [4, 2.4. Proposition] and [12, Theorem 1].

**Proposition 4.9** (a) \(\eta \circ \rho = |G| id_{\Omega(G, A)}\). (b) \(\rho \circ \eta = |G| id_{\Omega(G, A)}\).

**Corollary 4.10** The mark homomorphism \(\rho\) is injective.

4C Invariant of tensor induction

Let \(H \leq G\). By Example 3.13 and Proposition 3.14, we have

\[
\rho(\text{Map}_{H}(G, (H/H)_{\sigma})) = (\text{id}_G)_{K \leq G}
\]

(4.3)

for all \(\sigma \in Z^1(H, A)\). Let \(T\) be an \((H, A)\)-set. We are interested in the description of \(\rho(\text{Map}_{H}(G, T))\), which naturally extends Eq.(4.3) (see Proposition 4.14). For each \(K \leq G\), the \(K\)-component \([\text{Map}_{H}(G, T)]_K\) of \(\rho(\text{Map}_{H}(G, T))\) is also associated with a Mackey decomposition formula (see Proposition 3.9).

Let \(Y\) be a \((G, A)\)-set, and let \(K \leq G\). By Definition 4.2,

\[
[Y]_K = \frac{1}{|A|} \sum_{y \in \text{inv}_K^A(Y)} \text{res}_K^{G_A}(\sigma_y) = \sum_{yA \in \text{inv}_K^A(Y)/A} \text{res}_K^{G_A}(\sigma_y).
\]

Concerning this formula, we have

\[
[Y]_K = \frac{1}{|A|} \sum_{y \in \text{inv}_K^A(Y)} \text{res}_K^{G_A}(\sigma_y) = \frac{1}{|A|} \sum_{y \in \text{inv}_K^A(\text{res}_K^A(Y))} \sigma_y = [\text{res}_K^A(Y)]_K.
\]

(4.4)

Obviously, this fact implies that \(\rho_K^G(x) = \rho_K^G(\text{res}_K^A(x))\) for any \(x \in \Omega(G, A)\) which is applied to the following lemma.
Lemma 4.11 Let $H, K \leq G$. For any $x \in \Omega(H, A)$,
\[
\rho^K_G(\Map_H(G, x)) = \prod_{KgH \in K \setminus G/H} \rho^K_G(\Map_{KgH}(K, \res^{\Omega_H}_K(\Map_H(G, x)))).
\]

Proof. Since $\rho^K_G(\Map_H(G, x)) = \rho^K_G(\res^{\Omega_H}_K(\Map_H(G, x)))$ for any $x \in \Omega(H, A)$, the assertion follows from Proposition 3.24. This completes the proof. □

Definition 4.12 Let $H \leq G$. We define a map $-\otimes^G : ZH^1(H, A) \to ZH^1(G, A)$ by
\[
\sum_{\sigma \in H^1(H, A)} \ell_{\sigma} \mapsto \left( \sum_{\sigma \in H^1(H, A)} \ell_{\sigma} \right)^{-\otimes^G} := \sum_{\sigma \in H^1(H, A)} \ell_{\sigma} \sigma^{\otimes^G}
\]
for all $\ell_{\sigma} \in Z$ with $\sigma \in Z^1(H, A)$. (This map is well-defined; see Remark 3.12.)

Lemma 4.13 Let $H \leq G$, and let $T$ be an $(H, A)$-set. Then
\[
[\Map_H(G, T)]_G = \frac{1}{|A|} \sum_{f \in \inv^A_H(\Map_H(G, T))} \sigma_f = \frac{1}{|A|} \sum_{t \in \inv^A_H(T)} \sigma_t^{\otimes^G} = [T]^H \otimes^G.
\]

Proof. Fix a complete set $\{g_1, g_2, \ldots, g_n\}$ with $g_1 = \epsilon$ of representatives of $G/H$. Let $t \in \inv^A_H(T)$. We define an element $f(t)$ of $\Map_H(G, T)$ by
\[
f(t)(g_j^{-1}) = t
\]
for all $j \in [n]$. For any $g \in G$ and $j \in [n]$, if $g_H = gg_j H$, then
\[
(gf(t)(g_j^{-1}) = (g_j^{-1}gg_j)f(t)(g_j^{-1}) = (g_j^{-1}gg_j)t = f(t)(g_j^{-1})\sigma_t(g_j^{-1}gg_j).
\]
This, combined with Lemma 3.7, shows that $gf(t) = f(t)(\sigma_t^{\otimes^G})(g)$ for all $g \in G$ (see Definition 3.11). Hence $G = G_{f(t)A} : f(t) \in \inv^A_H(\Map_H(G, T))$, and $\sigma_{f(t)} = \sigma_t^{\otimes^G}$.

We now define a map $\hat{\Gamma} : \inv^A_H(T)/A \to \inv^A_G(\Map_H(G, T))/A$ by
\[
\hat{\Gamma}(tA) = f(t)A
\]
for all $t \in \inv^A_H(T)$. This map is well-defined, because, by Lemma 3.7, $f(tA) = f(t)b$ with $b = g_1g_2 \cdots g_n$ for any $a \in A$. If $\hat{\Gamma}(t_1A) = \hat{\Gamma}(t_2A)$ with $t_1, t_2 \in \inv^A_H(T)$, then $f(t_1) = f(t_2)a$ for some $a \in A$, and hence $t_1 = t_2b$ for some $b \in A$. Thus $\hat{\Gamma}$ is injective. Let $f \in \inv^A_H(\Map_H(G, T))$ with $f \in \Map_H(G, T)$, and let $g \in G$. Given $j \in [n]$, we have $(gf)(g_j^{-1}) = f(g_j^{-1})a_j(g)$ for some $a_j(g) \in A$. Set $t = f(\epsilon)$. Then
\[
h(t) = hf(\epsilon) = f(h) = (hf)(\epsilon) = ta_1(h)
\]
for all \( h \in H \), which yields \( t \in \text{inv}_H^A(T) \). Observe now that for any \( j \in [n] \),
\[
f(g_j^{-1}a_j(g_j)) = (g_jf)(g_j^{-1}) = f(\epsilon) = t = f_0(g_j^{-1}).
\]
By Lemma 3.7, we have \( \tilde{f} = f_0a \), where \( a = (g_0a_1(g_1)g_2a_2(g_2) \cdots g_na_n(g_n))^{-1} \), so that \( \tilde{\Gamma}(T^A) = f_0A \). Thus \( \tilde{\Gamma} \) is bijective. The assertion now follows from the fact that \( \sigma_{f_0} = \sigma_{t^G} \) for all \( t \in \text{inv}_H^A(T) \). This completes the proof. \( \square \)

The following proposition generalizes the equation in [32, p. 39] (see also [2, Lemma 9.2], [9, p. 149], and [30, p. 111, Eq.(2)]).

**Proposition 4.14** Let \( H, K \leq G \). For each \((H,A)\)-set \( T \),
\[
[\text{Map}_H(G,T)]_K = \prod_{KgH \in K \setminus G/H} [^gT]_{K \cap gH} \otimes K.
\]

**Proof.** Combining Lemma 4.13 with Lemma 4.11, we have
\[
[\text{Map}_H(G,T)]_K = \prod_{KgH \in K \setminus G/H} [\text{Map}_{K \cap gH}(K, \text{res}_{K \cap gH}[^gT])]_K
\]
\[
= \prod_{KgH \in K \setminus G/H} [\text{res}_{K \cap gH}[^gT]]_{K \cap gH} \otimes K.
\]
Hence the assertion follows from Eq.(4.4). This completes the proof. \( \square \)

How about the description of \( \rho(\text{Map}_H(G,x)) \) for any \( H \leq G \) and \( x \in \Omega(H,A) \)? By using Eq.(1.1), we are successful in proving Eq.(1.2) (see Theorem 4.16).

**Lemma 4.15** Let \( H \leq G \). For any \((H,A)\)-sets \( T_0 \) and \( T \),
\[
\rho_H^G(\text{Map}_H(G,\{T_0\} - [T])) = [\text{Map}_H(G,T_0)]_G - [\text{Map}_H(G,T)]_G.
\]

**Proof.** We may assume that \( H < G \). By Proposition 3.22,
\[
\text{Map}_H(G,\{T_0\} - [T]) = [\text{Map}_H(G,T_0)] + \sum_{i=1}^n (-1)^i[\text{Map}_H(G,T_0,T_1,\ldots,T_i)],
\]
where \( n = |G : H| \) and \( T = T_1 = \cdots = T_n \). If \( i \in [n] \) and \( i \geq 2 \), then obviously, \([\text{Map}_H(G,T_0,T_1,\ldots,T_i)]_G = 0 \). Moreover, we have
\[
\text{inv}_G^A(\text{Map}_H(G,T_0,T_1)) = \text{inv}_G^A(\text{Map}_H(G,\emptyset,T_1)) = \text{inv}_G^A(\text{Map}_H(G,T_1)),
\]
completing the proof. \( \square \)

The following theorem, which is equivalent to Eq.(1.2), is an extension of Proposition 4.14 and is a generalization of [32, §3(b.3)].
Theorem 4.16 Let $H \leq G$, and define a map $\text{ind}_H^G : \Omega(H, A) \to \Omega(G, A)$ by

$$(x_L)_{L \leq H} \mapsto \left( \prod_{KgH \in K \setminus G/H} \text{con}^g_{KgH}(x_{KgH}) \right)_{K \leq G}$$

for all $(x_L)_{L \leq H} \in \Omega(H, A)$. Then the diagram

$$
\begin{array}{ccc}
\Omega(G, A) & \xrightarrow{\rho_H} & \Omega(G, A) \\
\downarrow \text{Map}_H(G, -) & & \downarrow \text{ind}_H^G \\
\Omega(H, A) & \xrightarrow{\rho_H} & \Omega(H, A)
\end{array}
$$

is commutative, where $\rho_H : \Omega(H, A) \to \Omega(H, A)$ is the mark homomorphism.

Proof. We prove Eq.(1.2). Let $x \in \Omega(H, A)$. We may assume that $x = [T_0] - [T]$ for some $(H, A)$-sets $T_0$ and $T$. Let $K \leq G$. Then by Lemmas 4.11 and 4.15, we have

$$
\rho^K_G(\text{Map}_H(G, [T_0] - [T])) = \prod_{KgH \in K \setminus G/H} \rho^K_K(\text{Map}_{KgH}(K, [\text{res}^g_{KgH}(gT_0)] - [\text{res}^g_{KgH}(gT)]))
$$

Moreover, it follows from Eq.(4.4) and Lemma 4.13 that

$$
\rho^K_G(\text{Map}_H(G, [T_0] - [T])) = \prod_{KgH \in K \setminus G/H} \left\{ [gT_0]_{KgH} - [gT]_{KgH} \right\}.
$$

By Lemma 4.6(b), $[gT_1]_{KgH} = \text{con}^g_{KgH}(T_1)$, where $T_1 = T_0$ or $T_1 = T$, for all $g \in G$. Hence Eq.(1.2) follows from Eq.(4.2). This completes the proof. \[\square\]

Remark 4.17 Given $(H, \sigma) \in \mathcal{S}(G, A)$, it follows from Lemma 4.6 and Eq.(4.5) that

$$
\rho(\text{Map}_H(G, -[(H/H)_\sigma])) = \left( (-1)^{|K \setminus G/H|} \prod_{KgH \in K \setminus G/H} (g\sigma)_{KgH} \right)_{K \leq G}
$$

(see also Eq.(4.3)). Here we return to Remark 3.23. Deducing this fact directly from Lemma 3.7 and Proposition 3.22 requires the use of [25, (24d)] which provides a combinatorial explanation. Let $H, K \leq G$. When $A = \{e_A\}$, the $K$-component of $\rho(\text{Map}_H(G/H))$ is $(-1)^{|K \setminus G/H|}$ (see [29, Proposition 5.1] and [32, Lemma 3.6]).
For each $H \leq G$, we denote by $\Omega(H, A)^\times$ the unit group of $\Omega(H, A)$, and consider this abelian group as a $\mathbb{Z}$-module. Note that the $\mathbb{Z}$-module structure of $\Omega(H, A)^\times$ is different from that of $\Omega(H, A)$.

There is a fact relative to [2, Theorem 9.6] and [32, Lemma 3.1].

**Theorem 4.18** The family of $\mathbb{Z}$-modules $\Omega(H, A)^\times$ for $H \leq G$, together with conjugation, restriction, and multiplicative induction maps inherited from those on the family of $\mathbb{Z}$-algebras $\Omega(H, A)$ for $H \leq G$ defines a Mackey functor on $G$.

**Proof.** Let $\text{jnd}^H_K : \Omega(K, A)^\times \to \Omega(H, A)^\times$ with $K \leq H \leq G$ be the map inherited from $\text{Map}_K(H, -) : \Omega(K, A) \to \Omega(H, A)$. By [4, 1.1. Definition], Lemma 2.10, and Proposition 3.24, it suffices to verify that for any $U \leq V \leq H \leq G$ and $g \in G$,

$$\text{con}^g_H \circ \text{jnd}^H_U = \text{jnd}^V_H \circ \text{con}^g_U \quad \text{and} \quad \text{jnd}^V_U \circ \text{jnd}^H_U = \text{jnd}^H_U. \quad (4.6)$$

Given $H \leq G$ and $g \in G$, we define a map $\text{con}^g_H : \mathcal{U}(H, A) \to \mathcal{U}(^gH, A)$ by

$$(x_K)_{K \leq H} \mapsto (\text{con}^g_{K}(x_K))_{gK \leq ^gH}$$

for all $(x_K)_{K \leq H} \in \mathcal{U}(H, A)$. Let $U \leq V \leq H \leq G$, and let $g \in G$. Given $K \leq H$ and $(x_L)_{L \leq U} \in \mathcal{U}(U, A)$, we have

$$\text{con}^g_K(\text{con}^h_{K \cap U}(x_{Kh \cap U})^\otimes K) = (\text{con}^g_{K \cap U} \circ \text{con}^h_{Kh \cap U}(x_{Kh \cap U}))^\otimes ^gK$$

for all $h \in H$ and

$$\prod_{KhV \in K \setminus H/V} \text{con}^h_{K_h} \left( \prod_{KhU \in K_h \setminus U/V} \text{con}^h_{Kh \cap U}(x_{Kh \cap U})^\otimes K \right)^\otimes K = \prod_{KhU \in K \setminus H/U} \text{con}^h_{K \cap U}(x_{Kh \cap U})^\otimes K,$$

where $K_h = K^h \cap V$ (see Lemma 3.15). Relative to ‘$\text{jnd}$’ defined in Theorem 4.16, these equations enable us to obtain the equations

$$\text{con}^g_H \circ \text{jnd}^H_U = \text{jnd}^V_H \circ \text{con}^g_U \quad \text{and} \quad \text{jnd}^V_U \circ \text{jnd}^H_U = \text{jnd}^H_U.$$

By Lemma 4.6(b), $\text{con}^g_H \circ \rho_H = \rho_{^gH} \circ \text{con}^g_H$ and $\text{con}^g_U \circ \rho_U = \rho_{^gU} \circ \text{con}^g_U$. Hence Eq.(4.6) follows from Corollary 4.10 and Theorem 4.16. This completes the proof. □

5 **Fundamentals of monomial Burnside rings**

5A The Burnside homomorphism

The discussion in this section is a special case of [28, §9] (see also [27, §3, §4]).
For each \((U, \tau) \in \mathcal{S}(G, A)\), we set
\[
N_G(U, \tau) = \{g \in G \mid gU = U \text{ and } \text{con}_U^g(\tau) = \tau\}.
\]
By definition, the elements \((x_{H}^{(U, \tau)})_{H \leq G}\) for \((U, \tau) \in \mathcal{R}(G, A)\), where
\[
x_{H}^{(U, \tau)} = \begin{cases} 
\sum_{gN_G(U, \tau) \in N_G(U) / N_G(U, \tau)} \text{con}_U^g(\tau) & \text{if } H = rU \text{ with } r \in G, \\
0 & \text{otherwise},
\end{cases}
\]
form a free \(\mathbb{Z}\)-basis of the ghost ring \(\mathcal{O}(G, A)\). We define
\[
\tilde{\Omega}(G, A) := \prod_{(K, \nu) \in \mathcal{R}(G, A)} \mathbb{Z},
\]
so that there exists an isomorphism \(\kappa : \tilde{\Omega}(G, A) \sim \mathcal{O}(G, A)\) of \(\mathbb{Z}\)-lattices given by
\[
(\delta(U, \tau)(K, \nu))_{(K, \nu) \in \mathcal{R}(G, A)} \mapsto (x_{H}^{(U, \tau)})_{H \leq G}
\]
for all \((U, \tau) \in \mathcal{R}(G, A)\), where \(\delta\) is the Kronecker delta.

**Definition 5.1** We define an additive map \(\varphi : \Omega(G, A) \rightarrow \tilde{\Omega}(G, A)\) by
\[
\varphi([[(G/U)_{\tau}]]) = (|\text{inv}_{(K, \nu)}(G/U)_{\tau}|A_{\nu})_{(K, \nu) \in \mathcal{R}(G, A)}
\]
for all \((U, \tau) \in \mathcal{R}(G, A)\) (see Lemma 4.5), and call it the Burnside homomorphism.

**Proposition 5.2** The diagram
\[
\begin{array}{ccc}
\Omega(G, A) & \xrightarrow{\varphi} & \tilde{\Omega}(G, A) \\
\rho \downarrow & & \downarrow \kappa \\
\mathcal{O}(G, A)
\end{array}
\]
is commutative. In particular, the Burnside homomorphism \(\varphi\) is injective.

**Proof.** The assertion follows from Lemma 4.4 and Corollary 4.10. \(\square\)

Let \((U, \tau) \in \mathcal{R}(G, A)\). By Lemma 2.6, \(N_G(U, \tau)\) contains \(U\). Observe that for any \((K, \nu) \in \mathcal{R}(G, A)\), the \((K, \nu)\)-component of \(\varphi([[(G/U)_{\tau}]])\) is divisible by \([N_G(U, \tau)/U]\) (see Lemma 4.5). We define
\[
y^{(U, \tau)} := \frac{1}{[N_G(U, \tau)/U]} \varphi([[(G/U)_{\tau}]])
\]
for \((U, \tau) \in \mathcal{R}(G, A)\). The elements \(y^{(U, \tau)}\) for \((U, \tau) \in \mathcal{R}(G, A)\) form a free \(\mathbb{Z}\)-basis of the \(\mathbb{Z}\)-lattice \(\tilde{\Omega}(G, A)\).

**Proposition 5.3** The elements \(y^{(U, \tau)}\) for \((U, \tau) \in \mathcal{R}(G, A)\) form a free \(\mathbb{Z}\)-basis of the \(\mathbb{Z}\)-lattice \(\Omega(G, A)\).

**Proof.** The proof is completely analogous to that of [8, (80.15) Proposition]. \(\square\)
5B The Cauchy-Frobenius homomorphism

We aim to state a fundamental theorem for the monomial Burnside ring \( \Omega(G, A) \)
(see Theorem 5.9).

**Definition 5.4** For each \( (U, \tau) \in S(G, A) \), let \( W_G(U, \tau) \) denote the factor group \( N_G(U, \tau)/U \). We define

\[
\text{Obs} (G, A) := \prod_{(U, \tau) \in R(G, A)} \mathbb{Z}/|W_G(U, \tau)|\mathbb{Z},
\]

the obstruction group of \( \Omega(G, A) \).

The following fact is a corollary to Proposition 5.3.

**Corollary 5.5** \( \tilde{\Omega}(G, A)/\text{Im} \varphi \simeq \text{Obs} (G, A) \).

**Proof.** The proof is completely analogous to that of [27, Corollary 3.8]. □

Let \( p \) be a prime, and let \( \mathbb{Z}(p) \) be the localization of \( \mathbb{Z} \) at \( p \). For each \( \mathbb{Z} \)-module \( M \), we set \( M(p) = \mathbb{Z}(p) \otimes_{\mathbb{Z}} M \) and \( M(\infty) = M \). Let \( (U, \tau) \in S(G, A) \). We denote by \( W_G(U, \tau)_p \) a Sylow \( p \)-subgroup of \( W_G(U, \tau) \), and set \( W_G(U, \tau)_\infty = W_G(U, \tau) \).

Let \( p \) be a prime or the symbol \( \infty \) hereafter. By Proposition 2.14, the elements \( [(G/H)_{\sigma}] \) for \( (H, \sigma) \in R(G, A) \) form a free \( \mathbb{Z}(p) \)-basis of the \( \mathbb{Z}(p) \)-lattice \( \Omega(G, A)_p \).

We identify \( \tilde{\Omega}(G, A)_p \) and \( \text{Obs} (G, A)_p \) with

\[
\prod_{(K, \nu) \in R(G, A)} \mathbb{Z}(p) \quad \text{and} \quad \prod_{(U, \tau) \in R(G, A)} \mathbb{Z}(p)/|W_G(U, \tau)_p|\mathbb{Z}(p),
\]

respectively. Let \( \varphi(p) \) denote the monomorphism from \( \Omega(G, A)_p \) to \( \tilde{\Omega}(G, A)_p \) determined by \( \varphi \). (So \( \varphi(\infty) = \varphi \).) Then by Corollary 5.5, we have

\[
\tilde{\Omega}(G, A)_p/\text{Im} \varphi(p) \simeq \text{Obs} (G, A)_p. \quad (5.1)
\]

The expression \( `x \mod \ell ' \) with \( x, \ell \in \mathbb{Z}(p) \) denotes the coset \( x + \ell \mathbb{Z}(p) \) of \( \ell \mathbb{Z}(p) \) in \( \mathbb{Z}(p) \) containing \( x \). Let \( (U, \tau) \in S(G, A) \). Given \( (y_{(H, \sigma)})_{(H, \sigma) \in R(G, A)} \in \tilde{\Omega}(G, A)_p \),
\( y_{(U, \tau)} \) denotes \( y_{(H, \sigma)} \) for a representative \( (H, \sigma) \in R(G, A) \) of the \( F \)-orbit on \( S(G, A) \) containing \( (U, \tau) \). For each \( g \in N_G(U, \tau) \), we set

\[
H^1_F((g)U, A) = \{ \overline{v} \in H^1((g)U, A) \mid \text{res}^g_U(\overline{v}) = \overline{v} \}.
\]

**Definition 6.6** We define an additive map \( \psi(p) : \tilde{\Omega}(G, A)_p \to \text{Obs} (G, A)_p \) by

\[
(y_{(K, \nu)})_{(K, \nu) \in R(G, A)} \mapsto \left( \sum_{\substack{y_{(g)U, \nu) \mod |W_G(U, \tau)_p| \in H^1_F((g)U, A) \quad (U, \tau) \in R(G, A)}} y_{(g)U, \nu} \right)
\]

for each \( U \in \mathcal{U} \) and \( (U, \tau) \in R(G, A) \).
for all \((y_{(K,\nu)})_{(K,\nu)\in \mathcal{R}(G,A)} \in \tilde{\Omega}(G,A)_{(p)}\), and call it the Cauchy-Frobenius homomorphism.

Remark 5.7  (1) When \(p\) is a prime, \(\psi^{(p)}\) is independent of the choice of a Sylow \(p\)-subgroup \(W_G(U,\tau)_p\) of \(W_G(U,\tau)\) (cf. [28, §9]). (2) When \(p = \infty\), we write \(\psi = \psi^{(\infty)}\).

For each \((H,\sigma) \in \mathcal{R}(G,A)\), it follows from Lemma 4.5 that

\[
\psi^{(p)} \circ \varphi^{(p)}([G/H]_\sigma) = \left( \sum_{gU \in W_G(U,\tau)_p} |I_{gU,\tau}^{H,\sigma}| \mod |W_G(U,\tau)_p| \right)_{(U,\tau) \in \mathcal{R}(G,A)},
\]

where

\[
I_{gU,\tau}^{H,\sigma} = \{ rH \in G/H \mid (g)U \leq rH \text{ and } (r\sigma)|_U = A \tau \}.
\]

The following lemma, which is a special case of [28, Lemma 9.2], is a consequence of the Cauchy-Frobenius lemma (see, e.g., [33, 2.7 Lemma]).

Lemma 5.8 Let \((H,\sigma), (U,\tau) \in \mathcal{R}(G,A)\). For any \(V \leq N_G(U,\tau)\) with \(U \leq V\),

\[
\sum_{gU \in V/U} |I_{gU,\tau}^{H,\sigma}| \equiv 0 \pmod{|V/U|}.
\]

Proof. The proof is analogous to that of [28, Lemma 9.2], and is also analogous to part of the proof of [27, Lemma 4.1]. ∎

We are now in a position to show a special case of [28, Theorem 9.4], which is a generalization of [9, Proposition 1.3.5] and [32, Lemma 2.1].

Theorem 5.9 (Fundamental theorem) The sequence

\[
0 \rightarrow \Omega(G,A)_{(p)} \xrightarrow{\varphi^{(p)}} \tilde{\Omega}(G,A)_{(p)} \xrightarrow{\psi^{(p)}} \text{Obs } (G,A)_{(p)} \rightarrow 0
\]

of additive groups is exact.

Proof. By Proposition 5.2, \(\varphi^{(p)}\) is injective. Moreover, it is easily verified that \(\psi^{(p)}\) is surjective (see, e.g., the proof of [27, Lemma 4.3]). Using Eqs.(5.1) and (5.2) and Lemma 5.8, we have \(\text{Im } \varphi^{(p)} = \text{Ker } \psi^{(p)}\), completing the proof. ∎

5C Idempotents of Burnside rings

The Burnside ring \(\Omega(G)\) of \(G\), which is defined to be the Grothendieck ring of \(G\)-set, is the commutative unital ring consisting of all formal \(\mathbb{Z}\)-linear combinations of the symbols \([G/H]\) for \(H \in \mathcal{C}(G)\) with multiplication given by

\[
[G/H] \cdot [G/U] = \sum_{HgU \in H \setminus G/U} [G/(H \cap gU)]
\]

(5.3)
for all $H, U \in C(G)$, where $[G/(H \cap gU)] = [G/K]$ for a conjugate $K \in C(G)$ of $H \cap gU$ in $G$ (see, e.g., [33, 2.1]). The identity of $\Omega(G)$ is $[G/G]$.

We regard $\Omega(G)$ as $\Omega(G, A)$ with $A = \{e_A\}$. For each $X \in G$-set, the symbol $[X]$ denotes an element $\sum_{i=1}^{n}[G/H_i]$ of $\Omega(G)$ if $X \simeq \bigcup_{i \in [n]} G/H_i$ with $H_i \in C(G)$.

Remark 5.10 The product $X_1 \times X_2$ of $X_1, X_2 \in G$-set is their cartesian product with the componentwise action of $G$ (cf. [8, §80A]). Let $H, U \leq G$, and let $H \leq G/U$ be a complete set of representatives of $H \leq G/U$ (see, e.g., [33, 2.1]). The identity of $\Omega(G)$ is $[G/G].$

We regard $\Omega(G)$ as $\Omega(G; A)$ with $A = f \epsilon A g$. For each $X \in G$-set, the symbol $[X]$ denotes an element $\sum_{i=1}^{n}[G=H_i]$ of $\Omega(G)$ if $X \simeq \bigcup_{i \in [n]} G/H_i$ with $H_i \in C(G)$.

Remark 5.10 The product $X_1 \times X_2$ of $X_1, X_2 \in G$-set is their cartesian product with the componentwise action of $G$.

Definition 5.11 We define a ring homomorphism $\alpha : \Omega(G, A) \rightarrow \Omega(G)$ by

$$[(G/U)_\tau] \mapsto [G/U]$$

for all $(U, \tau) \in R(G, A)$ and define a ring homomorphism $\iota : \Omega(G) \rightarrow \Omega(G, A)$ by

$$[G/U] \mapsto [(G/U)_1]$$

for all $U \in C(G)$.

Since $\alpha \circ \iota = \text{id}_{\Omega(G)}$, the Burnside ring $\Omega(G)$ is identified with $\text{Im}\iota$. We define

$$\bar{\Omega}(G) := \prod_{H \in C(G)} \mathbb{Z}.$$  

There exists a ring monomorphism $\phi : \Omega(G) \rightarrow \bar{\Omega}(G)$ given by

$$[G/U] \mapsto ([\text{inv}_H(G/U)]_{H \in C(G)}$$

for all $U \in C(G)$ (cf. [8, (80.12) Proposition]), where $\text{inv}_H(G/U)$ is given by Eq.(4.1).

The ring homomorphism $\varepsilon : \mathbb{Z}H^1(H, A) \rightarrow \mathbb{Z}$ with $H \leq G$ given by

$$\sum_{\sigma \in H^1(H, A)} \ell_\sigma \tau \mapsto \sum_{\sigma \in H^1(H, A)} \ell_\sigma$$

for all $\ell_\sigma \in \mathbb{Z}$ where $\sigma \in Z^1(H, A)$ is called the augmentation map of $\mathbb{Z}H^1(H, A)$ (cf. [21, Definition 3.2.9]).
Definition 5.12 We define a ring homomorphism \( \tilde{\alpha} : \mathcal{U}(G, A) \to \mathcal{U}(G) \) by
\[
(x_H)_{H \leq G} \mapsto (\varepsilon(x_H))_{H \in \text{C}(G)}
\]
for all \( (x_H)_{H \leq G} \in \mathcal{U}(G, A) \) and define a ring homomorphism \( \tilde{\tau} : \mathcal{U}(G) \to \mathcal{U}(G, A) \) by
\[
(y_H)_{H \in \text{C}(G)} \mapsto (y_H)_{H \leq G},
\]
where \( \tilde{y}_H = y_K \) for a conjugate \( K \in \text{C}(G) \) of \( H \) in \( G \), for all \( (y_H)_{H \in \text{C}(G)} \in \mathcal{U}(G) \).

Obviously, \( \tilde{\alpha} \circ \tilde{\tau} = \text{id}_{\mathcal{U}(G)} \). We provide the following two lemmas.

Lemma 5.13 (a) The diagrams
\[
\begin{array}{ccc}
\Omega(G, A) & \xrightarrow{\rho} & \mathcal{U}(G, A) \\
\alpha & \downarrow & \tilde{\alpha} \\
\Omega(G) & \xrightarrow{\phi} & \mathcal{U}(G)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\Omega(G, A) & \xrightarrow{\rho} & \mathcal{U}(G, A) \\
\iota & \uparrow & \tilde{\tau} \\
\Omega(G) & \xrightarrow{\phi} & \mathcal{U}(G)
\end{array}
\]
are commutative.

(b) Let \( x \in \Omega(G, A) \). If \( \rho(x) = \tilde{\tau}(y) \) for some \( y \in \mathcal{U}(G) \), then \( \iota \circ \alpha(x) = x \).

Proof. The statement (a) is clear. We prove the statement (b). Since \( \tilde{\alpha} \circ \tilde{\tau} = \text{id}_{\mathcal{U}(G)} \), it follows from the statement (a) that
\[
\rho \circ \iota \circ \alpha(x) = \tilde{\tau} \circ \phi \circ \alpha(x) = \tilde{\tau} \circ \tilde{\alpha} \circ \rho(x) = \tilde{\tau} \circ \tilde{\alpha} \circ \tilde{\tau}(y) = \tilde{\tau}(y) = \rho(x).
\]
This, combined with Corollary 4.10, shows that \( \iota \circ \alpha(x) = x \), completing the proof.

Lemma 5.14 (a) \( \alpha \circ \eta \circ \tilde{\tau} \circ \phi = |G|\text{id}_{\Omega(G)} \). (b) \( \phi \circ \alpha \circ \eta \circ \tilde{\tau} = |G|\text{id}_{\mathcal{U}(G)} \).

Proof. The lemma follows from Proposition 4.9 and Lemma 5.13(a).

The rest of this section is devoted to the idempotents of \( \mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G) \).

Definition 5.15 Given \( U \leq G \), we define \( W_G(U) \) to be the factor group \( N_G(U)/U \).

Let \( p \) be a prime or the symbol \( \infty \). For each \( U \leq G \), we denote by \( W_G(U)_p \) a Sylow \( p \)-subgroup of \( W_G(U) \) provided \( p \) is a prime, and set \( W_G(U)_\infty = W_G(U) \).

The elements \( [G/H] \) for \( H \in \text{C}(G) \) form a free \( \mathbb{Z}(p) \)-basis of the \( \mathbb{Z}(p) \)-lattice \( \Omega(G)_{(p)} \). We identify \( \mathcal{U}(G)_{(p)} \) with \( \prod_{H \in \text{C}(G)} \mathbb{Z}(p) \). Let \( \phi(p) \) denote the ring monomorphism from \( \Omega(G)_{(p)} \) to \( \mathcal{U}(G)_{(p)} \) determined by \( \phi \).

We quote [9, Proposition 1.3.5] (see also [32, Lemma 2.1]).
Proposition 5.16 Let $\tilde{x} = (x_H)_{H \in C(G)} \in \mathcal{U}(G)_{(p)}$. Then $\tilde{x} \in \text{Im}\phi^{(p)}$ if and only if
\[ \sum_{gU \in W_G(U)_p} x_{(g)U} \equiv 0 \pmod{|W_G(U)_p|}, \]
where $x_{(g)U} = x_K$ for a conjugate $K \in C(G)$ of $(g)U$ in $G$, for all $U \in C(G)$. 

Proof. The assertion follows from Theorem 5.9 and Lemma 5.13(a). $\square$

By Lemma 5.14, the primitive idempotents of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$ are the elements
\[ e_H := \frac{1}{|G|} \alpha \circ \eta \circ \bar{\tau}(\delta_{H,K}K \in C(G)) = \frac{1}{|N_G(H)|} \sum_{U \leq H} |U| \mu(U,H)[G/U] \tag{5.4} \]
for $H \in C(G)$. This fact was shown by Gluck [14] and independently by Yoshida [31]. Obviously, $e_{HK} = \delta_{HK}e_H$ for all $H, K \in C(G)$, and $[G/G] = \sum_{H \in C(G)} e_H$.

Following [33], we present the primitive idempotents of $\Omega(G)$. Let $\sim_p$ be the equivalence relation on the set \{(H) \mid H \leq G\}, where $(H)$ is the set of conjugates of $H$ in $G$, generated by
\[ (gU) \sim_p (U) \]
for $U \leq G$ and $gU \in W_G(U)_p$ with $g \in N_G(U)$. We define an equivalence relation $\sim_p$ on the set $S(G)$ of subgroups of $G$ by
\[ H \sim_p K \iff (H) \sim_p (K). \]
Let $H \leq G$. When $p$ is a prime, we denote by $O^p(H)$ the smallest normal subgroup of $H$ such that $H/O^p(H)$ is a $p$-group (cf. [31]). Suppose that
\[ H = H^{(0)} \geq H^{(1)} \geq H^{(2)} \geq \cdots \geq H^{(i)} \geq \cdots \]
is the derived series of $H$ (cf. [26, Chapter 2, Definition 3.11]). Then we define $O^\infty(H) := \cap_{i=0}^\infty H^{(i)}$. The following lemma is well-known (cf. [33, p. 535]).

Lemma 5.17 Let $H, U \leq G$. Then $H \sim_p U$ if and only if $(O^p(H)) = (O^p(U))$. 

Proof. The ‘if’ part follows from [26, Chapter 2, Theorem 1.6]. To prove the ‘only if’ part, we may assume that $H = (g)U$ for some $gU \in W_G(U)_p$ with $g \in N_G(U)$. If $p$ is a prime, then $U \geq O^p(U) \geq O^p(H)$, and hence $O^p(U) = O^p(H)$. Suppose that $p = \infty$. We have $U^{(i-1)} \geq H^{(i)} \geq U^{(i)}$ for any $i \geq 1$. If $U^{(i-1)} = U^{(i)}$ for some $i$, then $U^{(i-1)} = H^{(i)} = U^{(i)}$. Thus we have $O^\infty(H) = O^\infty(U)$, completing the proof. $\square$

A subgroup $H$ of $G$ is said to be $p$-perfect if $H = O^p(H)$. For each $K \leq G$, $K \sim_p O^p(K)$ by Lemma 5.17, and $O^p(K)$ is $p$-perfect. Let $C^{(p)}(G)$ be a full set of non-conjugate $p$-perfect subgroups of $G$. For each $H \in C^{(p)}(G)$, we define
\[ e_H := \sum_{H \sim_p K \in C(G)} e_K, \]
The elements $e^p_H$ for $H \in C^p(G)$ are the primitive idempotents of $\Omega(G)_p$, and the elements $e^{(\infty)}_H$ for $H \in C^{(\infty)}(G)$ are also those of $\Omega(G,A)$.

Proof. For any idempotent $(x_H)_{H \in C(G)}$ of $\Omega(G)_p$, it follows from Proposition 5.16 that $(x_H)_{H \in C(G)} \in \text{Im} \phi(p)$ if and only if $x_K = x_U \in \{0, 1\}$ for all pairs $(K,U)$ of $K, U \in C(G)$ with $K \sim_p U$. Hence the elements $e^p_H$ for $H \in C^p(G)$ are the primitive idempotents of $\Omega(G)_p$. Let $x$ be an idempotent of $\Omega(G,A)$. According to [21, Corollary 7.2.4], $ZH^2(H,A)$ with $H \leq G$ contains only trivial idempotents, whence $\rho(x) = \overline{\ell}(y)$ for some $y \in \Omega(G)$. This, combined with Lemma 5.13(b), shows that $\psi \circ \alpha(x) = x$. By this fact, we may identify $x$ with $\alpha(x) \in \Omega(G)$. Since the map $\alpha : \Omega(G,A) \to \Omega(G)$ is a ring homomorphism, it follows that $\alpha(x)$ is an idempotent of $\Omega(G)$. Consequently, the idempotents of $\Omega(G,A)$ are those of $\Omega(G)$. This completes the proof. \(\square\)

There is an immediate consequence of Theorem 4.18 (see [4, 1.5. Proposition]).

**Proposition 5.19** The $\mathbb{Z}$-module $\Omega(G,A)^x$ has a structure of an $\Omega(G)$-module, namely,

$$\Omega(G) \otimes_{\mathbb{Z}} \Omega(G,A)^x \to \Omega(G,A)^x, \quad [G/H] \otimes_{\mathbb{Z}} x \mapsto \overline{\text{Map}}_H(G, \text{res}_H^G(x)).$$

Moreover,

$$\Omega(G,A)^x = \prod_{H \in C^{(\infty)}(G)} \{e^{(\infty)}_H x \mid x \in \Omega(G,A)^x\},$$

where $e^{(\infty)}_H x$ denotes the effect of $e^{(\infty)}_H$ on $x$.

### 6 Units of Burnside rings

6A The Yoshida criterion for the units of Burnside rings

We turn to the unit group $\Omega(G)^x$ of $\Omega(G)$. Let $\mathcal{U}(\Omega(G)^x)$ be the unit group of $\mathcal{U}(G)$, and let $\phi^{x} : \Omega(G)^x \to \mathcal{U}(\Omega(G)^x)$ be the map obtained by restriction of $\phi : \Omega(G) \to \mathcal{U}(G)$ from $\Omega(G)$ to $\Omega(G)^x$. Obviously, $\mathcal{U}(\Omega(G)^x) = \prod_{H \in C(G)} \langle -1 \rangle$, where $\langle -1 \rangle = \{\pm 1\}$, and hence $\Omega(G)^x$ is embedded in $\prod_{H \in C(G)} \langle -1 \rangle$. In particular, $\Omega(G)^x$ is an elementary abelian 2-group with identity $[G/G]$ (cf. [11, Proposition 3.1]). Thus $\Omega(G)^x$ consists of all $x \in \Omega(G)$ such that $(|G/G| \pm x)/2$ are idempotents of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$. 

where the sum is taken over all $K \in C(G)$ such that $H \sim_p K$.

The following theorem concerns [2, Theorem 7.3] and [33, 4.12 Theorem] (see also [14, Lemma 2] and [31, Theorem 3.1]).
Example 6.1 Suppose that $K \leq G$ and $|G : K| = 2$. Then $[G/K] \cdot [G/K] = 2[G/K]$, and hence $[G/G] - [G/K] \in \Omega(G)^\times$. We have $\phi^\times([G/G] - [G/K]) = ((-1)^{\zeta(H,K)})_{H \in C(G)}$, where $\zeta(H,K) = 1$ if $H \leq K$, and $\zeta(H,K) = 0$ otherwise.

Remark 6.2 According to Dress [10], $G$ is solvable if and only if 0 and $[G/G]$ are the only idempotents of $\Omega(G)$ (see also Lemma 5.17 and Theorem 5.18). Suppose that $G$ is of odd order. Then by Eq.(5.4), $\Omega(G)^\times$ consists of all $x \in \Omega(G)$ such that $([G/G] \pm x)/2$ are idempotents of $\Omega(G)$, whence $|\Omega(G)^\times|$ is the number of idempotents of $\Omega(G)$. Consequently, we have $\Omega(G)^\times = \langle -[G/G] \rangle$ because, by Feit-Thompson’s theorem, $G$ is solvable (cf. [9, Proposition 1.5.1]).

Definition 6.3 Given $\tilde{x} = (x_H)_{H \in C(G)} \in \tilde{\Omega}(G)^\times$ and $U \leq G$, we define a class function $\gamma_{\tilde{U}} : W_G(U) \to \langle -1 \rangle$ by

$$gU \mapsto x_U x_{(g)U}$$

for all $g \in N_G(U)$, where $x_{(g)U} = x_K$ for a conjugate $K \in C(G)$ of $(g)U$ in $G$.

We quote [32, Proposition 6.5] which is due to Yoshida.

Theorem 6.4 (The Yoshida criterion) The subgroup $\text{Im} \phi^\times$ of $\tilde{\Omega}(G)^\times$ consists of all $\tilde{x} = (x_H)_{H \in C(G)} \in \tilde{\Omega}(G)^\times$ such that $\gamma_{\tilde{U}} \in \text{Hom}(W_G(U), \langle -1 \rangle)$ for each $U \leq G$.

Example 6.5 Let $p$ be an odd prime, and suppose that $G$ is a finite $p$-group. Let $\tilde{x} = (x_H)_{H \in C(G)} \in \text{Im} \phi^\times$. If $G_r < \cdots < G_1 < G_0 = G$ is a sequence of subgroups of $G$ with $|G_{i-1} : G_i| = p$ for all $i \in [r]$, then by Theorem 6.4, $x_{G_0} = x_{G_1} = \cdots = x_{G_r}$. Thus it follows from [26, Chapter 2, Theorem 1.9] that $\tilde{x}$ is determined by $x_G$, and hence $\tilde{x} \in \langle (-1, -1, \ldots, -1) \rangle$. Consequently, $\Omega(G)^\times \approx \langle -[G/G] \rangle$ (see Remark 6.2).

Definition 6.6 For each $\tilde{x} \in \text{Im} \phi$, $\phi^{-1}(\tilde{x})$ denotes the unique element $x$ of $\Omega(G)$ such that $\tilde{x} = \phi(x)$. We define a subgroup $\Omega(G)^\times_1$ of $\Omega(G)^\times$ to be the product of the subgroups $\langle [G/K] - [G/G] \rangle$ for $K \leq G$ with $|G : K| = 2$, and define a subgroup $\Omega(G)^\times_1$ of $\Omega(G)^\times$ to be the group consisting of all $x = \phi^{-1}((x_H)_{H \in C(G)})$ with $(x_H)_{H \in C(G)} \in \text{Im} \phi^\times$ such that $x_H = 1$ whenever $H$ is cyclic.

The group $\text{Hom}(G, \langle -1 \rangle)$ with pointwise product is isomorphic to the factor group $G/G_2$ where $G_2$ is the intersection of all subgroups of index 2 in $G$.

Proposition 6.7 (a) $|\langle -[G/G] \rangle \times \Omega(G)^\times_0| = 2^{|\text{Hom}(G, \langle -1 \rangle)|}$.

(b) $\Omega(G)^\times_0 \approx \langle -[G/G] \rangle \times \Omega(G)^\times_0 \Omega(G)^\times_1 \simeq \Omega(G)^\times_0 \times \text{Hom}(G, \langle -1 \rangle) \times \Omega(G)^\times_1$.

Proof. Obviously, $\Omega(G)^\times_0$ is the direct product of the subgroups $\langle [G/K] - [G/G] \rangle$ for $K \leq G$ with $|G : K| = 2$. Thus the assertion (a) holds. We prove the assertion (b). For each $K \leq G$ with $|G : K| = 2$, if $\phi([G/K] - [G/G]) = \tilde{x} = (x_H)_{H \in C(G)}$, then
by Example 6.1 and Theorem 6.4, \( \gamma(K) := \gamma_{\{\epsilon\}} \in \text{Hom}(G, \langle -1 \rangle) \), \( \text{Ker}\gamma(K) = K \), and \( \gamma(K)(g) = x(g) \) for all \( g \in G \). Let \( y \in \Omega(G)^{\times} \), and suppose that the \( \{\epsilon\} \)-component of \( \phi(y) \) is 1. If \( \phi^{\times}(y) = \bar{y} = (y_H)_{H \in C\!(G)} \) with \( y_{\{\epsilon\}} = 1 \), then \( \gamma_{\{\epsilon\}}^{\bar{y}} \in \text{Hom}(G, \langle -1 \rangle) \) by Theorem 6.4, and \( \gamma_{\{\epsilon\}}^{\bar{y}}(g) = y(g) \) for all \( g \in G \). This, combined with the preceding argument, shows that \( ((G/K) - [G/G]) \cdot y \in \Omega(G)^{\times} \) with \( K = \text{Ker}\gamma_{\{\epsilon\}}^{\bar{y}} \), which yields \( y \in \Omega(G)^{\times} \Omega(G)^{\times} \). Hence \( \Omega(G)^{\times} \Omega(G)^{\times} \) consists of all \( x \in \Omega(G)^{\times} \) such that the \( \{\epsilon\} \)-component of \( \phi(x) \) is 1. We now obtain

\[
\Omega(G)^{\times} = \{ -[G/G] \times \Omega(G)^{\times} \Omega(G)^{\times} \}.
\]

Let \( K_1, K_2, \ldots, K_n \) be the subgroups of index 2 in \( G \). Then \( \Omega(G)^{\times} \) is the direct product of the subgroups \( ([G/K_i] - [G/G]) \) for \( i \in [n] \) and \( \text{Hom}(G, \langle -1 \rangle) \) is the group consisting of 1\(_G\) and the linear \( \mathbb{C} \)-characters \( \gamma(K_i) \) for \( i \in [n] \). Define a group epimorphism \( \gamma : \Omega(G)^{\times} \to \text{Hom}(G, \langle -1 \rangle) \) by

\[
\prod_{j=1}^{m} ([G/K_{i_j}] - [G/G]) \mapsto \prod_{j=1}^{m} \gamma(K_{i_j})
\]

for all sequences \((i_1, i_2, \ldots, i_m)\) with \( 1 \leq i_1 < i_2 < \cdots < i_m \leq n \) of natural numbers. Then it is obvious that \( \text{Ker}\gamma = \Omega(G)^{\times} \cap \Omega(G)^{\times} \). Consequently, we have

\[
\Omega(G)^{\times} \Omega(G)^{\times} \simeq \text{Hom}(G, \langle -1 \rangle) \times \Omega(G)^{\times},
\]

completing the proof. \( \Box \)

**Proposition 6.8** Let \( \hat{C}(G) \) be the set of all \( U \in C\!(G) \) such that \( |N_G(U) : U| \leq 2 \). For any \( \bar{x} = (x_H)_{H \in C\!(G)} \in \text{Im}\phi^{\times} \), the values \( x_H \) for \( H \in C\!(G) \) are determined by the values \( x_U \) for \( U \in \hat{C}(G) \). In particular, \( |\Omega(G)^{\times}| \leq 2|\hat{C}(G)| \).

**Proof.** Let \( \bar{x} = (x_H)_{H \in C\!(G)} \in \text{Im}\phi^{\times} \), and let \( H \leq G \). By Theorem 6.4, we have

\[
x_{(g_1)H}x_{(g_2)H}x_H = x_{(g_1g_2)H}
\]

for all \( g_1, g_2 \in N_G(H) \). Hence, if \( |N_G(H) : H| > 2 \), then the value \( x_H \) is determined by the values \( x_K \) with \( H < K \leq N_G(H) \) (cf. [7, p. 904]). This completes the proof. \( \Box \)

**Example 6.9** Assume that \( G \) is abelian. Then by Propositions 6.7 and 6.8, we have \( |\Omega(G)^{\times}| = 2^{\text{Hom}(G, \langle -1 \rangle)} \), because \( \hat{C}(G) \) is the set of all \( K \leq G \) such that \( |G : K| \leq 2 \) (cf. [32, Lemma 7.1]). This fact is due to Matsuda (cf. [18, Example 4.5]).
6B Structure of the unit groups of Burnside rings

We continue to discuss the structure of $\Omega(G)^\times$.

**Definition 6.10** We define a subset $\overline{C}(G)$ of $C(G)$ to be the set consisting of all subgroups $U$ which satisfy the following conditions.

(i) $|N_G(U) : U| \leq 2$.

(ii) If $L$ is a normal subgroup of $U$ and if $U/L$ is a non-trivial cyclic group, then $U/L$ is a cyclic 2-group and there exists a subgroup $K$ of index 2 in $N_G(L)$ containing $L$ such that

$$\{\langle g \rangle L \mid g \in N_G(L) - K \} = \{\langle g \rangle L \mid g \in N_G(L) \text{ and } (\langle g \rangle L) = (U)\}.$$

**Proposition 6.11** Let $U \in C(G)$, and set $\bar{x} = ((-1)^{y_{U}})_{H \in C(G)} \in \Omega(G)^\times$. Then $\bar{x} \in \text{Im } \phi^\times$ if and only if $U \in \overline{C}(G)$. In particular, if $U \in \overline{C}(G)$, then $2e_U \in \Omega(G)$, or equivalently, $[G/G] - 2e_U = \phi^{-1}(\bar{x}) \in \Omega(G)^\times$.

**Proof.** Assume that $\bar{x} \in \text{Im } \phi^\times$. For any $L \leq G$, it follows from Theorem 6.4 that the map $\gamma^\times_{L} : W_G(L) \to \langle -1 \rangle$ is a linear $C$-character of $W_G(L)$. Moreover, by assumption, $\gamma^\times_{U}(gU) = -1$ for any $g \in N_G(U) - U$. This means that $\text{Ker } \gamma^\times_{U} = U/U$. Consequently, $|N_G(U) : U| \leq 2$. Let $L$ be a normal subgroup of $U$, and suppose that $U/L$ is non-trivial cyclic. Set $U = \langle r \rangle L$ with $r \in N_G(L) - L$. Then for any $g \in N_G(L)$, $\gamma^\times_{L}(gL) = -1$ if and only if $\langle g \rangle L$ is a conjugate of $\langle r \rangle L$ in $G$. In particular, $rL$ must be a 2-element of $W_G(L)$, whence $U/L$ is a cyclic 2-group. Moreover, there exists a subgroup $K$ of index 2 in $N_G(L)$ containing $L$ such that $K/L = \text{Ker } \gamma^\times_{L} \langle g \rangle L$ and

$$\{\langle g \rangle L \mid g \in N_G(L) - K \} = \{\langle g \rangle L \mid g \in N_G(L) \text{ and } (\langle g \rangle L) = (U)\}.$$

Thus $U \in \overline{C}(G)$, as required. Conversely, if $U \in \overline{C}(G)$, then by Theorem 6.4, we have $\bar{x} \in \text{Im } \phi^\times$, completing the proof. $\square$

**Remark 6.12** Under the hypotheses of Proposition 6.11, it follows from Eq.(5.4) that $\bar{x} \in \text{Im } \phi^\times$ if and only if $[G/G] - 2e_U \in \Omega(G)^\times$.

**Corollary 6.13** Let $U \in \overline{C}(G)$, and suppose that $U$ is non-trivial cyclic. Then $U$ is a Sylow 2-subgroup of $G$, and $N_G(U) = U$.

**Proof.** Set $\bar{x} = ((-1)^{y_{U}})_{H \in C(G)} \in \Omega(G)^\times$. By Theorem 6.4 and Proposition 6.11, the map $\gamma^\times_{\{e\}} : G \to \langle -1 \rangle$ is a linear $C$-character of $G$. Since $U$ is non-trivial cyclic, it follows that $\gamma^\times_{\{e\}}$ is not the trivial character of $G$. If $K = \text{Ker } \gamma^\times_{\{e\}}$, then any cyclic subgroup $\langle g \rangle$ with $g \in G - K$ is a conjugate of $U$ in $G$ and

$$\frac{|G|}{2} = |K| = |G - K| = |G : N_G(U)| \cdot \frac{|U|}{2} = \frac{|G|}{2N_G(U) : U}.$$
because $U$ is a 2-group. Thus we have $|N_G(U) : U| = 1$. The corollary is now a consequence of [26, Chapter 2, Theorem 1.6]. This completes the proof. □

Let $\lambda = (\lambda_1, \ldots, \lambda_j, \ldots, \lambda_m, \lambda_{m+1}, \ldots)$, where $\lambda_1 > \cdots > \lambda_j > \cdots > \lambda_m > 0$ and $\lambda_\ell = 0$ for $\ell = m + 1$, $m + 2$, ..., be a partition of $n \in \mathbb{N}$. Such a partition is said to be strict. We set $S_\lambda = S_{(\lambda_1)} \times \cdots \times S_{(\lambda_j)} \times \cdots \times S_{(\lambda_m)}$, where each $S_{(\lambda_j)}$ is the symmetric group on $\{\sum_{i \geq j+1} \lambda_i + 1, \ldots, \sum_{i \geq j} \lambda_i\}$. Let $S_n$ be the symmetric group on $[n]$. Then $S_\lambda$ is a Young subgroup of $S_n$ associated with the strict partition $\lambda$.

**Proposition 6.14** For any strict partition $\lambda$ of $n$, the set $\overline{C}(S_n)$ contains a conjugate of the Young subgroup $S_\lambda$ of $S_n$ associated with $\lambda$.

**Proof.** We may assume that $S_\lambda \in C(S_n)$. Obviously, $N_{S_n}(S_\lambda) = S_\lambda$. We show that $S_\lambda \in \overline{C}(S_n)$. Under the preceding notation, let $A_{(\lambda_j)}$ with $j \in [m]$ be the subgroup of $S_{(\lambda_j)}$ consisting of all even permutations. Then the commutator subgroup of $S_\lambda$ is $A_{(\lambda_1)} \times \cdots \times A_{(\lambda_j)} \times \cdots \times A_{(\lambda_m)}$. Hence every normal subgroup $L$ of $S_\lambda$ such that $S_\lambda/L$ is a non-trivial cyclic is a subgroup of index 2 in $S_\lambda$. If $N_{S_n}(L) = S_\lambda$ for a subgroup $L$ of index 2 in $S_\lambda$, then $(g)L = S_\lambda$ for any $g \in N_{S_n}(L) - L$. Thus it suffices to verify that, if $N_{S_n}(L) \neq S_\lambda$ for a subgroup $L$ of index 2 in $S_\lambda$, then

$$\{(g)L \mid g \in N_{S_n}(L) - K\} = \{(g)L \mid g \in N_{S_n}(L) \text{ and } (g)L = (S_\lambda)\}$$

for a subgroup $K$ of index 2 in $N_{S_n}(L)$ containing $L$. Let $L \leq S_\lambda$ with $|S_\lambda : L| = 2$ and $N_{S_n}(L) \neq S_\lambda$. Then $\lambda_{m-1} = 2$, $\lambda_m = 1$, and every permutation in $L$ fixes both 2 and 3 in $[n]$. (In this case, $S_{(\lambda_{m-1})}$ is the symmetric group on $\{2, 3\}$). Hence it turns out that $L = S_{(\lambda_1)} \times \cdots \times S_{(\lambda_j)} \times \cdots \times S_{(\lambda_{m-2})}$, $S_\lambda = L \times S_{(\lambda_{m-1})} \times S_{(\lambda_m)}$, and $N_{S_n}(L) = L \times S_3$. Consequently, $L \leq L \times A_3 \leq N_{S_n}(L)$, $|N_{S_n}(L) : L \times A_3| = 2$, $(g)L \neq (S_\lambda)$ for any $g \in L \times A_3$, where $A_3$ is the alternating group on $[3]$, and the set of conjugates of $S_\lambda$ in $S_n$ includes the set $\{(g)L \mid g \in N_{S_n}(L) - (L \times A_3)\}$, as required. We now conclude that $S_\lambda \in \overline{C}(S_n)$, completing the proof. □

**Definition 6.15** For each $L \leq G$, we define a subset $S(G; L)$ of $S(G)$ to be the set consisting of all subgroups $U$ of $N_G(L)$ which satisfy the following conditions.

(i) $U/L$ is a non-trivial cyclic 2-group.

(ii) There exists a subgroup $K$ of index 2 in $N_G(L)$ containing $L$ such that

$$\{(g)L \mid g \in N_G(L) - K\} = \{(g)L \mid g \in N_G(L) \text{ and } (g)L = (U)\}.$$ 

Let $\approx$ be the equivalence relation on the set $\{(H) \mid G \geq H \neq \{\epsilon\}\}$ generated by

$$(g)L \approx (L)$$

for $L \in C(G)$ and $g \in N_G(L)$ such that $(g)L \notin S(G; L)$. We set $C(G)^0 = C(G) - \{\epsilon\}$, and define an equivalence relation $\approx$ on $C(G)^0$ by

$$H \approx K : \iff (H) \approx (K).$$
Proposition 6.16 If $|G| > 2$, then each $U \in \overline{C}(G)$ forms an equivalence class consisting of a single element with respect to the equivalence relation $\equiv$ on $C(G)^\circ$.

Proof. Suppose that $|G| > 2$, and let $U \in \overline{C}(G)$. Then $U \neq \{e\}$ and $|N_G(U) : U| \leq 2$. If $N_G(U) \neq U$, then $|N_G(U) : U| = 2$ and $N_G(U) \in S(G; U)$. Moreover, if $L$ is a normal subgroup of $U$ and if $U/L$ is a non-trivial cyclic group, then $U \in S(G; L)$. Thus $(U)$ is isolated with respect to $\equiv$. This completes the proof. □

Proposition 6.17 Suppose that $\tilde{y} = (yH)_{H \in C(G)} \in \text{Im} \phi^x$ and $\phi^{-1}(\tilde{y}) \in \Omega(G)_1^x$. Let $U \in C(G)^\circ$, and define $\tilde{x} = (xH)_{H \in C(G)} \in \overline{U}(G)^{\times}$ by

$$x_H = \begin{cases} y_H & \text{if } H \approx U, \\ 1 & \text{if } H \neq U \text{ or } H = \{e\}. \end{cases}$$

Then $\tilde{x} \in \text{Im} \phi^x$ and $\phi^{-1}(\tilde{x}) \in \Omega(G)_1^x$.

Proof. By the definition of $\tilde{x}$, the map $\gamma^x_L : G \to \langle -1 \rangle$ is the trivial character of $G$. Hence it suffices to verify that $\tilde{x} \in \text{Im} \phi^x$. Let $L \in C(G)^\circ$. We show that the map $\gamma^x_L : W_G(L) \to \langle -1 \rangle$ is a linear $C$-character of $W_G(L)$. By Theorem 6.4, the map $\gamma^y_L : W_G(L) \to \langle -1 \rangle$ is a linear $C$-character of $W_G(L)$. We may assume that $\gamma^x_L \notin \{ \gamma^y_L, 1_{W_G(L)} \}$. (If $\langle g \rangle L \notin S(G; L)$ for all $g \in N_G(L) - L$, then either $\gamma^x_L = \gamma^y_L$ or $\gamma^x_L = 1_{W_G(L)}$.) Obviously, $\gamma^x_L(L) = 1$. We analyze the values $\gamma^x_L(\langle g \rangle L)$ for $g \in N_G(L) - L$ in each of the cases where $L \approx U$ and $L \neq U$. Let $r$ be any element of $N_G(L) - L$ such that $\langle r \rangle L \in S(G; L)$. Then there exist a subgroup $K$ of index 2 in $N_G(L)$ containing $L$ such that for each $g \in N_G(L)$, $g \in N_G(L) - K$ if and only if $\langle g \rangle L$ is a conjugate of $\langle r \rangle L$ in $G$. We define a map $\beta_r : W_G(L) \to \langle -1 \rangle$ to be the linear $C$-character of $W_G(L)$ whose kernel is $K/L$.

Case 1. Assume that $L \approx U$. Let $X = \{ \langle r_i \rangle L \mid i \in [\ell] \}$ be a full set of non-conjugate subgroups of $G$ chosen from among the subgroups $\langle g \rangle L$ for $g \in N_G(L) - L$ with $\gamma^x_L(gL) \neq \gamma^y_L(gL)$. Then we have $\langle r_i \rangle L \neq (L)$ and $\langle r_i \rangle L \in S(G; L)$ for all $i \in [\ell]$. For any $g \in N_G(L) - L$,

$$\gamma^x_L(gL) = -\gamma^y_L(gL) = \gamma^y_L(gL) \prod_{i=1}^{\ell} \beta_{r_i}(gL)$$

if $\langle g \rangle L$ is a conjugate of some $\langle r_j \rangle L$ with $j \in [\ell]$ in $G$, and

$$\gamma^x_L(gL) = \gamma^y_L(gL) = \gamma^y_L(gL) \prod_{i=1}^{\ell} \beta_{r_i}(gL)$$

otherwise. Thus we have

$$\gamma^x_L = \gamma^y_L \prod_{i=1}^{\ell} \beta_{r_i}.$$
Case 2. Assume that \( L \not\approx U \). Then \( x_L \equiv 1 \). Let \( Y = \{ \langle r_i \rangle L \mid i \in [\ell] \} \) be a full set of non-conjugate subgroups of \( G \) chosen from among the subgroups \( \langle g \rangle L \) for \( g \in N_G(L) - L \) with \( \gamma \langle r_i \rangle L = 1 \). Then \( \langle r_i \rangle L \approx (U) \), whence \( \langle r_i \rangle L \not\approx (L) \) and \( \langle r_i \rangle L \in S(G; L) \) for all \( i \in [\ell] \). By an argument analogous to that in Case 1, we have \[
\gamma \langle r_i \rangle L = \prod_{i=1}^{\ell} \beta r_i.
\]

We now conclude that the map \( \gamma \langle r_i \rangle L : W_G(L) \to (-1) \) is a linear \( \mathbb{C} \)-character of \( W_G(L) \) in either case. Consequently, \( \gamma \langle r_i \rangle L \in \text{Hom}(W_G(L), (-1)) \) for any \( L \leq G \). This, combined with Theorem 6.4, shows that \( \beta \in \text{Im} \phi \), completing the proof. \( \square \)

**Corollary 6.18** Let \( C(G)^\circ / \cong \) be a complete set of representatives of equivalence classes with respect to the equivalence relation \( \approx \) on \( C(G)^\circ \). Set
\[
\Omega(G)_U^\chi = \{ \phi^{-1}(\bar{x}) \mid \bar{x} = (x_H)_{H \in C(G)} \in \text{Im} \phi^\chi \text{ and } x_H = 1 \text{ if } H \not\approx U \text{ or } H = \{\epsilon\} \}
\]
for each \( U \in C(G)^\circ / \cong \). Then
\[
\Omega(G)_U^\chi = \prod_{U \in C(G)^\circ / \cong} (\Omega(G)_U^\chi \cap \Omega(G)_U^\chi).
\]
Moreover, if \( U \in \overline{C}(G) \cap C(G)^\circ \), then \( U \in C(G)^\circ / \cong \) and \( \Omega(G)_U^\chi = \langle [G/G] - 2\epsilon_U \rangle \).

**Proof.** The assertion follows from Propositions 6.11, 6.16, and 6.17. \( \square \)

### 7 Units of monomial Burnside rings

#### 7A The unit groups of monomial Burnside rings

We continue assuming that \( A \) is abelian. Given a commutative unital ring \( R \), we denote by \( R^\times \) the unit group of \( R \), and denote by \( R^{\text{tor}} \) the group of torsion units of \( R \). For each \( H \leq G \), since \( H^1(H, A) \) is a finite abelian group, it follows from [21, Theorem 8.3.1] that \( (ZH^1(H, A))^\times \) is a finitely generated abelian group.

**Lemma 7.1** The group \( \mathcal{U}(G, A)^\times \) is a finitely generated abelian group.

**Proof.** Observe that \( \mathcal{U}(G, A) \simeq \prod_{H \in C(G)} (ZH^1(H, A))^{\text{Nc}(H)} \), where
\[
(ZH^1(H, A))^{\text{Nc}(H)} = \{ x_H \in ZH^1(H, A) \mid \text{con}_H^g(x_H) = x_H \text{ for all } g \in N_G(H) \}.
\]
Then we have \( \mathcal{U}(G, A)^\times \simeq \prod_{H \in C(G)} J_H \), where
\[
J_H = (ZH^1(H, A))^\times \cap (ZH^1(H, A))^{\text{Nc}(H)}.
\]
Hence it suffices to verify that the groups $J_H$ for $H \leq G$ are finitely generated. Let $H \leq G$, and assume that $(\mathbb{Z}H^1(H, A))^\times$ is generated by $x_1, \ldots, x_k$. We set $y_i = \prod_{g \in N_G(H)} \text{con}_H^g(x_i)$ for all $i$, and set $\widehat{J}_H = \langle y_1, \ldots, y_k \rangle$. Obviously, $\widehat{J}_H$ is a subgroup of $J_H$. We have

$$x^{[N_G(H)]} = \prod_{g \in N_G(H)} \text{con}_H^g(x) \in \widehat{J}_H$$

for any $x \in J_H$, so that $J_H/\widehat{J}_H$ is a torsion subgroup of $(\mathbb{Z}H^1(H, A))^\times/\widehat{J}_H$. Since $(\mathbb{Z}H^1(H, A))^\times/\widehat{J}_H$ is finitely generated, it follows from the fundamental theorem of abelian groups (see, e.g., [16, I, §10, Theorem 8]) that $J_H/\widehat{J}_H$ is a finite group. Thus $J_H$ is finitely generated, as desired. This completes the proof. □

**Proposition 7.2** The group $\Omega(G, A)^\times$ is a finitely generated abelian group. In particular, $\Omega(G, A)^\times$ is the direct product of $\Omega(G, A)^\omega$ and a free abelian group of finite rank, and $\Omega(G, A)^{\omega \times}$ is a finite abelian group.

**Proof.** By the fundamental theorem of abelian groups, it suffices to prove the first statement. Using Proposition 5.2 and Corollary 5.5, we have

$$\mathbb{Q} \otimes \mathbb{Z} \text{Im}\rho = \mathbb{Q} \otimes \mathbb{Z} \mathcal{U}(G, A).$$

This, combined with [21, Lemma 2.9.5], shows that $|\mathcal{U}(G, A)^\times : (\text{Im}\rho)^\times|$ is finite. Moreover, by Lemma 7.1, $\mathcal{U}(G, A)^\times$ is finitely generated. Hence it follows from [17, Corollary 2.7.1] that $(\text{Im}\rho)^\times$ is finitely generated. By Corollary 4.10, we have $\Omega(G, A)^{\times} \simeq (\text{Im}\rho)^{\times}$, completing the proof. □

### 7B Torsion units of monomial Burnside rings

From Higman’s theorem (cf. [21, Theorem 7.1.4]), we know that for any $H \leq G$,

$$(\mathbb{Z}H^1(H, A))^\omega = \langle -1 \rangle \times H^1(H, A) = \{ \pm \sigma \mid \sigma \in Z^1(H, A) \}.$$  \hspace{1cm} (7.1)

**Theorem 7.3** The necessary and sufficient condition for an element $\hat{x} = (x_H)_{H \leq G}$ of $\mathcal{U}(G, A)^\omega$ to be contained in $\text{Im}\rho$ is that $\gamma_{U}^G(\hat{x}) \in \text{Hom}(W_G(U), \langle -1 \rangle)$ for all $U \leq G$ and $(x(x_H)x_H)_{H \leq G} \in \Upsilon(G, A)$ (see Definitions 5.12 and 6.3), where

$$\Upsilon(G, A) = \left\{ (\sigma_H)_{H \leq G} \in \mathcal{U}(G, A)^\omega \left| \begin{array}{c} \sigma_U \in Z^1(U, A) \text{ and } \sigma_U = \text{res}_U^{(g)_U}(\sigma_{(g)_U}) \text{ for all } U \leq G \text{ and } g \in N_G(U) \end{array} \right. \right\}.$$  \hspace{1cm} 

**Proof.** Let $\hat{x} = (x_H)_{H \leq G} \in \mathcal{U}(G, A)^\omega$. Suppose that for each $H \leq G$, $\sigma_H = \varepsilon(x_H)x_H$ with $\sigma_H \in Z^1(H, A)$ (see Eq.(7.1)). We first prove ‘sufficient’ part. By assumption, $\text{con}_U^g(\sigma_U) = \sigma_U$ and $\sigma_U = \text{res}_U^{(g)_U}(\sigma_{(g)_U})$ for all $U \leq G$ and $g \in N_G(U)$, so that

$$\psi \circ \kappa^{-1}(\hat{x}) = (z_{(U, \tau)} \text{ mod } |W_G(U, \tau)|)_{(U, \tau) \in \mathcal{R}(G, A)},$$
where
\[
\zeta(U, \tau) = \sum_{gU \in W_G(U)} \varepsilon(x_{(g)U}) \text{ if } \tau = \sigma_U,
\]
For any \( U \leq G \), since \( \gamma^{(x)}_{U} \in \text{Hom}(W_G(U), \langle -1 \rangle) \), we have
\[
\frac{1}{|W_G(U)|} \sum_{gU \in W_G(U)} \varepsilon(x_U)\varepsilon(x_{(g)U}) \in \{0, 1\}
\]
by [8, (9.21) Proposition]. Hence either \( \zeta(U, \tau) = \varepsilon(x_U)|W_G(U)| \) or \( \zeta(U, \tau) = 0 \) for all \( (U, \tau) \in \mathcal{R}(G, A) \), which yields \( \psi \circ \kappa^{-1}(\tilde{x}) = 0 \in \text{Obs}(G, A) \). This, combined with Proposition 5.2 and Theorem 5.9, shows that \( \tilde{x} \in \text{Im} \rho \), as desired. We next prove ‘necessary’ part. Assume that \( \rho(x) = \tilde{x} \) with \( x \in \Omega(G, A)^\omega \). Then \( \alpha(x) \in \Omega(G)^\times \), because the map \( \alpha : \Omega(G, A) \to \Omega(G) \) is a ring homomorphism. By Lemma 5.13(a) and Theorem 6.4, \( \gamma^{(x)}_U \in \text{Hom}(W_G(U), \langle -1 \rangle) \) for all \( U \leq G \), and
\[
\rho(x \circ \alpha(x) \cdot x) = (\sigma_H)_{H \leq G} \in \Omega(G, A)^\omega.
\]
In particular, we have \( \text{con}^G_{(x)}(\sigma_U) = \sigma_U \) for all \( U \leq G \) and \( g \in N_G(U) \). For each \( (U, \tau) \in \mathcal{R}(G, A) \) with \( \tau = \sigma_U \), the \( (U, \tau) \)-component of \( \psi \circ \kappa^{-1}(\sigma_H)_{H \leq G} \) is
\[
\sum_{gU \in W_G(U), \sigma_U - \text{res}_{(g)U}^{(g)}(\sigma_{(g)U})} 1 \mod |W_G(U)|,
\]
where the sum is taken over all left cosets \( gU, g \in N_G(U) \), of \( U \) in \( N_G(U) \) such that \( \sigma_U = \text{res}_{(g)U}^{(g)}(\sigma_{(g)U}) \). Since \( (\sigma_H)_{H \leq G} \in \text{Im} \rho \), it follows from Proposition 5.2 and Theorem 5.9 that \( \sigma_U = \text{res}_{(g)U}^{(g)}(\sigma_{(g)U}) \) for all \( U \leq G \) and \( g \in N_G(U) \), as desired. This completes the proof. \( \square \)

In $\S$4A, the ring epimorphism \( \rho^G_G : \Omega(G, A) \to \mathbb{Z}H^1(G, A) \) is given by
\[
[(G/U)_{\tau}] \mapsto \begin{cases} 
\tau & \text{if } G = U, \\
0 & \text{otherwise}
\end{cases}
\]
for all \( (U, \tau) \in \mathcal{R}(G, A) \) (see Lemma 4.6(a)). Following [2, $\S$7], we define a ring monomorphism \( \nu : \mathbb{Z}H^1(G, A) \to \Omega(G, A) \) by
\[
\chi \mapsto [(G/G)_{\chi}]
\]
for all \( \chi \in \mathbb{Z}^1(G, A) \) (see Lemmas 2.6 and 2.15). There are group homomorphisms
\[
\nu^\omega : (\mathbb{Z}H^1(G, A))^\omega \to \Omega(G, A)^\omega \quad \text{and} \quad \theta^\omega : \Omega(G, A)^\omega \to (\mathbb{Z}H^1(G, A))^\omega
\]
inherited from \( \nu \) and \( \rho^G_G \), respectively (see Eq.(7.1)). Hence it turns out that
\[
\Omega(G, A)^\omega = \text{Im} \nu^\omega \times \text{Ker} \theta^\omega \simeq (-1) \times H^1(G, A) \times \text{Ker} \theta^\omega
\]
(cf. [2, $\S$8]), because \( \theta^\omega \circ \nu^\omega = \text{id}_{(\mathbb{Z}H^1(G, A))^\omega} \). We continue to describe \( \Omega(G, A)^\omega \).
Corollary 7.4 Identify the finite groups $\Omega(G)^\times$ and $H^1(G, A)$ with the subgroups 
$\{\iota(u) \mid u \in \Omega(G)^\times\}$ and $\{([G/G]_\chi) \mid \chi \in \mathbb{Z}^1(G, A)\}$ of $\Omega(G, A)^\omega$, respectively. Set 
\[
\nabla(G, A) = \left\{ \frac{1}{|G|} \sum_{H \leq G} \sum_{U \leq H} |U| \mu(U, H) ([G/U]_{\sigma_H}) \right\}_{\sigma_H \leq G \in \Upsilon(G, A) \text{ with } \sigma_G = 1_G}.
\]

Then 
\[
\Omega(G, A)^\omega = \Omega(G)^\times \times H^1(G, A) \times \nabla(G, A).
\]

Proof. Let $x \in \Omega(G, A)^\omega$, and suppose that $\rho(x) = (x_H)_{H \leq G}$. By Theorem 7.3, 
$\rho(\iota \circ \alpha(x) \cdot x) = (\varepsilon(x_H)x_H)_{H \leq G} \in \Upsilon(G, A)$. Since the map $\alpha : \Omega(G, A) \to \Omega(G)$ is a ring epimorphism, it follows from Proposition 4.9 and Theorem 7.3 that 
\[
\Omega(G, A)^\omega = \Omega(G)^\times \times \left\{ \frac{1}{|G|} \eta(\langle [G/H]_{H \leq G} \rangle) \right\}_{\langle [G/H]_{H \leq G} \rangle \in \Upsilon(G, A)}.
\]

Moreover, $\rho([([G/G]_\chi) \chi \in \mathbb{Z}^1(G, A) \times \langle [G/H]_{H \leq G} \rangle \in \Upsilon(G, A) \mid \sigma_G = 1_G\rangle$. 

The assertion now follows from Proposition 4.9. This completes the proof. \[\square\]

Remark 7.5 Suppose that $G$ is of odd order and that $G$ acts trivially on $A$. Then by Remark 6.2 and Corollary 7.4, we have 
\[
\Omega(G, A)^\omega = (-[([G/G]_1)] \times \Omega(G, A)^{\text{odd}},
\]
where $\Omega(G, A)^{\text{odd}}$ is the Hall $2'$-subgroup of $\Omega(G, A)^\omega$ (cf. [2, Proposition 8.2]).

Example 7.6 Suppose that $G$ is nilpotent. Then by [26, Chapter 4, Theorem 2.9], 
$\nabla(G, A) = ([([G/G]_1)]$ in Corollary 7.4, and hence 
\[
\Omega(G, A)^\omega \simeq \Omega(G)^\times \times H^1(G, A).
\]

REFERENCES


