PAPER

On Two Problems of Nano-PLA Design

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SUMMARY The logic mapping problem and the problem of finding a largest sub-crossbar with no defects in a nano-crossbar with nonprogrammable-crosspoint defects and disconnected-wire defects are known to be NP-hard. This paper shows that for nano-crossbars with only disconnected-wire defects, the former remains NP-hard, while the latter can be solved in polynomial time.

key words: *biclique problem, nano-crossbar, nano-PLA, orthogonal ray graphs, subraph isomorphism problem*

1. Introduction

Implementing a sum-of-product logic function in a conventional programmable logic array (PLA) is a straightforward task of arbitrarily assigning the literals and product terms to the wires of the crossbar and programming the appropriate crosspoints. However, in the case of nano-PLAs, this task is not trivial because of imperfections in the nano-wire crossbar. Defects in nano-wire crossbar have been broadly classified into two types: *nonprogrammable-crosspoint defects*, in which some crosspoints become unprogrammable, and *disconnected-wire defects*, in which each horizontal nanowire may not be connected to all vertical nano-wires [5]. The problem of mapping a sum-of-product logic function onto a defective nano-crossbar with nonprogrammable-crosspoint defects and disconnected-wire defects was first considered by Rao, Orailoglu, and Karri [5]. They proposed several heuristics since the problem is NP-hard. The problem of finding a maximum defect-free sub-crossbar in a nano-crossbar with nonprogrammable-crosspoint defects and disconnected-wire defects was first investigated by Tahoori [8]. Since the problem is also NP-hard, several heuristics have been proposed [1], [8].

This paper considers the complexity of the problems for nano-crossbars with only disconnected-wire defects.

1.1 LOGIC MAPPING

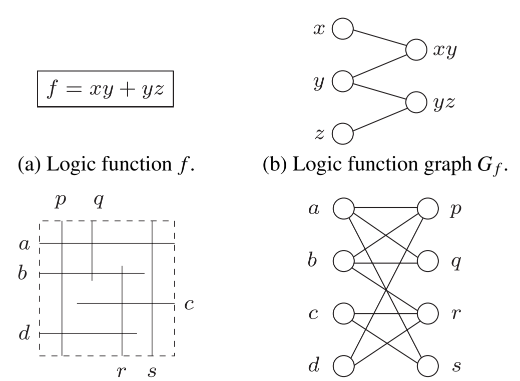
Let f be a logic function in a sum-of-product form. Let S be a nano-crossbar with disconnected-wire defects. The problem of implementing f in S is formulated as LOGIC MAPPING, which is the problem of assigning the literals and product terms of f to nano-wires of S so that containment relationships among the literals and product terms can be represented by crosspoint connections in S. A graph model of LOGIC MAPPING can be obtained as follows.

Let L_f be the set of literals of f, and P_f be the set of product terms of f. A logic function graph G_f for f is a bipartite graph defined as follows: $V(G_f) = L_f \cup P_f$, and (L_f, P_f) is a bipartition of G_f; vertices $l \in L_f$ and $p \in P_f$ are connected by an edge if and only if literal l is contained in product term p.

Let W_h be the set of horizontal nano-wires, and W_v be the set of vertical nano-wires of S. A crossbar graph G_S of S is a bipartite graph defined as follows: $V(G_S) = W_h \cup W_v$ and (W_h, W_v) is a bipartition of G_S; vertices $x \in W_h$ and $y \in W_v$ are connected by an edge if and only if nano-wires x and y have a crosspoint. Then, LOGIC MAPPING can be modeled as the subgraph isomorphism problem, which is to find a subgraph of G_S isomorphic to G_f. Examples of a logic function f, a defective crossbar S, and their corresponding bipartite graphs G_f and G_S are shown in Fig. 1.

1.2 SUB-CROSSBAR

SUB-CROSSBAR is the problem of finding a defect-free sub-crossbar consisting of given numbers of horizontal and vertical wires within the nano-crossbar with disconnected wire defects. SUB-CROSSBAR can be modeled as the $K_{m,n}$-biclique problem, which is to find a complete bipartite subgraph $K_{m,n}$ contained in a crossbar graph G_S.



(a) Logic function f. (b) Logic function graph G_f.

(c) Surviving crossbar S. (d) Surviving crossbar graph G_S.

Fig. 1 An instance of LOGIC MAPPING and the corresponding graphs.

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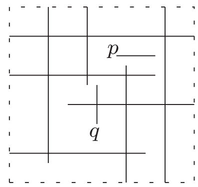

Fig. 2 Nano-wires such as p and q are unusable.

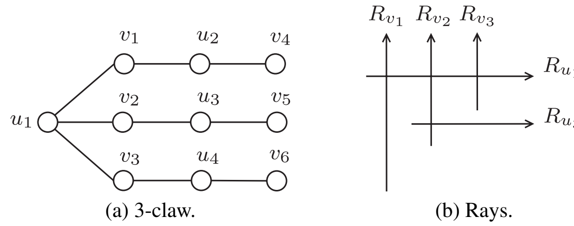

Fig. 3 (a) The 3-claw. (b) Rays defined in the proof of Lemma 1.

1.3 Our Results

Although it is well known that both the subgraph isomorphism problem and the $K_{m,n}$-biclique problem are NP-hard for bipartite graphs [2], [3], the complexity of LOGIC MAPPING and SUB-CROSSBAR is not immediately clear since the graphs representing surviving sub-crossbars are a special kind of bipartite graph.

A bipartite graph G with a bipartition (U, V) is called an *orthogonal ray graph* if there exist a set of non-intersecting rays (half-lines) $R_u, u \in U$, parallel to the x-axis in the xy-plane, and a set of non-intersecting rays $R_v, v \in V$, parallel to the y-axis such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if R_u and R_v intersect. An orthogonal ray graph G with a bipartition (U, V) is called a *two-directional orthogonal ray graph* if R_u is a rightward ray $\{(x, b_u) \mid x \geq a_u\}$ for each $u \in U$, and R_v is an upward ray $\{(a_v, y) \mid y \geq b_v\}$ for each $v \in V$, where a_w and b_w are real numbers for any $w \in U \cup V$.

Nano-wires such as p and q of a defective nano-crossbar shown in Fig. 2 cannot be controlled as they do not reach the boundary of the originally intended nano-crossbar. Since we cannot use such nano-wires, a graph representing a surviving sub-crossbar must be an orthogonal ray graph.

We show in Sect. 3 that LOGIC MAPPING is NP-hard by showing that the subgraph isomorphism problem is NP-hard even for orthogonal ray graphs. We show in Sect. 4 that SUB-CROSSBAR can be solved in polynomial time provided that the vertices of the orthogonal ray graph representing a surviving sub-crossbar are ordered to reflect the top-to-bottom order of horizontal nano-wires and left-to-right order of vertical nano-wires. This is a quite natural condition. We also show in Sect. 4 that in the case of two-directional orthogonal ray graphs, the $K_{m,n}$-biclique problem can be solved in polynomial time without the requirement of such an ordering, thereby providing a purely graph-theoretic solution for an interesting subproblem of SUB-CROSSBAR.

2. Orthogonal Ray Graphs

In this section, we shall discuss some properties of two-directional orthogonal ray graphs that will come in handy in the later sections. Some of the lemmas and theorems in this section also appear in our earlier work [7]. In order to make the paper self-contained, we revisit them and also provide direct explicit proofs for some of them.

The *3-claw* is a tree obtained from a complete bipartite graph $K_{1,3}$ by replacing each edge with a path of length 3. (See Fig, 3 (a).)

Lemma 1. *The 3-claw is not a 2-directional orthogonal ray graph.*

Proof. Assume to the contrary that the 3-claw is a 2-directional orthogonal ray graph. Let the vertices of the 3-claw be named as in Fig. 3 (a). We shall refer to the endpoint of the ray corresponding to a vertex v by (a_v, b_v). Without loss of generality, suppose R_{u_1} is a horizontal ray and that $R_{v_1}, R_{v_2}, R_{v_3}$ intersect with R_{u_1} such that R_{v_2} lies to the right of R_{v_1} and to the left of R_{v_3}. (See Fig. 3 (b)). It is easy to observe that $b_{v_3} > b_{v_2} > b_{v_1}$, or else it is not possible to define R_{u_2}, R_{u_3}, and R_{u_4}. Since R_{u_3} has to be defined such that $a_{u_3} > a_{v_1}$ and $b_{u_3} < b_{u_1}$, it is not possible to define R_{v_5} such that it intersects with R_{u_3} but not with R_{u_1}, a contradiction. □

A path P in a tree T is called a *spine* of T if every vertex of T is within distance two from at least one vertex of P.

Theorem 1. *A tree T has a spine if and only if T contains no 3-claw as a subtree.*

Proof. The necessity is obvious. To prove the sufficiency, assume T contains no 3-claw. Let P be a longest path in T, and let $V(P) = \{v_1, v_2, \ldots, v_p\}$ and $(v_i, v_{i+1}) \in E(P)$, $1 \leq i \leq p-1$. We claim that P is a spine. We distinguish two cases: $|V(P)| \leq 6$ and $|V(P)| > 6$.

For the former, it is easy to see that P is a spine because if there is a vertex $v \notin V(P)$ which is at a distance more than two from any vertex in P, then the assumption that P is a longest path is contradicted.

We next take the case of $|V(P)| > 6$. Assume P is not a spine. Let F be a forest obtained from T by deleting the edges in $E(P)$. Let T_i be a tree in F containing v_i, $1 \leq i \leq p$. Since P is a longest path in T, T_1 consists of only one vertex, v_1, and T_p consists of only one vertex, v_p. Also all vertices in T_2 and T_{p-1} are within distance one from v_2 and v_{p-1}, respectively; and all vertices in T_3 and T_{p-2} are within distance two from v_3 and v_{p-2}, respectively. Since we assumed that P is not a spine, there exists an integer j $(4 \leq j \leq p-3)$ such that T_j contains a vertex w_j whose distance from v_j is three. Let P' be the path from v_j to w_j. Then the subgraph of T induced by the vertices in $\{v_i \mid j-3 \leq i \leq j+3\} \cup V(P')$ is a 3-claw. This contradicts the assumption that T contains no 3-claw as a subtree, and therefore P is a spine. □

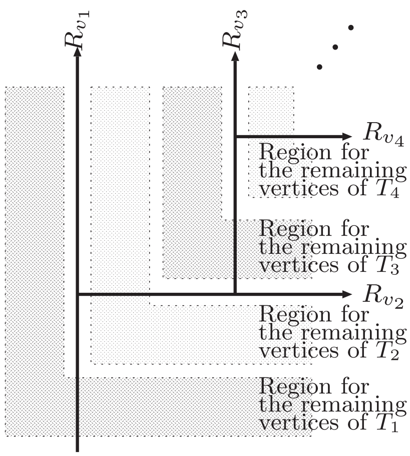

Fig. 4 Rays corresponding to the vertices of 3-claw-free tree T.

Theorem 2. *A tree T is a 2-directional orthogonal ray tree if and only if T contains no 3-claw as a subtree.*

Proof. The necessity follows from Lemma 1. We will show the sufficiency. Assume T contains no 3-claw as a subtree. Then from Theorem 1, T contains a spine P. Let $V(P) = \{v_1, v_2, \ldots, v_p\}$, and $(v_i, v_{i+1}) \in E(P)$, $1 \leq i \leq p-1$. Corresponding to each vertex v_i in P, define ray $R_{v_i} = \{(i, y) \mid y \geq i-1\}$ if i is odd, and define ray $R_{v_i} = \{(x, i) \mid x \geq i-1\}$ if i is even. Let F be a forest obtained from T by deleting the edges in $E(P)$. Let T_i be a tree in T containing v_i, $1 \leq i \leq p$. Consider T_i to be rooted at v_i. Let $w_{i1}, w_{i2}, \ldots, w_{iq(i)}$ be the children of v_i in T_i, where $q(i)$ is the number of children of v_i in T_i. Let $z_{ij1}, z_{ij2}, \ldots, z_{ijr(ij)}$ be the children of w_{ij} in T_i, where $r(ij)$ is the number of children of w_{ij} in T_i. The rays corresponding to w_{ij} and z_{ijk}, ($1 \leq i \leq p$, $1 \leq j \leq q(i)$, $1 \leq k \leq r(ij)$) can be placed in the region for T_i as shown in Fig. 4. Thus T is a 2-directional orthogonal ray graph. □

Lemma 2. *A cycle C_{2m} of length $2m$ is a two-directional orthogonal ray graph if and only if $m = 2$.*

Proof. It is easy to see that C_4 is a 2-directional orthogonal ray graph.

We show that C_{2m} is not a 2-directional orthogonal ray graph for any $m \geq 3$. Suppose to the contrary that C_{2m} is a 2-directional orthogonal ray graph for some $m \geq 3$. Let $V(C_{2m}) = \{0, 1, \ldots, 2m-1\}$ and $E(C_{2m}) = \{(i, i+1 \pmod{2m})) \mid 0 \leq i \leq 2m-1\}$. Suppose without loss of generality that $R_0 = \{(a_0, y) \mid y \geq b_0\}$, for some real numbers a_0 and b_0. Since $(0, 1) \in E(C_{2m})$, R_1 intersects with R_0 at some point. Similarly, R_2 intersects with R_1 at some other point. We distinguish two cases.

Case 1 When R_2 intersects with R_1 such that R_2 is to the left of R_0: Then R_3 must intersect with R_2 such that R_3 lies below the endpoint of R_0. Similarly R_4 must intersect with R_3 such that R_4 lies to the left of the endpoint of R_1. Continuing in this manner, R_i ($5 \leq i \leq 2m-1$) must lie below(to the left of) the endpoint of R_{i-3} for odd(even) i. Therefore R_{2m-1} lies in the region below the endpoint of R_4. However, R_0 is in the region right of R_2 and above R_3, making it impossible for R_0 to intersect with R_{2m-1} without intersecting with $R_3, R_5, \ldots, R_{2m-3}$, a contradiction.

Case 2 When R_2 intersects with R_1 such that R_2 is to the right of R_0: We further distinguish two cases.

Case 2-1 When R_3 intersects with R_2 such that R_3 is below R_1: Then R_4 must lie to the left of the endpoint of R_1. This confines R_0 within the region left of R_2 and above R_3, making it impossible for ray R_{2m-1} to intersect with R_0 without intersecting with R_2, a contradiction.

Case 2-2 When R_3 intersects with R_2 such that R_3 is above R_1: This case may be further broken down into two cases depending on whether R_4 is to the left of R_2 or right of R_2. In the former case, R_4 gets confined within the region left of R_2 and above R_1 making it impossible for R_5 to intersect with R_4 without intersecting with R_2, a contradiction. In the latter case, $R_5, \ldots, R_{2m-1}$ must lie in the region right of R_2 and above R_3, making it impossible for R_{2m-1} to intersect with R_0 without intersecting with $R_2, R_4, \ldots, R_{2m-2}$, a contradiction.

Thus we conclude that C_{2m} is not a 2-directional orthogonal ray graph for any $m \geq 3$. □

A bipartite graph is *chordal* if it contains no induced cycles of length greater than 4. A tree is chordal, by definition. Thus, by Lemma 2 and Theorem 2, we have:

Theorem 3. *A class of two-directional orthogonal ray graphs is a proper subset of the class of chordal bipartite graphs.* □

3. Intractability of LOGIC MAPPING

We show in this section the following.

Theorem 4. *LOGIC MAPPING is NP-hard.*

Theorem 4 follows from Theorem 5 below. A decision problem associated with the subgraph isomorphism problem is defined as follows:

SUBGRAPH ISOMORPHISM

INSTANCE: Graphs H and G.

QUESTION: Does G contain a subgraph isomorphic to H, that is, does there exist a one-to-one mapping $\phi : V(H) \to V(G)$ such that if $(u, v) \in E(H)$ then $(\phi(u), \phi(v)) \in E(G)$?

Theorem 5. *SUBGRAPH ISOMORPHISM is NP-complete even if G is a 2-directional orthogonal ray tree and H is a forest.*

Proof. It is easy to see that the problem is in NP. We show a polynomial-time reduction from 3-PARTITION, which has been shown to be strongly NP-complete in [2]. 3-PARTITION is defined as follows.

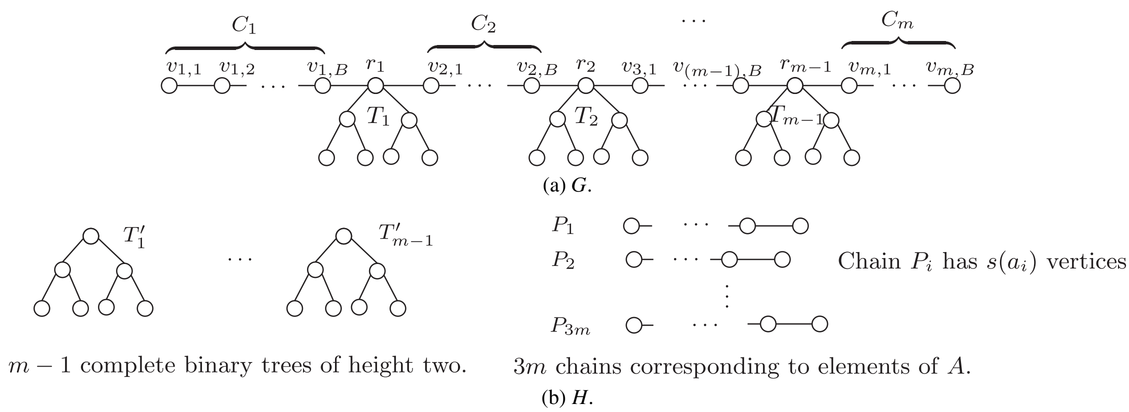

Fig. 5 Two-directional orthogonal ray tree G and forest H corresponding to the instance of 3-PARTITION.

3-PARTITION

INSTANCE: A finite set A of $3m$ elements, a bound $B \in \mathbb{Z}^+$, and a size $s(a) \in \mathbb{Z}^+$ for each $a \in A$, such that each $s(a)$ satisfies $B/4 < s(a) < B/2$ and such that $\sum_{a \in A} s(a) = mB$.

QUESTION: Does A have a 3-partition, that is, can A be partitioned into m disjoint sets $S_1, S_2, \ldots, S_m$ such that, for $1 \le i \le m$, $\sum_{a \in S_i} s(a) = B$?

Let $C_1, C_2, \ldots, C_m$ be B-vertex paths such that for each i $(1 \le i \le m)$, $V(C_i) = \{v_{i,j} \mid 1 \le j \le B\}$ and $E(C_i) = \{(v_{i,j}, v_{i,(j+1)}) \mid 1 \le j \le B-1\}$. Let $T_1, T_2, \ldots, T_{m-1}$ be complete binary trees of height two rooted at vertices $r_1, r_2, \ldots, r_{m-1}$, respectively. Let G be the graph defined as

$$V(G) = \left(\bigcup_{i=1}^{m} V(C_i)\right) \cup \left(\bigcup_{i=1}^{m-1} V(T_i)\right),$$

$$E(G) = \left(\bigcup_{i=1}^{m} E(C_i)\right) \cup \left(\bigcup_{i=1}^{m-1} E(T_i)\right) \cup \{(r_i, v_{i,B}), (r_i, v_{(i+1),1}) \mid 1 \le i \le m-1\}.$$

(See Fig. 5 (a).) Since the path in G from $v_{1,1}$ to $v_{m,B}$ is a spine of G, it follows from Theorems 1 and 2 that G is a two-directional orthogonal ray tree. Let H be a forest consisting of $m-1$ complete binary trees of height two $T'_1, T'_2, \ldots, T'_{m-1}$, and $3m$ paths $P_1, P_2, \ldots P_{3m}$, each P_j corresponding to element a_j of A and having $s(a_j)$ vertices. (See Fig. 5 (b).) G and H can be constructed in time polynomial in m and B.

We next prove that A has a 3-partition if and only if G contains a subgraph isomorphic to H.

Suppose first that A can be partitioned into m disjoint subsets $S_1, S_2, \ldots, S_m$ such that for each i $(1 \le i \le m)$, $\sum_{a \in S_i} s(a) = B$. An isomorphism from H to a subgraph of G can be obtained as follows. Since each path C_i contains B vertices, we can map the paths of H corresponding to the elements of S_i to the path C_i in G. Each T'_i in H can be mapped to T_i in G. It is easy to see that this is indeed an isomorphism from H to a subgraph of G.

Next suppose that H is isomorphic to a subgraph of G. Each $T'_j (1 \le j \le m-1)$ in H contains two vertices which have degree three and are at a distance two from each other. These vertices must be mapped to the children of vertex r_i of T_i for some i $(1 \le i \le m-1)$. Therefore, each T'_j in H must be mapped to some T_i in G. This means that paths $P_1, P_2, \ldots, P_{3m}$ in H are mapped to paths $C_1, C_2, \ldots, C_m$ in G. For $1 \le i \le m$, let S_i be the set of elements of A corresponding to the paths of H mapped to C_i. Since C_i has B vertices, $\sum_{a \in S_i} s(a) \le B$, for all i $(1 \le i \le m)$. Moreover, since the instance of 3-PARTITION satisfies $\sum_{a \in A} s(a) = mB$, we can conclude that $\sum_{a \in S_i} s(a) = B$ for all i $(1 \le i \le m)$. Therefore A has a 3-partition. □

4. Tractability of SQUARE SUB-CROSSBAR

Let $\mathcal{H}$ be a set of non-intersecting horizontal rays, and let $\mathcal{V}$ be a set of non-intersecting vertical rays. Let $\mathcal{K}_h \subseteq \mathcal{H}$ and $\mathcal{K}_v \subseteq \mathcal{V}$. $\mathcal{K}_h \cup \mathcal{K}_v$ is called a $|\mathcal{K}_h| \times |\mathcal{K}_v|$ sub-crossbar of $\mathcal{H} \cup \mathcal{V}$ if each $X \in \mathcal{K}_h$ intersects every $Y \in \mathcal{K}_v$. For a ray R, we shall denote the x and y-coordinates of its endpoints by $x(R)$ and $y(R)$, respectively. We associate with $\mathcal{H} \cup \mathcal{V}$, a sequence $X_{\mathcal{HV}}$ of the rays of $\mathcal{H} \cup \mathcal{V}$ sorted in the increasing order of x-coordinate values of the end points – ties are broken such that if a horizontal ray and a vertical ray have the same x-coordinate value, then the horizontal ray appears before the vertical ray in the sequence. We also associate with $\mathcal{H} \cup \mathcal{V}$, a sequence $Y_{\mathcal{HV}}$ of the rays of $\mathcal{H} \cup \mathcal{V}$ sorted in the increasing order of y-coordinate values of the end points – ties are broken such that if a vertical ray and a horizontal ray have the same y-coordinate value, then the vertical ray appears before the horizontal ray in the sequence.

Our earlier observation that a nano-wire crossbar can be represented by a set of orthogonal rays allows us to use the terms "nano-wires" and "rays" interchangeably. Then an alternate, equivalent definition of SUB-CROSSBAR is as follows:

SUB-CROSSBAR

INSTANCE: A set $\mathcal{H}$ of horizontal rays, a set $\mathcal{V}$ of vertical rays, and positive integers k_h and k_v.

Input: A set of rightward rays $\mathcal{R}$ and a set of upward rays $\mathcal{U}$, sequences $X_{\mathcal{RU}}$ and $Y_{\mathcal{RU}}$, and positive integers k_r and k_u.
Output: A $k_r \times k_u$ sub-crossbar of $\mathcal{R} \cup \mathcal{U}$, if one exists. NO, otherwise.
Step 1: Compute sequence $R = (R_1, R_2, \ldots, R_{|\mathcal{R}|})$ of the rays of $\mathcal{R}$ such that $y(R_i) < y(R_{i+1})$ $(1 \leq i \leq |\mathcal{R}| - 1)$;
Compute sequence $U = (U_1, U_2, \ldots, U_{|\mathcal{U}|})$ of the rays of $\mathcal{U}$ such that $x(U_j) < x(U_{j+1})$ $(1 \leq j \leq |\mathcal{U}| - 1)$;
Step 2: Set $\mathrm{uEnd}(1) = \{U_l \mid y(U_l) \leq y(R_1)\}$ and for each $i \in \{2, 3, \ldots, |\mathcal{R}|\}$, set $\mathrm{uEnd}(i) = \{U_l \mid y(R_{i-1}) < y(U_l) \leq y(R_i)\}$;
Set $\mathrm{rEnd}(1) = \{R_l \mid x(R_l) < x(U_1)\}$ and for each $j \in \{2, 3, \ldots, |\mathcal{U}|\}$, set $\mathrm{rEnd}(j) = \{R_l \mid x(U_{j-1}) < x(R_l) \leq x(U_j)\}$;
Step 3: Initialize $h = 0$, $v = 0$, rCross$= 0$, uCross$= 0$;
Step 4: if rCross $< k_r$ {
  $v = v + 1$;
  if $v > |\mathcal{U}| - k_u$ output NO and halt.
  Set rCross $=$ rCross $+ |\mathrm{rEnd}(v)|$;
}
Step 5: if uCross $< k_u$ {
  $h = h + 1$;
  if $h > |\mathcal{R}| - k_r$ output NO and halt.
  Set uCross $=$ uCross $+ |\mathrm{uEnd}(h)|$;
}
Step 6: if rCross $\geq k_r$ and uCross $\geq k_u$, then output $\bigcup_{l=1}^{v} \mathrm{rEnd}(l)$ and $\bigcup_{l=1}^{h} \mathrm{uEnd}(l)$ and halt;
Step 7: if rCross $< k_r$ {
  remove U_v from one of the sets $\mathrm{uEnd}(i)$ $(1 \leq i \leq |\mathcal{R}|)$ which contains it;
  if $y(U_v) < y(R_h)$, then uCross $=$ uCross $- 1$;
}
Step 8: if uCross $< k_u$ {
  remove R_h from one of the sets $\mathrm{rEnd}(j)$ $(1 \leq i \leq |\mathcal{U}|)$ which contains it;
  if $x(R_h) < x(U_v)$, then rCross $=$ rCross $- 1$;
}
Step 9: Return to Step 4;

Fig. 6 Algorithm 1.

QUESTION: Show a $k_h \times k_v$ sub-crossbar of $\mathcal{H} \cup \mathcal{V}$, if any.

An interesting subproblem of SUB-CROSSBAR in which the instance is restricted to rightward and upward rays can be defined as follows:

2-SUB-CROSSBAR

INSTANCE: A set $\mathcal{R}$ of rightward rays, a set $\mathcal{U}$ of upward rays, and positive integers k_r and k_u.

QUESTION: Show a $k_r \times k_u$ sub-crossbar of $\mathcal{R} \cup \mathcal{U}$, if any.

In the following subsections, we will discuss algorithms to solve these problems.

4.1 Algorithms for 2-SUB-CROSSBAR

Kloks and Kratsch [4] showed the following.

Lemma 3. [4] *A chordal bipartite graph with n vertices and m edges contains at most m maximal complete bipartite subgraphs which can be enumerated in $O(\min(m \log n, n^2))$ time.* □

From Lemma 3 and Theorem 3, we have:

Lemma 4. *The $K_{m,n}$ biclique problem can be solved in $O(\min(m \log n, n^2))$ time for n-vertex, m-edge 2-directional orthogonal ray graphs.* □

Since the graph representing $\mathcal{R} \cup \mathcal{U}$ is a 2-directional orthogonal ray graph, we have the following theorem from the above lemma.

Theorem 6. *2-SUB-CROSSBAR can be solved in $O(\min(m \log n, n^2))$ time for a crossbar, where $n = |\mathcal{R}| + |\mathcal{U}|$ and m is the number of crosspoints.* □

This is a purely graph theoretic approach, which assumes no information about the endpoints of rays. Takahashi [9] showed that a computational geometry approach utilizing the coordinates of the endpoints yields a faster algorithm of time complexity $O(n \log n)$. We present Algorithm 1 (See Fig. 6), which is a linear-time algorithm to solve 2-SUB-CROSSBAR given that sequences $X_{\mathcal{RU}}$ and $Y_{\mathcal{RU}}$ are provided. Since $X_{\mathcal{RU}}$ and $Y_{\mathcal{RU}}$ can be computed in $O(n \log n)$ time, Algorithm 1 can be easily extended to solve 2-SUB-CROSSBAR in $O(n \log n)$ time. However, the main purpose of introducing Algorithm 1 is to use it as a subroutine to solve SUB-CROSSBAR, as shown in the next subsection.

A brief description of Algorithm 1 follows. Algorithm 1 begins with some preprocessing operations, in which the sequences R, U and the sets $\mathrm{uEnd}(i)(1 \leq i \leq |\mathcal{R}|)$, $\mathrm{rEnd}(j)$ $(1 \leq j \leq |\mathcal{U}|)$ are computed (see Steps 1 and 2). To search for a $k_r \times k_u$ sub-crossbar, Algorithm 1 uses two sweep lines to perform a left-to-right, bottom-to-top scan of the rays. The horizontal sweep line stops at $R_1, R_2, \ldots$, and it is represented by variable h, which indicates that it is at the position of ray R_h. The vertical sweep line stops at $U_1, U_2, \ldots$, and it is represented by variable v, which indicates that it is at the position of ray U_v. At each stop, the following processes are carried out. The number rCross of horizontal rays that cross the vertical sweep line and lie in the area above, and including, the horizontal sweep line is

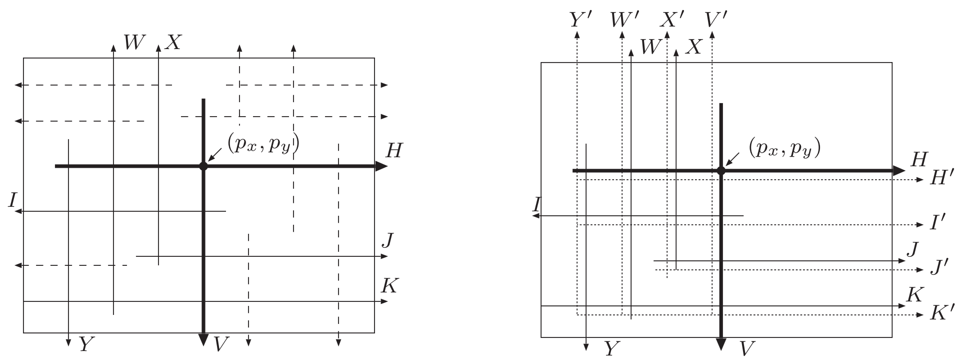

Fig. 7 An example showing $\mathcal{H}_{HV}$, $\mathcal{V}_{HV}$ (left figure) and $\mathcal{H}'_{HV}$, $\mathcal{V}'_{HV}$ (right figure) for a pair of intersecting rays H and V. In this example, $\mathcal{H}_{HV} = \{H, I, J, K\}; \mathcal{V}_{HV} = \{V, W, X, Y\}; \mathcal{H}'_{HV} = \{H', I', J', K'\}; \mathcal{V}'_{HV} = \{V', W', X', Y'\}$.

Input: A set of horizontal rays $\mathcal{H}$, a set of vertical rays $\mathcal{V}$, and positive integers k_h and k_v.
Output: A $k_h \times k_v$ sub-crossbar of $\mathcal{H} \cup \mathcal{V}$, if one exists. NO, otherwise.
Step 1 : Compute the sequences $X_{\mathcal{HV}}$ and $Y_{\mathcal{HV}}$;
Step 2 : Set $S = \{(H, V) \mid H \in \mathcal{H}, V \in \mathcal{V},$ and H and V intersect $\}$;
Step 3 : If S is empty, output NO and halt. Else arbitrarily choose a pair (H, V) from S;
Step 4 : Compute $\mathcal{H}'_{HV}, \mathcal{V}'_{HV}, X_{\mathcal{H}'_{HV}\mathcal{V}'_{HV}}$, and $Y_{\mathcal{H}'_{HV}\mathcal{V}'_{HV}}$;
Step 5 : With the items computed in the previous step and k_h, k_v as input, apply Algorithm 1;
Step 6 : If Algorithm 1 returns NO, set $S = S - \{(H, V)\}$ and return to Step 3.
Step 7 : Output the rays of $\mathcal{H} \cup \mathcal{V}$ corresponding to the rays output by Algorithm 1.

Fig. 8 Algorithm 2.

computed (Step 4). Similarly, the number uCross of vertical rays that cross the horizontal sweep line and lie in the area right of, and including, the vertical sweep line, is computed (Step 5). Evidently, if rCross $\geq k_r$ and uCross $\geq k_u$, then there exists a $k_r \times k_u$ subcrossbar, which is output (Step 6). If rCross $< k_r$, then ray U_v is not a part of any $k_r \times k_u$ subcrossbar, and therefore it is removed from further consideration by updating the appropriate set uEnd(i) and uCross (Step 7). The vertical sweep line then moves one step right to the position of ray U_{v+1}. If uCross $< k_u$, identical operations are carried out for the horizontal case (Step 8).

Let $n = |\mathcal{R}| + |\mathcal{U}|$. The items in Steps 1 and 2 can be obtained from the given sequences $X_{\mathcal{RU}}$ and $Y_{\mathcal{RU}}$ in $O(n)$ time. Each operation in Steps 3 through 8 can be performed in $O(1)$ time. Steps 4 through 9 are repeated until there are less than k_u vertical rays or less than k_r horizontal rays remaining to be scanned, which is $O(n)$ times. Thus the algorithm is linear in the order of the total number of rays. The correctness of Algorithm 1 is obvious, and therefore we have the following theorem.

Theorem 7. *Algorithm 1 solves 2-SUB-CROSSBAR in $O(|\mathcal{R}| + |\mathcal{U}|)$ time, provided that the sequences $X_{\mathcal{RU}}$ and $Y_{\mathcal{RU}}$ are given.* □

4.2 Algorithm for SUB-CROSSBAR

Let $\mathcal{H}$ be a set of non-intersecting horizontal rays and $\mathcal{V}$ be a set of non-intersecting vertical rays. For two rays $H \in \mathcal{H}$ and $V \in \mathcal{V}$ which intersect, say at point (p_x, p_y), define

$$\mathcal{H}_{HV} = \{R \mid R \in \mathcal{H}, R \text{ intersects } V, \text{ and } y(R) \leq p_y\}.$$

Similarly, define

$$\mathcal{V}_{HV} = \{R \mid R \in \mathcal{V}, R \text{ intersects } H, \text{ and } x(R) \leq p_x\}.$$

Let B be the bottommost ray in $\mathcal{H}_{HV}$, and let L be the leftmost ray in $\mathcal{V}_{HV}$. For each ray $R \in \mathcal{H}_{HV}$, define ray R' such that if R is a rightward ray, $R' = R$; and if R is a leftward ray, R' is a rightward ray with $x(R') = x(L)$ and $y(R') = y(R)$. And for each ray $R \in \mathcal{V}_{HV}$, define ray R' such that if R is an upward ray, $R' = R$; and if R is a downward ray, R' is an upward ray with $x(R') = x(R)$ and $y(R') = y(B)$. Finally, define

$$\mathcal{H}'_{HV} = \{R' \mid R \in \mathcal{H}_{HV}\}$$

and

$$\mathcal{V}'_{HV} = \{R' \mid R \in \mathcal{V}_{HV}\}.$$

Figure 7 shows an example of $\mathcal{H}_{HV}$, $\mathcal{V}_{HV}$, $\mathcal{H}'_{HV}$, and $\mathcal{V}'_{HV}$.

The following observation is obvious from the definitions above.

Observation 1. *Two rays in $\mathcal{H}'_{HV} \cup \mathcal{V}'_{HV}$ intersect if and only if their corresponding rays in $\mathcal{H}_{HV} \cup \mathcal{V}_{HV}$ intersect.* □

Observation 2. *$\mathcal{H} \cup \mathcal{V}$ contains a $k_h \times k_v$ sub-crossbar if and only if there exists a pair of intersecting rays $H \in \mathcal{H}$ and $V \in \mathcal{V}$ such that $\mathcal{H}'_{HV} \cup \mathcal{V}'_{HV}$ contains a $k_h \times k_v$ sub-crossbar.*

Proof. The sufficiency is immediate from Observation 1. To see the necessity, set H and V to be the topmost and rightmost rays, respectively of a $k_h \times k_v$ sub-crossbar of $\mathcal{H} \cup \mathcal{V}$. □

Since $\mathcal{H}'_{HV}$ contains only rightward rays and $\mathcal{V}'_{HV}$ contains only upward rays, we can use Algorithm 1 to find a $k_h \times k_v$ sub-crossbar in $\mathcal{H}'_{HV} \cup \mathcal{V}'_{HV}$. Algorithm 2 which solves SUB-CROSSBAR is shown in Fig. 8. It exhaustively checks all pairs of intersecting rays to determine if there exists a pair $H \in \mathcal{H}$ and $V \in \mathcal{V}$ such that $\mathcal{H}'_{HV} \cup \mathcal{V}'_{HV}$ contains a $k_h \times k_v$ sub-crossbar.

Let $n = |\mathcal{H}|+|\mathcal{V}|$. Step 1 can be performed in $O(n \log n)$ time. Step 2 takes $O(n^2)$ time. The items in Step 4 can be computed in $O(n)$ time from the sequences obtained in Step 1. Step 5 takes $O(n)$ time. Steps 3 through 6 are repeated $O(n^2)$ time. Then it follows from Observation 2 and Theorem 7 that:

Theorem 8. *Algorithm 2 solves SUB-CROSSBAR in* $O((|\mathcal{H}| + |\mathcal{V}|)^3)$ *time.* □

5. Concluding Remarks

The complexity of SUBGRAPH ISOMORPHISM in which G is a 2-directional orthogonal ray graph and H is a connected graph is open. Note that if both G and H are trees, then SUBGRAPH ISOMORPHISM is polynomial-time solvable [2]. Reducing the time complexity of SUB-CROSSBAR is another interesting open question.

A preliminary version of this paper has appeared in [6].

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