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One-parameter families of Legendre curves and plane line congruences

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Summary

Families of curves in the Euclidean plane naturally contain singular curves, where the frame of classical differential geometry does not work well. We introduce the notions of one-parameter family of Legendre curves in the Euclidean plane, congruent equivalence and curvature. Especially, a one-parameter family of Legendre curves can contain singular curves, and is determined by the curvature up to congruence. We also give properties of one-parameter families of Legendre curves. As applications, we give a relation between one-parameter families of Legendre curves and Legendre surfaces. Moreover, we study plane line congruences (one-parameter families of lines in plane) in terms of the curvatures as one-parameter families of Legendre curves.

KEYWORDS
One-parameter family of Legendre curves, curvature, plane to plane map, plane line congruence

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58K05, 57R45, 53A55

1 | INTRODUCTION

We are interested in differential geometrical aspects of families of curves in the Euclidean plane. In the method of classical curve theory, one can never deal with families of curves to which degenerate curves belong: for example, a family of parallel curves in plane where curves naturally become non-immersive as in Figure 1; a one-parameter family of lines in plane as in Figure 2 (the curvature of a line is identically zero). This paper aims to give a new framework to deal with such a general one-parameter family of curves.

As smooth plane curves with singular points, that is, singular plane curves, we may consider frontals and Legendre curves in the unit tangent bundle over the Euclidean plane. In [7], we gave existence and uniqueness theorems of the curvature of Legendre curves. In the present paper, as plane to plane maps, we consider one-parameter families of Legendre curves. We define the notions of a congruent equivalence and a curvature such that the one-parameter family of Legendre curves is determined by the curvature up to congruence, which is a natural expansion of the theory for Legendre curves in [7].

We study a relation between the curvature of the map as a one-parameter family of Legendre curves and differential topological invariants of the map some of which are induced from singularity theory. Notice that the curvature is a kind of differential geometrical invariant. In the usual sense, differential geometry means the geometry of a map $\mathbb{R}^n \to \mathbb{R}^p$ where the dimension number p of the target space is bigger than the dimension number n of the source space. We emphasize that our approach implies a new direction of differential geometry: differential geometry of a map with general dimensions of the source and target spaces. Note that K. Saji showed a different approach to this idea from the viewpoint of normal forms of singularities in his talk at Kobe,

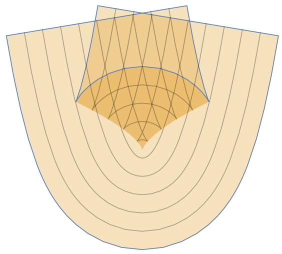

FIGURE 1 The family of parallel curves to a parabola in a plane. Some curves have cusp singularities.

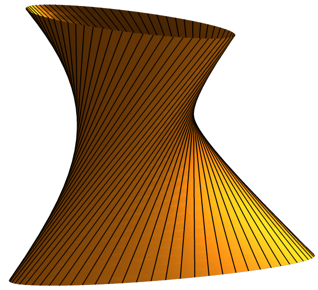

FIGURE 2 A ruled surface. The linear projection to a plane gives a plane line congruence.

2017 (similar method for singularities of surfaces are shown in [22, 23, 28]). Also, his papers about criteria for singularities suggest another approach [26, 27].

One of the most typical classes in one-parameter families of Legendre curves is the class of one-parameter families of lines in plane (called the *plane line congruences* in the present paper). Line geometry is a classical subject (cf. [21, 31]), and recently singularity theory provides it with new insights (cf. [6, 16, 17, 18, 30]). For example, singularities generically appearing in line congruences (two-parameter families of lines) in 3-space or ruled surfaces (one-parameter families of lines) in 3-space are classified in [17, 18]. We deal with a plane line congruence, and compare the curvature of it with the types of singularities of maps and functions related to it in §5. Especially, in §6, the exact $\mathcal{A}$-equivalent types (up to $\mathcal{A}_e$-codimension two) of plane line congruences as a map $\mathbb{R}^2 \to \mathbb{R}^2$ are geometrically characterized in terms of the curvature as a one-parameter families of Legendre curves. Note that the singularities of plane line congruences as plane to plane maps are considered as the envelope or evolute in a generalized sense (cf. Figures 8-15).

Moreover, one-parameter families of curves naturally appear when we project surfaces equipped with families of curves into planes. For example, the projections of ruled surfaces to planes give plane line congruences, see Figure 2. There have been a lot of works on the application of singularity theory to the area of vision science (cf. [4, 5, 16, 20]), while they have been mainly concerned with the apparent contour of a surface (the discriminant of a projection mapping restricted to the surface). On the other hand, if the surface is equipped with a family of curves suitably, we have the curvature of the families of the projected curves (as a one-parameter families of Legendre curves) at points even outside the apparent contour. For instance, in §5.1.1, §6.2 and §7.1, we investigate a local nature of a plane (normal) line congruence around a point at which the Jacobian of the map is not equal to zero, but the differential vanishes. Namely, the point is not a singularity of the map but a singularity of the Jacobian. Note also that the Jacobian is characterized by the curvature of the plane line congruence. Thus our method possibly gives new tools to the area of vision science. On the other hand, as smooth surfaces with singular points, that is, singular surfaces, we may consider frontals or framed surfaces in the Euclidean space [1, 2, 11, 12]. Hence we can investigate the differential geometrical relation between a surface and its projected image in a more general setting than ever before. This application will be discussed in somewhere else by the authors.

The paper is organized as follows: We give the existence and uniqueness theorems of the curvatures of one-parameter families of Legendre curves in §2. We also give properties of one-parameter families of Legendre curves in §3. As applications, we give a relation between one-parameter families of Legendre curves and Legendre surfaces in §4. Moreover, in §5-7, we study local geometry of plane line congruences from the viewpoint of the curvatures as one-parameter families of Legendre curves, where normal line congruences are mainly dealt with. In §5, we study what kind of geometrical information is given from $\mathcal{R}$-types of functions in the curvatures of normal line congruences. In §6, we study the relation between higher order information of functions in the curvatures and several geometrical properties of normal line congruences: some new notions (the index of a function, Jacobian constant curve etc.) are defined; and several unstable $\mathcal{A}$-equivalent types of normal line congruences are precisely studied. In §7, we show several examples of plane line congruences with figures.

All maps and manifolds considered here are differential of class C^∞.

2 | LEGENDRE CURVES AND ONE-PARAMETER FAMILIES OF LEGENDRE CURVES

Let $\mathbb{R}^2$ be the Euclidean plane equipped with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$, where $\boldsymbol{a} = (a_1, a_2), \boldsymbol{b} = (b_1, b_2) \in \mathbb{R}^2$. We denote the norm of $\boldsymbol{a}$ by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$.

We review on the theory of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$, in detail see [7]. We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma, \nu)^*\theta = 0$ for all $t \in I$, where θ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$ (cf. [1, 2]). This condition is equivalent to $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^2$ is a *frontal* if there exists $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve. Examples of Legendre curves see [14, 15]. We denote by $J(\boldsymbol{a}) = (-a_2, a_1)$ the anticlockwise rotation by $\pi/2$ of a vector $\boldsymbol{a} = (a_1, a_2)$. We have the Frenet formula of a frontal γ as follows. We put on $\boldsymbol{\mu}(t) = J(\nu(t))$. Then we call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame of a frontal* $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet formula of the frontal (or, Legendre curve),

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \ \dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t),$$

where $\ell(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$ and $\beta(t) = \dot{\gamma}(t) \cdot \boldsymbol{\mu}(t)$. We call the pair (ℓ, β) *the curvature of the Legendre curve*.

Definition 2.1. Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are *congruent as Legendre curves* if there exist a constant rotation $A \in SO(2)$ and a translation $\boldsymbol{a}$ on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t) = A(\gamma(t)) + \boldsymbol{a}$ and $\widetilde{\nu}(t) = A(\nu(t))$ for all $t \in I$.

Theorem 2.2 (Existence Theorem for Legendre curves)**.** Let $(\ell, \beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is (ℓ, β).

Theorem 2.3 (Uniqueness Theorem for Legendre curves)**.** Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves with the curvatures of Legendre curves (ℓ, β) and $(\widetilde{\ell}, \widetilde{\beta})$. Then (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are congruent as Legendre curves if and only if (ℓ, β) and $(\widetilde{\ell}, \widetilde{\beta})$ coincide.

We now consider one-parameter families of Legendre curves in the unit tangent bundle $T_1\mathbb{R}^2$ over $\mathbb{R}^2$. Let U be a simply connected domain in $\mathbb{R}^2$.

Definition 2.4. Let $(f, \nu) : U \to \mathbb{R}^2 \times S^1$ be a smooth mapping. We say that (f, ν) is a *one-parameter family of Legendre curves with respect to u* (respectively, *with respect to v*) if $f_u(u, v) \cdot \nu(u, v) = 0$ (respectively, $f_v(u, v) \cdot \nu(u, v) = 0$) for all $(u, v) \in U$.

If (f, ν) is a one-parameter family of Legendre curves with respect to u, then $(f(\cdot, v), \nu(\cdot, v))$ is a Legendre curve for each fixed parameter v, that is, $(f(\cdot, v), \nu(\cdot, v))$ is an integrable curve with respect to the canonical contact 1-form on $\mathbb{R}^2 \times S^1$. Therefore, $f : U \to \mathbb{R}^2$ is a one-parameter family of frontals.

In this paper, we deal with one-parameter families of Legendre curves with respect to u. We define $\boldsymbol{\mu}(u, v) = J(\nu(u, v))$. Since $\{\nu(u, v), \boldsymbol{\mu}(u, v)\}$ is a moving frame along $f(u, v)$ on $\mathbb{R}^2$, we have the Frenet type formula.

$$\begin{aligned}
\begin{pmatrix} \nu_u(u, v) \\ \boldsymbol{\mu}_u(u, v) \end{pmatrix} &= \begin{pmatrix} 0 & \ell(u, v) \\ -\ell(u, v) & 0 \end{pmatrix} \begin{pmatrix} \nu(u, v) \\ \boldsymbol{\mu}(u, v) \end{pmatrix}, \\
\begin{pmatrix} \nu_v(u, v) \\ \boldsymbol{\mu}_v(u, v) \end{pmatrix} &= \begin{pmatrix} 0 & L(u, v) \\ -L(u, v) & 0 \end{pmatrix} \begin{pmatrix} \nu(u, v) \\ \boldsymbol{\mu}(u, v) \end{pmatrix}, \\
f_u(u, v) &= \beta(u, v)\boldsymbol{\mu}(u, v), \\
f_v(u, v) &= A(u, v)\nu(u, v) + B(u, v)\boldsymbol{\mu}(u, v),
\end{aligned}$$

where

$$\begin{cases}
\ell(u, v) = \nu_u(u, v) \cdot \boldsymbol{\mu}(u, v), \\
L(u, v) = \nu_v(u, v) \cdot \boldsymbol{\mu}(u, v), \\
\beta(u, v) = f_u(u, v) \cdot \boldsymbol{\mu}(u, v), \\
A(u, v) = f_v(u, v) \cdot \nu(u, v), \\
B(u, v) = f_v(u, v) \cdot \boldsymbol{\mu}(u, v).
\end{cases} \tag{2.1}$$

By the integrability conditions $\nu_{uv}(u,v) = \nu_{vu}(u,v)$ and $f_{uv}(u,v) = f_{vu}(u,v)$, (ℓ, L, β, A, B) satisfies the conditions

$$\begin{cases} L_u(u,v) = \ell_v(u,v), \\ A_u(u,v) = B(u,v)\ell(u,v) - L(u,v)\beta(u,v), \\ B_u(u,v) = \beta_v(u,v) - A(u,v)\ell(u,v) \end{cases} \tag{2.2}$$

for all $(u,v) \in U$. We call the mapping (ℓ, L, β, A, B) with the integrability condition (2.2) the *curvature of the one-parameter family of Legendre curves* (f, ν).

Each component in the above curvature is geometrically defined as in equations in (2.1). We state some additional properties of them in the following remarks:

Remark 2.5. For a one-parameter family of Legendre curves (f, ν) with respect to u, $(f,\nu)(u, v_0)$ is a Legendre curve for a fixed value v_0, and $(\ell, \beta)(u, v_0)$ is the curvature of it in the sense of Theorem 2.2.

Remark 2.6. For a smooth mapping $f : \mathbb{R}^2 \to \mathbb{R}^2, (u,v) \mapsto (f_1(u,v), f_2(u,v))$, the Jacobian λ of f is defined as

$$\lambda(u,v) = J_f(u,v) = \begin{vmatrix} \frac{\partial f_1}{\partial u}(u,v) & \frac{\partial f_1}{\partial v}(u,v) \\ \frac{\partial f_2}{\partial u}(u,v) & \frac{\partial f_2}{\partial v}(u,v) \end{vmatrix},$$

where $\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$ means the determinant of a matrix. The Jacobian λ is related to information of the local density of the image of the mapping f. Especially, $\lambda(p) = 0$ for $p \in \mathbb{R}^2$ means p is a singularity of f. The Jacobian plays an important role in characterizations of singularities (see [13, 19, 26]). For a one-parameter family of Legendre curves (f, ν), the Jacobian of f is written as $\lambda = -\beta A$, where β and A are components of the curvature of (f, ν). Note also that the differential of the Jacobian with respect to u or v measures a ratio of the change of the density of families of curves along special curves.

Remark 2.7. Let $(f, \nu) : U \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with the curvature (ℓ, L, β, A, B). Then $(f, -\nu)$ is also a one-parameter family of Legendre curves with the curvature $(\ell, L, -\beta, -A, -B)$. Moreover, $(-f, \nu)$ is also a one-parameter family of Legendre curves with the curvature $(\ell, L, -\beta, -A, -B)$.

Definition 2.8. Let (f, ν) and $(\widetilde{f}, \widetilde{\nu}) : U \to \mathbb{R}^2 \times S^1$ be one-parameter families of Legendre curves. We say that (f, ν) and $(\widetilde{f}, \widetilde{\nu})$ are *congruent as one-parameter family of Legendre curves* if there exist a constant rotation $A \in SO(2)$ and a constant vector $\boldsymbol{a} \in \mathbb{R}^2$ such that $\widetilde{f}(u,v) = A(f(u,v)) + \boldsymbol{a}$ and $\widetilde{\nu}(u,v) = A(\nu(u,v))$ for all $(u,v) \in U$.

We gave the existence and uniqueness theorems for one-parameter families of Legendre curves in [24, 29]. However, we give here an explicit construction of one-parameter families of Legendre curves by using the curvatures.

Theorem 2.9 (Existence Theorem for one-parameter families of Legendre curves)**.** Let $(\ell, L, \beta, A, B) : U \to \mathbb{R}^5$ be a smooth mapping with the integrability condition. There exists a one-parameter family of Legendre curves $(f, \nu) : U \to \mathbb{R}^2 \times S^1$ whose associated curvature is (ℓ, L, β, A, B).

Proof. Let $(u_0, v_0) \in U$ be fixed. We define a smooth mapping $\theta : I \times \Lambda \to \mathbb{R}$ by

$$\theta(u,v) = \int_{u_0}^{u} \ell(u,v)du + \int_{v_0}^{v} L(u_0, v)dv.$$

Then θ satisfies the conditions $\theta_u(u,v) = \ell(u,v)$ and $\theta_v(u,v) = L(u,v)$ for all $(u,v) \in U$. We define $\nu(u,v) = (\cos\theta(u,v), \sin\theta(u,v))$ and hence $\boldsymbol{\mu}(u,v) = (-\sin\theta(u,v), \cos\theta(u,v))$. We also define a smooth mapping $f : U \to \mathbb{R}^2$ by

$$f(u,v) = \int_{u_0}^{u} \beta(u,v)\boldsymbol{\mu}(u,v)du + \int_{v_0}^{v} (A(u_0,v)\nu(u_0,v) + B(u_0,v)\boldsymbol{\mu}(u_0,v))dv.$$

By a direct calculation, $f_u(u,v) = \beta(u,v)\boldsymbol{\mu}(u,v)$ and $f_v(u,v) = A(u,v)\nu(u,v) + B(u,v)\boldsymbol{\mu}(u,v)$. It follows that $(f,\nu) : U \to \mathbb{R}^2 \times S^1$ is a one-parameter family of Legendre curves with the curvature (ℓ, L, β, A, B). $\square$

Theorem 2.10 (Uniqueness Theorem for one-parameter families of Legendre curves)**.** Let (f, ν) and $(\widetilde{f}, \widetilde{\nu}) : U \to \mathbb{R}^2 \times S^1$ be one-parameter families of Legendre curves with the curvatures (ℓ, L, β, A, B) and $(\widetilde{\ell}, \widetilde{L}, \widetilde{\beta}, \widetilde{A}, \widetilde{B})$ respectively. Then (f, ν) and $(\widetilde{f}, \widetilde{\nu})$ are congruent as one-parameter family of Legendre curves if and only if (ℓ, L, β, A, B) and $(\widetilde{\ell}, \widetilde{L}, \widetilde{\beta}, \widetilde{A}, \widetilde{B})$ coincide.

Proof. Suppose that (f, ν) and $(\widetilde{f}, \widetilde{\nu})$ are congruent as one-parameter family of Legendre curves. Then there exist a constant rotation $A \in SO(2)$ and a constant vector $\boldsymbol{a} \in \mathbb{R}^2$ such that $\widetilde{f}(u, v) = A(f(u, v)) + \boldsymbol{a}$ and $\widetilde{\nu}(u, v) = A(\nu(u, v))$ for all $(u, v) \in U$. It follows that $\widetilde{\boldsymbol{\mu}}(u, v) = A(\boldsymbol{\mu}(u, v))$. By a direct calculation, (ℓ, L, β, A, B) and $(\widetilde{\ell}, \widetilde{L}, \widetilde{\beta}, \widetilde{A}, \widetilde{B})$ coincide.

Conversely, let $(u_0, v_0) \in U$ be fixed. By using congruence as one-parameter family of Legendre curves, we may $(f, \nu)(u_0, v_0) = (\widetilde{f}, \widetilde{\nu})(u_0, v_0)$. By the construction in the proof of Theorem 2.9, $(f, \nu)(u, v) = (\widetilde{f}, \widetilde{\nu})(u, v)$ for all $(u, v) \in U$. $\square$

3 | PROPERTIES OF ONE-PARAMETER FAMILIES OF LEGENDRE CURVES

Let $(f, \nu) : U \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with respect to u and (ℓ, L, β, A, B) be the curvature.

We say that $\phi : \widetilde{U} \to U$ is a *one-parameter parameter change* if ϕ is a diffeomorphism of the form $\phi(p, q) = (u(p, q), v(q))$.

Proposition 3.1. Under the above notations, $(\widetilde{f}, \widetilde{\nu}) = (f \circ \phi, \nu \circ \phi) : \widetilde{U} \to \mathbb{R}^2 \times S^1$ is a one-parameter family of Legendre curves with respect to p and the curvature $(\widetilde{\ell}, \widetilde{L}, \widetilde{\beta}, \widetilde{A}, \widetilde{B})$ is given by

$$\begin{aligned}
\widetilde{\ell}(p, q) &= \ell(\phi(p, q))u_p(p, q), \\
\widetilde{L}(p, q) &= \ell(\phi(p, q))u_q(p, q) + L(\phi(p, q))v_q(q), \\
\widetilde{\beta}(p, q) &= \beta(\phi(p, q))u_p(p, q), \\
\widetilde{A}(p, q) &= A(\phi(p, q))v_q(q), \\
\widetilde{B}(p, q) &= \beta(\phi(p, q))u_q(p, q) + B(\phi(p, q))v_q(q).
\end{aligned}$$

Proof. Since $\widetilde{f}_p(p, q) \cdot \widetilde{\nu}(p, q) = f_u(\phi(p, q))u_p(p, q) \cdot \nu(\phi(p, q)) = 0$ for all $(p, q) \in \widetilde{U}$, $(\widetilde{f}, \widetilde{\nu})$ is a one-parameter family of Legendre curves with respect to p. By a direct calculation, we have the curvature $(\widetilde{\ell}, \widetilde{L}, \widetilde{\beta}, \widetilde{A}, \widetilde{B})$. $\square$

Next we consider a diffeomorphism on the target $\mathbb{R}^2$.

Proposition 3.2. Let $(f, \nu) : U \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with respect to u and (ℓ, L, β, A, B) be the curvature. Suppose that $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism. Then there exists a smooth mapping $\widetilde{\nu} : U \to S^1$ such that $(\Phi \circ f, \widetilde{\nu}) : U \to \mathbb{R}^2 \times S^1$ is a one-parameter family of Legendre curves with respect to u.

Proof. We denote $\Phi(x, y) = (\phi_1(x, y), \phi_2(x, y))$, $f(u, v) = (x(u, v), y(u, v))$ and $\nu(u, v) = (a(u, v), b(u, v))$. By the Frenet type formula, $f_u(u, v) = \beta(u, v)\boldsymbol{\mu}(u, v)$, that is, $x_u(u, v) = -\beta(u, v)b(u, v)$ and $y_u(u, v) = \beta(u, v)a(u, v)$. We define

$$\begin{aligned}
\overline{\nu}(u, v) = \;& (\phi_{2y}(x(u, v), y(u, v))a(u, v) - \phi_{2x}(x(u, v), y(u, v))b(u, v), \\
& -\phi_{1y}(x(u, v), y(u, v))a(u, v) + \phi_{1x}(x(u, v), y(u, v))b(u, v))
\end{aligned}$$

and $\widetilde{\nu}(u, v) = \overline{\nu}(u, v)/|\overline{\nu}(u, v)|$. Note that $\overline{\nu}(u, v) = J_\Phi(u, v)^t D_\Phi^{-1}(f(u, v))\nu(u, v)$, where

$$D_\Phi(u, v) = \begin{pmatrix} \phi_{1x} & \phi_{1y} \\ \phi_{2x} & \phi_{2y} \end{pmatrix}(u, v), \quad J_\Phi(u, v) = \det D_\Phi(u, v),$$

and ${}^tA$ is the transpose matrix of A, and $|\overline{\nu}(u, v)| \neq 0$ for all $(u, v) \in U$. Then $\widetilde{\nu} : U \to S^1$ is a smooth mapping.

Since $\Phi \circ f(u, v) = (\phi_1(x(u, v), y(u, v)), \phi_2(x(u, v), y(u, v)))$, we have

$$\begin{aligned}
(\Phi \circ f)_u(u, v) = \;& \Big(\phi_{1x}(x(u, v), y(u, v))x_u(u, v) + \phi_{1y}(x(u, v), y(u, v))y_u(u, v), \\
& \phi_{2x}(x(u, v), y(u, v))x_u(u, v) + \phi_{2y}(x(u, v), y(u, v))y_u(u, v)\Big) \\
= \;& \beta(u, v)\Big(-\phi_{1x}(x(u, v), y(u, v))b(u, v) + \phi_{1y}(x(u, v), y(u, v))a(u, v), \\
& -\phi_{2x}(x(u, v), y(u, v))b(u, v) + \phi_{2y}(x(u, v), y(u, v))a(u, v)\Big).
\end{aligned}$$

By a direct calculation, we have $(\Phi\circ f)_u(u,v)\cdot\widetilde{\nu}(u,v)=0$ for all $(u,v)\in U$. Hence $(\Phi\circ f,\widetilde{\nu})$ is a one-parameter family of Legendre curves with respect to u. □

Remark 3.3. By a direct calculation, we have the curvature $(\widetilde{\ell},\widetilde{L},\widetilde{\beta},\widetilde{A},\widetilde{B})$ of $(\Phi\circ f,\widetilde{\nu})$ as follows:

$$
\begin{aligned}
\widetilde{\ell} &= (1/|\overline{\nu}|^2)\Big(\beta\big((\phi_{2xx}b^2-2\phi_{2xy}ab+\phi_{2yy}a^2)(\phi_{1y}a-\phi_{1x}b)\\
&\quad -(\phi_{1xx}b^2-2\phi_{1xy}ab+\phi_{1yy}a^2)(\phi_{2y}a-\phi_{2x}b)\big)\\
&\quad +\ell(\phi_{1x}\phi_{2y}-\phi_{2x}\phi_{1y})\Big),\\
\widetilde{L} &= (1/|\overline{\nu}|^2)\Big(\big((\phi_{2xy}x_v+\phi_{2yy}y_v)a-(\phi_{2xx}x_v+\phi_{2xy}y_v)b\big)(\phi_{1y}a-\phi_{1x}b)\\
&\quad +\big(-(\phi_{1xy}x_v+\phi_{1yy}y_v)a+(\phi_{1xx}x_v+\phi_{1xy}y_v)b\big)(\phi_{2y}a-\phi_{2x}b)\\
&\quad +L(\phi_{1x}\phi_{2y}-\phi_{2x}\phi_{1y})\Big),\\
\widetilde{\beta} &= |\overline{\nu}|\beta,\\
\widetilde{A} &= (1/|\overline{\nu}|)A(\phi_{1x}\phi_{2y}-\phi_{2x}\phi_{1y}),\\
\widetilde{B} &= |\overline{\nu}|B+(1/|\overline{\nu}|)A\Big((\phi_{1x}a+\phi_{1y}b)(\phi_{1y}a-\phi_{1x}b)+(\phi_{2x}a+\phi_{2y}b)(\phi_{2y}a-\phi_{2x}b)\Big).
\end{aligned}
$$

We now consider existence conditions for a one-parameter family of Legendre curves of a given map $f:U\to\mathbb{R}^2$.

Proposition 3.4. Let $f:U\to\mathbb{R}^2$, $f(u,v)=(x(u,v),y(u,v))$ be a smooth mapping and $p=(u_0,v_0)\in U$.

(1) If rank $df=2$ at $p\in U$, then there exists ν around p such that (f,ν) is a one-parameter family of Legendre curves with respect to u around p.

(2) If rank $df=1$ at $p\in U$, then there exists ν around p such that (f,ν) is a one-parameter family of Legendre curves with respect to u or with respect to v around p.

(3) Let rank $df=0$ at $p\in U$. Suppose that there exist smooth map germs $\lambda_1,\lambda_2:(U,p)\to\mathbb{R}$ with $\lambda_1(p)\neq 0,\lambda_2(p)\neq 0$, k_1,k_2 are natural numbers and ℓ_1,ℓ_2 are non-negative integers such that $x(u,v)=\lambda_1(u,v)(u-u_0)^{k_1}(v-v_0)^{\ell_1}$ and $y(u,v)=\lambda_2(u,v)(u-u_0)^{k_2}(v-v_0)^{\ell_2}$. If $1\le k_1\le k_2$ and $\ell_1\le\ell_2$ (respectively, $k_1\ge k_2\ge 1$ and $\ell_1\ge\ell_2$), then there exists ν around p such that (f,ν) is a one-parameter family of Legendre curves with respect to u around p.

(4) Suppose that $f(u,v)$ is given by the form $f(u,v)=(x(u,v),y(v))$. Then there exists ν such that (f,ν) is a one-parameter family of Legendre curves with respect to u.

Proof. (1) Put $\widetilde{f}(u,v)=(u,v)$. If we take $\nu:U\to S^1,\nu(u,v)=(0,1)$, then $(\widetilde{f},\nu)$ is a one-parameter family of Legendre curves with respect to u around p. From the assumption, taking a suitable coordinate of the target space $\mathbb{R}^2$, $f(u,v)$ is expressed as $\widetilde{f}(u,v)$. Thus the statement follows from Proposition 3.2.

(2) From the assumption, by taking suitable coordinates of the source and target spaces, $f(u,v)$ is expressed as $\widetilde{f}(u,v)=(u,\psi(u,v))$ or $(v,\psi(u,v))$. By Propositions 3.1 and 3.2, in order to prove the statement, it is enough to show that $\widetilde{f}(u,v)$ is a one-parameter family of Legendre curves with respect to u or v around p. Suppose that $\widetilde{f}(u,v)=(u,\psi(u,v))$. If we take

$$\nu:U\to S^1,\nu(u,v)=(1/\sqrt{\psi_u^2(u,v)+1})(-\psi_u(u,v),1),$$

then $(\widetilde{f},\nu)$ is a one-parameter family of Legendre curves with respect to u around p. On the other hand, suppose that $\widetilde{f}(u,v)=(v,\psi(u,v))$. If we take

$$\nu:U\to S^1,\nu(u,v)=(1/\sqrt{\psi_v^2(u,v)+1})(-\psi_v(u,v),1),$$

then $(\widetilde{f},\nu)$ is a one-parameter family of Legendre curves with respect to v around p.

(3) We may assume that $p=(u_0,v_0)=(0,0)$. By a direct calculation, we have

$$f_u(u,v)=((\lambda_{1u}(u,v)u+\lambda_1(u,v)k_1)u^{k_1-1}v^{\ell_1},(\lambda_{2u}(u,v)u+\lambda_2(u,v)k_2)u^{k_2-1}v^{\ell_2}).$$

If $1\le k_1\le k_2$ and $\ell_1\le\ell_2$, then there exist non-negative integers n_1,n_2 such that $k_2=k_1+n_1$ and $\ell_2=\ell_1+n_2$. If we take $\nu:U\to S^1$ by

$$\nu(u,v)=\frac{(-(\lambda_{2u}(u,v)u+\lambda_2(u,v)k_2)u^{n_1}v^{n_2},\lambda_{1u}(u,v)u+\lambda_1(u,v)k_1)}{\sqrt{(\lambda_{2u}(u,v)u+\lambda_2(u,v)k_2)^2u^{2n_1}v^{2n_2}+(\lambda_{1u}(u,v)u+\lambda_1(u,v)k_1)^2}},$$

then (f, ν) is a one-parameter family of Legendre curves with respect to u around p. In the case $k_1 \geq k_2 \geq 1$ and $\ell_1 \geq \ell_2$, we can prove similarly.

(4) Suppose that $f(u,v) = (x(u,v), y(v))$. Then $f_u(u,v) = (x_u(u,v), 0)$. If we take $\nu : U \to S^1, \nu(u,v) = (0,1)$, (f,ν) is a one-parameter family of Legendre curves with respect to u. $\square$

Example 3.5. Let $(f,\nu) : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(u,v) = (uv, v^2), \nu(u,v) = (0,1)$. Since $f_u(u,v) = (v,0)$, (f,ν) is a one-parameter family of Legendre curves with respect to u (cf. Proposition 3.4 (4)).

On the other hand, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(u,v) = (uv, u^2)$. Then $f_u(u,v) = (v, 2u)$. There does not exist $\nu : \mathbb{R}^2 \to S^1$ such that (f,ν) is a one-parameter family of Legendre curves with respect to u, see Example 4.3. The figures are in Figure 3.

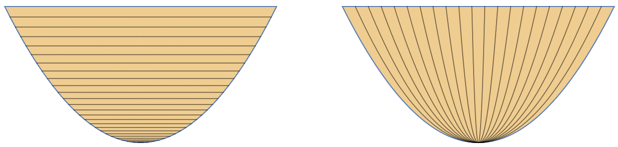

FIGURE 3 The left figure shows the one-parameter family of Legendre curves on $f(u,v) = (uv, v^2)$ with respect to u. The right figure shows the family of curves on $f(u,v) = (uv, u^2)$ with respect to u, which is not a one-parameter family of frontals.

Example 3.6. (Parallel curves of Legendre curves) Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature (ℓ, β). The parallel curve $\gamma^k : I \to \mathbb{R}^2$ is given by $\gamma^k(t) = \gamma(t) + k\nu(t)$ for fixed $k \in \mathbb{R}$ (cf. [8, 9]). Then $(\gamma^k, \nu) : I \to \mathbb{R}^2 \times S^1$ is also a Legendre curve with the curvature $(\beta + k\ell, \ell)$. We define $(f, \widetilde{\nu}) : I \times \mathbb{R} \to \mathbb{R}^2 \times S^1$ by

$$f(t,k) = \gamma^k(t) = \gamma(t) + k\nu(t),\ \widetilde{\nu}(t,k) = \nu(t).$$

Then $(f, \widetilde{\nu})$ is a one-parameter family of Legendre curves with respect to t with the curvature $(\ell, 0, \beta + k\ell, 1, 0)$. Figure 1 shows an example of a family of parallel curves.

Moreover, we define $(f, \widetilde{\nu}) : \mathbb{R} \times I \to \mathbb{R}^2 \times S^1$ by

$$f(k,t) = \gamma^k(t) = \gamma(t) + k\nu(t),\ \widetilde{\nu}(k,t) = \boldsymbol{\mu}(t).$$

Then $(f, \widetilde{\nu})$ is a one-parameter family of Legendre curves with respect to k with the curvature $(0, \ell, -1, \beta + k\ell, 0)$. The mapping f is an example of plane line congruences, see section 5.

4 | RELATIONS BETWEEN ONE-PARAMETER FAMILIES OF LEGENDRE CURVES AND LEGENDRE SURFACES

We give a relation between one-parameter families of Legendre curves and Legendre surfaces.

We say that $(\boldsymbol{x}, \boldsymbol{n}) : U \to \mathbb{R}^3 \times S^2$ is *a Legendre surface* if $(\boldsymbol{x}, \boldsymbol{n})^*\theta = 0$ for all $(u,v) \in U$, where θ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^3 = \mathbb{R}^3 \times S^2$ over $\mathbb{R}^3$ (cf. [1, 2]). This condition is equivalent to $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) = 0$ and $\boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) = 0$ for all $(u,v) \in U$. We say that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a *frontal* if there exists $\boldsymbol{n} : U \to S^2$ such that $(\boldsymbol{x}, \boldsymbol{n})$ is a Legendre surface.

Proposition 4.1. (1) Let $(f,\nu) : U \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with respect to u, where $f(u,v) = (x(u,v), y(u,v))$ and $\nu(u,v) = (a(u,v), b(u,v))$. Then $\boldsymbol{x} : U \to \mathbb{R}^3, \boldsymbol{x}(u,v) = (x(u,v), y(u,v), v)$ is a frontal. More precisely, $(\boldsymbol{x}, \boldsymbol{n}) : U \to \mathbb{R}^3 \times S^2$ is a Legendre surface, where

$$\boldsymbol{n}(u,v) = \frac{(a(u,v), b(u,v), -b(u,v)y_v(u,v) - a(u,v)x_v(u,v))}{\sqrt{1 + (b(u,v)y_v(u,v) + a(u,v)x_v(u,v))}}.$$

(2) Let $(\boldsymbol{x}, \boldsymbol{n}) : U \to \mathbb{R}^3 \times S^2$ be a Legendre surface of the form $\boldsymbol{x}(u,v) = (x(u,v), y(u,v), v)$ and $\boldsymbol{n}(u,v) = (a(u,v), b(u,v), c(u,v))$. Then $(f, \nu) : U \to \mathbb{R}^2 \times S^1$ is a one-parameter family of Legendre curves with respect to u, where

$$f(u,v) = (x(u,v), y(u,v)),\ \nu(u,v) = \frac{(a(u,v), b(u,v))}{\sqrt{a^2(u,v) + b^2(u,v)}}.$$

Proof. (1) Since $\boldsymbol{x}_u(u,v) = (x_u(u,v), y_u(u,v), 0)$ and $\boldsymbol{x}_v(u,v) = (x_v(u,v), y_v(u,v), 1)$, we have $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) = 0$ and $\boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) = 0$. It follows that $(\boldsymbol{x}, \boldsymbol{n})$ is a Legendre surface.

(2) Since $\boldsymbol{x}_u(u,v) = (x_u(u,v), y_u(u,v), 0)$ and $\boldsymbol{x}_v(u,v) = (x_v(u,v), y_v(u,v), 1)$, we have $x_u(u,v)a(u,v) + y_u(u,v)b(u,v) = 0$ and $x_v(u,v)a(u,v) + y_v(u,v)b(u,v) + c(u,v) = 0$. If $a(u,v) = b(u,v) = 0$, then $c(u,v) = 0$. It is a contradiction the fact that $\boldsymbol{n}(u,v) \in S^2$. Therefore, we have $(a(u,v), b(u,v)) \neq (0,0)$ for all $(u,v) \in U$. It follows that (f, ν) is a one-parameter family of Legendre curves. □

Remark 4.2. Let $(f, \nu) : U \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with respect to u. Then $(\boldsymbol{x}, \widetilde{\nu}_1, \widetilde{\nu}_2) : U \to \mathbb{R}^3 \times \Delta$ is also a one-parameter family of framed curves with respect to u, where $\boldsymbol{x}(u,v) = (f(u,v), v), \widetilde{\nu}_1(u,v) = (\nu(u,v), 0)$ and $\widetilde{\nu}_2(u,v) = (0,0,1)$ (cf. [12, 24]). Moreover, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface, where

$$\boldsymbol{n}(u,v) = \frac{(a(u,v), b(u,v), -b(u,v)y_v(u,v) - a(u,v)x_v(u,v))}{\sqrt{1 + (b(u,v)y_v(u,v) + a(u,v)x_v(u,v))}}, \boldsymbol{s}(u,v) = (-b(u,v), a(u,v), 0)$$

(cf. [11, 12]).

Example 4.3 (Example 3.5)**.** Let $(f, \nu) : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(u,v) = (uv, v^2), \nu(u,v) = (0,1)$. Then $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3, \boldsymbol{x}(u,v) = (uv, v^2, v)$ is a frontal by Proposition 4.1.

On the other hand, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(u,v) = (uv, u^2)$. Then $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3, \boldsymbol{x}(u,v) = (uv, u^2, v)$ is a cross cap. It is an example which is not a frontal.

5 | PLANE LINE CONGRUENCES

We deal with local geometry of plane line congruences (one-parameter families of lines in plane). Let I be an open interval, and $\boldsymbol{\gamma} : I \to \mathbb{R}^2$ and $\boldsymbol{e} : I \to S^1 (\subset \mathbb{R}^2)$ be smooth mappings. We define *a plane line congruence* as a map of the following form:

$$f : \mathbb{R} \times I \to \mathbb{R}^2,\ (u,v) \mapsto \boldsymbol{\gamma}(v) + u\boldsymbol{e}(v).$$

$\boldsymbol{\gamma}$ and $\boldsymbol{e}$ are respectively called *the base and direction curves of the plane line congruence* f, and the pair $(\boldsymbol{\gamma}, \boldsymbol{e})$ is often regarded as the plane line congruence f itself.

Proposition 5.1. The mapping $(f, \nu) : \mathbb{R} \times I \to \mathbb{R}^2 \times S^1, f(u,v) = \boldsymbol{\gamma}(v) + u\boldsymbol{e}(v), \nu(u,v) = J(\boldsymbol{e}(v))$ is a one-parameter family of Legendre curves with respect to u, and the curvature is given as follows:

$$\begin{aligned}
\ell(u,v) &= 0,\\
L(u,v) &= |\boldsymbol{e}(v)\ \boldsymbol{e}'(v)|,\\
\beta(u,v) &= -1,\\
A(u,v) &= \overline{A}(v) + uL(v),\\
B(u,v) &= |J(\boldsymbol{e}(v))\ \boldsymbol{\gamma}'(v)|,
\end{aligned}$$

where $'$ means d/dv, $|\cdot,\cdot|$ means the determinant of the vectors in $\mathbb{R}^2$ and $\overline{A}(v) = |\boldsymbol{e}(v)\ \boldsymbol{\gamma}'(v)|$. Especially $L, \overline{A}, B$ are functions depending only on the parameter v.

Proof. Since $f_u(u,v) \cdot \nu(u,v) = \boldsymbol{e}(v) \cdot J(\boldsymbol{e}(v)) = 0$ for all $(u,v) \in \mathbb{R} \times I$, (f, ν) is a one-parameter family of Legendre curves with respect to u. We calculate the curvature of (f, ν). Put $\boldsymbol{\mu}(u,v) = J(\nu(v)) = -\boldsymbol{e}(v)$. Since $\nu_u(u,v) = 0$ and $f_u(u,v) = \boldsymbol{e}(v)$, we have $\ell(u,v) = 0$ and $\beta(u,v) = -1$. By $\boldsymbol{\mu}_v(u,v) = -\boldsymbol{e}'(v)$,

$$L(u,v) = |\boldsymbol{\mu}(u,v)\ \boldsymbol{\mu}_v(u,v)| = |\boldsymbol{e}(v)\ \boldsymbol{e}'(v)|.$$

Finally, we have $f_v(u,v) = \boldsymbol{\gamma}'(v) + u\boldsymbol{e}'(v)$, thus

$$A(u,v) = |f_v(u,v)\ \ \boldsymbol{\mu}(u,v)| = |\boldsymbol{e}(v)\ \boldsymbol{\gamma}'(v)| + u|\boldsymbol{e}(v)\ \boldsymbol{e}'(v)|$$

and

$$B(u,v) = |\nu(u,v)\ f_v(u,v)| = |J(\boldsymbol{e}(v))\ \boldsymbol{\gamma}'(v)|.$$

□

Conversely, we can construct the mapping of a plane line congruence for given smooth functions $L, \overline{A}, B : I \to \mathbb{R}$ and a fixed point $v_0 \in I$, following the construction in the proof of Theorem 2.9: First put $\theta(v) = \int_{v_0}^{v} L(v)dv$, $\nu(v) = (\cos\theta(v), \sin\theta(v))$ and hence $\boldsymbol{\mu}(v) = (-\sin\theta(v), \cos\theta(v))$. Then we get mappings $\boldsymbol{\gamma}, \boldsymbol{e} : I \to \mathbb{R}^2$ as follows

$$\boldsymbol{\gamma}(v) = \int_{v_0}^{v} (\overline{A}(v)\nu(v) + B(v)\boldsymbol{\mu}(v))dv, \quad \boldsymbol{e}(v) = -\boldsymbol{\mu}(v).$$

Then $(f,\nu) : \mathbb{R} \times I \to \mathbb{R}^2 \times S^1$, $f(u,v) = \boldsymbol{\gamma}(v) + u\boldsymbol{e}(v)$, $\nu(u,v) = J(\boldsymbol{e}(v))$ is a one-parameter family of Legendre curves with respect to u with the curvature $(0, L, -1, \overline{A} + uL, B)$.

Remark 5.2. The congruent type of a plane line congruence is distinguished even by the choice of the coordinate of the parameter v, though the family of lines in the plane are the same.

We are interested in local geometry of plane line congruences, thus deal with germs of direction and base curves $\boldsymbol{\gamma}, \boldsymbol{e} : (I,0) \to \mathbb{R}^2$ constructed from function germs $L, \overline{A}, B : (I,0) \to \mathbb{R}$. Recall that $L, \overline{A}, B$ are clearly characterized by geometry of $\boldsymbol{\gamma}, \boldsymbol{e}$. Note also that A and L coincide with the following functions which are meaningful with respect to a geometry of the mapping $f(u,v) = \boldsymbol{\gamma}(v) + u\boldsymbol{e}(v)$.

<u>The Jacobian</u>: For a plane line congruence $f(u,v) = \boldsymbol{\gamma}(v) + u\boldsymbol{e}(v)$, the Jacobian $\lambda(u,v)$ coincides with the function $A(u,v)$:

$$\lambda(u,v) = |\boldsymbol{e}(v)\ \boldsymbol{\gamma}'(v)| + u|\boldsymbol{e}(v)\ \boldsymbol{e}'(v)| = \overline{A}(v) + uL(v) = A(u,v).$$

<u>The differential of the Jacobian with respect to u</u>: u is a special parameter of the map of a plane line congruence parametrizing each line for fixed v, and the derivative of the Jacobian with respect to u coincides with the function L:

$$\frac{\partial\lambda}{\partial u}(u,v) = L(v).$$

Remark 5.3. The above characterizations say that the congruent invariants L and A of the mapping f of a plane line congruence is determined just by a function λ satisfying $\lambda_{uu} = 0$.

Two map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are said to be $\mathcal{A}$-equivalent (written as $f \sim_{\mathcal{A}} g$) if there exist diffeomorphism germs $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $\tau : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ such that $f = \tau \circ g \circ \phi$. When τ is the identity in the above, f and g are said to be $\mathcal{R}$-equivalent (written as $f \sim_{\mathcal{R}} g$). A lot of works on the classification of germs with the above equivalences have been done in the context of singularity theory. See [1, 2, 3, 13, 16] for basic idea of singularity theory, and [19, 25, 26] for classification of map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with respect to $\mathcal{A}$-equivalence. The $\mathcal{A}$ or $\mathcal{R}$-equivalent class of mappings or functions are worth studying from the viewpoint of differential topology. Especially, we investigate the relation between the congruent invariants of a plane line congruence and the $\mathcal{A}$ or $\mathcal{R}$-equivalent classes of mappings or functions related to the plane line congruence.

A one-variable function $f : (I, t_0) \to (\mathbb{R}, 0)$ has type A_k at $t_0 \in I$ if $f^{(i)}(t_0) = 0$ for $i = 1, \dots, k$ and $f^{(k+1)}(t_0) \neq 0$. Then f is $\mathcal{R}$-equivalent to $\pm v^{k+1}$ (cf. [3]). The following relation between $\boldsymbol{e}$ and L immediately holds:

Proposition 5.4. For $k \geq 1$, $\boldsymbol{e} : (I,0) \to S^1$ is $\mathcal{R}$-equivalent to the germ of type A_k (whose normal form is $\pm v^{k+1}$) if and only if $L : (I,0) \to \mathbb{R}$ is $\mathcal{R}$-equivalent to the germ of type A_{k-1} and $L(0) = 0$. In addition, when $L(0) \neq 0$, $\boldsymbol{e}$ is always of type A_0.

Proof. Taking a suitable coordinate, $\boldsymbol{e}$ is locally expressed as $\theta(v) = \int_0^v L(v)dv$ around $0 \in I$. □

In addition, the next proposition says that we can locally parametrize any plane line congruence by a normal line congruence that is studied in the next subsection:

Proposition 5.5. For a given germ $(\boldsymbol{\gamma}, \boldsymbol{e}) \colon (I, 0) \to \mathbb{R}^2$, put $\epsilon(v) = \int_0^v B(v)dv$ and $\widehat{\boldsymbol{\gamma}} = \boldsymbol{\gamma} + \epsilon \boldsymbol{e} \colon (\mathbb{R}, 0) \to \mathbb{R}^2$. Then $(\widehat{\boldsymbol{\gamma}}, \boldsymbol{e})$ satisfies the following:

1. $(\boldsymbol{\gamma}, \boldsymbol{e})(v)$ and $(\widehat{\boldsymbol{\gamma}}, \boldsymbol{e})(v)$ define the same line for any v.
2. $|J(\boldsymbol{e})\ \widehat{\boldsymbol{\gamma}}'|(v) = 0$ for any v.

Proof. Since $\widehat{\boldsymbol{\gamma}}(v) + u\boldsymbol{e}(v) = \boldsymbol{\gamma}(v) + (u + \epsilon(v))\boldsymbol{e}(v)$, the statement (1) is clear. The statement (2) follows from a direct calculation:

$$|J(\boldsymbol{e})\ \widehat{\boldsymbol{\gamma}}'| = |J(\boldsymbol{e})\ \boldsymbol{\gamma}'| + \epsilon'|J(\boldsymbol{e})\ \boldsymbol{e}| + \epsilon|J(\boldsymbol{e})\ \boldsymbol{e}'| = B + B(-1) + \epsilon \cdot 0 = 0.$$

□

Remark 5.6. Let $(\widehat{\boldsymbol{\gamma}}, \boldsymbol{e})$ in Proposition 5.5 have the curvature $(\widehat{\ell}, \widehat{L}, \widehat{\beta}, \widehat{A}, \widehat{B})$. Then the curvature is expressed in terms of (ℓ, L, β, A, B) (the curvature of $(\boldsymbol{\gamma}, \boldsymbol{e})$) as follows:

$$\widehat{\ell} = \ell = 0,\ \ \widehat{L} = L,\ \ \widehat{\beta} = \beta = -1,\ \ \widehat{A} = A + \epsilon L,\ \ \widehat{B} = 0.$$

Especially, in the same use of the notation "hat", $\widehat{\overline{A}} = \overline{A} + \epsilon L$.

5.1 | Normal line congruences

We deal with *a normal line congruence*: a plane line congruence satisfying $B = |J(\boldsymbol{e})\ \boldsymbol{\gamma}'| \equiv 0$. Therefore the congruent type of a normal line congruence and thus the corresponding base and direction curves $\boldsymbol{\gamma}, \boldsymbol{e}$ are uniquely determined by the invariants $\overline{A}$ and L. It is also remarkable that the pair $(L, \overline{A})$ coincides with the curvature $(\ell_\gamma, \beta_\gamma)$ of the Legendre curve $(\boldsymbol{\gamma}, \boldsymbol{e}) : I \to \mathbb{R}^2 \times S^1$. Especially, the non-inflective case with $L = \ell_\gamma \neq 0$ is well studied in [8].

Remark 5.7. Conversely, let a function $\lambda(u, v)$ satisfying $\lambda_{uu} = 0$ be given. Then we can uniquely determine functions $A(u, v)$ and $L(v)$ thus a congruent type of normal line congruences. Especially λ coincides with the Jacobian of the congruent type (precisely, the representatives).

The following are the Taylor expansion (or normal form) of such $\boldsymbol{\gamma}(v) = (\gamma_1(v), \gamma_2(v))$ and $\boldsymbol{e}(v) = (e_1(v), e_2(v))$ with $L_i = (d^i L/dv^i)(0)$ and $\overline{A}_i = (d^i \bar{A}/dv^i)(0)$:

$$\begin{aligned}
\gamma_1(v) &= \overline{A}_0 v + \frac{1}{2!}\overline{A}_1 v^2 + \frac{1}{3!}(-\overline{A}_0 L_0^2 + \overline{A}_2)v^3 + \frac{1}{4!}(-3\overline{A}_0 L_0 L_1 - 3\overline{A}_1 L_0^2 + \overline{A}_3)v^4 + \cdots, \\
\gamma_2(v) &= \frac{1}{2!}\overline{A}_0 L_0 v^2 + \frac{1}{3!}(\overline{A}_0 L_1 + 2\overline{A}_1 L_0)v^3 + \frac{1}{4!}(\overline{A}_0(L_2 - L_0^3) + 3\overline{A}_1 L_1 + 3\bar{A}_2 L_0)v^4 + \cdots, \\
e_1(v) &= L_0 v + \frac{1}{2!}L_1 v^2 + \frac{1}{3!}(-L_0^3 + L_2)v^3 + \frac{1}{4!}(-6L_0^2 L_1 + L_3)v^4 + \cdots, \\
e_2(v) &= -1 + \frac{1}{2!}L_0^2 v^2 + \frac{3}{3!}L_0 L_1 v^3 + \frac{1}{4!}(-L_0^4 + 3L_1^2 + 4L_0^2 L_2)v^4 + \cdots.
\end{aligned}$$

Recall that the Jacobian λ of the map of the plane line congruence $f(u, v) = \boldsymbol{\gamma}(v) + u\boldsymbol{e}(v)$ is expressed as

$$\lambda(u, v) = \overline{A}(v) + uL(v).$$

From the above expressions, the following is easily seen.

Proposition 5.8. The following are equivalent:

1. $\lambda(0) = 0$, that is, f is not immersive at 0.
2. $\overline{A}(0) = |\boldsymbol{e}(0)\ \boldsymbol{\gamma}'(0)| = 0$.
3. $\boldsymbol{\gamma}$ is not immersive at 0.

Since $\overline{A}$ and L are function germs of one variable, they are $\mathcal{R}$-equivalent to one of the A_k-types. From now on, we fix the $\mathcal{R}$-types of $\overline{A}$ and L respectively, and study the $\mathcal{A}$ or $\mathcal{R}$-types of singularities in some functions or mappings which give us geometric information of normal line congruences. We must remark that the types of $\overline{A}$ and L are not always independent of the coordinate change of the source space to $(\boldsymbol{\gamma}, \boldsymbol{e})$ (see §6.1). Note also that normal line congruences are divided into four types depending on $L_0, \overline{A}_0$:

$$(1)\ \overline{A}_0, L_0 \neq 0,\ (2)\ \overline{A}_0 \neq 0, L_0 = 0,\ (3)\ \overline{A}_0 = 0, L_0 \neq 0,\ (4)\ \overline{A}_0 = L_0 = 0.$$

Since neither $\boldsymbol{e}, \boldsymbol{\gamma}, f, \lambda$ nor λ_u are singular, we do not deal with the case (1).

5.1.1 | Inflective regular base curves

Consider the case (2) $\overline{A}_0 \neq 0$ and $L_0 = 0$. Then the map of a normal line congruence f is immersive at 0; but $\boldsymbol{e}$ and λ_u are not regular, and $\boldsymbol{\gamma}$ is immersive but inflective or more degenerated. Note that $\boldsymbol{e}$ and λ_u depend only on L, while λ and $\boldsymbol{\gamma}$ depend on both $\overline{A}$ and L.

Let $\overline{A}$ be of type A_{k_1-1} and L be of type A_{k_2-1}, and write

$$\overline{A}(v) = \overline{A}_0 + \frac{\overline{A}_{k_1}}{k_1!}v^{k_1} + \cdots,\ L(v) = \frac{L_{k_2}}{k_2!}v^{k_2} + \cdots$$

where $\overline{A}_{k_1}, L_{k_2} \neq 0$ $(k_1, k_2 \geq 1)$. In the following we show types of leading terms of certain functions or mappings.

Proposition 5.9 (λ with $\overline{A}_0 \neq 0$ and $L_0 = 0$)**.** In the above setting, there are 3 cases with respect to the Jacobian λ:

1. for $k_1 = 1$, λ is regular;
2. for $k_1 \geq k_2 + 1$,
$$\lambda(u, v) = \overline{A}_0 + v^{k_2}\left(\frac{L_{k_2}}{k_2!}u + \cdots\right) \sim_{\mathcal{R}} \overline{A}_0 + uv^{k_2};$$
3. for $k_2 + 1 > k_1 \geq 2$,
$$\lambda(u, v) = \overline{A}_0 + \frac{\overline{A}_{k_1}}{k_1!}v^{k_1}(1 + \cdots) \sim_{\mathcal{R}} \overline{A}_0 \pm v^{k_1}.$$

Proof. Since the Jacobian of the map of a normal line congruence is expressed as

$$\lambda(u, v) = \left(\overline{A}_0 + \frac{\overline{A}_{k_1}}{k_1!}v^{k_1} + \cdots\right) + u\left(\frac{L_{k_2}}{k_2!}v^{k_2} + \cdots\right),$$

we have the result. □

Proposition 5.10 ($\boldsymbol{\gamma}$ with $\overline{A}_0 \neq 0$ and $L_0 = 0$)**.** In the above setting, $\boldsymbol{\gamma}$ has an inflection point at 0. Especially, it has the contact type of $(k_2 + 1)$-th order with the tangent line at 0.

Proof. Recall that $\boldsymbol{\gamma}$ is constructed as

$$\boldsymbol{\gamma}(v) = (\gamma_1(v), \gamma_2(v)) = \int_0^v \left(\overline{A}(v)\cos\theta(v), \overline{A}(v)\sin\theta(v)\right) dv$$

with $\theta(v) = \int_0^v L(v)dv$. Thus we have

$$\gamma_1(v) = \int_0^v \left(\overline{A}_0 + \frac{\overline{A}_{k_1}}{k_1!}v^{k_1} + \cdots\right)\cos\left(\frac{L_{k_2}}{(k_2+1)!}v^{k_2+1} + \cdots\right) dv = \overline{A}_0 v + \cdots,$$

$$\gamma_2(v) = \int_0^v \left(\overline{A}_0 + \frac{\overline{A}_{k_1}}{k_1!}v^{k_1} + \cdots\right)\sin\left(\frac{L_{k_2}}{(k_2+1)!}v^{k_2+1} + \cdots\right) dv = \overline{A}_0 L_{k_2}\frac{1}{(k_2+2)!}v^{k_2+2} + \cdots.$$

□

Summing up the above results for relatively small numbers of k_1 and k_2, we get the Table 1.

$\overline{A}$	$L = \ell_\gamma = \lambda_u$	λ	$\boldsymbol{e}$	$\boldsymbol{\gamma}$	$L/\overline{A}$
A_0	A_0	regular	A_1	$(1,3)$	$(1,\geq 2)$
	A_1	regular	A_2	$(1,4)$	$(1,\geq 2)$
	A_2	regular	A_3	$(1,5)$	$(1,\geq 2)$
	$A_{\geq 3}$	regular	$A_{\geq 4}$	$(1,\geq 6)$	$(1,\geq 2)$
A_1	A_0	uv	A_1	$(1,3)$	$(2,\geq 3)$
	A_1	v^2	A_2	$(1,4)$	$(2,\geq 3)$
	A_2	v^2	A_3	$(1,5)$	$(2,\geq 3)$
	$A_{\geq 3}$	v^2	$A_{\geq 4}$	$(1,\geq 6)$	$(2,\geq 3)$
A_2	A_0	uv	A_1	$(1,3)$	$(3,\geq 4)$
	A_1	uv^2	A_2	$(1,4)$	$(3,\geq 4)$
	A_2	v^3	A_3	$(1,5)$	$(3,\geq 4)$
	$A_{\geq 3}$	v^3	$A_{\geq 4}$	$(1,\geq 6)$	$(3,\geq 4)$

TABLE 1 The types of $\lambda, \boldsymbol{e}, \boldsymbol{\gamma}$ are shown for given types of $\overline{A}$, L. f and $\boldsymbol{\gamma}$ are regular, but $\boldsymbol{\gamma}$ has an inflection point at the origin, and the order is determined by the type of $L = \ell_\gamma$; Especially, (m_1, m_2) in the column means the degrees of leading terms of $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$. The last column shows 2-multi indices (see §6.1) of $L/\overline{A}$.

5.1.2 | Singular mappings and base curves

Consider the case (3) $\overline{A}_0 = 0$ and $L_0 \neq 0$. Then the germs f and $\boldsymbol{\gamma}$ are not immersive; on the other hand $\boldsymbol{e}$ and λ are regular. Note that this is non-inflective case in the sense of [8], and is well studied with respect to the base curve and the evolute. Let $\overline{A}$ be of type A_{k_1-1} and L be of type A_{k_2-1}, and write

$$\overline{A}(v) = \frac{\overline{A}_{k_1}}{k_1!}v^{k_1} + \cdots, \; L(v) = L_0 + \frac{L_{k_2}}{k_2!}v^{k_2} + \cdots$$

where $\overline{A}_{k_1}, L_{k_2} \neq 0$ $(k_1, k_2 \geq 1)$.

Proposition 5.11 ($\boldsymbol{\gamma}$ with $\overline{A}_0 = 0$ and $L_0 \neq 0$)**.** In the above setting, $\boldsymbol{\gamma}$ is $\mathcal{A}$-equivalent to a map germ of the form $(t^{k_1+1}, t^{k_1+2} + h.o.t)$.

Proof. The base curve $\boldsymbol{\gamma}(v) = (\gamma_1(v), \gamma_2(v))$ is as follows:

$$\gamma_1(v) = \int_0^v \frac{\bar{A}_{k_1}}{k_1!}v^{k_1}\cos\left(L_0 v + \frac{L_{k_2}}{(k_2+1)!}v^{k_2+1} + \cdots\right)dv = \frac{\bar{A}_{k_1}}{(k_1+1)!}v^{k_1+1} + \cdots,$$

$$\gamma_2(v) = \int_0^v \frac{\bar{A}_{k_1}}{k_1!}v^{k_1}\sin\left(L_0 v + \frac{L_{k_2}}{(k_2+1)!}v^{k_2+1} + \cdots\right)dv = L_0\frac{\bar{A}_{k_1}(k_1+1)}{(k_1+2)!}v^{k_1+2} + \cdots.$$

□

Proposition 5.12 (f with $\overline{A}_0 = 0$ and $L_0 \neq 0$)**.** In the above setting, f is $\mathcal{A}$-equivalent to a map germ of the form

$$(u, uv + v^{k_1+1} + h.o.t)$$

for the map of the normal line congruence $f(u,v) = \boldsymbol{\gamma}(v) + u\boldsymbol{e}(v) = (ue_1(v) + \gamma_1(v), ue_2(v) + \gamma_2(v))$. Especially, f is $\mathcal{A}$-equivalent to of type fold (u, v^2) for $k_1 = 1$; cusp $(u, uv + v^3)$ for $k_1 = 2$; swallowtail $(u, uv + v^4)$ for $k_1 = 4$.

Proof. Remark that the leading terms of e_1 and e_2 are expressed as

$$\begin{aligned} e_1(v) &= L_0 v + h.o.t, \\ e_2(v) &= -1 + \frac{1}{2!}L_0^2 v^2 + h.o.t. \end{aligned}$$

$\overline{A}$	$L = \ell_\gamma = \lambda_u$	f	$\boldsymbol{\gamma}$	$\overline{A}/L$
A_0	A_0	fold	$(2,3)$	$(1,\geq 2)$
	A_1	fold	$(2,3)$	$(1,\geq 2)$
	A_2	fold	$(2,3)$	$(1,\geq 2)$
	$A_{\geq 3}$	fold	$(2,3)$	$(1,\geq 2)$
A_1	A_0	cusp	$(3,4)$	$(2,\geq 3)$
	A_1	cusp	$(3,4)$	$(2,\geq 3)$
	A_2	cusp	$(3,4)$	$(2,\geq 3)$
	$A_{\geq 3}$	cusp	$(3,4)$	$(2,\geq 3)$
A_2	A_0	swallowtail	$(4,5)$	$(3,\geq 4)$
	A_1	swallowtail	$(4,5)$	$(3,\geq 4)$
	A_2	swallowtail	$(4,5)$	$(3,\geq 4)$
	$A_{\geq 3}$	swallowtail	$(4,5)$	$(3,\geq 4)$

TABLE 2 The types of $f, \boldsymbol{\gamma}$ are shown for given types of $\overline{A}, L$. λ and $\boldsymbol{e}$ are regular. $\boldsymbol{\gamma}$ is singular and (m_1, m_2) in the column means the degrees of leading terms of $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$. The last column shows 2-multi indices (see §6.1) of $\overline{A}/L$.

Since $e_2(0) \neq 0$, we can replace $\overline{u}(u,v) = ue_2(v) + \gamma_2(v)$. Then

$$f(\overline{u}, v) = \left(\frac{(\overline{u} - \gamma_2(v))}{e_2(v)} e_1(v) + \gamma_1(v), \overline{u} \right)$$

and exchanging the components it is $\mathcal{A}$-equivalent to

$$\left(\overline{u}, \overline{u}\frac{e_1(v)}{e_2(v)} - \frac{\gamma_2(v)e_1(v)}{e_2(v)} + \gamma_1(v) \right).$$

Here

$$\frac{e_1(v)}{e_2(v)} = -L_0 v + \text{h.o.t}, \quad \frac{\gamma_2(v)e_1(v)}{e_2(v)} + \gamma_1(v) = \frac{\overline{A}_{k_1}}{k_1+1} v^{k_1+1} + \text{h.o.t},$$

thus f is $\mathcal{A}$-equivalent to

$$(u, uv + v^{k_1+1} + h.o.t).$$

In addition, whether the germs are $\mathcal{A}$-equivalent to of type fold, cusp or swallowtail is determined by the leading terms of the second component in the above form (cf. [25]). □

Summing up the above results for relatively small numbers of k_1 and k_2, we get the Table 2. When $k_1 \geq 5$, we need analyze higher order terms in the germ in order to determine the exact $\mathcal{A}$-type. Refer to §6.4 for the detail with some examples.

5.1.3 | Singular mappings, base curves and direction curves

Consider the case (4) $\overline{A}_0 = L_0 = 0$. Then $\boldsymbol{\gamma}, f, \lambda$ and λ_u are singular. Let $\overline{A}$ be of type A_{k_1-1} and L be of type A_{k_2-1}, and write

$$\overline{A}(v) = \frac{\overline{A}_{k_1}}{k_1!} v^{k_1} + \cdots, \quad L(v) = \frac{L_{k_2}}{k_2!} v^{k_2} + \cdots$$

where $\overline{A}_{k_1}, L_{k_2} \neq 0$ $(k_1, k_2 \geq 1)$.

Proposition 5.13 (λ with $\overline{A}_0 = L_0 = 0$)**.** In the above setting, there are 3 cases with respect to the Jacobian λ:

1. for $k_1 = 1$, λ is regular;
2. for $k_1 \geq k_2 + 1$, $\lambda(u,v) \sim_{\mathcal{R}} uv^{k_2}$;
3. for $k_2 + 1 > k_1 \geq 2$, $\lambda(u,v) \sim_{\mathcal{R}} \pm v^{k_1}$.

Proof. Since the Jacobian of the map of a normal line congruence is expressed as

$$\lambda(u,v) = \left(\frac{\overline{A}_{k_1}}{k_1!}v^{k_1} + \cdots\right) + u\left(\frac{L_{k_2}}{k_2!}v^{k_2} + \cdots\right),$$

we have the result. □

Proposition 5.14 ($\boldsymbol{\gamma}$ with $\overline{A}_0 = L_0 = 0$)**.** In the above setting, the leading terms of the Taylor expansions of γ_1 and γ_2 are

$$\gamma_1(v) = \frac{\overline{A}_{k_1}}{(k_1+1)!}v^{k_1+1} + \cdots, \quad \gamma_2(v) = \frac{\overline{A}_{k_1}L_{k_2}}{k_1!(k_2+1)!(k_1+k_2+2)}v^{(k_1+k_2+2)} + \cdots.$$

Proof. The base curve $\boldsymbol{\gamma}(v) = (\gamma_1(v), \gamma_2(v))$ is as follows:

$$\gamma_1(v) = \int_0^v \left(\frac{\overline{A}_{k_1}}{k_1!}v^{k_1} + \cdots\right)\cos\left(\frac{L_{k_2}}{(k_2+1)!}v^{k_2+1} + \cdots\right)dv = \frac{\overline{A}_{k_1}}{(k_1+1)!}v^{k_1+1} + \cdots,$$

$$\begin{aligned}\gamma_2(v) &= \int_0^v \left(\frac{\overline{A}_{k_1}}{k_1!}v^{k_1} + \cdots\right)\sin\left(\frac{L_{k_2}}{(k_2+1)!}v^{k_2+1} + \cdots\right)dv \\ &= \frac{\overline{A}_{k_1}L_{k_2}}{k_1!(k_2+1)!(k_1+k_2+2)}v^{(k_1+k_2+2)} + \cdots.\end{aligned}$$

□

Proposition 5.15 (f with $\overline{A}_0 = L_0 = 0$)**.** In the above setting, there are 3-cases for the map of the normal line congruence $f(u,v) = \boldsymbol{\gamma}(v) + u\boldsymbol{e}(v) = (ue_1(v) + \gamma_1(v), ue_2(v) + \gamma_2(v))$:

1. for $k_1 \le k_2$, f is $\mathcal{A}$-equivalent to (u, v^{k_1+1});
2. for $k_1 - 1 = k_2$, f is $\mathcal{A}$-equivalent to $\big(u, (u+v)\,v^{k_1} + h.o.t\big)$. Especially, when $k_1 = 2$, f is $\mathcal{A}$-equivalent to of type beaks $(u, u^2v - v^3)$;
3. for $k_1 > k_2 + 1$, f is $\mathcal{A}$-equivalent to $(u, uv^{k_2+1} + v^{k_1+1} + h.o.t)$.

Proof. First, remark that the leading terms of e_1 and e_2 are $\frac{L_{k_2}}{(k_2+1)!}v^{k_2+1}$ and -1, respectively. Thus $f(u,v)$ is $\mathcal{A}$-equivalent to the following form:

$$\left(\overline{u}, \overline{u}\frac{e_1(v)}{e_2(v)} - \frac{\gamma_2(v)e_1(v)}{e_2(v)} + \gamma_1(v)\right) = \left(\overline{u}, -\frac{L_{k_2}}{(k_2+1)!}\overline{u}v^{k_2+1}(1+\cdots) + \frac{\overline{A}_{k_1}}{(k_1+1)!}v^{k_1+1}(1+\cdots)\right).$$

Note that when $k_1 = 2, k_2 = 1$,

$$f \sim_{\mathcal{A}} \big(u, (u+v)\,v^2 + h.o.t\big) \sim_{\mathcal{A}} (u, u^2v - v^3 + h.o.t).$$

From 3-$\mathcal{A}$-determinacy of the beaks-type map germ (refer to [25]), we can determine the $\mathcal{A}$-type of the above germ as beaks $(u, v^3 - u^2v)$. □

Summing up the above results for relatively small numbers of k_1 and k_2, we get the Table 3.

6 | SEVERAL GEOMETRICAL ASPECTS OF PLANE LINE CONGRUENCES

In §5.1, we focused on properties of normal line congruences which depend on the $\mathcal{R}$-types of functions in the curvature. In this section, we show results on geometry of normal line congruences depending on higher order information of the functions.

For later use, we define a detailed type of a function germ:

Definition 6.1. Take a one-variable smooth function germ $f : (I, x_0) \to \mathbb{R}, \quad x \mapsto f(x)$.

$\overline{A}$	$L = \ell_\gamma = \lambda_u$	λ	f_2	e	γ
A_0	A_0	regular	v^2	A_1	$(2,4)$
	A_1	regular	v^2	A_2	$(2,5)$
	A_2	regular	v^2	A_3	$(2,6)$
	$A_{\geq 3}$	regular	v^2	$A_{\geq 4}$	$(2, \geq 7)$
A_1	A_0	uv	$u^2 v - v^3$	A_1	$(3,5)$
	A_1	v^2	v^3	A_2	$(3,6)$
	A_2	v^2	v^3	A_3	$(3,7)$
	$A_{\geq 3}$	v^2	v^3	$A_{\geq 4}$	$(3, \geq 8)$
A_2	A_0	uv	$*\ uv^2 (+v^4)$	A_1	$(4,6)$
	A_1	uv^2	$*\ uv^3 + v^4$	A_2	$(4,7)$
	A_2	v^3	v^4	A_3	$(4,8)$
	$A_{\geq 3}$	v^3	v^4	$A_{\geq 4}$	$(4, \geq 9)$

TABLE 3 The types of λ, f, e, γ are shown for given types of $\overline{A}, L$. The 4-th column shows the second components of a mapping $(u, f_2(u, v))$ which is a representative of the $\mathcal{A}$-type of the mapping of the normal line congruence f. Note that an item with $*$ means the type of a jet, thus several $\mathcal{A}$-types exist over the type of the jet.

1. We say that *the 1st index of f* with respect to x is a non-negative integer ℓ_1 if $f^{(i)}(x_0) = 0$ for any integer i with $0 \leq i \leq \ell_1 - 1$ and $f^{(\ell_1)}(x_0) \neq 0$ (here we define $f^{(0)}(x_0) := f(x_0)$); and the 1st index is ∞ if $f^{(i)}(x_0) = 0$ for any integer $i \geq 0$.
2. For an integer $n \geq 2$, we say that *the n-th index of f* with respect to x is $\ell_n \in \mathbb{N}$ if $f^{(i)}(x_0) = 0$ for any integer i with $\ell_{n-1} < i < \ell_n$ and $f^{(\ell_n)}(x_0) \neq 0$, where ℓ_{n-1} is an $n-1$-th index of f;
3. Suppose f has the finite n-th index with respect to x, and the i-th index is ℓ_i for an integer i with $0 \leq i \leq n$. The tuple $(\ell_1, \cdots, \ell_n)$ is called *the n-multi index of f* with respect to x.

Example 6.2. Let $f : (I, 0) \to \mathbb{R}$ be a one-variable smooth function germ which is written as

$$f(x) = \frac{a_{\ell_1}}{\ell_1!} x^{\ell_1} + \frac{a_{\ell_2}}{\ell_2!} x^{\ell_2} + \frac{a_{\ell_3}}{\ell_3!} x^{\ell_3} + \cdots + \frac{a_{\ell_m}}{\ell_m!} x^{\ell_m} + h.o.t.$$

with integers $0 \leq \ell_1 < \ell_2 < \ell_3 < \cdots < \ell_m$ and non-zero real values $a_{\ell_1}, a_{\ell_2}, \cdots, a_{\ell_m}$. The i-th index of f is ℓ_i for an integer i with $1 \leq i \leq m$, and the m-multi index is $(\ell_1, \cdots, \ell_m)$.

Remark 6.3. The multi-index for $f : (I, x_0) \to \mathbb{R}$ is not coordinate free (in other words, not invariant under $\mathcal{R}$-equivalence), however it plays an interesting role in a context of local differential geometry. This notion is a natural expansion of the type of the curvature function of a plane curve at a degenerate point such as a vertex or inflection point (cf. [16]). Clearly, a vertex on a plane curve can be characterized by that the curvature function at the point has the 2-multi index $(0, m)$ for $m \geq 2$ ($m = 2$ for an ordinary vertex). In addition, an inflection point can be characterized by that the curvature function at the point has the 2-multi index (ℓ_1, ℓ_2) for $\ell_1 \geq 1$ ($\ell_1 = 1$ for an ordinary inflection). In the above settings, m measures the degree of the contact of the curve with circles; ℓ_1 measures the degree of the contact of the curve with lines. One aim of introducing multi-index is focusing on the second index ℓ_2 (especially for the case $\ell_1 \geq 1$) to the components of the curvature of a Legendre curve appearing in our setting.

The notion of multi index is characterized by the following equivalence of functions.

Definition 6.4. Let $f, g : (I, x_0) \to \mathbb{R}$ be smooth functions.

1. When the 1st indices of f, g are 0, f, g are $\mathcal{R}_{(0,\ell_2,\cdots,\ell_n)}$*-equivalent* if

$$f - j^{\ell_{i-1}} f(x_0) \sim_{\mathcal{R}} g - j^{\ell_{i-1}} g(x_0)$$

for any integer i with $1 < i \leq n$.

2. When the 1st indices of f, g are more than 0, f, g are $\mathcal{R}_{(\ell_1,\cdots,\ell_n)}$*-equivalent* if
$$f - j^{\ell_{i-1}} f(x_0) \sim_{\mathcal{R}} g - j^{\ell_{i-1}} g(x_0)$$
for any integer i with $1 \le i \le n$.

The next proposition follows from the definition (see also Example 6.2).

Proposition 6.5. Suppose $f : (I, x_0) \to \mathbb{R}$ is a smooth function germ.

1. The following are equivalent:
   (a) f has a n-multi index $(0, \ell_2, \cdots, \ell_n)$.
   (b) $f - j^{\ell_{i-1}} f(x_0)$ is $\mathcal{R}_{(0,\ell_2,\cdots,\ell_n)}$-equivalent to $A_{\ell_i - 1}$-type for a non-negative integer i with $2 \le i \le n$.
2. The following are equivalent:
   (a) f has a n-multi index $(\ell_1, \cdots, \ell_n)$ for a positive integer ℓ_1.
   (b) $f - j^{\ell_{i-1}} f(x_0)$ is $\mathcal{R}_{(\ell_1,\cdots,\ell_n)}$-equivalent to $A_{\ell_i - 1}$-type for a non-negative integer i with $1 \le i \le n$.

As an example, the function f in Example 6.2 is $\mathcal{R}_{(\ell_1,\cdots,\ell_m)}$-equivalent to
$$\pm x^{\ell_1} \pm x^{\ell_2} \pm x^{\ell_3} \pm \cdots \pm x^{\ell_m}.$$

6.1 | The 2-multi indices of $L/\overline{A}$ and $\overline{A}/L$

Let $(\boldsymbol{\gamma}, \boldsymbol{e}) : (I, 0) \to \mathbb{R}^2 \times S^1$ and $(\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{e}}) : (\widetilde{I}, 0) \to \mathbb{R}^2 \times S^1$ express plane line congruence germs. Assume that $(\boldsymbol{\gamma}, \boldsymbol{e}) \sim_{\mathcal{R}} (\widetilde{\boldsymbol{\gamma}}, \widetilde{\boldsymbol{e}})$, then we have the relations of curvatures between (ℓ, L, β, A, B) of $f(u, v) = \boldsymbol{\gamma}(v) + u\boldsymbol{e}(v)$ and $(\widetilde{\ell}, \widetilde{L}, \widetilde{\beta}, \widetilde{A}, \widetilde{B})$ of $\widetilde{f}(u, q) = \widetilde{\boldsymbol{\gamma}}(q) + u\widetilde{\boldsymbol{e}}(q)$ as follows:
$$\begin{aligned}
\widetilde{\ell}(u, q) &= 0, \\
\widetilde{L}(u, q) &= L(u, v(q)) v_q(q), \\
\widetilde{\beta}(u, q) &= -1, \\
\widetilde{A}(u, q) &= A(u, v(q)) v_q(q) = \overline{A}(v(q)) v_q(q) + uL(u, v(q)) v_q(q), \\
\widetilde{B}(u, q) &= B(u, v(q)) v_q(q)
\end{aligned} \tag{6.1}$$
where $\phi(u, q) = (u, v(q))$ is a one-parameter parameter change of a special form in the source space (see Proposition 3.1). Thus the $\mathcal{R}$-types of function germs in the curvature of a plane line congruence germs $(\boldsymbol{\gamma}, \boldsymbol{e}) : (I, 0) \to \mathbb{R}^2 \times S^1$ as a one-parameter families of Legendre curves depends on the coordinate of $(\boldsymbol{\gamma}, \boldsymbol{e})$. On the other hand, it is easily seen that the ratio of two functions in $L, A, B, \overline{A}$ is invariant. Especially, we study functions of the form $L/\overline{A}$ or $\overline{A}/L$ in the following.

From now on, we consider a normal line congruence $(\boldsymbol{\gamma}, \boldsymbol{e}) : (I, 0) \to \mathbb{R}^2 \times S^1$ which is characterized by function germs $L, \overline{A} : (I, 0) \to \mathbb{R}$. If $\overline{A}(0) \neq 0$, the base curve $\boldsymbol{\gamma}$ is regular and [8] shows that $-L/\overline{A}$ is equal to the curvature κ of $\boldsymbol{\gamma}$ as a regular curve. Thus the ratio of L and $\overline{A}$ plays an important role, and we want to expand the notion.

Let $\overline{A}, L : (I, 0) \to \mathbb{R}$ be function germs with the 2-multi indices (a_1, a_2) and (ℓ_1, ℓ_2), respectively. If $\ell_1 \ge a_1$ (resp. $a_1 \ge \ell_1$), then we can define a function germ $-L/\overline{A}$ (resp. $-\overline{A}/L$). Remark that, from the equation (6.1), the above $-L/\overline{A}$ (resp. $-\overline{A}/L$) is invariant under the coordinate change of $(\boldsymbol{\gamma}, \boldsymbol{e})$.

Write
$$\overline{A}(v) = \frac{\overline{A}_{a_1}}{a_1!} v^{a_1} + \frac{\overline{A}_{a_2}}{a_2!} v^{a_2} + \cdots, \quad L(v) = \frac{L_{\ell_1}}{\ell_1!} v^{\ell_1} + \frac{L_{\ell_2}}{\ell_2!} v^{\ell_2} + \cdots. \tag{6.2}$$
Then we have the following formula for the 2-multi index of the above newly defined functions.

Proposition 6.6. 1. When $\ell_1 \ge a_1$,
$$\text{2-multi index of } -\frac{L}{\overline{A}} = \left\{ \begin{array}{cc} (\ell_1 - a_1, \ell_2 - a_1) & (\ell_1 - \ell_2 > a_1 - a_2) \\ (\ell_1 - a_1, *) & (\ell_1 - \ell_2 = a_1 - a_2) \\ (\ell_1 - a_1, \ell_1 + a_2 - 2a_1) & (\ell_1 - \ell_2 < a_1 - a_2) \end{array} \right\}.$$

Here $*$ is a number more than or equal to $\ell_2 - a_1$, where the equality holds when

$$a_2!\ell_1!L_{\ell_2}A_{a_1} - a_1!\ell_2!L_{\ell_1}A_{a_2} = 0.$$

2. When $a_1 \geq \ell_1$,

$$\text{2-multi index of } -\frac{\overline{A}}{L} = \left\{ \begin{array}{cc} (a_1 - \ell_1, a_2 - \ell_1) & (a_1 - a_2 > \ell_1 - \ell_2) \\ (a_1 - \ell_1, *) & (a_1 - a_2 = \ell_1 - \ell_2) \\ (a_1 - \ell_1, a_1 + \ell_2 - 2\ell_1) & (a_1 - a_2 < \ell_1 - \ell_2) \end{array} \right\}.$$

Here $*$ is a number more than or equal to $a_2 - \ell_1$, where the equality holds when

$$\ell_2!a_1!\overline{A}_{a_2}L_{\ell_1} - \ell_1!a_2!\overline{A}_{a_1}L_{\ell_2} = 0.$$

Proof. Proving the statement (1) is enough. When $\ell_1 \geq a_1$, the generalized curvature is written as

$$\frac{L}{\overline{A}} = \frac{a_1!}{\ell_1!}\frac{L_{\ell_1}}{\overline{A}_{a_1}}v^{\ell_1 - a_1} + \left\{ \begin{array}{cc} \frac{a_1!}{\ell_2!}\frac{L_{\ell_2}}{\overline{A}_{a_1}}v^{\ell_2 - a_1} & (\ell_1 - \ell_2 > a_1 - a_2) \\ \left(\frac{a_1!}{\ell_2!}\frac{L_{\ell_2}}{\overline{A}_{a_1}} - \frac{(a_1!)^2}{\ell_1!a_2!}\frac{L_{\ell_1}\overline{A}_{a_2}}{\overline{A}^2_{a_1}} \right) v^{\ell_2 - a_1} & (\ell_1 - \ell_2 = a_1 - a_2) \\ -\frac{(a_1!)^2}{\ell_1!a_2!}\frac{L_{\ell_1}\overline{A}_{a_2}}{\overline{A}^2_{a_1}}v^{\ell_1 + a_2 - 2a_1} & (\ell_1 - \ell_2 < a_1 - a_2) \end{array} \right\} + \cdots;$$

especially, when $\ell_1 - \ell_2 = a_1 - a_2$, the coefficient of the second term is written as

$$\frac{a_1!}{\ell_1!\ell_2!a_2!}\frac{1}{A^2_{a_1}}(a_2!\ell_1!L_{\ell_2}A_{a_1} - a_1!\ell_2!L_{\ell_1}A_{a_2}).$$

Thus we have the statement (1). $\square$

Remark 6.7. Generally, the second indices of $L/\overline{A}$ and $\overline{A}/L$ depend on the choice of coordinate v. However, in the case $a_1 = \ell_1$ the second indices of $L/\overline{A}$ and $\overline{A}/L$ become independent of the choice of the coordinate v, like the degree of a vertex of a regular curve.

6.2 | Jacobian constant curves

According to §5, for a normal line congruence $f(u, v) = \boldsymbol{\gamma}(v) + u\boldsymbol{e}(v)$, the Jacobian coincides with the function $A(u, v)$:

$$\lambda(u, v) = \overline{A}(v) + uL(v) = A(u, v).$$

In general, $\lambda^{-1}(0)$ (or $f(\lambda^{-1}(0))$) is called the set of singularities of the mapping f, at which f is not immersive. The singular set of a mapping is a main object to study of singularity theory. On the other hand, even when $\lambda(0,0) \neq 0$, the level set $\lambda^{-1}(\lambda(0,0))$ is sometimes an important geometrical feature of a normal line congruence (cf. §7.1).

Write

$$\overline{A}(v) = \overline{A}(0) + \frac{\overline{A}_{a_1}}{a_1!}v^{a_1} + \cdots, \quad L(v) = \frac{L_{\ell_1}}{\ell_1!}v^{\ell_1} + \cdots$$

for integers $a_1 \geq 1, \ell_1 \geq 0$ and real values $\overline{A}_{a_1}, L_{\ell_1} \neq 0$. The diffeomorphic types of the level set germ $\lambda^{-1}(\lambda(0,0))$ at $(0,0)$ in the uv-plane are divided into the following three cases:

(i) When $\ell_1 = 0$,

$$\lambda(u, v) = \lambda(0, 0) \Longleftrightarrow u = -\frac{\overline{A}(v) - \overline{A}(0)}{L(v)}$$

and $f(\lambda^{-1}(\lambda(0,0)))$ is a Legendre curve (see Proposition 6.8);

(ii) When $\ell_1 \geq a_1 \geq 1$,

$$\lambda(u, v) = \lambda(0, 0) \Longleftrightarrow v^{a_1}\left(\frac{\overline{A}_{a_1}}{a_1!} + h.o.t. \right) = 0 \Longleftrightarrow v = 0$$

and $f(\lambda^{-1}(\lambda(0,0)))$ is the straight line along the direction $\boldsymbol{e}(0)$;

(iii) When $a_1 > \ell_1 \geq 1$,

$$\lambda(u,v) = \lambda(0,0) \Longleftrightarrow v^{\ell_1}\left(u + \frac{\overline{A}(v) - \overline{A}(0)}{L(v)}\right) = 0 \Longleftrightarrow v = 0 \text{ or } u = -\frac{\overline{A}(v) - \overline{A}(0)}{L(v)},$$

and $f(\lambda^{-1}(\lambda(0,0)))$ consists of the straight line along the direction $\boldsymbol{e}(0)$ and another Legendre curve (see Proposition 6.8).

In the cases (i) and (iii), there exists a branch of $f(\lambda^{-1}(\lambda(0,0)))$ expressed by

$$\widehat{\boldsymbol{\gamma}} := \boldsymbol{\gamma} - \frac{\overline{A} - \overline{A}(0)}{L}\boldsymbol{e}.$$

If there exists the above germ $\widehat{\boldsymbol{\gamma}}$ which goes through $f(0,0)$, we call it *the Jacobian constant curve (for short, JC curve) of the normal line congruence f at* $f(0,0)$. Put

$$\widehat{\boldsymbol{\gamma}}(v) := \boldsymbol{\gamma}(v) - \frac{\overline{A}(v) - \overline{A}(0)}{L(v)}\boldsymbol{e}(v)$$

$$\widehat{\boldsymbol{e}}(v) := \begin{cases} \dfrac{\overline{A}(0)\boldsymbol{e}(v) + \frac{d}{dv}\left(\frac{\overline{A}(v)-\overline{A}(0)}{L(v)}\right)J(\boldsymbol{e}(v))}{\sqrt{\left(\frac{d}{dv}\left(\frac{\overline{A}(v)-\overline{A}(0)}{L}\right)\right)^2 + \overline{A}^2(0)}} & \text{when } \overline{A}(0) \neq 0, \\ \boldsymbol{e}(v) & \text{when } \overline{A}(0) = 0. \end{cases}$$

The next statement immediately follows from direct calculations (see [10] for the case $\overline{A}(0) = 0$).

Proposition 6.8. $(\widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{e}})$ is a Legendre curve, and the curvature (ℓ, β) of it is given as follows:

$$\ell(v) = \begin{cases} L(v) + \dfrac{\overline{A}(0)\frac{d^2}{dv^2}\left(\frac{\overline{A}(v)-\overline{A}(0)}{L}\right)}{\left(\frac{d}{dv}\left(\frac{\overline{A}(v)-\overline{A}(0)}{L}\right)\right)^2 + \overline{A}^2(0)} & \text{when } \overline{A}(0) \neq 0, \\ L(v) & \text{when } \overline{A}(0) = 0, \end{cases}$$

$$\beta(v) = \begin{cases} \sqrt{\left(\dfrac{d}{dv}\left(\dfrac{\overline{A}(v)-\overline{A}(0)}{L}\right)\right)^2 + \overline{A}^2(0)} & \text{when } \overline{A}(0) \neq 0, \\ \dfrac{d}{dv}\left(\dfrac{\overline{A}(v)}{L(v)}\right) & \text{when } \overline{A}(0) = 0. \end{cases}$$

Remark 6.9. The JC curve of a normal line congruence f at a point is exactly the evolute (or envelope) when it is a branch of the singular value set of f (i.e. $f(\lambda^{-1}(0))$). It is well known that the evolute of a curve locally never has intersections with normal lines at inflection points of the curve; while general JC curves at the points can exist. Especially, the existence of JC curves is local invariant under the congruent equivalence, see Table 1 and §7.1.

6.3 | Evolutes and normal line congruences

We consider the cotangent bundle $\pi : T^*\mathbb{R}^2 \to \mathbb{R}^2$ over $\mathbb{R}^2$. Let (x, y, p, q) be the canonical coordinate and $\omega = dp\wedge dx + dq\wedge dy$ be a canonical symplectic form on $T^*\mathbb{R}^2$.

For a plane line congruence $f : \mathbb{R}\times I \to \mathbb{R}^2$, $f(u,v) = \gamma(v) + u\boldsymbol{e}(v)$, we define $\widetilde{f} : \mathbb{R}\times I \to T^*\mathbb{R}^2$ by $\widetilde{f}(u,v) = (f(u,v), \boldsymbol{e}(v))$. Then $\widetilde{f}^*\omega = \boldsymbol{e}'(v)\cdot\boldsymbol{e}(v)dv \wedge du = 0$. Hence $\widetilde{f}$ is a Lagrangian mapping (cf. [1, 2, 16]). Moreover, if $L(v) \neq 0$ for all $v \in I$, then $\widetilde{f}$ is a Lagrangian immersion. The caustic C_f of $\widetilde{f}$ is defined by the set of the critical value of $\pi\circ\widetilde{f} = f$. In this case, the caustic C_f is given by $\{\gamma(v) - (\overline{A}(v)/L(v))\boldsymbol{e}(v) | v \in I\}$, under the assumption $L(v) \neq 0$ for all $v \in I$. Moreover, if we consider $\widehat{f}(u,v) = \widehat{\gamma}(v) + u\boldsymbol{e}(v)$, where $\widehat{\gamma}$ is in Proposition 5.5, then $C_{\widehat{f}} = C_f$. Since $\widehat{f}$ is a normal line congruence, $(\widehat{\gamma}, \boldsymbol{e}) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve (a Legendre immersion when $L \neq 0$) with the curvature $(\ell_{\widehat{\gamma}}, \beta_{\widehat{\gamma}}) = (L, \overline{A} + \epsilon L)$. It follows that the caustic C_f is given by the evolute of $\widehat{\gamma}$ (cf. [8]). In fact, when $L(v) \neq 0$ for all $v \in I$, the evolute of $\widehat{\gamma}$, $\mathcal{E}v(\widehat{\gamma}) : I \to \mathbb{R}^2$ is given by

$$\begin{aligned} \mathcal{E}v(\widehat{\gamma})(v) &= \widehat{\gamma}(v) - \frac{\beta_{\widehat{\gamma}}(v)}{\ell_{\widehat{\gamma}}(v)}\boldsymbol{e}(v) \\ &= \gamma(v) + \epsilon(v)\boldsymbol{e}(v) - \frac{\overline{A}(v) + \epsilon(v)L(v)}{L(v)}\boldsymbol{e}(v) \\ &= \gamma(v) - \frac{\overline{A}(v)}{L(v)}\boldsymbol{e}(v). \end{aligned}$$

6.4 | $\mathcal{A}$-types of normal line congruences

In §5.1, we studied the $\mathcal{A}$-types of germs of normal line congruences, where conditions to determine just first or second terms of the Taylor expansions of the map germs are given for some cases. In this section, we show the conditions for singularities of $\mathcal{A}$-codimension ≤ 4 to appear in map germs of normal line congruences. Here the $\mathcal{A}$-codimension means a codimension of the $\mathcal{A}$-orbit in the space of map germs (see [16, 25] for details).

6.4.1 | Butterfly

Set $\overline{A}_0 = 0$, $L_0 \neq 0$, $\overline{A} \sim_{\mathcal{R}} A_3$, then $f \sim_{\mathcal{A}} (u, uv + v^5 + h.o.t)$ (see section 5.1.2). The normal forms of the $\mathcal{A}$-orbits over this type are either $(u, uv + v^5 \pm v^7)$ or $(u, uv + v^5)$, which are 7-$\mathcal{A}$-determined [25]. The first type is called butterfly. The difference is determined by higher order terms of the given germ, that is, higher order terms of $\overline{A}$, L.

Proposition 6.10. Assume that $\overline{A}$, L are written as

$$\overline{A}(v) = \frac{\overline{A}_4}{4!}v^4 + \frac{\overline{A}_5}{5!}v^5 + \frac{\overline{A}_6}{6!}v^6 + \cdots, \ L(v) = L_0 + \frac{L_1}{1!}v + \frac{L_2}{2!}v^2 + \cdots$$

where $A_4, L_0 \neq 0$. Then f is $\mathcal{A}$-equivalent to

- $(u, uv + v^5 \pm v^7)$ if and only if $\quad P \gtrless 0$;
- $(u, uv + v^5)$ if and only if $\quad P = 0$

where

$$P := -15(160L_0^4 - 147L_1^2 + 112L_0L_2)\overline{A}_4^2 + 6L_0(7L_1\overline{A}_5 + 8L_0\overline{A}_6)\overline{A}_4 - 35L_0^2\overline{A}_5^2.$$

Especially, f is always equivalent to $(u, uv + v^5 - v^7)$ if the 2nd index of L is more than 2 (that is, $L_1 = L_2 = 0$) and that of $\overline{A}$ is more than 6 (that is, $\overline{A}_5 = \overline{A}_6 = 0$).

Proof. The claim follows from direct calculations using criteria of $\mathcal{A}$-types in [19]. First, by routine coordinate changes, f is $\mathcal{A}$-equivalent to the form

$$(x, xy + y^5 + \sum_{7\geq i+j\geq 6} c_{ij}x^i y^j + h.o.t)$$

where c_{ij} are polynomials consisting of A_i, L_i as variables. Especially, the data of the following coefficients are important:

$$c_{06} = \frac{-15L_1A_4 + L_0\overline{A}_5}{6\overline{A}_4L_0}, \quad c_{07} = \frac{-5(10L_0^4 - 42L_1^2 + 7L_0L_2)\overline{A}_4 - 21L_0L_1\overline{A}_5 + L_0^2\overline{A}_6}{42\overline{A}_4L_0^2}.$$

From criteria (2) of Proposition 3.3 in [19], we see that the value

$$c_{07} - \frac{5}{8}c_{06}^2 = \frac{-15(160L_0^4 - 147L_1^2 + 112L_0L_2)\overline{A}_4^2 + 6L_0(7L_1\overline{A}_5 + 8L_0\overline{A}_6)\overline{A}_4 - 35L_0^2\overline{A}_5^2}{2016\overline{A}_4^2L_0^2}$$

determines the $\mathcal{A}$-type of f. □

Remark 6.11. The remark on index types of $\overline{A}$, L in Proposition 6.10 can be stated also as follows: f is always equivalent to $(u, uv + v^5 - v^7)$ if L is $\mathcal{R}_{(0,\ell_2)}$-equivalent to $L_0 \pm x^{\ell_2}$ for $\ell_2 \geq 3$ and $\overline{A}$ is $\mathcal{R}_{(4,\overline{\ell}_2)}$-equivalent to $\pm x^4 \pm x^{\overline{\ell}_2}$ for $\overline{\ell}_2 \geq 7$.

6.4.2 | Gulls

Set $\overline{A}_0 = 0$, $L_0 = 0$, $\overline{A} \sim_{\mathcal{R}} A_2$ and $L \sim_{\mathcal{R}} A_0$, then $f \sim_{\mathcal{A}} (u, uv^2 + v^4 + h.o.t)$(see Section 5.1.3). The $\mathcal{A}$-orbits over the above form have the representatives $(u, uv^2 + v^4 + v^{2p+1})$ with $p \geq 2$ which are $2p + 1$-determined (refer to [25]). The type with $p = 2$ is called gulls.

Proposition 6.12. Assume that $\overline{A}$, L are written as

$$\overline{A}(v) = \frac{\overline{A}_3}{3!}v^3 + \frac{\overline{A}_4}{4!}v^4 + \cdots, \ L(v) = \frac{L_1}{1!}v + \frac{L_2}{2!}v^2 + \cdots$$

$\mathcal{A}_e$-cod	$\mathcal{A}$-cod	type	characterization
0	1	fold : (u, v^2)	$\overline{A} \sim_{\mathcal{R}} A_0$
0	2	cusp : $(u, uv + v^3)$	$\overline{A} \sim_{\mathcal{R}} A_1$ and $L_0 \neq 0$
1	3	swallowtail : $(u, uv + v^4)$	$\overline{A} \sim_{\mathcal{R}} A_2$ and $L_0 \neq 0$
1	3	beaks : $(u, u^2v - v^3)$	$\overline{A} \sim_{\mathcal{R}} A_1, L_0 = 0$ and $L \sim_{\mathcal{R}} A_0$
2	4	butterfly : $(u, uv + v^5 \pm v^7)$	$\overline{A} \sim_{\mathcal{R}} A_3, L_0 \neq 0$ and a condition in Proposition 6.10
2	4	gulls : $(u, uv^2 + v^4 + v^5)$	$\overline{A} \sim_{\mathcal{R}} A_2, L \sim_{\mathcal{R}} A_0, L_0 = 0$ and a condition in Proposition 6.12

TABLE 4 $\mathcal{A}$-types of $\mathcal{A}_e$ (resp. $\mathcal{A}$)-codimension ≤ 2 (resp. 4) appearing in maps of plane line congruences. See [25] for the definition of $\mathcal{A}_e$-codimension.

where $\overline{A}_3, L_1 \neq 0$. Then f is $\mathcal{A}$-equivalent to

- $(u, uv^2 + v^4 + v^5)$ if and only if $-10L_2\overline{A}_3 + 3L_1\overline{A}_4 \neq 0$;
- $(u, uv^2 + v^4 + v^7)$ if and only if $-10L_2\overline{A}_3 + 3L_1\overline{A}_4 = 0, Q \neq 0$

where

$$Q := -7(10L_2^3 + 10L_1L_2L_3 - 3L_1^2L_4)\overline{A}_3 + 21L_1^2L_2\overline{A}_5 - 3L_1^3\overline{A}_6.$$

Especially, f is not equivalent to $(u, uv^2 + v^4 + v^5)$ if the 2nd index of L is more than 2 (that is, $L_2 = 0$) and that of $\overline{A}$ is more than 4 (that is, $\overline{A}_4 = 0$).

Proof. As in the proof of Proposition 6.10, the claim also follows from direct calculations using criteria of $\mathcal{A}$-types in [19]. By routine coordinate changes, f is $\mathcal{A}$-equivalent to the form

$$(x, xy^2 + y^4 + \sum_{7 \geq i+j \geq 5} c_{ij}x^iy^j + h.o.t)$$

where c_{ij} are polynomials consisting of L_i as variables. Especially,

$$c_{23} = 0, \quad c_{05} = \frac{-10L_2\overline{A}_3 + 3L_1\overline{A}_4}{15\overline{A}_3L_1}, \quad c_{15} = \frac{5L_2^3 - 10L_1L_2L_3 + 3L_1^2L_4}{180L_1^3},$$
$$c_{07} = \frac{-35L_2^3A_3 + 105L_1L_2^2\overline{A}_4 - 63L_1^2L_2\overline{A}_5 + 9L_1^3\overline{A}_6}{1890\overline{A}_3L_1^3}.$$

From criteria (2) of Proposition 3.5 in [19], we see that if $c_{05} \neq 0$, then f is of type gulls; and if $c_{05} = 0$ and $c_{07} - 2c_{15} + 4c_{23} \neq 0$, then f is $\mathcal{A}$-equivalent to $(x, xy^2 + y^4 + y^7)$. □

Remark 6.13. The remark on index types of $\overline{A}, L$ in Proposition 6.12 can be stated also as follows: f is not equivalent to $(u, uv^2 + v^4 + v^5)$ if L is $\mathcal{R}_{(1,\ell_2)}$-equivalent to $\pm x \pm x^{\ell_2}$ for $\ell_2 \geq 3$ and $\overline{A}$ is $\mathcal{R}_{(3,\overline{\ell}_2)}$-equivalent to $\pm x^3 \pm x^{\overline{\ell}_2}$ for $\overline{\ell}_2 \geq 5$.

6.4.3 | Summary

Summing up the above results including parts of propositions in §5.1, we have the following characterizations of singular germs of $\mathcal{A}$-codimension up to 4 appearing in the maps of normal line congruences. Examples of the figures for those $\mathcal{A}$-types are seen in §7.2 and §7.3.

Theorem 6.14. The following table shows all $\mathcal{A}$-types and characterizations of germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with $\mathcal{A}$-codimension ≤ 4 appearing in the maps of normal line congruences ($\overline{A}_0 = 0$ is always assumed). Especially, the $\mathcal{A}$-types of lips: $(u, u^2v + v^3)$ ($\mathcal{A}$-codimension $= 3$), goose: $(u, v^3 + u^3v)$ ($\mathcal{A}$-codimension $= 4$) or of corank 2 never appear.

Proof. The absences of lips and goose types follow from Proposition 5.15. The absences of germs of corank 2 immediately follow from the form of the map of a plane line congruence. □

Remark 6.15. The absences of lips and goose types can be explained also in terms of the Jacobian λ. In [19, 26], it is shown that a map germ is $\mathcal{A}$-equivalent to lips, then the Jacobian is $\mathcal{R}$-equivalent to $A_1^+ : x^2 + y^2$; or to an $\mathcal{A}$-type of the form $(x, y^3 + x^k y)$ for $k \geq 3$ (the germ is goose when $k = 3$), then the Jacobian is $\mathcal{R}$-equivalent to $A_k : x^2 \pm y^k$. With the above facts, since the Jacobian λ for the map of a plane line congruence satisfies $\lambda_{uu} = 0$ as seen in Remark 5.3, the map is never $\mathcal{A}$-equivalent to lips or an $\mathcal{A}$-type of the form $(x, y^3 + x^k y)$ for $k \geq 3$. Note also that, according to Proposition 3.4, the absent germs of corank 1 in normal line congruences can be realized by one-parameter families of Legendre curves.

7 | EXAMPLES OF NORMAL LINE CONGRUENCES WITH FIGURES

In this section, we show several figures of normal line congruences as examples.

7.1 | Inflective regular base curves

We deal with normal line congruences of types in §5.1.1. The Figures 4 - 7 show the figures of normal line congruences to base curves of the form $\boldsymbol{\gamma}_{(a,b)}(v) = (v + v^{a+2}, 2v^{b+3})$ for pairs of non-negative integers (a, b). $\overline{A}(0) \neq 0$ and $L(0) = 0$ hold for the curvatures $(\overline{A}, L)$ to the normal line congruences $f_{(a,b)}(u, v) = \boldsymbol{\gamma}_{(a,b)}(v) + u\boldsymbol{e}(v)$, where $\boldsymbol{e}$ is the normal vector of $\boldsymbol{\gamma}_{(a,b)}$ constructed as in §5. The blue curve expresses $\boldsymbol{\gamma}_{(a,b)}$, which has an inflection point at $v = 0$. The red curve expresses the set

$$f_{(a,b)}(\lambda^{-1}(\lambda(0,0))) - \{\text{the normal line of } \boldsymbol{\gamma}_{(a,b)} \text{ at } \boldsymbol{\gamma}_{(a,b)}(0)\}.$$

In Figures 4 - 5, the red curves never go through the origin, that is, the JC curves at $f_{(a,b)}(0,0)$ (defined in §6.2) never exist; while the red curves in Figures 6 - 7 are the JC curves at $f_{(a,b)}(0,0)$. Note that all figures are drawn as the image of the domain with $-0.9 \leq u \leq 0.9$ and $-0.5 \leq v \leq 0.5$ in the (u, v)-plane, and the straight lines parametrized by u are plotted par $1/30$ intervals to the domain of v.

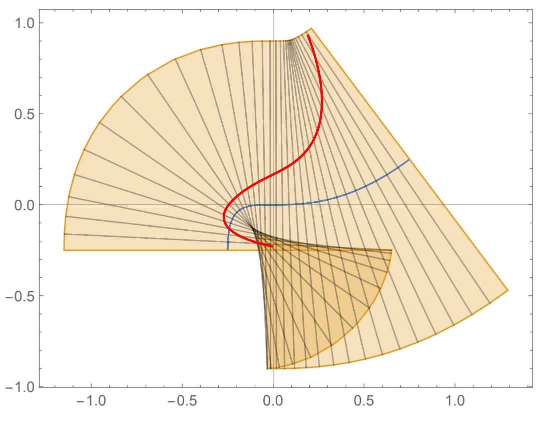

FIGURE 4 The normal line congruence to $\boldsymbol{\gamma}_{(0,0)}(v) = (v + v^2, 2v^3)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. L) at $v = 0$ is A_0 (resp. A_0).

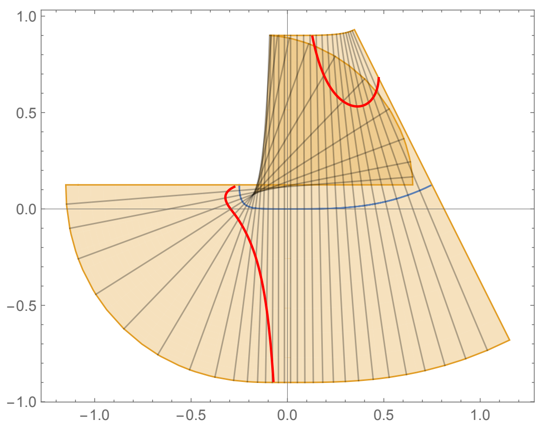

FIGURE 5 The normal line congruence to $\boldsymbol{\gamma}_{(0,1)}(v) = (v + v^2, 2v^4)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. L) at $v = 0$ is A_0 (resp. A_1).

7.2 | Singular mappings and base curves

We deal with normal line congruences of types in §5.1.2. The Figures 8 - 11 show the figures of normal line congruences to base curves of the form $\boldsymbol{\gamma}_a(v) = (v^{a+2}, v^{a+3} + v^{a+4})$ for non-negative integers a. $\overline{A}(0) = 0$ and $L(0) \neq 0$ hold for the curvatures

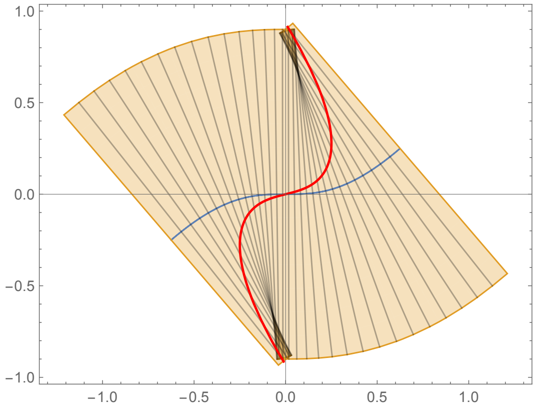

FIGURE 6 The normal line congruence to $\boldsymbol{\gamma}_{(1,0)}(v) = (v + v^3, 2v^3)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. L) at $v = 0$ is A_1 (resp. A_0).

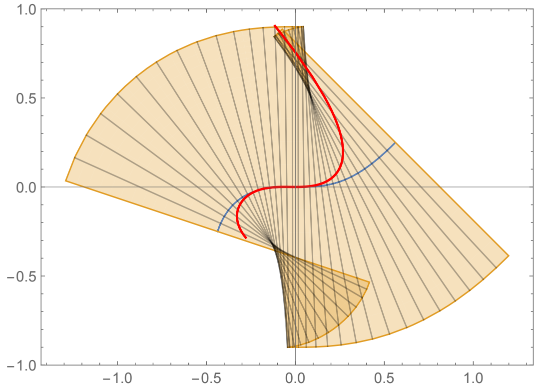

FIGURE 7 The normal line congruence to $\boldsymbol{\gamma}_{(2,0)}(v) = (v + v^4, 2v^3)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. L) at $v = 0$ is A_2 (resp. A_0).

$(\overline{A}, L)$ to the normal line congruences $f_a(u, v) = \boldsymbol{\gamma}_a(v) + u\boldsymbol{e}(v)$, where $\boldsymbol{e}$ is the normal vector of $\boldsymbol{\gamma}_a$ constructed as in §5. The blue curve expresses $\boldsymbol{\gamma}_a$. Note that all figures are drawn as the image of the domain with $-0.05 \le u \le 0.05$ and $-0.5 \le v \le 0.5$ in the (u, v)-plane, and the straight lines parametrized by u are plotted par $1/30$ intervals to the domain of v.

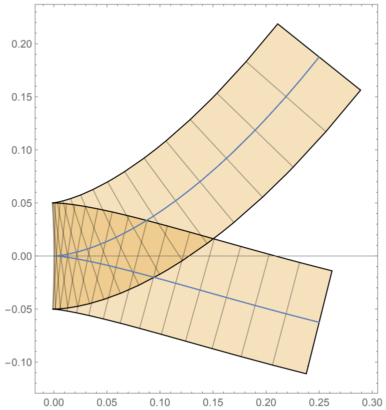

FIGURE 8 The normal line congruence to $\boldsymbol{\gamma}_0(v) = (v^2, v^3 + v^4)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. L) at $v = 0$ is A_0 (resp. A_0). The $\mathcal{A}$-equivalent type of the map of the normal line congruence is fold.

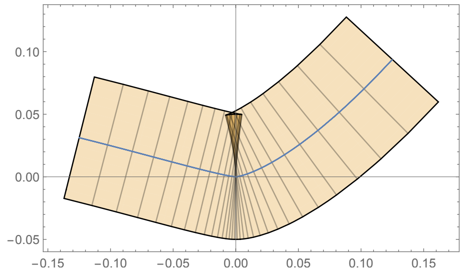

FIGURE 9 The normal line congruence to $\boldsymbol{\gamma}_1(v) = (v^3, v^4 + v^5)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. L) at $v = 0$ is A_1 (resp. A_0). The $\mathcal{A}$-equivalent type of the map of the normal line congruence is cusp.

7.3 | Singular mappings, base curves and direction curves

We deal with normal line congruences of types in 5.1.3. The Figures 12 - 14 show the figures of normal line congruences to base curves of the form $\boldsymbol{\gamma}_a(v) = (v^{a+2}, -2v^{a+4} + v^{a+5})$ for non-negative integers a, and the Figure 15 shows the figure of a normal line congruence to a base curve of the form $\widetilde{\boldsymbol{\gamma}}(v) = (v^4 + v^6, -2v^6 + v^9)$. $\overline{A}(0) = L(0) = 0$ hold for the curvatures $(\overline{A}, L)$ to the normal line congruences. The blue curve expresses $\boldsymbol{\gamma}_a$ or $\widetilde{\boldsymbol{\gamma}}$. The red curve expresses the JC curve of the normal line congruence at the origin (see (2) of Proposition 5.13 or case (ii) of §6.2). Remark also that the map of the normal line congruence to $\boldsymbol{\gamma}_2$ is $\mathcal{A}$-equivalent to the germ of type $(u, uv^2 + v^4 + v^5)$, which is called gulls, at $(u, v) = (0, 0)$; while that to $\widetilde{\boldsymbol{\gamma}}$ is $\mathcal{A}$-equivalent to the germ of type $(u, uv^2 + v^4 + v^7)$ at $(u, v) = (0, 0)$ (see §6.4.2). Note that all figures are drawn as the image of the domain with

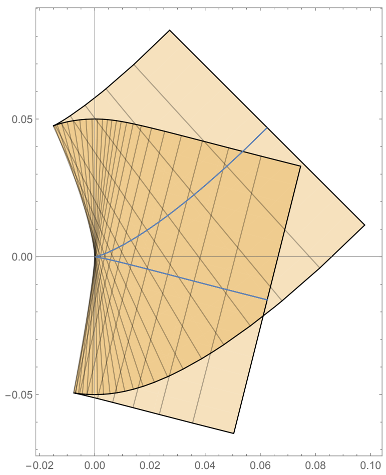

FIGURE 10 The normal line congruence to $\gamma_2(v) = (v^4, v^5 + v^6)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. L) at $v = 0$ is A_2 (resp. A_0). The $\mathcal{A}$-equivalent type of the map of the normal line congruence is swallowtail.

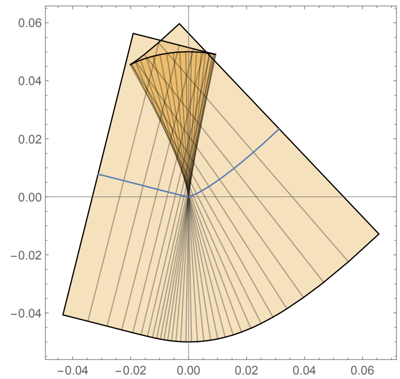

FIGURE 11 The normal line congruence to $\gamma_3(v) = (v^5, v^6 + v^7)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. L) at $v = 0$ is A_3 (resp. A_0). The $\mathcal{A}$-equivalent type of the map of the normal line congruence is butterfly.

$-0.5 \leq u \leq 0.5$ and $-0.6 \leq v \leq 0.6$ in the (u, v)-plane, and the straight lines parametrized by u are plotted par $1/30$ intervals to the domain of v.

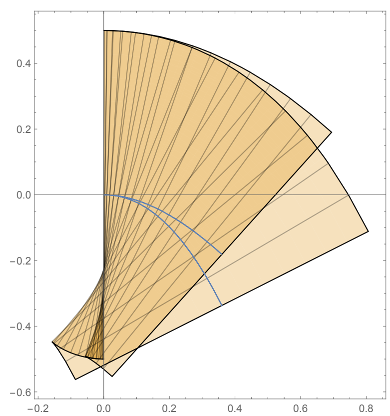

FIGURE 12 The normal line congruence to $\gamma_0(v) = (v^2, -2v^4 + v^5)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. L) at $v = 0$ is A_0 (resp. A_0). The $\mathcal{A}$-equivalent type of the map of the normal line congruence is fold. Here the JC curve at the origin does not exist.

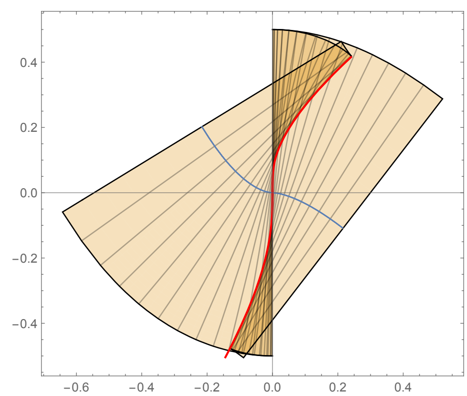

FIGURE 13 The normal line congruence to $\gamma_1(v) = (v^3, -2v^5 + v^6)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. L) at $v = 0$ is A_1 (resp. A_0). The $\mathcal{A}$-equivalent type of the map of the normal line congruence is beaks.

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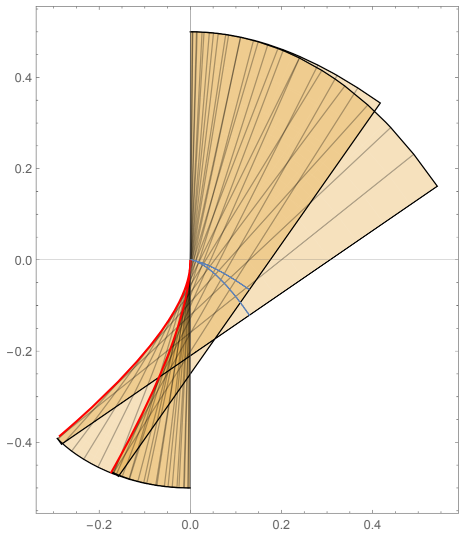

FIGURE 14 The normal line congruence to $\boldsymbol{\gamma}_2(v) = (v^4, -2v^6+v^7)$. The $\mathcal{R}$-type of $\overline{A}$ (resp. L) at $v = 0$ is A_2 (resp. A_0). The $\mathcal{A}$-equivalent type of the map of the normal line congruence is gulls.

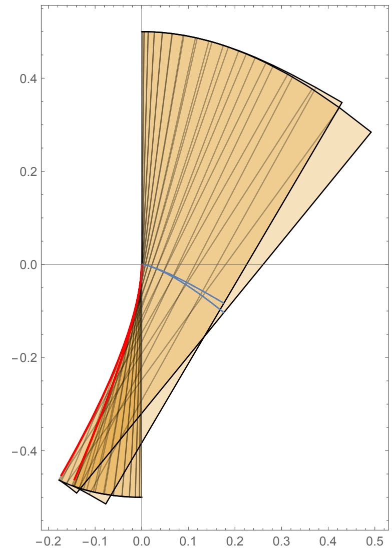

FIGURE 15 The normal line congruence to $\boldsymbol{\gamma}_d(v) = (v^4+v^6, -2v^6+v^9)$. The $\mathcal{R}$-types of $\overline{A}$ and L are the same with those to $\boldsymbol{\gamma}_2(v) = (v^4, -2v^6+v^7)$, however the $\mathcal{A}$-type of the map of the normal line congruence to $\boldsymbol{\gamma}_d$ is different from that to $\boldsymbol{\gamma}_2$.

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