Complexity of Hamiltonian Cycle Reconfiguration

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Abstract: The Hamiltonian cycle reconfiguration problem asks, given two Hamiltonian cycles C_0 and C_t of a graph G, whether there is a sequence of Hamiltonian cycles $C_0, C_1, \ldots, C_t$ such that C_i can be obtained from C_{i-1} by a switch for each i with $1 \leq i \leq t$, where a switch is the replacement of a pair of edges uv and wz on a Hamiltonian cycle with the edges uw and vz of G, given that uw and vz did not appear on the cycle. We show that the Hamiltonian cycle reconfiguration problem is PSPACE-complete, settling an open question posed by Ito et al. (2011) and van den Heuvel (2013). More precisely, we show that the Hamiltonian cycle reconfiguration problem is PSPACE-complete for chordal bipartite graphs, strongly chordal split graphs, and bipartite graphs with maximum degree 6. Bipartite permutation graphs form a proper subclass of chordal bipartite graphs, and unit interval graphs form a proper subclass of strongly chordal graphs. On the positive side, we show that, for any two Hamiltonian cycles of a bipartite permutation graph and a unit interval graph, there is a sequence of switches transforming one cycle to the other, and such a sequence can be obtained in linear time.

Keywords: bipartite permutation graphs; chordal bipartite graphs; combinatorial reconfiguration; Hamiltonian cycle; PSPACE-complete; split graphs; strongly chordal graphs; unit interval graphs

1. Introduction

A *reconfiguration problem* asks, given two feasible solutions of a combinatorial problem together with some transformation rules between the solutions, whether there is a step-by-step transformation from one solution to the other such that all intermediate states are also feasible. The reconfiguration problems have attracted much attention recently because of their applications as well as theoretical interest. See, for example, a survey [1] and references of [2,3].

In this paper, we study a reconfiguration problem for Hamiltonian cycles. A *Hamiltonian cycle* of a graph is a cycle that contains all the vertices of the graph. Given two Hamiltonian cycles C_0 and C_t of a graph G, the *Hamiltonian cycle reconfiguration problem* asks whether there is a sequence of Hamiltonian cycles $C_0, C_1, \ldots, C_t$ such that C_i and C_{i+1} differ in two edges for each i with $0 \leq i < t$. Such a sequence of Hamiltonian cycles is called a *reconfiguration sequence*. The Hamiltonian cycle reconfiguration problem also can be defined in terms of the transformation rule, which is called *switch* (Switches are also used for sampling and counting perfect matchings [4,5] and transforming graphs with the same degree sequence ([6,7], p.46)). Let C be a Hamiltonian cycle of a graph G. A *switch* is the replacement of a pair of edges uv and wz on C with the edges uw and vz of G, given that uw and vz did not appear on C. The Hamiltonian cycle reconfiguration problem asks whether there is a sequence of switches transforming one cycle to the other such that all intermediate cycles are also Hamiltonian.

The complexity of the reconfiguration problem for Hamiltonian cycles has been implicitly posed as an open question by Ito et al. [8] (Precisely, they asked the complexity of the reconfiguration of the travelling salesman problem, which is a generalization of the Hamiltonian cycle problem) and revisited by van den Heuvel [1]. The Hamiltonian cycle problem, which asks whether a given graph has a Hamiltonian cycle, is one of the well-known NP-complete problems [9], but the complexity of its reconfiguration version still seems to be open.

1.1. Our Contribution

In this paper, we show that the Hamiltonian cycle reconfiguration problem is PSPACE-complete, even for chordal bipartite graphs, strongly chordal split graphs, and bipartite graphs with maximum degree 6. Our reduction for PSPACE-hardness follows from the reduction by Müller [10] for proving the NP-hardness of the Hamiltonian cycle problem for chordal bipartite graphs. However, while Müller shows a polynomial-time reduction from the satisfiability problem, we show a reduction from the nondeterministic constraint logic problem [11], which is used to show the PSPACE-hardness of some reconfiguration problems [11,12].

Unit interval graphs form a proper subclass of strongly chordal graphs, and bipartite permutation graphs form a proper subclass of chordal bipartite graphs (See [13] for example). A Hamiltonian cycle of a unit interval graph and a bipartite permutation graph can be obtained in linear time [14–17]. On the positive side, we show that, for any two Hamiltonian cycles of a unit interval graph and a bipartite permutation graph, there is a sequence of switches transforming one cycle to the other. Moreover, we show that such a sequence can be obtained in linear time. In order to show these results, we introduce the *canonical Hamiltonian cycle* (*canonical cycle* for short) of a unit interval graph and a bipartite permutation graph, using vertex ordering characterizations of these graphs [14,17]. We then show that each Hamiltonian cycle of a unit interval graph (resp. a bipartite permutation graph) can be transformed into the canonical cycle with at most $n-2$ switches (resp. at most $n-3$ switches), where n is the number of vertices of the graph. It follows that, for any two Hamiltonian cycles of a unit interval graph (resp. a bipartite permutation graph), there is a sequence of at most $2n-4$ switches (resp. at most $2n-6$ switches) from one cycle to the other.

1.2. Notation

In this paper, we will deal only with finite graphs having no loops and multiple edges. Unless stated otherwise, graphs are assumed to be undirected, but we also deal with directed graphs. We write uv for the *undirected edge* joining a vertex u and a vertex v, and we write (u,v) for the *directed edge* from u to v. For a graph $G=(V,E)$, we sometimes write $V(G)$ for the vertex set V of G and write $E(G)$ for the edge set E of G.

An *independent set* of a graph $G=(V,E)$ is a subset $S \subseteq V$ such that $uv \notin E$ for any two vertices $u,v \in S$. A graph G is a *bipartite graph* if its vertex set V can be partitioned into two independent set U and W. The independent sets U and W are called *color classes* of G, and the pair (U,W) is called *bipartition* of G. We sometimes use the notation $G=(U,W,E)$ for the bipartite graph with bipartition (U,W).

An *orientation* of an undirected graph $G=(V,E)$ is a graph obtained from G by orienting each edge in E, that is, replacing each edge $uv \in E$ with either (u,v) or (v,u). An *oriented graph* is an orientation of some graph. Notice that an oriented graph contains no pair of edges (u,v) and (v,u) for some vertices u,v. We will denote an orientation of a graph only by its edge set, since the vertex set is clear from the context.

2. PSPACE-Completeness

We can observe that the Hamiltonian cycle reconfiguration problem is in PSPACE ([8], Theorem 1). In this section, we show the reduction from the nondeterministic constraint logic problem, which is known to be PSPACE-complete [11], to the Hamiltonian cycle reconfiguration problem.

2.1. Nondeterministic Constraint Logic

Let G be a 3-regular graph with edge weights among $\{1, 2\}$. A vertex of G is an *AND vertex* if exactly one incident edge has weight 2, and a vertex of G is an *OR vertex* if all the incident edges have weight 2. A graph G is a *constraint graph* if it consists of only AND vertices and OR vertices. An orientation F of G is *legal* if for every vertex v of G, the sum of weights of in-coming edges of

v is at least 2. A *legal move* from a legal orientation is the reversal of a single edge that results in another legal orientation. Figure 1 illustrates all the possible orientations of edges incident to an AND vertex. We can also verify that all the possible legal move of an incident edge of the AND vertex are those depicted by the arrows in Figure 1. Given a constraint graph G and two legal orientation F_0 and F_t of G, the *nondeterministic constraint logic problem* asks whether there is a sequence of legal orientations $F_0, F_1, \ldots, F_t$ such that F_i is obtained from F_{i-1} by a legal move for each i with $1 \leq i \leq t$. Such a sequence of legal orientations is called a *reconfiguration sequence*. The nondeterministic constraint logic problem is known to be PSPACE-complete even if the constraint graph is planar [11]. See [18] for more information on constraint logic.

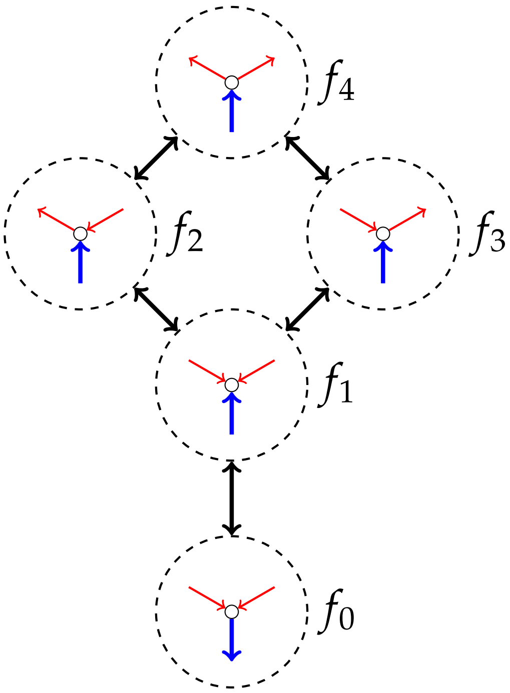


Figure 1. All the possible orientations of edges incident to an AND vertex, where (blue) thick arrows denote the edges with weight 2, and (red) thin arrows denote the edges with weight 1. Each dotted circle represents a possible orientation of the edges, and two circles are joined by an arrow if one is obtained from the other by reversing the direction of a single edge.

For convenience of the reduction, we define a problem slightly different from the nondeterministic constraint logic problem. Let G be a bipartite graph with bipartition (A, B) such that every vertex of A has degree 3 and every vertex of B has degree 2 or 3. The graph G has edge weights among $\{1, 2\}$ such that for every vertex of A, exactly one incident edge has weight 2. An orientation F of G is *legal* if

- for every vertex $v \in A$, the sum of weights of in-coming edges of v is at least 2, and
- every vertex of B has one or two in-coming edges, but at most one vertex of B has two in-coming edges.

A *legal move* from a legal orientation is the reversal of a single edge that results in another legal orientation. Notice that, in the legal moves, the vertices of A behave in the same way as the AND vertices of the nondeterministic constraint logic problem, that is, as shown in Figure 1. Given such a bipartite graph G and two legal orientation F_0 and F_t of G, the problem Π asks whether there is a sequence of legal orientations $F_0, F_1, \ldots, F_t$ such that F_i is obtained from F_{i-1} by a legal move for each i with $1 \leq i \leq t$. We further add a constraint to the instance of the problem Π so that every vertex of B has exactly one in-coming edge in F_0 and F_t.

Lemma 1. *The problem Π is PSPACE-complete.*

Proof. We can observe that the problem Π is in PSPACE ([8], Theorem 1). We thus show a polynomial-time reduction from the nondeterministic constraint logic problem. Let (G, F_0, F_t) be an instance of the problem, that is, G is a constraint graph, consisting of AND vertices and OR vertices, and F_0 and F_t are two legal orientations of G. We construct an instance (G', F'_0, F'_t) of the problem Π such that (G, F_0, F_t) is a yes-instance if and only if (G', F'_0, F'_t) is a yes-instance.

Let G'' be the bipartite graph obtained from G by replacing each edge uv with two edges uw and wv so that uw and wv have the same weight as uv, where w is a newly added vertex. The bipartite graph G' with bipartition (A, B) is obtained from G'' by replacing each OR vertex with a subgraph shown in Figure 2, where A consists of the AND vertices of G and the white points in the subgraphs (see Figure 2) while B consists of the newly added vertices of G'' and the gray points in the subgraphs. We can check that all the vertices of A are incident to one weight-2 edge and two weight-1 edges.

Let F be a legal orientation of G. We define a legal orientation F' of G' *associated with* F. Let F'' be the orientation of G'' obtained from F by replacing each edge $(u, v) \in F$ with two edges (u, w) and (w, v), where w is the newly added vertex. Let F''' be an orientation of G' obtained from F'' by replacing each OR vertex with the subgraph in Figure 2 such that if L is directed inward (resp. outward) in F'' then the edges L_0 and L_1 and the weight-1 edges between them are directed inward (resp. outward) in F''' (and similarly for the edges R and D). The legal orientation F' is obtained from F''' by reversing the direction of the edges incident to the OR vertices so that exactly one edge of $\{L_1, R_1, D_1\}$ is directed inward for each OR vertex. Notice that at least one edge of $\{L_1, R_1, D_1\}$ can be directed inward, since at least one edge of $\{L, R, D\}$ is directed inward in F. We can see that F' has no vertex of B having two in-coming edges. The legal orientations F'_0 and F'_t are the orientations associated with F_0 and F_t, respectively. This completes the construction of the instance (G', F'_0, F'_t) of the problem Π.

Assume that there is a reconfiguration sequence $F_0, F_1, \ldots, F_t$ from F_0 to F_t. Let F'_i be a legal orientation of G' associated with F_i. If F_{i+1} is obtained from F_i by a legal move of an edge joining two AND vertices, we have a reconfiguration sequence from F'_i to F'_{i+1}. Suppose that F_{i+1} is obtained by a legal move of an edge incident to an OR vertex. Let L, R, and D be the edges incident to the OR vertex. We assume without loss of generality that F_{i+1} is obtained by a legal move of the edge L. When L is directed inward in F_i, the edge L is directed outward in F_{i+1}, and thus the edges R or D are directed inward in F_i. Hence, in F'_i the edge R_1 or D_1 can be directed inward (see Figure 2). Therefore, the edges L_0 and L_1 together with the weight-1 edges between them can be directed outward to obtain F'_{i+1}. When L is directed outward in F_i and inward in F_{i+1}, in F'_i the edges L_0 and L_1 together with the weight-1 edges between them can be directed inward to obtain F'_{i+1}. Since there is a reconfiguration sequence from F'_i to F'_{i+1} for any i with $0 \le i < t$, the instance (G', F'_0, F'_t) is a yes-instance if (G, F_0, F_t) is a yes-instance. Notice that, in the subgraph shown in Figure 2, if two edges of $\{L_0, R_0, D_0\}$ are directed outward, then the remaining edge must be directed inward. Thus, a reconfiguration sequence from F_0 to F_t can be obtained from a reconfiguration sequence from F'_0 to F'_t. It follows that the instance (G, F_0, F_t) is a yes-instance if (G', F'_0, F'_t) is a yes-instance.

Since the graph G' and the legal orientations F'_0 and F'_t can be obtained in polynomial time, we have the claim. □

We can further see from the proof of Lemma 1 that the problem Π is PSPACE-complete for planar graphs, since the nondeterministic constraint logic problem is PSPACE-complete even if the constraint graph is planar [11]. We can also see the following observation, which we will use in the proof of Lemma 2.

Proposition 1. *Let (G, F_0, F_t) be an instance of the problem Π with a reconfiguration sequence $F_0, F_1, \ldots, F_t$ from F_0 to F_t. If i is even, then F_i has no vertex of B having two in-coming edges, while F_i has one vertex of B having two in-coming edges if otherwise. If a vertex $b_i \in B$ has two in-coming edges (a_i, b_i) and (a'_i, b_i) in F_i, then we can assume without loss of generality that F_i is obtained from F_{i-1} by reversing the direction of the edge $a_i b_i$, while F_{i+1} is obtained from F_i by reversing the direction of the edge $a'_i b_i$.*

Proof. Let F_i be a legal orientation such that every vertex of B has exactly one in-coming edge. Suppose that F_{i+1} is obtained from F_i by reversing the direction of an edge a_ib_i, where a_i and b_i are the vertices of A and B, respectively. Since all the vertices of B has one in-coming edge in F_i, we have $(b_i, a_i) \in F_i$ and $(a_i, b_i) \in F_{i+1}$. Now, b_i has two in-coming edges in F_{i+1}. Let $(a_i', b_i) \in F_i$ be the in-coming edge of b_i in F_i. If we reverse the direction of an edge other than a_ib_i or $a_i'b_i$, then the orientation is no longer legal. Thus, we can reverse the direction of either a_ib_i or $a_i'b_i$ to obtain F_{i+2}, in which every vertex of B has exactly one in-coming edge. However, if we reverse the direction of a_ib_i, then we have the same orientation as F_i. Thus, we can assume without loss of generality that $(a_i', b_i) \in F_{i+1}$ and $(b_i, a_i') \in F_{i+2}$. Now, we have the claim. □

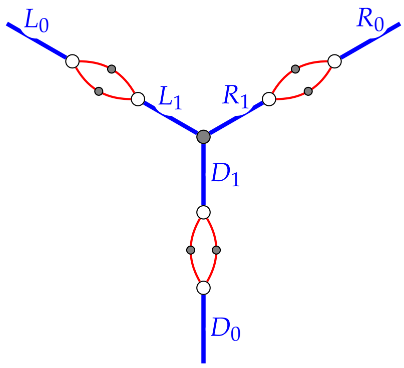
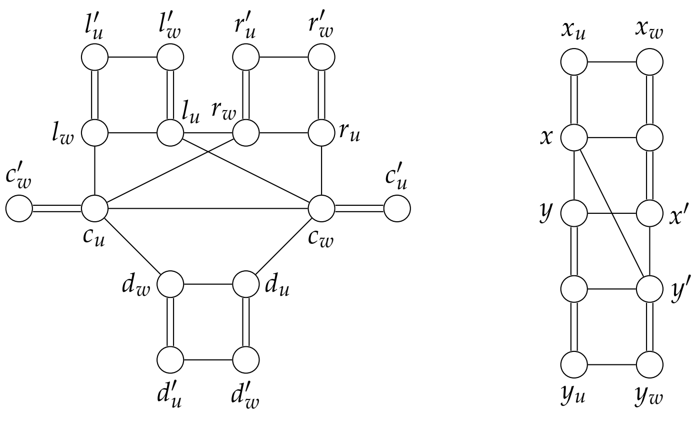


Figure 2. The reduction from the nondeterministic constraint logic problem to the problem Π. White points denote the vertices of A, and gray points denote the vertices of B. Thick (blue) lines denote the edges with weight 2, and thin (red) lines denote the edges with weight 1.

2.2. Reduction

Let (G, F_0, F_t) be an instance of the problem Π. In this section, we construct a reduction graph H together with two Hamiltonian cycles C_0 and C_t such that there is a reconfiguration sequence from F_0 to F_t if and only if there is a reconfiguration sequence from C_0 to C_t. That is, (G, F_0, F_t) is a yes-instance if and only if (H, C_0, C_t) is a yes-instance of the Hamiltonian cycle reconfiguration problem.

We use three types of gadgets corresponding to the vertices in A, the vertices in B, and the edges of G. A gadget for a vertex in A and a gadget for an edge of G is shown in Figure 3a,b respectively. Double lines in the figures denote *edges with ears*, where an *ear* of an edge uw is a path of length 3 joining u and w. Recall that, in the legal moves, the vertices in A behave in the same way as the AND vertices. We thus refer to the gadgets for the vertices in A as AND gadgets. Let b be a vertex in B of degree k, and recall that k is 2 or 3. A gadget for b is a cycle $(u_0, w_0, u_1, w_1, \ldots, u_{k-1}, w_{k-1})$ of length $2k$ such that the edge w_iu_{i+1} has a ear for each i with $0 \leq i < k$ (indices are modulo k).



(**a**) (**b**)

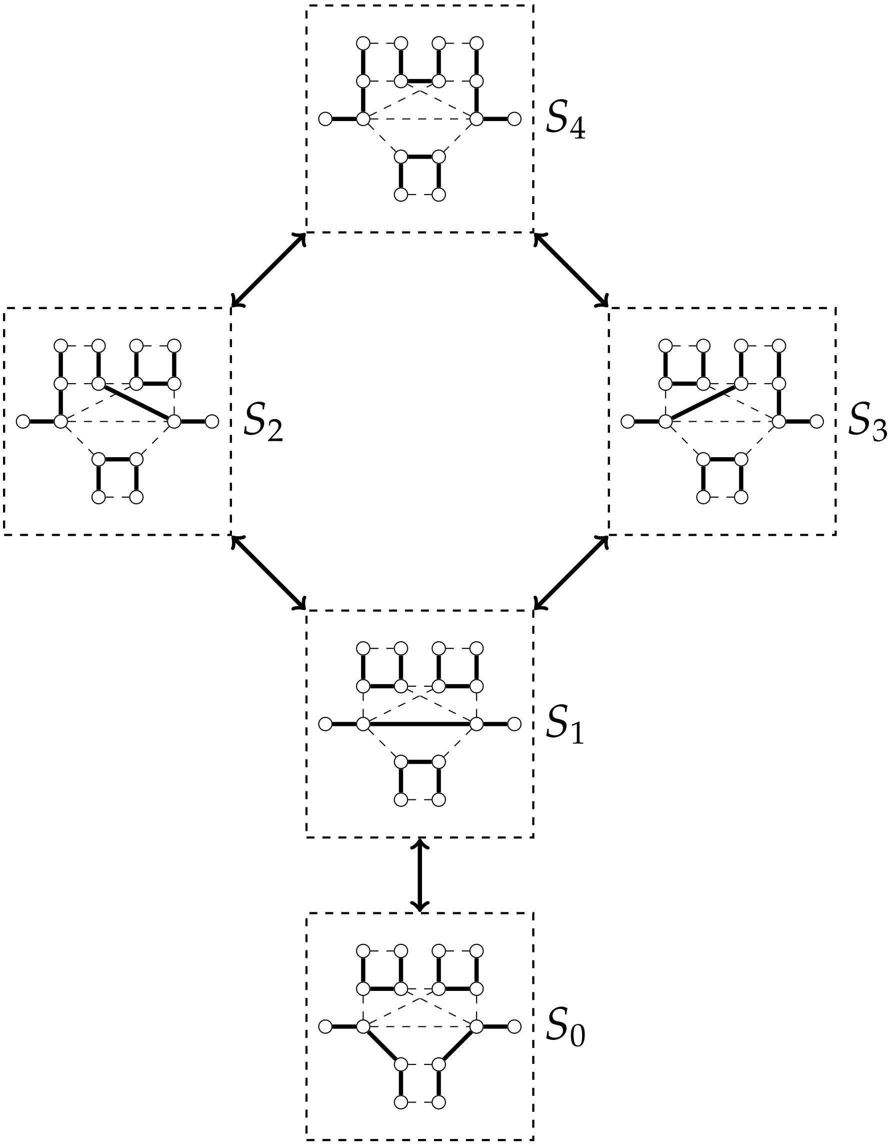
Figure 3. Gadgets. Double lines denote edges with ears. (**a**) an AND gadget; (**b**) an edge gadget.

We construct the reduction graph H from G as follows: (1) Let a be a vertex in A, and let e_l, e_r, e_d be the edges of G incident to a such that e_l and e_r have weight 1 and e_d has weight 2. We identify the vertices l'_u and l'_w of the gadget for a with the vertices x_u and x_w of the gadget for e_l, respectively. Similarly, we identify the vertices r'_u and r'_w of the gadget for a with the vertices x_u and x_w of the gadget for e_r, respectively. Moreover, we identify the vertices d'_u and d'_w of the gadget for a with the vertices x_u and x_w of the gadget for e_d, respectively. (2) Let b be a vertex in B of degree k, and let $e_0, e_1, \ldots, e_{k-1}$ be the edges of G incident to b. We identify, for each i with $0 \leq i < k$, the vertices u_i and w_i of the gadget for b with the vertices y_u and y_w of the gadget for e_i, respectively. (3) We finally concatenate the gadgets for the vertices in A cyclically using edges with ears joining the vertices c'_u and c'_w of the gadgets.

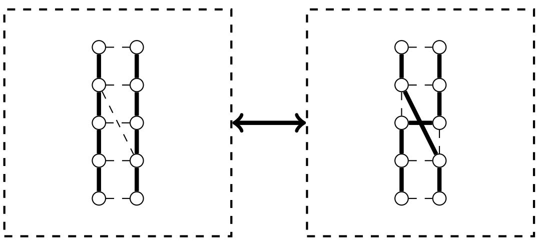
Before describing the construction of the Hamiltonian cycles C_0 and C_t, we consider the possible configurations of a Hamiltonian cycle of the reduction graph H passing through the gadgets. We will show that all the possible configurations in an AND gadget and an edge gadget are shown in Figure 4a,b, respectively. We can also verify that all the possible transformations of Hamiltonian cycles by a single switch occurred in a gadget are those depicted by the arrows in the figures. Let C be a Hamiltonian cycle. We first consider the configurations of C in an AND gadget. The Hamiltonian cycle C passes through all the edges on the ears, since interior vertices of an ear has degree 2. Thus, C passes through any of the edges $c_u d_w$, $c_u c_w$, $c_u r_w$, or $c_u l_w$. We also have that C does not pass through the edges $l'_u l'_w$, $r'_u r'_w$, or $d'_u d'_w$, since when we construct the reduction graph H the vertices l'_u, l'_w, r'_u, r'_w, d'_u and d'_w are identified with the vertices of the edge gadgets incident to the edges with ears. Suppose that C passes through $c_u d_w$. Since C cannot pass through $d_u d_w$, it passes through $d_u c_w$. Since C cannot pass through $c_u l_w$, it passes through $l_u l_w$. Since C cannot pass through $l_u r_w$, it passes through $r_u r_w$, and we have the configuration S_0 in Figure 4a. Suppose that C passes through $c_u c_w$. Since C cannot pass through $c_u d_w$, it passes through $d_u d_w$. Since C cannot pass through $c_u l_w$, it passes through $l_u l_w$. Since C cannot pass through $l_u r_w$, it passes through $r_u r_w$, and we have the configuration S_1 in Figure 4a. Suppose that C passes through $c_u r_w$. Since C cannot pass through $c_u d_w$, it passes through $d_u d_w$. Since C cannot pass through $c_u l_w$, it passes through $l_u l_w$. Since C cannot pass through $r_u r_w$, it passes through $r_u c_w$, and we have the configuration S_3 in Figure 4a. Suppose that C passes through $c_u l_w$. Since C cannot pass through $c_u d_w$, it passes through $d_u d_w$. Since C cannot pass through $l_u l_w$, it passes through either $l_u r_w$ or $l_u c_w$. If C passes through $l_u r_w$, then it passes through $r_u c_w$ since it cannot pass through $r_u r_w$, and we have the configuration S_4 in Figure 4a. If C passes through $l_u c_w$, then it passes through $r_u r_w$ since it cannot pass through $l_u r_w$, and we have the configuration S_2 in Figure 4a. Therefore, all the possible configurations in an AND gadget are shown in Figure 4a. We next consider the configurations of the Hamiltonian cycle C in an edge gadget. Since C passes through all the edges on the ears, it passes through either xy or xy'. If C passes through xy then it passes through $x'y'$, while if C passes through xy', then it passes through $x'y$. We also have that C does not pass through the edges $x_u x_w$ or $y_u y_w$, since when we construct the reduction graph H the vertices x_u, x_w, y_u, and y_w are identified with the vertices of the AND gadgets incident to the edges with ears. Therefore, all the possible configurations in an edge gadget are shown in Figure 4b.

Let v be a vertex of A. We next make a correspondence between the possible configurations of a Hamiltonian cycle in the gadget for v and the possible orientations of the edges incident to v such that the configuration S_i in Figure 4a corresponds to the orientation f_i in Figure 1 for each $i \in \{0, 1, \ldots, 4\}$. We also make a correspondence between switches occurred in the gadget for v and legal moves of the edges incident to v such that switching the configuration from S_i to S_j in the gadget for v corresponds to the legal move from f_i to f_j of the edges of v, where $i, j \in \{0, 1, \ldots, 4\}$.

We define a legal orientation F of G *associated with* a Hamiltonian cycle C of H so that for each vertex $v \in A$, the edges incident to v are oriented *according to* the configuration of C in the gadget for v. That is, the edges of v are oriented as f_i in F if the configuration of C in the gadget for v looks like S_i (see Figures 1 and 4a). Notice that a Hamiltonian cycle C of H has exactly one legal orientation of G associated with C, but a legal orientation F may have some Hamiltonian cycles that are associated with F, due to the two possible configurations in an edge gadget shown in Figure 4b.



(**a**) The five possible configurations of a Hamiltonian cycle in an AND gadget.



(**b**) The two possible configurations of a Hamiltonian cycle in an edge gadget.

Figure 4. All the possible configurations of a Hamiltonian cycle passing through gadgets. The edges on the cycle are indicated by thick lines, but the ears are omitted; the edges out of the cycle are indicated by dotted lines. Each dotted square represents a possible configuration, and two squares are joined by an arrow if one is obtained from the other by a single switch.

Now, we construct the Hamiltonian cycle C_0 from F_0 as follows, and C_t is constructed similarly from F_t. (1) For each vertex $v \in A$, we take the configuration in the gadget for v according to the orientations of the edges incident to v. That is, we take the configuration S_i in Figure 4a for the gadget for v if the edges of v are oriented as f_i in Figure 1. (2) We choose the configuration in each edge gadget arbitrarily among those in Figure 4b. (3) The remaining parts are uniquely determined, since any Hamiltonian cycle pass through all the edges on the ears. Figure 5b illustrates the Hamiltonian cycle constructed in this way from the legal orientation in Figure 5a. Recall that every vertex of B has exactly one in-coming edge in F_0 and F_t. This guarantees that C_0 and C_t are Hamiltonian. This completes the construction of the instance (H, C_0, C_t) of the Hamiltonian cycle reconfiguration problem. We remark two facts, which we use in the proof of the following lemma. First, we can see that C_0 and C_t are associated with F_0 and F_t, respectively. Second, if every vertex of B has exactly one in-coming edge in a legal orientation F, then for any two Hamiltonian cycles that are associated with F_t, there is a reconfiguration sequence from one to the other, in which the switches occur only in edge gadgets.

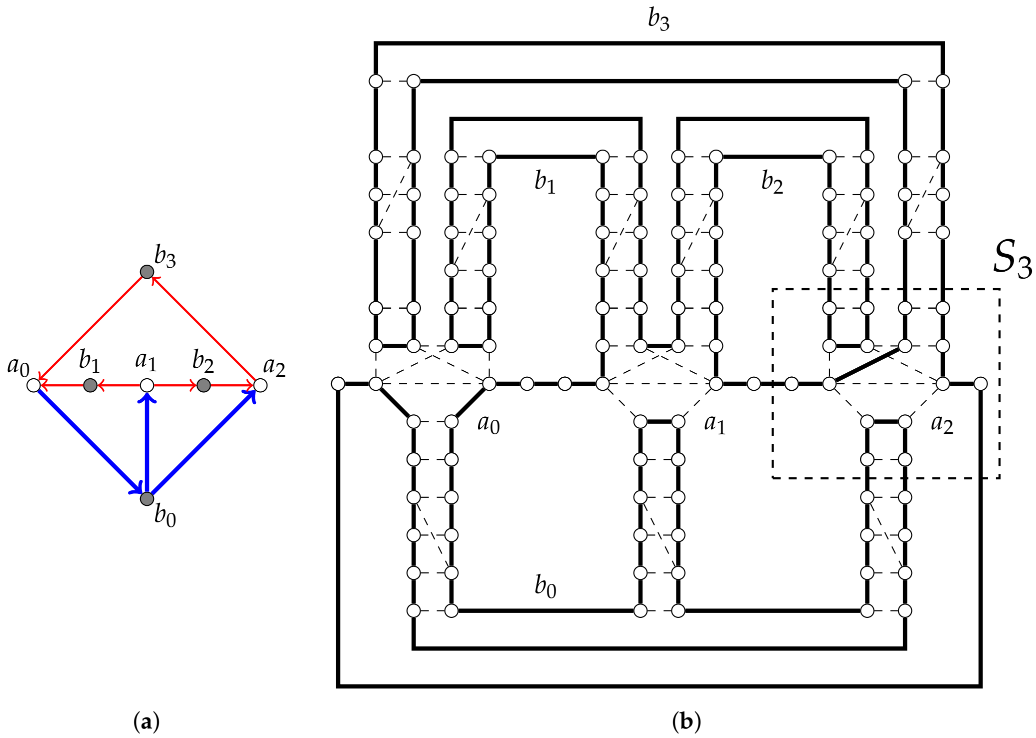


Figure 5. (**a**) a legal orientation of the problem Π. White points denote the vertices of A, and gray points denote the vertices of B. Thick (blue) lines denote the edges with weight 2, and thin (red) lines denote the edges with weight 1; (**b**) the Hamiltonian cycle obtained from the legal orientation in Figure 5a. We take the configuration S_3 for the gadget for a_2, since the edges of a_2 are oriented as f_3 in Figure 5a. Notice that, when we replace the configuration from S_3 to S_4, we have two cycles.

Lemma 2. *The instance (G, F_0, F_t) of the problem Π is a yes-instance if and only if (H, C_0, C_t) of the Hamiltonian cycle reconfiguration problem is a yes-instance.*

Proof. We first prove the if direction. Assume that there is a reconfiguration sequence $C_0, C_1, \ldots, C_t$ from C_0 to C_t. Let F_i be the legal orientation of G associated with C_i (Recall that a Hamiltonian cycle C of H has exactly one legal orientation associated with C). Notice that $F_i = F_{i+1}$ if and only if C_{i+1} is obtained from C_i by a switch occurred in an edge gadget. When $F_i = F_{i+1}$ for some i with $0 \leq i < t$, we remove F_{i+1} from the sequence $F_0, F_1, \ldots, F_t$ to obtain the reconfiguration sequence from F_0 to F_t.

We next prove the only-if direction. Assume that there is a reconfiguration sequence $F_0, F_1, \ldots, F_t$ from F_0 to F_t. Recall that, for any two Hamiltonian cycles that are associated with F_t, there is a reconfiguration sequence from one to the other, since every vertex of B has exactly one in-coming edge in F_t. Thus, it suffices to show that for each Hamiltonian cycle C_i with $0 \leq i < t$, there is a Hamiltonian cycle C_{i+1} together with a reconfiguration sequence from C_i to C_{i+1}, where C_i and C_{i+1} are Hamiltonian cycles associated with F_i and F_{i+1}, respectively. Suppose that F_{i+1} is obtained from F_i by reversing the direction of an edge $a_i b_i$, where a_i and b_i are the vertices of A and B, respectively.

We first consider the case when $(b_i, a_i) \in F_i$ and $(a_i, b_i) \in F_{i+1}$. We have from Proposition 1 that F_i has no vertex of B having two in-coming edges. Let C be a graph obtained from C_i by switching the configuration in the gadget for a_i according to the legal move. If C is a Hamiltonian cycle, the claim holds. However, there is some possibility that C is disconnected. (In Figure 5b, for example, when we replace the configuration in the gadget for a_2 from S_3 to S_4, we have two cycles, while, in Figure 5a, this replacement corresponds to the reversal of the edge (b_2, a_2) that results in another legal orientation). In this case, we use two steps as follows: Let C' be a graph obtained from C_i by switching the configuration in the edge gadget for $a_i b_i$ as shown in Figure 4b. Let C'' be a graph obtained from C' by switching the configuration in the gadget for a_i according to the legal move. We show that C' and C'' are Hamiltonian cycles. Suppose that C is obtained from C_i by switching edges v_1v_2 and v_3v_4 with edges v_1v_3 and v_2v_4. Suppose also that C' is obtained from C by switching edges v_5v_6 and v_7v_8 with edges v_5v_7 and v_6v_8. Since C is disconnected while C_i is Hamiltonian, the vertices v_1, v_2, v_3, and v_4 appear on C_i as $C_i = (v_1, v_2, \ldots, v_4, v_3, \ldots)$. Since $(b_i, a_i) \in F_i$ and the switch occurs in the edge gadget, we can assume without loss of generality that the vertices v_5, v_6, v_7, and v_8 appear on C_i as

$$C_i = (v_1, v_2, \ldots, v_5, v_6, \ldots, v_4, v_3, \ldots, v_7, v_8, \ldots).$$

Thus, C' and C'' are the following Hamiltonian cycles.

$$C' = (v_1, v_2, \ldots, v_5, v_7, \ldots, v_3, v_4, \ldots, v_6, v_8, \ldots),$$
$$C'' = (v_1, v_3, \ldots, v_7, v_5, \ldots, v_2, v_4, \ldots, v_6, v_8, \ldots).$$

We can see that C' is also associated with F_i since the switch occurs in an edge gadget. Hence, C'' is associated with F_{i+1}, and the claim holds.

We then consider the case when $(a_i, b_i) \in F_i$ and $(b_i, a_i) \in F_{i+1}$. Let C be a graph obtained from C_i by switching the configuration in the gadget for a_i according to the legal move. We show that C is the Hamiltonian cycle. We have from Proposition 1 that there is the vertex $a'_i \in A$ with $a'_i \neq a_i$ such that $(a'_i, b_i) \in F_i$ while $(b_i, a'_i) \in F_{i-1}$. Let C' be the Hamiltonian cycle associated with F_{i-1} from which C_i is obtained by a single switch. We can see that this switch occurs in the gadget for a'_i. Suppose that C is obtained from C_i by switching edges v_1v_2 and v_3v_4 with edges v_1v_3 and v_2v_4. Suppose also that C_i is obtained from C' by switching edges v_5v_6 and v_7v_8 with edges v_5v_7 and v_6v_8. Since (a_i, b_i) is the only in-coming edge of b_i in F_{i-1}, the vertices v_1, v_2, v_3, and v_4 appear on C' as $C' = (v_1, v_2, \ldots, v_4, v_3, \ldots)$. Since $(b_i, a'_i) \in F_{i-1}$, we can assume without loss of generality that the vertices v_5 and v_6 appear on C' as $C' = (v_1, v_2, \ldots, v_5, v_6, \ldots, v_4, v_3, \ldots)$. Since C_i is also a Hamiltonian cycle, the vertices v_7 and v_8 appear on C' as

$$C' = (v_1, v_2, \ldots, v_5, v_6, \ldots, v_4, v_3, \ldots, v_7, v_8, \ldots).$$

Thus, C_i and C are the following Hamiltonian cycles.

$$C_i = (v_1, v_2, \ldots, v_5, v_7, \ldots, v_3, v_4, \ldots, v_6, v_8, \ldots),$$
$$C = (v_1, v_3, \ldots, v_7, v_5, \ldots, v_2, v_4, \ldots, v_6, v_8, \ldots).$$

Since C is associated with F_{i+1}, the claim holds. □

Obviously, the reduction graph H is bipartite. We can easily check that H has maximum degree 6 (The vertices c_v and c_w of each AND gadget have degree 6). Since the instance (H, C_0, C_t) can be constructed from (G, F_0, F_t) in polynomial time, we have the following.

Theorem 1. *The Hamiltonian cycle reconfiguration problem is PSPACE-complete for bipartite graphs with maximum degree 6.*

A bipartite graph is *chordal bipartite* if each cycle in the graph of length greater than 4 has a chord, that is, an edge joining two vertices that are not consecutive on the cycle. Let D be the vertices of the reduction graph H incident with two edges having ears. We construct a graph H' from H by adding edges uv for all vertices $u \in D$ and all vertices v of H that is in the color class different from u and is not an interior vertex of any ear. It is obvious that H' is bipartite. Suppose that H' has a chordless cycle Z of length greater than 4. Clearly, Z has no interior vertices of any ear. We also have that Z has no vertices in D, for otherwise Z would have a chord. Thus, Z is a cycle in a single AND gadget or a single edge gadget, but these gadgets contains no chordless cycle of length greater than 4. Therefore, H' is a chordal bipartite graph.

Since every added edges in H' is incident to a vertex in D, any Hamiltonian cycle does not pass through the added edges. Thus, there is a reconfiguration sequence from C_0 to C_t in H if and only if there is a reconfiguration sequence from C_0 to C_t in H'. Now, we have the following.

Theorem 2. *The Hamiltonian cycle reconfiguration problem is PSPACE-complete for chordal bipartite graphs.*

2.3. Strongly Chordal Split Graphs

A graph is *chordal* if each cycle in the graph of length greater than 3 has a chord. A *clique* of $G = (V, E)$ is a subset $S \subseteq V$ such that $uv \in E$ for any two vertices $u, v \in S$. A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set. A chordal graph is *strongly chordal* [19] if each cycle of even length at least 6 has an odd chord, that is, an edge joining two vertices having odd distance on the cycle. Strongly chordal graphs are closely related to chordal bipartite graphs. Let $G = (U, W, E)$ be a bipartite graph. We define a split graph $S(G) = (U \cup W, E \cup E_U)$, where $E_U = \{uu' : u, u' \in U\}$. It is known that a bipartite graph G is a chordal bipartite graph if and only if $S(G)$ is strongly chordal. See ([20,21], Lemma 12.4).

Let $G = (U, W, E)$ be a bipartite graph with $|U| = |W|$. Obviously, any Hamiltonian cycle of $S(G)$ does not pass through the edges in E_U. Thus, there is a reconfiguration sequence from a Hamiltonian cycle C_0 of G to another Hamiltonian cycle C_t of G if and only if there is a reconfiguration sequence from C_0 to C_t in $S(G)$. Now, we have the following from Theorem 2.

Theorem 3. *The Hamiltonian cycle reconfiguration problem is PSPACE-complete for strongly chordal split graphs.*

3. Canonical Hamiltonian Cycles

Unit interval graphs form a proper subclass of strongly chordal graphs, and bipartite permutation graphs form a proper subclass of chordal bipartite graphs (See [13], for example). In this section, we introduce the *canonical Hamiltonian cycle* (*canonical cycle* for short) of a unit interval graph and the canonical cycle of a bipartite permutation graph. We then show that each Hamiltonian cycle of a unit interval graph and a bipartite permutation graph can be transformed into the canonical cycle by a sequence of switches.

3.1. Unit Interval Graphs

A graph is an *interval graph* if each vertex can be assigned an interval on the real line so that two vertices are adjacent if and only if their assigned intervals intersect. An interval graph is a *unit interval graph* if each vertex can be assigned an interval of unit length. There are some linear-time algorithms to find a Hamiltonian cycle of a unit interval graph [14–16]. We follow the algorithm of Chen et al. [14], which uses the following vertex ordering characterization.

Theorem 4 ([14,22])**.** *A* consecutive ordering *of a graph G is a sequence of vertices $v_0, v_1, \ldots, v_{n-1}$ of G such that for any three vertices v_i, v_j, v_k with $i < j < k$, if $v_iv_k \in E(G)$ then $v_iv_j, v_jv_k \in E(G)$. A graph is* *a unit interval graph if and only if it has a consecutive ordering. Moreover, a consecutive ordering of a unit interval graph can be obtained in linear time.*

Notice that, in the consecutive ordering of a graph G, the vertices in $N[v]$ are consecutive for every vertex $v \in V(G)$, where $N[v] = \{v\} \cup \{u\colon\ uv \in E(G)\}$.

It is known that a unit interval graph has a Hamiltonian cycle if and only if it is biconnected [14–16]. Biconnected unit interval graphs are characterized as follows.

Theorem 5 ([14]). *A unit interval graph G with a consecutive ordering $v_0, v_1, \ldots, v_{n-1}$ is biconnected if and only if $v_iv_j \in E(G)$ for every i and j with $1 \leq |i-j| \leq 2$.*

We can observe that such a unit interval graph G has a Hamiltonian cycle consisting of the edges v_0v_1, $v_{n-2}v_{n-1}$, and v_iv_{i+2} for every i with $0 \leq i \leq n-3$ [14]; we define it as the *canonical Hamiltonian cycle* (*canonical cycle* for short) of G.

Theorem 6. *Let G be a unit interval graph. For each Hamiltonian cycle of G, there is a sequence of at most $n-2$ switches transforming it to the canonical cycle of G.*

The following is a useful fact about consecutive orderings.

Lemma 3. *Let v_i, v_j, v_k, v_h be four vertices of G with $i < j < k$ and $i < h$. If $v_iv_k, v_jv_h \in E(G)$, then $v_iv_j, v_kv_h \in E(G)$.*

Proof. We have that v_iv_k implies $v_iv_j \in E(G)$ by the definition of consecutive orderings. If $h < k$, then $v_iv_k \in E(G)$ and $i < h$ implies $v_kv_h \in E(G)$. If $k < h$, then $v_jv_h \in E(G)$ implies $v_kv_h \in E(G)$. □

Proof of Theorem 6. We assume $n \geq 4$, since the claim trivially holds when $n \leq 3$. Let G have a consecutive ordering $v_0, v_1, \ldots, v_{n-1}$, and let C_t be the canonical cycle of G. Let C_0 be a Hamiltonian cycle of G. It suffices to show a sequence of Hamiltonian cycles $C_0, C_1, \ldots, C_{n-2}$ that satisfy the following conditions for each i with $1 \leq i \leq n-2$:

- C_i contains the edges on C_t induced by $\{v_0, v_1, \ldots, v_i\}$,
- C_i is obtained from C_{i-1} by *at most one* switch.

Notice that C_{n-2} is the canonical cycle C_t by the following reason: since C_{n-2} is Hamiltonian, $v_{n-3}v_{n-2} \notin E(C_{n-2})$; we thus have $v_{n-3}v_{n-1}, v_{n-2}v_{n-1} \in E(C_{n-2})$.

We first construct C_1 from C_0. When $v_0v_1 \in E(C_0)$, we define C_0 as C_1. We then consider the case when $v_0v_1 \notin E(C_0)$. Let v_j, v_k, v_h, v_l be the vertices of G such that

$$C_0 = (v_0, v_j, \ldots, v_k, v_1, v_h, \ldots, v_l).$$

Note that there is some possibility that $v_j = v_k$ or $v_h = v_l$. It is clear that $j, k, h, l \geq 2$. Since $v_0v_j, v_1v_h \in E(G)$, we have $v_0v_1, v_jv_h \in E(G)$ by Lemma 3. We define that C_1 is the Hamiltonian cycle obtained from C_0 by switching the edges v_0v_j and v_1v_h with the edges v_0v_1 and v_jv_h, that is,

$$C_1 = (v_0, v_1, v_k, \ldots, v_j, v_h, \ldots, v_l).$$

We now construct C_i from C_{i-1} with $i \geq 2$. Recall that C_{i-1} contains the edges on C_t induced by $\{v_0, \ldots, v_{i-2}, v_{i-1}\}$. When $v_{i-2}v_i \in E(C_{i-1})$, we define C_{i-1} as C_i. We then consider the case when $v_{i-2}v_i \notin E(C_{i-1})$. Let v_j, v_k, v_h be the vertices of G such that

$$C_{i-1} = (v_{i-1}, \ldots, v_{i-2}, v_j, \ldots, v_k, v_i, v_h, \ldots).$$

Note that there is some possibility that $v_j = v_k$ or $v_{i-1} = v_h$. We have $j > i - 2$ by the definition of C_{i-1}. Since C_{i-1} is Hamiltonian, $v_{i-2}v_{i-1} \notin E(C_{i-1})$, and thus $j \neq i - 1$. We also have $j > i$ from $v_{i-2}v_i \notin E(C_{i-1})$. Moreover, we have $k, h > i - 2$ by the definition of C_{i-1} and $v_{i-2}v_i \notin E(C_{i-1})$. Since $v_{i-2}v_j, v_iv_h \in E(G)$, we have $v_{i-2}v_i, v_jv_h \in E(G)$ by Lemma 3. We define that C_i is the Hamiltonian cycle obtained from C_{i-1} by switching the edges $v_{i-2}v_j$ and v_iv_h with the edges $v_{i-2}v_i$ and v_jv_h, that is,

$$C_i = (v_{i-1}, \ldots, v_{i-2}, v_i, v_k, \ldots, v_j, v_h, \ldots).$$

Therefore, we have the sequence of at most $n - 2$ switches transforming C_0 into the canonical cycle C_t. □

We also have the following from Theorem 6.

Corollary 1. *For each Hamiltonian cycle C_0 of a unit interval graph G, we can compute a sequence of switches transforming C_0 to the canonical cycle of G in $O(n)$ time, provided that a consecutive ordering of G is given.*

Proof. The algorithm follows the steps of the proof of Theorem 6. We analyze the implementation details and the running time. We store C_0 in a circular doubly linked list L as a sequence of vertices; we store the consecutive ordering $v_0, v_1, \ldots, v_{n-1}$ in an array A, in which the element of position i has a pointer to the vertex v_i in L for each i with $0 \leq i < n$. In order to compute the Hamiltonian cycle C_1 from C_0, it suffices to take the vertices v_0, v_1, v_j, v_h in L, where v_j and v_h is the successor or the predecessor of v_0 and v_1, respectively. Similarly in order to compute C_i from C_{i-1} with $i \geq 2$, it suffices to take the vertices v_{i-2}, v_i, v_j, v_h in L, where v_j and v_h is the successor or the predecessor of v_{i-2} and v_i, respectively. Since one iteration takes a constant time, we have the claim. □

Now, we have the following from Theorem 6 and Corollary 1.

Corollary 2. *For any two Hamiltonian cycles of a unit interval graph, there is a sequence of at most $2n - 4$ switches transforming one cycle to the other. Moreover, we can compute such a sequence in $O(n)$ time, provided that a consecutive ordering of G is given.*

3.2. Bipartite Permutation Graphs

A graph G with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is a *permutation graph* if there is a permutation π on $\{1, 2, \ldots, n\}$ such that $v_iv_j \in E(G)$ if and only if $(i - j)(\pi(i) - \pi(j)) < 0$ for every $i, j \in \{1, 2, \ldots, n\}$. A permutation graph is a *bipartite permutation graph* [17] if it is bipartite. A Hamiltonian cycle of a bipartite permutation graph can be obtained in linear time [17]. We follow this algorithm, which uses the following vertex ordering characterization.

Theorem 7 ([17])**.** *A* strong ordering *of a bipartite graph $G = (U, W, E)$ is a pair of total orderings $u_0, u_1, \ldots, u_{|U|-1}$ of U and $w_0, w_1, \ldots, w_{|W|-1}$ of W such that for every i, j, k, h with $0 \leq i < j < |U|$ and $0 \leq k < h < |W|$, if $u_iw_h \in E$ and $u_jw_k \in E$ then $u_iw_k \in E$ and $u_jw_h \in E$. A bipartite graph is a bipartite permutation graph if and only if it has a strong ordering. Moreover, a strong ordering of a bipartite permutation graph can be obtained in linear time.*

A bipartite graph $G = (U, W, E)$ is *balanced* if $|U| = |W|$. Notice that, if a bipartite permutation graph G has a Hamiltonian cycle, then G is biconnected and balanced with $|U| = |W| \geq 2$, but the converse does not hold. See Figure 6 for example. Bipartite permutation graphs having a Hamiltonian cycle are characterized as follows.

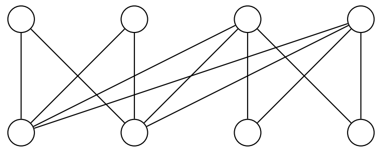


Figure 6. A biconnected bipartite permutation graph having no Hamiltonian cycles.

Theorem 8 ([17])**.** *Let $G = (U, W, E)$ be a bipartite permutation graph with $|U| = |W| = p \geq 2$, and let G have a strong ordering $u_0, u_1, \ldots, u_{p-1}$ of U and $w_0, w_1, \ldots, w_{p-1}$ of W. The graph G has a Hamiltonian cycle if and only if the vertices $u_i, w_i, u_{i+1}, w_{i+1}$ form a cycle of length 4 for every i with $0 \leq i \leq p-2$.*

We can observe that such a bipartite permutation graph G has a Hamiltonian cycle consisting of the edges u_0w_0, $u_{p-1}w_{p-1}$, u_iw_{i+1}, and $u_{i+1}w_i$ for every i with $0 \leq i \leq p-2$ [17]; we define it as the *canonical Hamiltonian cycle* (*canonical cycle* for short) of G.

Theorem 9. *Let $G = (U, W, E)$ be a bipartite permutation graph with $|U| = |W| = p \geq 2$. For each Hamiltonian cycle of G, there is a sequence of at most $n-3$ switches transforming it to the canonical cycle of G.*

Proof. We assume $p \geq 3$, since the claim trivially holds when $p \leq 2$. Let G have a strong ordering $u_0, u_1, \ldots, u_{p-1}$ of U and $w_0, w_1, \ldots, w_{p-1}$ of W, and let C_t be the canonical cycle of G. Let C_0 be a Hamiltonian cycle of G. It suffices to show a sequence of Hamiltonian cycles $C_0, C_1, \ldots, C_{n-3}$ that satisfy the following conditions for each i with $1 \leq i \leq n-3$:

- C_i contains the edges on C_t induced by $\{v_0, v_1, \ldots, v_i\}$, where $v_0 = u_0$, $v_1 = w_0$, $v_2 = u_1$, $v_3 = w_1$, $\ldots$, $v_{n-2} = u_{p-1}$, $v_{n-1} = w_{p-1}$;
- C_i is obtained from C_{i-1} by *at most one* switch.

Notice that C_{n-3} is the canonical cycle C_t by the following reason: since C_{n-3} is Hamiltonian, $u_{p-2}w_{p-2} \notin E(C_{n-3})$; we thus have $u_{p-2}w_{p-1}, u_{p-1}w_{p-2}, u_{p-1}w_{p-1} \in E(C_{n-3})$.

We first construct C_1 from C_0. When $u_0w_0 \in E(C_0)$, we define C_0 as C_1. We then consider the case when $u_0w_0 \notin E(C_0)$. Let w_j, u_k, u_h, w_l be the vertices of G such that

$$C_0 = (u_0, w_j, \ldots, u_k, w_0, u_h, \ldots, w_l).$$

It is clear that $j, k, h, l \geq 0$. Since $u_0w_j, u_hw_0 \in E(G)$, we have $u_0w_0, u_hw_j \in E(G)$ by the definition of strong orderings. We define that C_1 is the Hamiltonian cycle obtained from C_0 by switching the edges u_0w_j and u_hw_0 with the edges u_0w_0 and u_hw_j, that is,

$$C_1 = (u_0, w_0, u_k, \ldots, w_j, u_h, \ldots, w_l).$$

We next construct C_i from C_{i-1} with $i = 2q \geq 2$. Recall that C_{i-1} contains the edges on C_t induced by $\{u_0, \ldots, u_{q-1}, w_{q-1}\}$. When $u_qw_{q-1} \in E(C_{i-1})$, we define C_{i-1} as C_i. We then consider the case when $u_qw_{q-1} \notin E(C_{i-1})$. Let u_j, w_k, w_h be the vertices of G such that

$$C_{i-1} = (u_{q-1}, \ldots, w_{q-1}, u_j, \ldots, w_k, u_q, w_h, \ldots).$$

We have $j > q-2$ by the definition of C_{i-1}. Since C_{i-1} is Hamiltonian, $u_{q-1}w_{q-1} \notin E(C_{i-1})$, and thus $j \neq q-1$. We also have $j > q$ from $u_qw_{q-1} \notin E(C_{i-1})$. We have $k, h > q-2$ by the definition of C_{i-1}. Since $u_qw_{q-1} \notin E(C_{i-1})$, we have $k, h \neq q-1$, and thus $k, h > q-1$. Since $u_qw_h, u_jw_{q-1} \in E(G)$, we have $u_qw_{q-1}, u_jw_h \in E(G)$ by the definition of strong orderings. We define that C_i is the Hamiltonian cycle obtained from C_{i-1} by switching the edges u_qw_h and u_jw_{q-1} with the edges u_qw_{q-1} and u_jw_h, that is,

$$C_i = (u_{q-1}, \ldots, w_{q-1}, u_q, w_k, \ldots, u_j, w_h, \ldots).$$

We finally construct C_i from C_{i-1} with $i = 2q + 1 \geq 3$. Recall that C_{i-1} contains the edges on C_t induced by $\{u_0, \ldots, u_{q-1}, w_{q-1}, u_q\}$, When $u_{q-1}w_q \in E(C_{i-1})$, we define C_{i-1} as C_i. We then consider the case when $u_{q-1}w_q \notin E(C_{i-1})$. Let w_j, u_k, u_h be the vertices of G such that

$$C_{i-1} = (u_q, w_{q-1}, \ldots, u_{q-1}, w_j, \ldots, u_k, w_q, u_h, \ldots).$$

We have $j > q - 1$ by the definition of C_{i-1}. Since $u_{q-1}w_q \notin E(C_{i-1})$, we have $j > q$. We also have $k, h > q - 2$ by the definition of C_{i-1}. Since $u_{q-1}w_q \notin E(C_{i-1})$, we have $k, h \neq q - 1$, and thus $k, h > q - 1$. Since $u_{q-1}w_j, u_h w_q \in E(G)$, we have $u_{q-1}w_q, u_h w_j \in E(G)$ by the definition of strong orderings. We define that C_i is the Hamiltonian cycle obtained from C_{i-1} by switching the edges $u_{q-1}w_j$ and $u_h w_q$ with the edges $u_{q-1}w_q$ and $u_h w_j$, that is,

$$C_i = (u_q, w_{q-1}, \ldots, u_{q-1}, w_q, u_k, \ldots, w_j, u_h, \ldots).$$

Therefore, we have the sequence of at most $n - 3$ switches transforming C_0 into the canonical cycle C_t. □

We also have the following from Theorem 9.

Corollary 3. *For each Hamiltonian cycle of a bipartite permutation graph G, we can compute a sequence of switches transforming it to the canonical cycle of G in $O(n)$ time, provided that a strong ordering of G is given.*

Proof. The proof is similar to that of Corollary 1, and is omitted. □

Now, we have the following from Theorem 9 and Corollary 3.

Corollary 4. *For any two Hamiltonian cycles of a bipartite permutation graph, there is a sequence of at most $2n - 6$ switches transforming one cycle to the other. Moreover, we can compute such a sequence in $O(n)$ time, provided that a strong ordering of G is given.*

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