A Note on the Intersection of Alternately Orientable Graphs and Cocomparability Graphs

PAPER *Special Section on Discrete Mathematics and Its Applications*

A Note on the Intersection of Alternately Orientable Graphs and Cocomparability Graphs

Asahi TAKAOKA[†a)], ***Member***

SUMMARY We studied whether a statement similar to the Ghouila-Houri's theorem might hold for alternating orientations of cocomparability graphs. In this paper, we give the negative answer. We prove that it is NP-complete to decide whether a cocomparability graph has an orientation that is alternating and acyclic. Hence, cocomparability graphs with an acyclic alternating orientation form a proper subclass of alternately orientable cocomparability graphs. We also provide a separating example, that is, an alternately orientable cocomparability graph such that no alternating orientation is acyclic.

key words: *alternately orientable graphs, cocomparability graphs, recognition problem, simple-triangle graphs*

1. Introduction

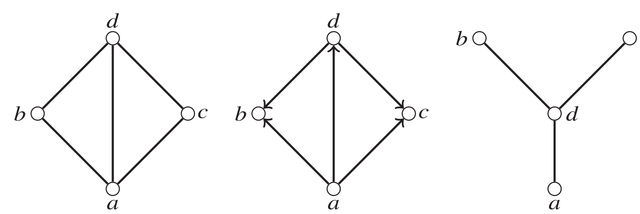
All graphs in this paper are finite without loops or multiple edges. Unless stated otherwise, graphs are assumed to be undirected, but we also deal with directed graphs. We write uv for the *undirected edge* joining two vertices u and v, and we write (u, v) for the *directed edge* from u to v. For a graph $G = (V, E)$, we sometimes write $V(G)$ for the vertex set V and write $E(G)$ for the edge set E.

Let G be an undirected graph. An *orientation* of G is a directed graph obtained from G by orienting each edge of G, that is, replacing each edge $uv \in E(G)$ with either (u, v) or (v, u). We will denote an orientation only by its edge set because the vertex set is clear from the context.

An orientation F of a graph G is *transitive* if $(u, v) \in F$ and $(v, w) \in F$ imply $(u, w) \in F$ for any three vertices u, v, w of G, see Fig. 1(a) for example. A graph is *transitively orientable* if it has a transitive orientation. Transitively orientable graphs are also called *comparability graphs*. The complement of a comparability graph is a *cocomparability graph*, where the *complement* of a graph G is the graph $\overline{G}$ such that $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G}) \iff uv \notin E(G)$ for any two vertices u, v of G.

Comparability graphs and cocomparability graphs are two of the most fundamental classes in graph theory, see, e.g., [1]–[3]. They can be recognized and a transitive orientation can be obtained in polynomial time, see, e.g., [2], [4].

Transitive orientations naturally correspond to posets. A *partially ordered set* (*poset* for short) is a pair $P = (V, \leq)$, where V is a ground set and $\leq$ is a binary relation on V that



(a) A graph G. (b) An orientation F of G. (c) A poset P.

Fig. 1 An example of transitive orientation. The poset is depicted by its Hasse diagram. The orientation F is a transitive orientation of G. The graph G is the comparability graph of P, and the poset P corresponds to F.

is reflexive, transitive, and antisymmetric. We denote $u < v$ if $u \leq v$ and $u \neq v$. Two elements $u, v \in V$ are *comparable* in P if $u < v$ or $u > v$. The *comparability graph of a poset* P is the graph $G = (V, E)$ such that $uv \in E$ if u and v are comparable in P, see Figs. 1(c) and 1(a) for example. Note that a poset P can be viewed as the transitive orientation F of its comparability graph such that $(u, v) \in F \iff u < v$ in P for any two elements u, v of P, see Figs. 1(c) and 1(a) for example. The complement of comparability graph of a poset P is the *cocomparability graph of* P.

An orientation F of a graph G is *quasi-transitive* if $(u, v) \in F$ and $(v, w) \in F$ imply $uw \in E(G)$, that is, $(u, w) \in F$ or $(w, u) \in F$. In other words, an orientation F is quasi-transitive if for any vertices u, v, w with $uv, vw \in E(G)$ and $uw \notin E(G)$, we have $(u, v), (w, v) \in F$ or $(v, u), (v, w) \in F$.

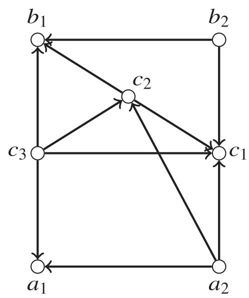
An orientation is transitive if and only if it is quasi-transitive and acyclic. Thus, every transitively orientable graph is quasi-transitively orientable. The classical theorem of Ghouila-Houri [5] states that the converse also holds. We note that another proof of the theorem is shown in [6].

Theorem 1. *If a graph has a quasi-transitive orientation, then it has quasi-transitive orientation that is acyclic.*

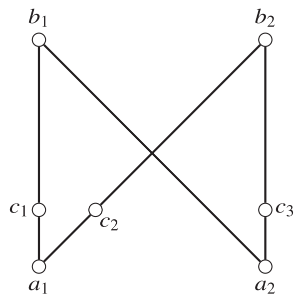
An orientation of a graph is *alternating* [7] if it is transitive on every chordless cycle of length greater than or equal to 4, that is, the directions of the edges alternate on the cycles, see Fig. 2(a) for example. A graph is *alternately orientable* if it has an alternating orientation. Note that, by definition, every transitively orientable graph is alternately orientable. Alternately orientable graphs can be recognized and an alternating orientation can be obtained in polynomial time [7].

It was conjectured that a statement similar to the

Manuscript received August 6, 2021.
Manuscript revised January 19, 2022.
Manuscript publicized March 7, 2022.
[†]The author is with the College of Information and Systems, Muroran Institute of Technology, Muroran-shi, 050-8585 Japan.
a) E-mail: takaoka@mmm.muroran-it.ac.jp
DOI: 10.1587/transfun.2021DMP0001



(a) An orientation F.



(b) A poset P.

Fig. 2 An example of alternating orientation. The poset is depicted by its Hasse diagram. The orientation F is an alternating orientation of the cocomparability graph of P. Note that F is not transitive since $(a_2, c_2), (c_2, b_1) \in F$ but $(a_2, b_1) \notin F$. The union of F and the transitive orientation corresponding to P is acyclic.

Ghouila-Houri's theorem (i.e., Theorem 1) might hold for alternating orientation [7], that is, it was conjectured that every alternately orientable graph has an alternating orientation that is acyclic. Later, however, a counterexample was provided [8]. Thus, graphs with an acyclic alternating orientation form a proper subclass of alternately orientable graphs. Moreover, it is NP-complete to decide whether a graph has an alternating orientation that is acyclic [9].

Recall that a cocomparability graph is the complement of a transitively orientable graph. A cocomparability graph is a permutation graph if and only if it is transitively orientable, see, e.g., [2]. Permutation graphs can be recognized in linear time [4].

A cocomparability graph is a trapezoid graph if it is alternately orientable, but the converse does not hold [10]. Thus, alternately orientable cocomparability graphs form a proper subclass of trapezoid graphs. Alternately orientable cocomparability graphs can be recognized in polynomial time [9].

A cocomparability graph G is a simple-triangle graph [11] if and only if there is an alternating orientation F of G and a transitive orientation D of $\overline{G}$ such that $F \cup D$ is acyclic [12]. Figure 2 shows an example from [12]. Thus, simple-triangle graphs form a subclass of alternately orientable cocomparability graphs. This inclusion is known to be proper [12]. Simple-triangle graphs can be recognized in polynomial time [9], [13], [14].

It was recently shown that, for simple-triangle graphs, a statement similar to the Ghouila-Houri's theorem holds as follows. Let G be a cocomparability graph, and let D be a transitive orientation of $\overline{G}$. We say that an orientation F of G is *Δ-free* if $F \cup D$ contains no directed cycle (u, v, w) with $(u, v), (v, w) \in F$ and $(w, u) \in D$. Note that $F \cup D$ is acyclic if and only if F is acyclic and Δ-free. Thus, G is a simple-triangle graph if and only if G has an alternating orientation F that is acyclic and Δ-free. The following statement holds [9].

Theorem 2. *If a cocomparability graph has an alternating orientation that is Δ-free, then it has an alternating orientation that is acyclic and Δ-free.*

Now, it is natural to ask whether every alternately orientable cocomparability graph has an alternating orientation that is acyclic. We give a negative answer to this question.

2. Preliminaries

A *cycle* of a graph G is a sequence of distinct vertices $(v_1, v_2, \dots, v_k)$ of G with $v_1v_2, v_2v_3, \dots, v_{k-1}v_k, v_kv_1 \in E(G)$. The *length* of the cycle is the number k of the edges on the cycle. A *chord* of a cycle is an edge joining two vertices that are not consecutive on the cycle. A cycle is *chordless* if it contains no chords.

Let F be an orientation of G. A *directed cycle* of F is a sequence of distinct vertices $(v_1, v_2, \dots, v_k)$ of G with $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k), (v_k, v_1) \in F$. The length of a directed cycle is defined analogously to the undirected case.

Any cocomparability graph contains no chordless cycle of length greater than or equal to 5, see, e.g., [15], [16]. Thus, we have the following.

Lemma 3. *An orientation of a cocomparability graph is alternating if and only if it alternates on every chordless cycle of length 4.*

Any chordless cycle of odd length n has no alternating orientation if $n \geq 5$. Since the directions of the edges alternate on every cycles of even length, we have the following.

Lemma 4. *An alternating orientation is acyclic if and only if it contains no directed cycles of length 3.*

3. NP-Completeness

The following is our main result.

Theorem 5. *It is NP-complete to decide whether a graph has an alternating orientation that is acyclic, even if the graph is a cocomparability graph.*

Proof. We can verify in polynomial time whether an orientation is alternating and acyclic. Thus, the problem is in NP. We now show a polynomial-time reduction from the betweenness problem, which is known to be NP-complete [17].

The *betweenness problem* is as follows. We are given a positive integer n and a set T of m ordered triples of distinct elements of $[n]$, where $[n]$ denotes the set $\{1, 2, \dots, n\}$. Each triple of T is called a *betweenness constraint*. A permutation π on $[n]$ *satisfies* a constraint $(a, b, c) \in T$ if $\pi(a) < \pi(b) < \pi(c)$ or $\pi(a) > \pi(b) > \pi(c)$. The betweenness problem is to decide whether there is a permutation on $[n]$ that satisfies all constraints of T.

Assume that the constraints of T are numbered from 1 to m, and let $(a(k), b(k), c(k))$ denote the k-th constraint. We construct a poset $P = (V, \preceq)$ as follows, see Fig. 3. Let p, q, r, s be functions from $[m]$ to $[4m]$ such that

$$p(k) = 4k - 3, q(k) = 4k - 2,$$
$$r(k) = 4k - 1, \text{ and } s(k) = 4k.$$

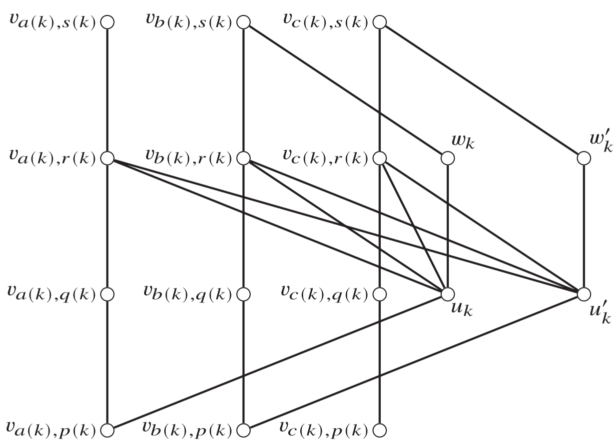


Fig. 3 An induced subposet of P depicted by its Hasse diagram. We omit the relations between V_h and $V_{h'}$ with $h+2 \le h'$ for simplicity.

The elements of P are partitioned so that

$$V = \bigcup_{h \in [4m]} V_h, \text{ where}$$
$$V_h = \begin{cases} \tilde{V}_h & \text{if } h = p(k) \text{ or } h = s(k), \\ \tilde{V}_h \cup \{u_k, u'_k\} & \text{if } h = q(k), \\ \tilde{V}_h \cup \{w_k, w'_k\} & \text{if } h = r(k), \end{cases}$$
$$\tilde{V}_h = \{v_{i,h} \colon\ i \in [n]\}.$$

The relations of P is as follows:

1. $v_{i,h} \prec v_{i,h+1}$ for any $i \in [n]$ and $h \in [4m-1]$.
2. For any $k \in [m]$,
$$v_{a(k),p(k)} \prec u_k \prec w_k \prec v_{b(k),s(k)},$$
$$v_{b(k),p(k)} \prec u'_k \prec w'_k \prec v_{c(k),s(k)}.$$
3. $u_k \prec v$ and $u'_k \prec v$ for any $k \in [m]$ and $v \in \tilde{V}_{r(k)}$.
4. $v \prec v'$ for any $v \in V_h$ and $v' \in V_{h'}$ with $h+2 \le h'$.

It is easy to verify that P is transitive, and hence, a poset.

Let G be the cocomparability graph of P. It is clear that P and G can be constructed in time polynomial in n and m. It remains to show that G has an acyclic alternating orientation if and only if there is a permutation on $[n]$ satisfying the betweenness constraints.

Suppose that G has an alternating orientation F that is acyclic. It alternates on $(v_{i,h}, v_{j,h}, v_{i,h+1}, v_{j,h+1})$ for any $i, j \in [n]$ and $h \in [4m-1]$. Hence, $(v_{i,0}, v_{j,0}) \in F \iff (v_{i,h}, v_{j,h}) \in F$. For any $k \in [m]$, the orientation F alternates on the following chordless cycles:

$$(v_{a(k),p(k)}, v_{b(k),p(k)}, u_k, u'_k), (u_k, u'_k, w_k, w'_k),$$
$$(w_k, w'_k, v_{b(k),s(k)}, v_{c(k),s(k)}).$$

Then we would have the following equivalences:

$$\begin{aligned}(v_{a(k),p(k)}, v_{b(k),p(k)}) \in F &\iff (u_k, u'_k) \in F \\ &\iff (w_k, w'_k) \in F \\ &\iff (v_{b(k),s(k)}, v_{c(k),s(k)}) \in F.\end{aligned}$$

Thus, $(v_{a(k),0}, v_{b(k),0}) \in F \iff (v_{b(k),0}, v_{c(k),0}) \in F$.

We now define a permutation π by $\pi(i) < \pi(j) \iff (v_{i,0}, v_{j,0}) \in F$. Since F is acyclic, π is well-defined. Since $\pi(a(k)) < \pi(b(k)) \iff \pi(b(k)) < \pi(c(k))$ for any $k \in [m]$, the permutation π satisfies the betweenness constraints.

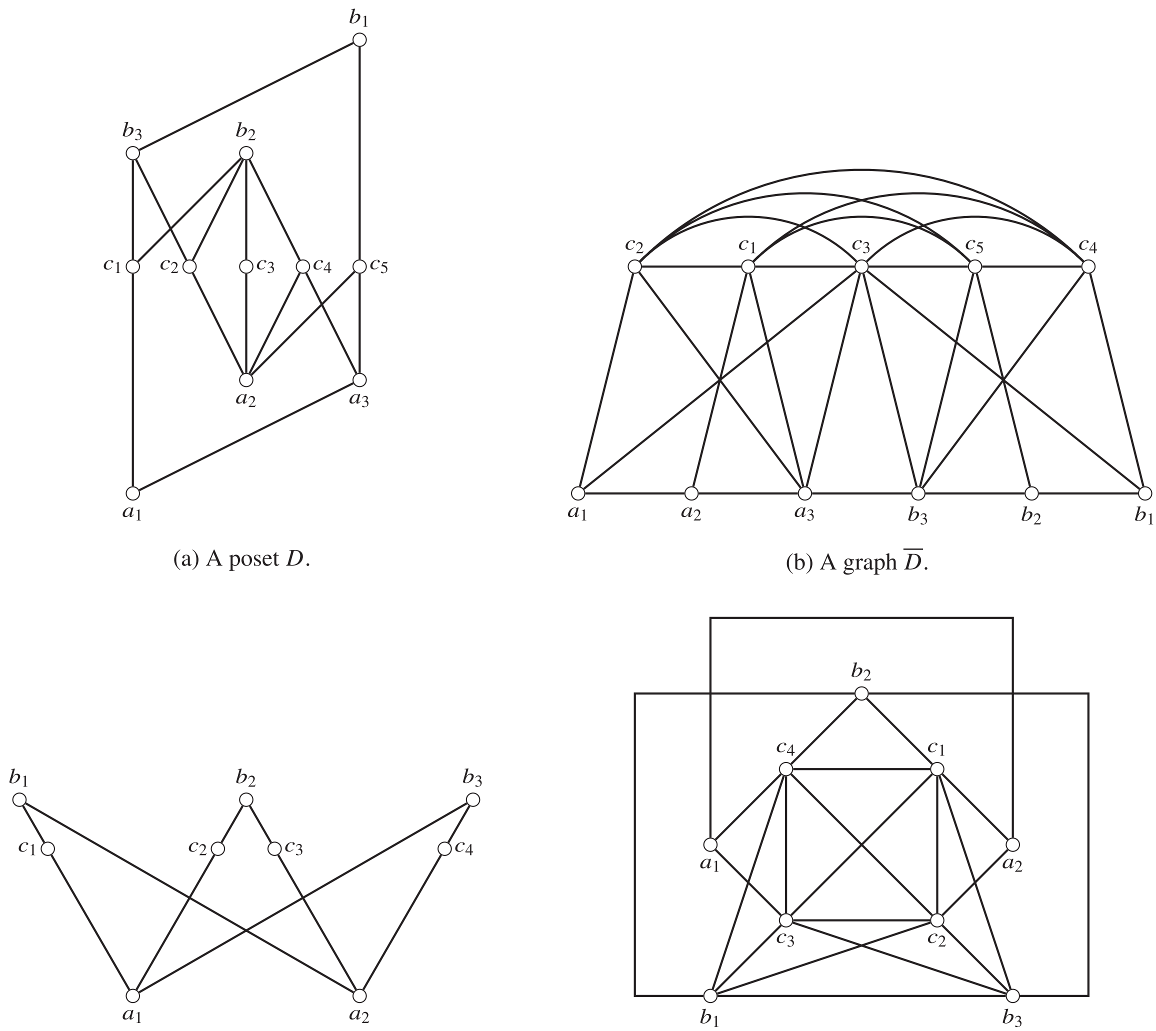
Suppose that there is a permutation π on $[n]$ that satisfies the betweenness constraints. We assume without loss of generality $\pi(1) < \pi(2) < \cdots < \pi(n)$. We define an orientation F of G as follows:

1. For any $i, j \in [n]$ with $i < j$,
   - $(v_{i,h}, v_{j,h}) \in F$ for any $h \in [4m]$,
   - $(v_{i,h}, v_{j,h+1}), (v_{i,h+1}, v_{j,h}) \in F$ for any $h \in [4m-1]$.
2. For any $i \in [n]$ and $k \in [m]$,
   - $(v_{i,p(k)}, u_k) \in F$ if $i < a(k)$ and $(u_k, v_{i,p(k)}) \in F$ if $a(k) < i$,
   - $(v_{i,q(k)}, u_k) \in F$ if $i \le a(k)$ and $(u_k, v_{i,q(k)}) \in F$ if $a(k) < i$,
   - $(v_{i,p(k)}, u'_k) \in F$ if $i < b(k)$ and $(u'_k, v_{i,p(k)}) \in F$ if $b(k) < i$,
   - $(v_{i,q(k)}, u'_k) \in F$ if $i \le b(k)$ and $(u'_k, v_{i,q(k)}) \in F$ if $b(k) < i$,
   - $(v_{i,q(k)}, w_k), (v_{i,r(k)}, w_k) \in F$ if $i \le b(k)$ and $(w_k, v_{i,q(k)}), (w_k, v_{i,r(k)}) \in F$ if $b(k) < i$,
   - $(v_{i,s(k)}, w_k) \in F$ if $i < b(k)$ and $(w_k, v_{i,s(k)}) \in F$ if $b(k) < i$,
   - $(v_{i,q(k)}, w'_k), (v_{i,r(k)}, w'_k) \in F$ if $i \le c(k)$ and $(w'_k, v_{i,q(k)}), (w'_k, v_{i,r(k)}) \in F$ if $c(k) < i$,
   - $(v_{i,s(k)}, w'_k) \in F$ if $i < c(k)$ and $(w'_k, v_{i,s(k)}) \in F$ if $c(k) < i$.
3. For any $k \in [m]$,
   - $(u_k, u'_k), (u_k, w'_k), (w_k, u'_k), (w_k, w'_k) \in F$ if $a(k) < b(k) < c(k)$,
   - $(u'_k, u_k), (u'_k, w_k), (w'_k, u_k), (w'_k, w_k) \in F$ if $a(k) > b(k) > c(k)$.

We first prove that F is alternating. By Lemma 3, it suffices to show that F alternates on every chordless cycle of length 4. Note that each cycle consists of two vertices from V_h and two vertices from V_{h+1} for some $h \in [4m-1]$, because V_h induces a clique for each $h \in [4m]$ and there is no edge between V_h and $V_{h'}$ with $|h - h'| \ge 2$. Let $G[V_{p(k)} \cup V_{q(k)}]$ denote the subgraph of G induced by $V_{p(k)} \cup V_{q(k)}$. The subgraph $G[V_{p(k)} \cup V_{q(k)}]$ contains the following chordless cycles:

$$(v_{i,p(k)}, v_{j,p(k)}, v_{i,q(k)}, v_{j,q(k)}) \text{ with } i, j \in [n],$$
$$(v_{i,p(k)}, v_{a(k),p(k)}, v_{i,q(k)}, u_k) \text{ with } i \in [n] \setminus \{a(k)\},$$
$$(v_{i,p(k)}, v_{b(k),p(k)}, v_{i,q(k)}, u'_k) \text{ with } i \in [n] \setminus \{b(k)\},$$
$$(v_{a(k),p(k)}, v_{b(k),p(k)}, u_k, u'_k).$$

Then we can check that F alternates on the cycles. Similarly, F alternates on every chordless cycle of $G[V_{r(k)} \cup V_{s(k)}]$ and $G[V_{s(k)} \cup V_{p(k+1)}]$. The subgraph $G[V_{q(k)} \cup V_{r(k)}]$ contains the following chordless cycles:



(a) A poset D.

(b) A graph $\overline{D}$.

(c) A poset W.

(d) A graph $\overline{W}$.

Fig. 4 Separating examples. Posets are depicted by their Hasse diagrams.

$(v_{i,q(k)}, v_{j,q(k)}, v_{i,r(k)}, v_{j,r(k)})$ with $i, j \in [n]$,
(u_k, u'_k, w_k, w'_k).

Then we can check that F alternates on the cycles. Thus, F is an alternating orientation.

We now prove that F is acyclic. Suppose that F contains a directed cycle. By Lemma 4, the cycle consists of three vertices. Since there is no edge between V_h and $V_{h'}$ with $|h - h'| \geq 2$, the cycle consists of vertices from $V_h \cup V_{h+1}$ for some $h \in [4m-1]$. It is clear that $G[\bigcup_{h\in[4m]} \tilde{V}_h]$ contains no directed cycles. We also have that $G[V_{q(k)}]$ and $G[V_{r(k)}]$ contains no directed cycles for every $k \in [m]$. Suppose that $G[V_{p(k)} \cup V_{q(k)}]$ contains a directed cycle. Note that $(u_k, v_{i,p(k)}) \in F \iff (u_k, v_{i,q(k)}) \in F$ for any $i \in [n] \setminus \{a(k)\}$. Similarly, $(u'_k, v_{i,p(k)}) \in F \iff (u'_k, v_{i,q(k)}) \in F$ for any $i \in [n] \setminus \{b(k)\}$. Note also that $(v_{i,p(k)}, v_{j,p(k)}) \in F \iff (v_{i,p(k)}, v_{j,q(k)}) \in F \iff (v_{i,q(k)}, v_{j,p(k)}) \in F \iff (v_{i,q(k)}, v_{j,q(k)}) \in F$ for any $i, j \in [n]$. Therefore, $G[V_{q(k)}]$ contains a directed cycle, a contradiction. By similar arguments, we have that $G[V_{r(k)} \cup V_{s(k)}]$ contains no directed cycles. Since $(w_k, v_{i,q(k)}) \in F \iff (w_k, v_{i,r(k)}) \in F$ and $(w'_k, v_{i,q(k)}) \in F \iff (w'_k, v_{i,r(k)}) \in F$ for any $i \in [n]$, if $G[V_{q(k)} \cup V_{r(k)}]$ contains a directed cycle, then $G[V_{r(k)}]$ contains a directed cycle, a contradiction. Therefore, F is acyclic, and the theorem holds. □

4. Separating Examples

Theorem 5 indicates that cocomparability graphs with an acyclic alternating orientation form a proper subclass of alternately orientable cocomparability graphs, because alternately orientable cocomparability graphs can be recognized in polynomial time [9]. We now show a separating example.

Example 6. The graph $\overline{D}$ in Fig. 4(b), which is the cocomparability graph of the poset D in Fig. 4(a), is alternately orientable, but no alternating orientation is acyclic.

Proof. Suppose that $\overline{D}$ has an alternating orientation F, which alternates on the following chordless cycles:

$$(c_1, c_5, b_3, a_3), (c_1, a_3, b_3, c_4),$$
$$(b_3, c_4, b_1, b_2), (b_1, b_2, c_5, c_3).$$

Then we would have the following equivalences:

$$\begin{aligned}(c_1, c_5) \in F &\iff (c_1, a_3) \in F \iff (b_3, c_4) \in F \\ &\iff (b_1, b_2) \in F \iff (c_5, c_3) \in F.\end{aligned}$$

The orientation F also alternates on the following chordless cycles:

$$(c_1, c_5, b_3, a_3), (b_3, c_5, c_2, a_3),$$
$$(c_2, a_3, a_2, a_1), (a_2, a_1, c_3, c_1).$$

Then we would have the following equivalences:

$$\begin{aligned}(c_1, c_5) \in F &\iff (b_3, c_5) \in F \iff (c_2, a_3) \in F \\ &\iff (a_2, a_1) \in F \iff (c_3, c_1) \in F.\end{aligned}$$

Thus, F contains a directed cycle (c_1, c_5, c_3) if $(c_1, c_5) \in F$ and a directed cycle (c_5, c_1, c_3) if $(c_5, c_1) \in F$. Therefore, no alternating orientation of $\overline{D}$ is acyclic. It is routine work to see that $\overline{D}$ is alternately orientable. □

Theorem 5 also indicates that simple-triangle graphs form a proper subclass of cocomparability graphs with an acyclic alternating orientation. A separating example can be found in [12].

Example 7. The graph $\overline{W}$ in Fig. 4(d), which is the cocomparability graph of the poset W in Fig. 4(c), has an orientation that is alternating and acyclic. The graph $\overline{W}$ is not a simple-triangle graph.

Proof. As shown in [12], the graph $\overline{W}$ is not a simple-triangle graph. It is straight to see that $\overline{W}$ has a unique (up to reversal) alternating orientation that is acyclic. □

5. Concluding Remarks

Inspired by the recent result for simple-triangle graphs (i.e., Theorem 2), we studied whether a statement similar to the Ghouila-Houri's theorem (i.e., Theorem 1) might hold for alternating orientations of cocomparability graphs. In this paper, we give the negative answer. We prove that it is NP-complete to recognize cocomparability graphs with an acyclic alternating orientation, indicating that simple-triangle graphs form a proper subclass of cocomparability graphs with an acyclic alternating orientation, which is a proper subclass of alternately orientable cocomparability graphs. We also provide the separating examples.

Acknowledgments

We are grateful to the anonymous referees for their time and suggestions. A part of this work was done while the author was in Kanagawa University.

References

[1] A. Brandstädt, V.B. Le, and J.P. Spinrad, Graph Classes: A Survey, SIAM, Philadelphia, PA, USA, 1999.
[2] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, 2 ed., Ann. Discrete Math., vol.57, Elsevier, 2004.
[3] M.C. Golumbic and A.N. Trenk, Tolerance Graphs, Cambridge University Press, 2004.
[4] R.M. McConnell and J.P. Spinrad, "Modular decomposition and transitive orientation," Discrete Math., vol.201, no.1-3, pp.189–241, 1999.
[5] A. Ghouila-Houri, "Caractérisation des graphes non orientés dont on peut orienter les arêtes de manière à obtenir le graphe d'une relation d'ordre," C.R. Acad. Sci., vol.254, no.1, pp.1370–1371, 1962 (in French).
[6] P. Hell and J. Huang, "Lexicographic orientation and representation algorithms for comparability graphs, proper circular arc graphs, and proper interval graphs," J. Graph Theory, vol.20, no.3, pp.361–374, 1995.
[7] C.T. Hoàng, "Alternating orientation and alternating colouration of perfect graphs," J. Combin. Theory Ser. B, vol.42, no.3, pp.264–273, 1987.
[8] S. Hougardy, "Counterexamples to three conjectures concerning perfect graphs," Discrete Math., vol.117, no.1-3, pp.245–251, 1993.
[9] A. Takaoka, "A recognition algorithm for simple-triangle graphs," Discret. Appl. Math., vol.282, pp.196–207, 2020.
[10] S. Felsner, "Tolerance graphs, and orders," J. Graph Theory, vol.28, no.3, pp.129–140, 1998.
[11] D.G. Corneil and P.A. Kamula, "Extensions of permutation and interval graphs," Congr. Numer., vol.58, pp.267–275, 1987.
[12] A. Takaoka, "A vertex ordering characterization of simple-triangle graphs," Discrete Math., vol.341, no.12, pp.3281–3287, 2018.
[13] G.B. Mertzios, "The recognition of simple-triangle graphs and of linear-interval orders is polynomial," SIAM J. Discrete Math., vol.29, no.3, pp.1150–1185, 2015.
[14] A. Takaoka, "Recognizing simple-triangle graphs by restricted 2-chain subgraph cover," Discret. Appl. Math., vol.279, pp.154–167, 2020.
[15] T. Gallai, "Transitiv orientierbare graphen," Acta Math. Acad. Sci. Hung., vol.18, no.1–2, pp.25–66, 1967.
[16] W.T. Trotter, Combinatorics and Partially Ordered Sets: Dimension Theory, 1992.
[17] J. Opatrny, "Total ordering problem," SIAM J. Comput., vol.8, no.1, pp.111–114, 1979.

Asahi Takaoka received the B.E. degree in computer science and M.E. and D.E. degrees in communications and integrated systems from Tokyo Institute of Technology, Japan, in 2010, 2012, and 2015, respectively. From 2016 to 2020, he was an assistant professor in Kanagawa University, Japan. He is currently an assistant professor in Muroran Institute of Technology. His research interests are in algorithmic graph theory and structure of graph classes. He is a member of IEICE and IEEE.