LETTER

A Note on Irreversible 2-Conversion Sets in Subcubic Graphs

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SUMMARY Irreversible k-conversion set is introduced in connection with the mathematical modeling of the spread of diseases or opinions. We show that the problem to find a minimum irreversible 2-conversion set can be solved in $O(n^2 \log^6 n)$ time for graphs with maximum degree at most 3 (subcubic graphs) by reducing it to the graphic matroid parity problem, where n is the number of vertices in a graph. This affirmatively settles an open question posed by Kynčl et al. (2014).

key words: *feedback vertex set, graphic matroid parity problem, irreversible threshold process, subcubic graphs*

1. Introduction

As a mathematical model of the spread of diseases or opinions, an irreversible k-threshold process is introduced by Dreyer and Roberts [4]. Let $G = (V, E)$ be an undirected simple graph with vertices colored either black or white. An *irreversible k-threshold process* is a discrete-time process on G where a white vertex becomes black at time t if it is adjacent to at least k black vertices at time $t-1$. An *irreversible k-conversion set* of G (*IkCS* for short) is a vertex set $S \subseteq V$ such that if the vertices in S are set to be black and the other vertices are set to be white at time 0, the irreversible k-threshold process will change all white vertices to be black after a finite number of steps. The IkCS problem is to find a minimum IkCS in a given graph.

Dreyer and Roberts [4] show that a vertex set $S \subseteq V$ of a k-regular graph $G = (V, E)$ is an IkCS if and only if $V \setminus S$ is an independent set, and as a result, the IkCS problem is NP-hard even for k-regular graphs if $k \geq 3$. Since the IkCS problem is trivial if $k = 1$, they asked the complexity of the I2CS problem.

Later, Kynčl et al. show in [9] that the I2CS problem is NP-hard even for graphs with maximum degree 4 (The NP-hardness of the I2CS problem is also proved in [3]). They also mention that the I2CS problem is trivial for graphs with maximum degree at most 2, since the size of I2CS of a path and cycle is known [4]. A vertex set $S \subseteq V$ of a graph $G = (V, E)$ is called a *feedback vertex set* of G if $G - S$ contains no cycle, where $G - S$ denotes the graph obtained from G by deleting all vertices in S. Kynčl et al. mention in [9] that for 3-regular graphs, the I2CS problem is equivalent to the problem of finding a minimum feedback vertex set, which can be solved in polynomial time [12], [13].

Kynčl et al. recently showed in [10] that the I2CS problem for graphs with maximum degree 3 (*subcubic graphs*) can be solved in $O(n^{1+\omega})$ time by reducing it to the linear matroid parity problem, which can be solved in polynomial time [7], [11], where n is the number of vertices of a graph and $O(n^\omega)$ is the time to multiply two $n \times n$ matrices (The best known upper bound of the exponent is $\omega < 2.3728639$ [8]). They asked whether the problem can be efficiently reduced to the cographic matroid parity problem. This paper settles the question affirmatively by showing the following.

Theorem 1. *The I2CS problem can be solved in $O(n^2 \log^6 n)$ time for subcubic graphs.* □

2. Proof of Theorem 1

We show that the I2CS problem for subcubic graphs can be reduced to the graphic matroid parity problem in linear time. Since the graphic matroid parity problem can be solved in $O(nm \log^6 n)$ time [6], [7], we have the theorem, where m is the number of edges of a graph.

It remains to show the reduction of the I2CS problem for subcubic graphs to the graphic matroid parity problem. We employ a reduction similar to that used in [2], [5], [12]. Let $G = (V, E)$ be a subcubic graph, and let V_1, V_2, and V_3 be the set of vertices with degree 1, 2, and 3, respectively.

Lemma 2. *A vertex set $S \subseteq V$ is an I2CS if and only if*

(1) $V_1 \subseteq S$,
(2) *$G - S$ contains no cycle, and*
(3) *each connected component of $G - S$ has at most one vertex in $V_2 \setminus S$.*

Proof. Let $S \subseteq V$ be an I2CS of G. We show that S satisfies conditions (1)-(3). Since a white vertex needs two black adjacent vertices to become black, every vertex of degree 1 must be in S.

Suppose a cycle of G has no vertex in S. Every vertex on the cycle has at most one adjacent vertex outside the cycle. Since every vertex on the cycle is white at the beginning of the process, they remain white forever, contradicting the assumption that S is an I2CS. Hence, any cycle of G has at least one vertex in S, that is, $G - S$ contains no cycle.

Suppose a component of $G - S$ has at least two vertices in V_2. We have a path in the component connecting two of

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them. Every vertex on the path has at most one adjacent vertex outside the path, since two end-vertices of the path has degree 2 in G. Since every vertex on the path is white at the beginning of the process, they remain white forever, contradicting the assumption that S is an I2CS. Hence, each component of $G - S$ has at most one vertex in V_2.

Conversely, let $S \subseteq V$ be a vertex set of G satisfying conditions (1)-(3). We show that S is an I2CS. Each connected component of $G-S$ is a tree consisting of vertices in V_3 and at most one vertex in V_2. Hence, at any time of the process, each component of the subgraph of G induced by the white vertices is a tree consisting of vertices in V_3 and at most one vertex in V_2. A leaf of the components becomes black if it is in V_3, since it has two black adjacent vertices. Since each component has at most one vertex in V_2, all white vertices in V_3 will be changed to be black. Finally, the white vertices in V_2 become black, since all its adjacent vertices are black. Hence, S is an I2CS. □

Let $G^* = (V^*, E^*)$ be a multigraph obtained from G by

- adding a vertex v^*,
- adding an edge joining v^* and v for each $v \in V_2$, and
- adding two edges joining v^* and v for each $v \in V_1$.

Notice that every vertex in $V^* \setminus \{v^*\}$ has degree 3. The following lemma indicates that the I2CS problem can be reduced to a variant of the feedback vertex set problem.

Lemma 3. *A vertex set $S \subseteq V^* \setminus \{v^*\}$ is an I2CS of G if and only if S is a feedback vertex set of G^*.*

Proof. Let $S \subseteq V^* \setminus \{v^*\}$ be an I2CS of G. We show that S is a feedback vertex set of G^*. Notice that $G^* - S$ can be obtained from $G - S$ by adding v^* and adding an edge joining v^* and each vertex of $V_2 \setminus S$. We have by Lemma 2 that each connected component of $G - S$ is a tree that has at most one vertex in V_2. Hence, $G^* - S$ contains no cycle, and S is a feedback vertex set of G^*.

Conversely, let $S \subseteq V^* \setminus \{v^*\}$ be a feedback vertex set of G^*. We show by Lemma 2 that S is an I2CS of G. For each $v \in V_1$, two edges joining v and v^* form a cycle. Since $v^* \notin S$, we conclude that $v \in S$, and hence, $V_1 \subseteq S$.

Since $G^* - S$ contains no cycle by definition, and $G - S$ is a subgraph of $G^* - S$, we conclude that $G - S$ contains no cycle.

Recall that v^* and each vertex in $V_2 \setminus S$ are joined by an edge in $G^* - S$. Each connected component of $G - S$ contains at most one vertex in $V_2 \setminus S$, for otherwise v^* and the vertices on the path connecting two vertices in $V_2 \setminus S$ induce a cycle, contradicting the assumption that S is a feedback vertex set of G^*. Hence, we have from Lemma 2 that S is an I2CS. □

The graphic matroid parity problem can be stated as follows by using terminology only from graph theory. Let $G = (V, E)$ be a graph whose edges are partitioned into pairs, that is, every edge $e \in E$ has a unique *mate* $\bar{e} \in E$. A *parity set* $F \subseteq E$ is an edge set such that for any edge $e \in E$, $e \in F$ if and only if $\bar{e} \in F$. The *graphic matroid parity problem* is to find a largest parity set containing no cycle.

It should be noted that $V^* = V \cup \{v^*\}$ and $E^* = E \cup \Gamma_{G^*}(v^*)$, where $\Gamma_{G^*}(v^*)$ is the set of edges of G^* incident to v^*. Let $G' = (V', E')$ be the graph obtained from G^* by replacing each edge in $E^* \setminus \Gamma_{G^*}(v^*)$ by a path of length 2. Notice that $\{\Gamma_{G'}(v)\}_{v \in V}$ is a partition of E'. Let $G'' = (V'', E'')$ be the graph obtained from G' by applying the following procedure for each $v \in V$:

- Let $\Gamma_{G'}(v) = \{e_0, e_1, e_2\}$;
- Replace each edge in $\Gamma_{G'}(v)$ by a path of length 2, and let $e_{i,1}$ and $e_{i,2}$ be the two edges on the path corresponding to e_i, $0 \le i \le 2$;
- Let $e_{i,1}$ be the mate of $e_{i+1,2}$ for any i (subscripts are modulo 3).

We say that $v \in V$ is *associated* with each of these three pairs, and vice versa.

Lemma 4. *Let F be a parity set of G'', and let S be the set of vertices associated with pairs in $E'' \setminus F$. Then, F contains no cycle in G'' if and only if S is a feedback vertex set of G^*.*

Proof. If F contains a cycle, all edges associated with the vertices on the cycle are in F. Hence, no vertex on the cycle is in S, and S is not a feedback vertex set of G^*. Conversely, if $G - S$ contains a cycle, all edges associated with the vertices on the cycle are in F, and F contains the cycle. □

Lemma 5. *Let $S \subseteq V$ be a vertex set of G^*, and let F be a parity set of G'' obtained from E'' by removing one of three pairs associated with each vertex in S. Then, S is a feedback vertex set of G^* if and only if F contains no cycle in G''.*

Proof. The proof is similar to that of Lemma 4, and is omitted. □

Now, we have the following.

Lemma 6. *Let f be the number of edges in a largest parity set of G'' containing no cycle, and let s be the number of vertices in a smallest feedback vertex set of G^* that is also a subset of V. Then, $f + 2s = |E''|$.*

Proof. Let F be a largest parity set of G'' containing no cycle, that is, $|F| = f$. Let S be the set of vertices associated with pairs in $E'' \setminus F$. We have by Lemma 4 that S is a feedback vertex set of G^*. For each $v \in V$, $E'' \setminus F$ contains at most one pair associated with v, for otherwise F is not largest since adding one of pairs in $E'' \setminus F$ associated with v makes no cycle in F. Hence, for each vertex in S, $E'' \setminus F$ has a unique pair associated with the vertex, and we have $f + 2s \le |F| + 2|S| = |E''|$.

Let $S \subseteq V$ be a smallest feedback vertex set of G^* that is also a subset of V, that is, $|S| = s$. Let F be a parity set of G'' obtained from E'' by removing one of three pairs associated with each vertex in S. We have by Lemma 5 that F contains no cycle. By the construction of F, we have $f + 2s \ge |F| + 2|S| = |E''|$.

Now, we have $f + 2s = |E''|$. □

By Lemmas 4 and 6, we can see that if F is a largest parity set containing no cycle, then the set of vertices associated with pairs in $E'' \setminus F$ is a minimum feedback vertex set of G^* that is also a subset of V, and hence, it is a minimum I2CS of G by Lemma 3. Since it is obvious that G'' can be obtained in linear time from the original graph G, we have the theorem.

3. Concluding Remarks

It should be noted that the theorem also holds for multigraphs. Other results on IkCS can be found in [1], [3], [4], [9].

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