Enveloids of Legendre Curves in the Unit Tangent Bundle over the Euclidean Plane

Enveloids of Legendre curves in the unit tangent bundle over the Euclidean plane

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Abstract

We define $\boldsymbol{\theta}$-enveloids for one-parameter families of Legendre curves. The $\boldsymbol{\theta}$-enveloid for a given one-parameter family of Legendre curves is a plane curve that cuts each member of the family in the same constant angle $\boldsymbol{\theta}$. As an application, we consider the definition of involutoids of frontals from the view point of $\boldsymbol{\theta}$-enveloids. Moreover, we consider the properties of normal envelopes.

Keywords: $\boldsymbol{\theta}$-Enveloids · Envelopes · Normal Envelopes · Legendre Curves · Involutoids

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1 Introduction

Isogonal trajectory is a classical topic in mathematics. There are many applications of isogonal trajectories to differential geometry, differential equations and algebra [5, 7, 13, 20, 21, 23]. An isogonal trajectory of a family of plane curves is a plane curve that cuts each member of the family in the same constant angle θ at some points. When θ equals 0 and $\pi/2$, the isogonal trajectories are envelopes and orthogonal trajectories, respectively. Many geometric objects can be explained from the perspective of isogonal trajectories. For instance, involutes and evolutes of curves are closely related to the $\pi/2$-isogonal trajectories (orthogonal trajectories) and 0-isogonal trajectories (envelopes) of the families of tangent lines and normal lines of the original curves [2–4, 6, 10–12, 16–18, 20, 25]. Pedal curves, which have important applications in other subjects, can also be described by using the language of isogonal trajectories [4, 17, 25]. From the view point of slant geometry (cf. [2, 4, 12, 17, 18]), the isogonal trajectories are the generalizations of envelopes and orthogonal trajectories [5, 13, 19, 21, 23]. If the family of curves are regular, then the angle between isogonal trajectories and the curves of the family at the intersections is well-defined. But we can not define isogonal trajectories for singular curves. In reality, however, singularities always exist on curves. In Takahashi [24], envelopes for Legendre curves in the Euclidean plane are defined. They are the generalizations of the classical envelopes for regular plane curves. In this paper, we will define θ-enveloids for one-parameter families of Legendre curves, which are the generalizations of the isogonal trajectories for regular plane curves. The definition of enveloids is a generalization of the classical envelopes from the view point of "angle", such as "evolute" to "evolutoid", "pedal" to "pedaloid" and "primitive" to "primitivoid", see [12, 17, 18]. Although there is only a difference in "angle" between envelopes and θ-enveloids ($\theta \neq 0$) in geometric meaning, there is a great difference between the two, especially the θ-enveloids are closely related to ordinary differential equations, see Theorems 5 and 6.

In section 2, we review the definitions of Legendre curves and one-parameter families of Legendre curves. We also give a moving frame and a Frenet type formula (cf. [22, 24]). In section 3, we define a θ-enveloid for a one-parameter family of Legendre curves. We find that the θ-enveloid is a frontal. As a main result, we give the θ-enveloid theorem, see Theorem 2. Moreover, we also consider the existence and uniqueness of enveloids with initial values under conditions, see Theorem 6. In section 4, from the view point of θ-enveloids, we define involutoids of frontals (singular curves). The involutoids are not only the generalizations of the classical tanvolutes for regular plane curves, but also the opposite processes of the evolutoids (cf. [2, 4, 12, 17]). We can see the involutoids of a front without inflection points are fronts, so we give the curvature. We also give the relationships between involutoids and evolutoids. In section 5, as a special case, we consider the properties of $\pi/2$-enveloids (normal envelops). The basic results on the singularity theory see [3, 6, 14, 15].

All maps considered in this paper are differentiable of class C^∞.

2 Legendre curves and one-parameter families of Legendre curves

In this section, we introduce the definitions of Legendre curves and one-parameter families of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$. For more details about Legendre curves and one-parameter families of Legendre curves see [9, 22, 24].

Let I and Λ be intervals in $\mathbb{R}$. We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve if $\gamma'(t) \cdot \nu(t) = 0$ for all $t \in I$. Moreover, if (γ, ν) is an immersion, we call (γ, ν) a Legendre immersion. We say that $\gamma : I \to \mathbb{R}^2$ is a frontal (respectively, a front) if there exists a smooth map $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve (respectively, a Legendre immersion).

We denote $J(\mathbf{x}) = (-x_2, x_1)$ the anticlockwise rotation by $\pi/2$ of $\mathbf{x} = (x_1, x_2)$. Then we define $\mu(t) = J(\nu(t))$, thus $\{\nu(t), \mu(t)\}$ is a moving frame along $\gamma(t)$. The Frenet type formula of (γ, ν) is given by

$$\begin{aligned} \begin{pmatrix} \nu'(t) \\ \mu'(t) \end{pmatrix} &= \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \mu(t) \end{pmatrix}, \\ \gamma'(t) &= \beta(t)\mu(t). \end{aligned}$$

The pair (ℓ, β) is called the curvature of (γ, ν). In this paper, we call t_0 an inflection point of the Legendre curve (γ, ν) if $\ell(t_0) = 0$, see [10, 11].

Definition 1 (cf. [22, 24]) Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a smooth mapping. We say that (γ, ν) is a one-parameter family of Legendre curves if $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$ for all $(t, \lambda) \in I \times \Lambda$.

By definition, $(\gamma(\cdot, \lambda), \nu(\cdot, \lambda)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve for each fixed $\lambda \in \Lambda$ and $\gamma : I \times \Lambda \to \mathbb{R}^2$ is a one-parameter family of frontals.

We define $\mu(t, \lambda) = J(\nu(t, \lambda))$. Then $\{\nu(t, \lambda), \mu(t, \lambda)\}$ is a moving frame along $\gamma(t, \lambda)$ and the Frenet type formula is given by

$$\begin{aligned} \begin{pmatrix} \nu_t(t,\lambda) \\ \mu_t(t,\lambda) \end{pmatrix} &= \begin{pmatrix} 0 & \ell(t,\lambda) \\ -\ell(t,\lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t,\lambda) \\ \mu(t,\lambda) \end{pmatrix}, \\ \begin{pmatrix} \nu_\lambda(t,\lambda) \\ \mu_\lambda(t,\lambda) \end{pmatrix} &= \begin{pmatrix} 0 & m(t,\lambda) \\ -m(t,\lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t,\lambda) \\ \mu(t,\lambda) \end{pmatrix}, \\ \gamma_t(t,\lambda) &= \beta(t,\lambda)\mu(t,\lambda), \\ \gamma_\lambda(t,\lambda) &= A(t,\lambda)\nu(t,\lambda) + B(t,\lambda)\mu(t,\lambda), \end{aligned}$$

where

$$\ell(t,\lambda) = \nu_t(t,\lambda)\cdot\mu(t,\lambda),\ m(t,\lambda) = \nu_\lambda(t,\lambda)\cdot\mu(t,\lambda),\ \beta(t,\lambda) = \gamma_t(t,\lambda)\cdot\mu(t,\lambda),$$

$$A(t,\lambda) = \gamma_\lambda(t,\lambda)\cdot\nu(t,\lambda),\ B(t,\lambda) = \gamma_\lambda(t,\lambda)\cdot\mu(t,\lambda).$$

By the integrability condition $\gamma_{t\lambda}(t, \lambda) = \gamma_{\lambda t}(t, \lambda)$, $\nu_{t\lambda}(t, \lambda) = \nu_{\lambda t}(t, \lambda)$ and $\mu_{t\lambda}(t, \lambda) = \mu_{\lambda t}(t, \lambda)$, $\ell(t, \lambda)$, $m(t, \lambda)$, $\beta(t, \lambda)$, $A(t, \lambda)$ and $B(t, \lambda)$ satisfy

$$\begin{aligned} \ell_\lambda(t,\lambda) &= m_t(t,\lambda), \\ A_t(t,\lambda) &= B(t,\lambda)\ell(t,\lambda) - \beta(t,\lambda)m(t,\lambda), \\ B_t(t,\lambda) &= \beta_\lambda(t,\lambda) - A(t,\lambda)\ell(t,\lambda) \end{aligned}$$

for all $(t,\lambda) \in I \times \Lambda$. We call the tuple (ℓ, m, β, A, B) with the integreability condition a curvature of the one-parameter family of Legendre curves (γ, ν).

Remark 1 Let $(\gamma,\nu): I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ, m, β, A, B). Then $(\gamma, -\nu)$ is also a one-parameter family of Legendre curves with curvature $(\ell, m, -\beta, -A, -B)$.

Example 1 Let $(\gamma,\nu): \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \times S^1$ be given by

$$\gamma(t,\lambda) = (t^2 - \lambda, t^3 + \lambda),\ \nu(t,\lambda) = (4+9t^2)^{-1/2}(-3t, 2).$$

Since

$$\gamma_t(t,\lambda) = (2t, 3t^2),\ \gamma_\lambda(t,\lambda) = (-1,1),$$
$$\nu_t(t,\lambda) = 6(4+9t^2)^{-3/2}(-2,-3t),\ \nu_\lambda(t,\lambda) = (0,0)$$

and

$$\mu(t,\lambda) = (4+9t^2)^{-1/2}(-2,-3t),$$

(γ,ν) is a one-parameter family of Legendre curves and the curvature is given by

$$\begin{aligned}&(\ell, m, \beta, A, B)(t,\lambda)\\ &= (6(4+9t^2)^{-1}, 0, -t(4+9t^2)^{1/2}, (4+9t^2)^{-1/2}(3t+2), (4+9t^2)^{-1/2}(2-3t)).\end{aligned}$$

In [22, 24], the existence and uniqueness theorem for one-parameter families of Legendre curves are given.

3 θ-enveloids for Legendre curves

Let $(\gamma,\nu): I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ, m, β, A, B) and $e: U \to I \times \Lambda, e(u) = (t(u), \lambda(u))$ be a smooth curve, where I, Λ, U are intervals in $\mathbb{R}$. Denote $E(u) = \gamma \circ e(u)$.

Definition 2 We call E a θ-enveloid and e a pre-θ-enveloid ($\theta \in [0,\pi)$) for the family of Legendre curves (γ,ν), when the conditions (1) and (2) are satisfied.

(1) The map $\lambda: U \to \Lambda$ is surjective and non-constant on any non-trivial subinterval of U (The variability condition).

(2) For all $u \in U$, $E'(u)$ and $\cos\theta\mu(t(u),\lambda(u)) + \sin\theta\nu(t(u),\lambda(u))$ are linearly dependent (The θ-parallel condition).

If we clarify θ, then we denote $e[\theta]$ and $E[\theta]$ respectively. Actually, we can prove that $E[\theta]$ is a frontal, see Proporsition 1. We remark that the θ-parallel condition is equivalent to

$$E[\theta]'(u) \cdot (-\cos\theta\nu(e[\theta](u)) + \sin\theta\mu(e[\theta](u))) = 0$$

for all $u \in U$. When $\theta = 0$, $e[0]$ is a pre-envelope and $E[0]$ is an envelope of (γ, ν) respectively [24], when $\theta = \pi/2$, we call $e[\pi/2]$ a *pre-normal envelope* and $E[\pi/2]$ a *normal envelope* of (γ, ν), respectively. When γ is regular, we can easily see that they are envelopes and orthogonal trajectories of one-parameter families of regular curves [5, 13, 19, 22–24].

However, different from the classical envelopes, the θ-enveloids ($\theta \neq 0$) of a one-parameter family of Legendre curves are closely related to the solutions of ordinary differential equations, so the θ-enveloids often appear in the form of a one-parameter family of curves, see Theorem 5 and Example 2. In this case, we call them the one-parameter family of θ-enveloids for the one-parameter family of Legendre curves.

Remark 2 By definition, $E[\theta] = E[\theta + \pi]$. Therefore, we only consider $\theta \in [0, \pi)$.

Remark 3 For a fixed $u \in U$, $E[\theta](u) = \gamma(t(u), \lambda(u))$ is not only a point on the θ-enveloid $E[\theta]$ but also a point on the frontal $\gamma(\cdot, \lambda(u))$. Since $\lambda : U \to \Lambda$ is surjective, $E[\theta]$ cuts each member of the family of frontals γ at some points.

Proposition 1 Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ, m, β, A, B). If $e[\theta] : U \to I \times \Lambda$ is a pre-θ-enveloid and

$$E[\theta] = \gamma \circ e[\theta] : U \to \mathbb{R}^2$$

is a θ-enveloid of (γ, ν), respectively. Then

$$(E[\theta], -\cos\theta\nu \circ e[\theta] + \sin\theta\mu \circ e[\theta]) : U \to \mathbb{R}^2 \times S^1$$

is a Legendre curve with the curvature

$$\begin{aligned} \ell[\theta](u) &= t'(u)\ell(e[\theta](u)) + \lambda'(u)m(e[\theta](u)), \\ \beta[\theta](u) &= -\cos\theta t'(u)\beta(e[\theta](u)) - \sin\theta\lambda'(u)A(e[\theta](u)) - \cos\theta\lambda'(u)B(e[\theta](u)). \end{aligned}$$

Proof Denote $e[\theta](u) = (t(u), \lambda(u))$. Since $e[\theta]$ is a pre-θ-enveloid of (γ, ν), we have

$$E[\theta]'(u) \cdot (-\cos\theta\nu \circ e[\theta](u) + \sin\theta\mu \circ e[\theta](u)) = 0$$

for all $u \in U$ by the θ-parallel condition. Then

$$(E[\theta], -\cos\theta\nu \circ e[\theta] + \sin\theta\mu \circ e[\theta]) : U \to \mathbb{R}^2 \times S^1$$

is a Legendre curve. The moving frame of $(E[\theta](u), -\cos\theta\nu \circ e[\theta](u) + \sin\theta\mu \circ e[\theta](u))$ is given by

$$\{-\cos\theta\nu \circ e[\theta](u) + \sin\theta\mu \circ e[\theta](u), -\cos\theta\mu \circ e[\theta](u) - \sin\theta\nu \circ e[\theta](u)\}.$$

Therefore, we have the curvature

$$\begin{aligned} \ell[\theta](u) &= \frac{d}{du}(\sin\theta\mu \circ e[\theta](u) - \cos\theta\nu \circ e[\theta](u)) \cdot (-\cos\theta\mu \circ e[\theta](u) - \sin\theta\nu \circ e[\theta](u)) \\ &= (\sin\theta\mu_t(e[\theta](u))t'(u) + \sin\theta\mu_\lambda(e[\theta](u))\lambda'(u) - \cos\theta\nu_t(e[\theta](u))t'(u) \\ &\quad - \cos\theta\nu_\lambda(e[\theta](u))\lambda'(u)) \cdot (-\cos\theta\mu \circ e[\theta](u) - \sin\theta\nu \circ e[\theta](u)) \\ &= t'(u)\ell(e[\theta](u)) + \lambda'(u)m(e[\theta](u)), \\ \beta[\theta](u) &= E[\theta]^{'}(u) \cdot (-\cos\theta\mu \circ e[\theta](u) - \sin\theta\nu \circ e[\theta](u)) \end{aligned}$$

$$= (t'(u)\gamma_t(e[\theta](u)) + \lambda'(u)\gamma_\lambda(e[\theta](u))) \cdot (-\cos\theta\mu \circ e[\theta](u) - \sin\theta\nu \circ e[\theta](u))$$
$$= -\cos\theta t'(u)\beta(e[\theta](u)) - \sin\theta\lambda'(u)A(e[\theta](u)) - \cos\theta\lambda'(u)B(e[\theta](u)).$$

□

Now we give the θ-enveloid theorem for one-parameter families of Legendre curves.

Theorem 2 Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ, m, β, A, B) and $e[\theta] : U \to I \times \Lambda$ be a smooth curve which satisfies the variability condition. Then $e[\theta]$ is a pre-θ-enveloid of (γ, ν) if and only if

$$\sin\theta t'(u)\beta(e[\theta](u)) - \cos\theta\lambda'(u)A(e[\theta](u)) + \sin\theta\lambda'(u)B(e[\theta](u)) = 0 \tag{1}$$

for all $u \in U$.

Proof Let $e[\theta]$ be a pre-θ-enveloid and $E[\theta] = \gamma \circ e[\theta]$ be a θ-enveloid of (γ, ν), respectively. By differentiate $E[\theta](u) = \gamma \circ e[\theta](u)$, we have

$$E[\theta]'(u) = t'(u)\gamma_t(e[\theta](u)) + \lambda'(u)\gamma_\lambda(e[\theta](u)).$$

Since

$$\gamma_t(t, \lambda) = \beta(t, \lambda)\mu(t, \lambda), \ \gamma_\lambda(t, \lambda) = A(t, \lambda)\nu(t, \lambda) + B(t, \lambda)\mu(t, \lambda),$$

we have

$$E[\theta]'(u) = t'(u)\beta(e[\theta](u))\mu(e[\theta](u)) + \lambda'(u)(A(e[\theta](u))\nu(e[\theta](u)) + B(e[\theta](u))\mu(e[\theta](u))).$$

By the θ-parallel condition

$$E[\theta]'(u) \cdot (-\cos\theta\nu(e[\theta](u)) + \sin\theta\mu(e[\theta](u))) = 0,$$

we have

$$\sin\theta t'(u)\beta(e[\theta](u)) - \cos\theta\lambda'(u)A(e[\theta](u)) + \sin\theta\lambda'(u)B(e[\theta](u)) = 0$$

for all $u \in U$.

On the other hand, suppose that

$$\sin\theta t'(u)\beta(e[\theta](u)) - \cos\theta\lambda'(u)A(e[\theta](u)) + \sin\theta\lambda'(u)B(e[\theta](u)) = 0$$

for all $u \in U$. Since

$$E[\theta]'(u) \cdot (-\cos\theta\nu(e[\theta](u)) + \sin\theta\mu(e[\theta](u)))$$
$$= \sin\theta t'(u)\beta(e[\theta](u)) - \cos\theta\lambda'(u)A(e[\theta](u)) + \sin\theta\lambda'(u)B(e[\theta](u)) = 0,$$

$e[\theta]$ satisfies the θ-parallel condition. Therefore $e[\theta]$ is a pre-θ-enveloid of (γ, ν). □

Remark 4 By Theorem 2, the pre-envelope $e[0]$ satisfies $A(e[0](u)) = 0$ for all $u \in U$, see [24].

Proposition 3 Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves. If $e[\theta] : U \to I \times \Lambda$ is a pre-θ-enveloid and $E[\theta] = \gamma \circ e[\theta] : U \to \mathbb{R}^2$ is a θ-enveloid of (γ, ν), respectively. Then $e[\theta]$ and $E[\theta]$ are also a pre-θ-enveloid and a θ-enveloid of $(\gamma, -\nu)$, respectively.

Proof By Remark 1, $(\gamma, -\nu)$ is a one-parameter family of Legendre curves. By the θ-enveloid theorem, we have the same pre-θ-enveloids and θ-enveloids of (γ, ν) and $(\gamma, -\nu)$. □

Let $\Phi : \widetilde{I} \times \widetilde{\Lambda} \to I \times \Lambda, \Phi(s, k) = (\phi(s, k), \varphi(k))$ be a one-parameter family of parameter changes, that is, Φ is a diffeomorphism by the above form.

Proposition 4 Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ, m, β, A, B), $e[\theta] : U \to I \times \Lambda$ be a pre-θ-enveloid and $E[\theta] = \gamma \circ e[\theta] : U \to \mathbb{R}^2$ be a θ-enveloid of (γ, ν), respectively. If $\Phi : \widetilde{I} \times \widetilde{\Lambda} \to I \times \Lambda$ is a one-parameter family of parameter changes, then

$$(\widetilde{\gamma}, \widetilde{\nu}) = (\gamma \circ \Phi, \nu \circ \Phi) : \widetilde{I} \times \widetilde{\Lambda} \to \mathbb{R}^2 \times S^1$$

is also a one-parameter family of Legendre curves. Moreover, $\Phi^{-1} \circ e[\theta] : U \to \widetilde{I} \times \widetilde{\Lambda}$ is a pre-θ-enveloid of $(\widetilde{\gamma}, \widetilde{\nu})$ and $E[\theta]$ is also a θ-enveloid of $(\widetilde{\gamma}, \widetilde{\nu})$.

Proof We denote

$$\widetilde{\gamma}(s, k) = \gamma(\Phi(s, k)),\ \widetilde{\nu}(s, k) = \nu(\Phi(s, k)),\ \widetilde{\mu}(s, k) = \mu(\Phi(s, k)).$$

Since

$$\widetilde{\gamma}_s(s, k) = \phi_s(s, k)\gamma_t(\Phi(s, k)),\ \gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$$

for all $(t, \lambda) \in I \times \Lambda$, we have $\widetilde{\gamma}_s(s, k) \cdot \widetilde{\nu}(s, k) = 0$ for all $(s, k) \in \widetilde{I} \times \widetilde{\Lambda}$. Then $(\widetilde{\gamma}, \widetilde{\nu})$ is a one-parameter family of Legendre curves with the moving frame $\{\widetilde{\nu}(s, k), \widetilde{\mu}(s, k)\}$. Since Φ is a diffeomorphism and $\Phi(s, k) = (\phi(s, k), \varphi(k))$, $\Phi^{-1} : I \times \Lambda \to \widetilde{I} \times \widetilde{\Lambda}$ is given by $\Phi^{-1}(t, \lambda) = (\psi(t, \lambda), \varphi^{-1}(\lambda))$. We denote $e[\theta](u) = (t(u), \lambda(u))$. It follows that $\Phi^{-1} \circ e[\theta](u) = (\psi(t(u), \lambda(u)), \varphi'(\lambda(u)))$. Since $\lambda : U \to \Lambda$ is surjective and $\varphi^{-1} : \Lambda \to \widetilde{\Lambda}$ is a diffeomorphism, $\varphi^{-1} \circ \lambda : U \to \widetilde{\Lambda}$ is also surjective. Moreover, $d(\varphi^{-1}(\lambda(u)))/du = (\varphi^{-1})'(\lambda(u))\lambda'(u)$, thus the variability condition still holds. The curvature of $(\widetilde{\gamma}, \widetilde{\nu})$ is given by

$$\begin{aligned}
\widetilde{\ell}(s, k) &= \widetilde{\nu}_s(s, k) \cdot \widetilde{\mu}(s, k) = \phi_s(s, k)\ell(\Phi(s, k)),\\
\widetilde{m}(s, k) &= \widetilde{\gamma}_k(s, k) \cdot \widetilde{\mu}(s, k) = \phi_k(s, k)\ell(\Phi(s, k)) + \varphi'(k)m(\Phi(s, k)),\\
\widetilde{\beta}(s, k) &= \widetilde{\gamma}_s(s, k) \cdot \widetilde{\mu}(s, k) = \phi_s(s, k)\beta(\Phi(s, k)),\\
\widetilde{A}(s, k) &= \widetilde{\gamma}_k(s, k) \cdot \widetilde{\nu}(s, k) = \varphi'(k)A(\Phi(s, k)),\\
\widetilde{B}(s, k) &= \widetilde{\gamma}_k(s, k) \cdot \widetilde{\mu}(s, k) = \phi_k(s, k)\beta(\Phi(s, k)) + \varphi'(k)B(\Phi(s, k)).
\end{aligned}$$

By a direct calculation, we have

$$\begin{aligned}
&\sin\theta \left(\frac{d}{du}(\psi(t, \lambda) \circ e[\theta](u))\right) \widetilde{\beta}(\Phi^{-1} \circ e[\theta](u))\\
&- \cos\theta \left(\frac{d}{du}(\varphi^{-1}(\lambda) \circ e[\theta](u))\right) \widetilde{A}(\Phi^{-1} \circ e[\theta](u))\\
&+ \sin\theta \left(\frac{d}{du}(\varphi^{-1}(\lambda) \circ e[\theta](u))\right) \widetilde{B}(\Phi^{-1} \circ e[\theta](u))\\
= &\sin\theta(\psi_t(t, \lambda) \circ e[\theta](u)t'(u) + \psi_\lambda(t, \lambda) \circ e[\theta](u)\lambda'(u))\phi_s(\Phi^{-1} \circ e[\theta](u))\beta(e[\theta](u))\\
&- \cos\theta((\varphi^{-1})'(\lambda(u))\lambda'(u))\varphi'(\varphi^{-1}(\lambda(u)))A(e[\theta](u))\\
&+ \sin\theta((\varphi^{-1})'(\lambda(u))\lambda'(u))(\phi_k(\Phi^{-1} \circ e[\theta](u))\beta(e[\theta](u)) + \varphi'(\varphi^{-1}(\lambda(u)))B(e[\theta](u)))
\end{aligned}$$

$= \sin\theta t'(u)\beta(e[\theta](u)) - \cos\theta\lambda'(u)A(e[\theta](u)) + \sin\theta\lambda'(u)B(e[\theta](u)).$

Since $e[\theta]$ is a pre-θ-enveloid of (γ,ν), by Theorem 2, we have

$$\sin\theta t'(u)\beta(e[\theta](u)) - \cos\theta\lambda'(u)A(e[\theta](u)) + \sin\theta\lambda'(u)B(e[\theta](u)) = 0$$

for all $u \in U$. Thus, $\Phi^{-1} \circ e[\theta] : U \to \widetilde{I} \times \widetilde{\Lambda}$ is a pre-θ-enveloid of $(\widetilde{\gamma},\widetilde{\nu})$ and $E[\theta] = \gamma \circ e[\theta] = \gamma \circ \Phi \circ \Phi^{-1} \circ e[\theta]$ is a θ-enveloid of $(\widetilde{\gamma},\widetilde{\nu})$. □

By Theorem 2, if $\theta \neq 0$, the pre-θ-enveloids are actually the solutions of specific differential equations that are related to the curvature. Therefore, the θ-enveloids often appear in the form of one-parameter families of curves, which is different from the classical envelopes (cf. [24]). In this case, we call them the one-parameter families of θ-enveloids for the one-parameter families of Legendre curves. Moreover, we can also consider the existence and uniqueness of θ-enveloids with initial values under conditions, see Theorem 6. In the following theorem, we first give the relationships between the existence of θ-enveloids and the existence of the solutions of ordinary differential equations related to the curvature, and the theorem also gives a general method to find θ-enveloids.

Theorem 5 Let $(\gamma,\nu) : \mathbb{R} \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ, m, β, A, B). If $\theta \neq 0$, then there exists a pre-θ-enveloid $e[\theta] : U \to \mathbb{R} \times \Lambda$, $e[\theta](u) = (t(u), \lambda(u))$ of (γ,ν), where $\lambda : U \to \Lambda$ is a diffeomorphism, if and only if the following ordinary differential equation

$$\beta(y,u)\frac{dy}{du} = \cot\theta A(y,u) - B(y,u) \tag{2}$$

has a solution $y = y(u)$ on the whole interval Λ, where $\cot\theta = 1/\tan\theta = \cos\theta/\sin\theta$.

Proof By Theorem 2, if $e[\theta] : U \to \mathbb{R} \times \Lambda$, $e[\theta](u) = (t(u), \lambda(u))$ is a pre-θ-enveloid of (γ,ν), we have

$$\sin\theta t'(u)\beta(e[\theta](u)) - \cos\theta\lambda'(u)A(e[\theta](u)) + \sin\theta\lambda'(u)B(e[\theta](u)) = 0,$$

that is, $e[\theta](u) = (t(u), \lambda(u))$ satisfies (1). If $t(u) = y$ and $\lambda(u) = x$, then we have an ordinary differential equation (2). Since $\lambda : U \to \Lambda$, $\lambda(u) = x$ is a diffeomorphism, we have the inverse mapping $\lambda^{-1} : \Lambda \to U$, $\lambda^{-1}(x) = u$. Then we have $y = t(u) = t(\lambda^{-1}(x))$, and we denote $y = y(x) = t(\lambda^{-1}(x))$.

Next we will verify that $y = y(x)$ is a solution of (2) on the whole interval Λ. By $y = y(x) = t(\lambda^{-1}(x))$, $\lambda^{-1}(x) = u$, we have

$$\frac{dy}{dx} = t'(\lambda^{-1}(x))(\lambda^{-1})'(x) = t'(u)\frac{1}{\lambda'(u)}.$$

By (1), it is easy to obtain that $y = y(x)$ satisfies (2), which means $y = y(x)$ is a solution of (2). Moreover, since $\lambda : U \to \Lambda$ is a diffeomorphism, we have $y = y(x)$ is a solution of (2) on the whole interval Λ.

Conversely, if (2) has a solution $y = y(u)$ on the whole interval Λ, then we have

$$\beta(y(u),u)y'(u) = \cot\theta A(y(u),u) - B(y(u),u).$$

Let $e[\theta] : \Lambda \to \mathbb{R} \times \Lambda$, $e[\theta](u) = (y(u), u)$, then $e[\theta]$ satisfies the variability condition. Moreover, we have

$$\sin\theta y'(u)\beta(y(u),u) - \cos\theta A(y(u),u) + \sin\theta B(y(u),u) = 0$$

for all $u \in \Lambda$. By Theorem 2, $e[\theta]$ is a pre-θ-enveloid of (γ,ν). □

Remark 5 Sometimes for the sake of calculation, the pre-θ-enveloids might be of the form $(u, y(u))$, see Example 2.

By Theorem 5, $e[\theta] : \Lambda \to \mathbb{R} \times \Lambda,\ e[\theta](u) = (y(u), u)$ is a pre-θ-enveloid of $(\gamma, \nu) : \mathbb{R} \times \Lambda \to \mathbb{R}^2 \times S^1$ if $y = y(u)$ is a solution of

$$\beta(y, u)\frac{dy}{du} = \cot\theta A(y, u) - B(y, u)$$

on the whole interval Λ. Then we consider the standard form of the above ordinary differential equation, that is

$$\frac{dy}{du} = \frac{\cot\theta A(y, u) - B(y, u)}{\beta(y, u)}.$$

We denote $F(y, u) = (\cot\theta A(y, u) - B(y, u))/\beta(y, u)$. Since we consider the Legendre curves, $\beta(y, u)$ may be equal to 0 at some points, which means $F(y, u)$ may not be well defined at some points. Therefore, we consider the condition that $F : \mathbb{R} \times \Lambda \to \mathbb{R}$ is a smooth function. In the following theorem, by the Lipschitz condition of ordinary differential equations with initial values (cf.[1]), we give the existence and uniqueness theorem of θ-enveloids with initial values.

Theorem 6 Let $(\gamma, \nu) : \mathbb{R} \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ, m, β, A, B). Suppose that $\gamma(y_0, u_0)$ is an initial point of the one-parameter family of frontals γ, where $y_0 \in \mathbb{R},\ u_0 \in \Lambda$. If $F(y, u) = (\cot\theta A(y, u) - B(y, u))/\beta(y, u)$ satisfies the globally Lipschitz condition on the strip area $S = \{(y, u) \mid y \in \mathbb{R}, u \in \Lambda\}$, that is

$$\mid F(y_1, u) - F(y_2, u) \mid \leq K \mid y_1 - y_2 \mid$$

for some constant $K > 0$ and for all $(y_1, u),\ (y_2, u)$ in S, then there exists a unique solution $y = y(u)$ of the initial value problem

$$\frac{dy}{du} = F(y, u),\ y(u_0) = y_0,$$

on the whole interval Λ. Moreover, there exists a unique pre-θ-enveloid $e[\theta] : \Lambda \to \mathbb{R} \times \Lambda,\ e[\theta](u) = (y(u), u)$ of (γ, ν) which satisfies $y(u_0) = y_0$.

Proof By the Lipschitz condition of ordinary differential equations with initial values, see [1], we obtain the theorem. □

Example 2 Let $(\gamma, \nu) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves given by

$$\gamma(t, \lambda) = (t^2 - \lambda, t^3 + \lambda),\ \nu(t, \lambda) = (4 + 9t^2)^{-1/2}(-3t, 2),$$

see Fig. 1. We consider $\theta = \pi/4$. By Example 1, the curvature of (γ, ν) is

$$\begin{aligned}&(\ell, m, \beta, A, B)(t, \lambda)\\ &= (6(4 + 9t^2)^{-1}, 0, -t(4 + 9t^2)^{1/2}, (4 + 9t^2)^{-1/2}(3t + 2), (4 + 9t^2)^{-1/2}(2 - 3t)).\end{aligned}$$

By Theorem 5, we have the following ordinary differential equation

$$-y\sqrt{4+9y^2}\frac{dy}{du} = \frac{6y}{\sqrt{4+9y^2}}. \tag{3}$$

By a direct calculation, we have

$$\frac{dy}{du} = \frac{-6}{4+9y^2}. \tag{4}$$

Let $F(y,u) = -6/(4+9y^2)$, we have

$$\begin{aligned} \mid F(y_1,u) - F(y_2,u) \mid &= \frac{\mid -54y_2^2 + 54y_1^2 \mid}{(4+9y_1^2)(4+9y_2^2)} \\ &= 54 \mid y_1 - y_2 \mid \frac{\mid y_1 + y_2 \mid}{(4+9y_1^2)(4+9y_2^2)} \\ &\leq 54 \mid y_1 - y_2 \mid \frac{\mid y_1 \mid + \mid y_2 \mid}{(4+9y_1^2)(4+9y_2^2)} \\ &\leq 54 \mid y_1 - y_2 \mid \left(\frac{\mid y_1 \mid}{(4+9y_1^2)} + \frac{\mid y_2 \mid}{(4+9y_2^2)} \right) \\ &\leq 9 \mid y_1 - y_2 \mid. \end{aligned}$$

Therefore, $F(y,u)$ satisfies the Lipschitz condition, and there exists a unique pre-θ-enveloid $e[\theta] : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $e[\theta](u) = (y(u), u)$ if we choose an initial point by Theorem 6.

However, the forms of the solutions of (4) are complicated. Hence we consider the ordinary differential equation (5),

$$\frac{du}{dy} = -\frac{2}{3} - \frac{3}{2}y^2. \tag{5}$$

Then the general solution of (5) is given by

$$u = -\frac{2}{3}y - \frac{1}{2}y^3 + c,$$

where c is a constant. Therefore, the one-parameter family of pre-$\pi/4$-enveloids of (γ, ν) is given by

$$e^c[\pi/4] : \mathbb{R} \to \mathbb{R} \times \mathbb{R},\ e^c[\pi/4](u) = \left(u, -\frac{2}{3}u - \frac{1}{2}u^3 + c \right),$$

where c is a constant, see Remark 5. Note that $y = 0$ is also a solution of (3), then

$$e[\pi/4] : \mathbb{R} \to \mathbb{R} \times \mathbb{R},\ e[\pi/4](u) = (0, u)$$

is also a pre-$\pi/4$-enveloid of (γ, ν). Therefore, we have

$$E^c[\pi/4](u) = \gamma \circ e^c[\pi/4](u) = \left(\frac{2}{3}u + u^2 + \frac{1}{2}u^3 - c, -\frac{2}{3}u + \frac{1}{2}u^3 + c \right),$$

$$E[\pi/4](u) = \gamma \circ e[\pi/4](u) = (-u, u)$$

are $\pi/4$-enveloids of (γ, ν), see Figs. 2 and 3. Where $E^c[\pi/4]$ is a one-parameter family of $\pi/4$-enveloids of (γ, ν). Note that $E[\pi/4]$ is passing through 3/2-cusps of (γ, ν), see Fig. 4.

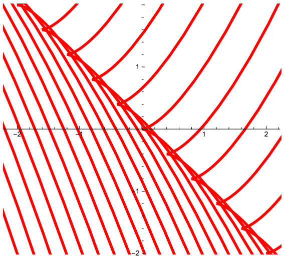

Fig. 1 One-parameter family of frontals γ (the red curves).

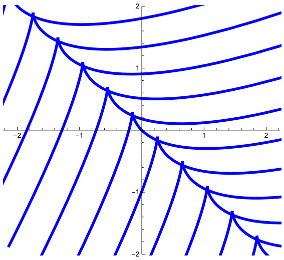

Fig. 2 The one-parameter family of $\pi/4$-enveloids $E^c[\pi/4]$ (the blue curves).

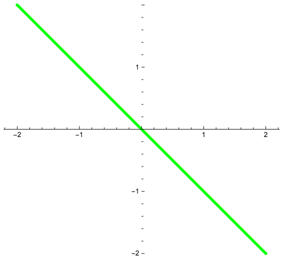

Fig. 3 The $\pi/4$-enveloid $E[\pi/4]$ (the green curve).

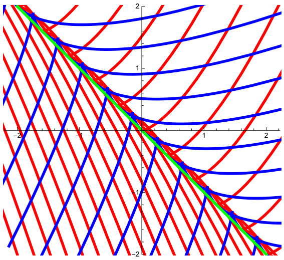

Fig. 4 γ (the red curves), the one-parameter family of $\pi/4$-enveloids $E^c[\pi/4]$ (the blue curves) and the $\pi/4$-enveloid $E[\pi/4]$ (the green curve).

4 Involutoids of frontals in the Euclidean plane from the view point of θ-enveloids

In this section, we introduce involutoids of frontals in the Euclidean plane from the view point of θ-enveloids. The notion of involutes and evolutes for regular plane curves are familiar objects in differential geometry. The evolute of a regular plane curve can be described as an envelope of the family of normal lines of the original curve, conversely, the original curve is the evolute of the involute. In Giblin and Warder [12], for plane curves, the notion of evolutoid was introduced. It is a generalization of the classical evolute. They consider the process of when the envelope of the tangent lines changes to the normal lines of the given curve. In Apostol and Mnatsakanian [2], the notion of tanvolutes for regular plane curves was introduced. The tanvolute of a regular plane curve is not only the generalization of involute but also the opposite process of the evolutoid. In Izumiya and Takeuchi [17], the evolutoids of frontals (singular curves) were introduced, the authors find some relationships between evolutoids and pedaloids, they also find some relationships between primitivoids and inversions of plane curves [18]. In Aydın Şekerci and Izumiya [4], the evolutoids and pedaloids in the Minkowshi plane are also investigated.

However, for singular plane curves, the definition of tanvolutes is vague. We will define involutoids of frontals (singular curves) in the Euclidean plane from the view point of θ-enveloids. Actually, for regular plane curves, the involutoids are the tanvolutes. Hence the involutoids are the generalizations of the classical tanvolutes for regular plane curves. Firstly, we review the definitions of evolutes and involutes of fronts without inflection points. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with curvature (ℓ, β). Suppose that $\ell(t) \neq 0$ for all $t \in I$. Then (γ, ν) is a Legendre immersion.

In Fukunaga and Takahashi [10], the evolute of a front γ without inflection points is defined by

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \alpha(t)\nu(t),$$

where $\alpha(t) = \beta(t)/\ell(t)$. Moreover, $\mathcal{E}v(\gamma)$ is also a front.

In Fukunaga and Takahashi [11], the involute of a front γ without inflection points at t_0 $(t_0 \in I)$ is defined by

$$Inv(\gamma, t_0)(t) = \gamma(t) - \left(\int_{t_0}^{t} \beta(t)dt\right)\mu(t).$$

Moreover, $Inv(\gamma, t_0)$ is also a front.

The definition of θ-evolutoid of a Legendre curve is as follows.

Definition 3 (cf. [17]) Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with curvature (ℓ, β). Suppose that there exists $\alpha(t)$ such that $\beta(t) = \alpha(t)\ell(t)$ for any $t \in I$. Then the θ-evolutoid of γ is given by

$$\mathcal{E}v(\gamma)[\theta](t) = \gamma(t) - \alpha(t)\sin\theta(\cos\theta\mu(t) + \sin\theta\nu(t)).$$

By definition, we have

$$\mathcal{E}v(\gamma)[0](t) = \gamma(t),\ \mathcal{E}v(\gamma)[\pi/2](t) = \mathcal{E}v(\gamma)(t).$$

In Izumiya and Takeuchi [17], for a Legendre immersion (γ, ν) without inflection points, the θ-evolutoid $\mathcal{E}v(\gamma)[\theta]$ is a front, more precisely, $(\mathcal{E}v(\gamma)[\theta], \nu_{\mathcal{E}}[\theta]) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the moving frame $\{\nu_{\mathcal{E}}[\theta], \mu_{\mathcal{E}}[\theta]\}$, where

$$\nu_{\mathcal{E}}[\theta](t) = -\cos\theta\nu(t) + \sin\theta\mu(t),\ \mu_{\mathcal{E}}[\theta](t) = -\sin\theta\nu(t) - \cos\theta\mu(t),$$

and the curvature of $(\mathcal{E}v(\gamma)[\theta], \nu_{\mathcal{E}}[\theta])$ is given by

$$\ell_{\mathcal{E}}[\theta](t) = \ell(t),\ \beta_{\mathcal{E}}[\theta](t) = \alpha'(t)\sin\theta - \beta(t)\cos\theta.$$

Now we give the definition of θ-involutoids of frontals.

Definition 4 Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with curvature (ℓ, β). If $\theta \neq 0$, then the θ-involutoid of γ at $t_0 \in I$ is defined by

$$Inv(\gamma, t_0)[\theta](t) = \gamma(t) - \left(e^{-\int_{t_0}^t \cot\theta\ell(t)dt} \int_{t_0}^t \beta(t)e^{\int_{t_0}^t \cot\theta\ell(t)dt}dt\right)\mu(t),$$

where $\cot\theta = 1/\tan\theta = \cos\theta/\sin\theta$. If $\theta = 0$, $Inv(\gamma, t_0)[0](t) = \gamma(t)$.

When $\theta = \pi/2$, the $\pi/2$-involutoid is the classical involute given by

$$Inv(\gamma, t_0)[\pi/2](t) = \gamma(t) - \left(\int_{t_0}^t \beta(t)dt\right)\mu(t).$$

Let Λ be an interval of $\mathbb{R}$ and $(\gamma, \nu) : \Lambda \to \mathbb{R}^2 \times S^1, \lambda \mapsto (\gamma(\lambda), \nu(\lambda))$ be a Legendre curve with curvature (ℓ, β). The family of tangent lines of γ at $\gamma(\lambda)$ is given by $\widetilde{\gamma} : \mathbb{R} \times \Lambda \to \mathbb{R}^2$, $\widetilde{\gamma}(t, \lambda) = \gamma(\lambda) + t\mu(\lambda)$. Since $\widetilde{\gamma}_t(t, \lambda) = \mu(\lambda)$, we have $\widetilde{\gamma}_t(t, \lambda) \cdot \nu(\lambda) = 0$ for all $(t, \lambda) \in \mathbb{R} \times \Lambda$. Therefore, $(\widetilde{\gamma}, \widetilde{\nu}) : \mathbb{R} \times \Lambda \to \mathbb{R}^2 \times S^1$ is a one-parameter family of Legendre curves, where $\widetilde{\nu}(t, \lambda) = \nu(\lambda)$. Moreover, we define $\widetilde{\mu}(t, \lambda) = J(\widetilde{\nu}(t, \lambda)) = \mu(\lambda)$. We call $(\widetilde{\gamma}, \widetilde{\nu})$ a one-parameter family of Legendre tangent lines of (γ, ν) and the curvature of $(\widetilde{\gamma}, \widetilde{\nu})$ is given by

$$\begin{aligned}
\widetilde{\ell}(t, \lambda) &= \widetilde{\nu}_t(t, \lambda) \cdot \widetilde{\mu}(t, \lambda) = 0, \\
\widetilde{m}(t, \lambda) &= \widetilde{\nu}_\lambda(t, \lambda) \cdot \widetilde{\mu}(t, \lambda) = m(\lambda), \\
\widetilde{\beta}(t, \lambda) &= \widetilde{\gamma}_t(t, \lambda) \cdot \widetilde{\mu}(t, \lambda) = 1, \\
\widetilde{A}(t, \lambda) &= \widetilde{\gamma}_\lambda(t, \lambda) \cdot \widetilde{\nu}(t, \lambda) = -t\ell(\lambda), \\
\widetilde{B}(t, \lambda) &= \widetilde{\gamma}_\lambda(t, \lambda) \cdot \widetilde{\mu}(t, \lambda) = \beta(\lambda).
\end{aligned}$$

As an application of θ-enveloids, we consider θ-involutoids as the θ-enveloids of the one-parameter family of Legendre tangent lines $(\widetilde{\gamma}, \widetilde{\nu})$. Suppose that

$e[\theta] : U \to \mathbb{R} \times \Lambda$, $e[\theta](u) = (t(u), \lambda(u))$ is a pre-θ-enveloid of $(\widetilde{\gamma}, \widetilde{\nu})$. When $\theta = 0$, we assume that (γ, ν) has no inflection points.

Proposition 7 Under the above notations, we have the following.

(1) If $\theta = 0$, then $\widetilde{\gamma} \circ e[0](u) = \gamma(u) = Inv(\gamma, u_0)[0](u)$, where $e[0] : \Lambda \to \mathbb{R} \times \Lambda$, $e[0](u) = (0, u)$ is the pre-envelope of $(\widetilde{\gamma}, \widetilde{\nu})$ and $u_0 \in \Lambda$.

(2) If $\theta \neq 0$, then

$$\begin{aligned}\widetilde{\gamma} \circ e[\theta](u) &= \gamma(u) - \left(e^{-\int_{u_0}^{u} \cot\theta \ell(u) du} \int_{u_0}^{u} \beta(u) e^{\int_{u_0}^{u} \cot\theta \ell(u) du} du \right) \mu(u) \\ &= Inv(\gamma, u_0)[\theta](u),\end{aligned}$$

where $e[\theta] : \Lambda \to \mathbb{R} \times \Lambda$,

$$e[\theta](u) = \left(-e^{-\int_{u_0}^{u} \cot\theta \ell(u) du} \int_{u_0}^{u} \beta(u) e^{\int_{u_0}^{u} \cot\theta \ell(u) du} du, u \right)$$

is the pre-θ-enveloid of $(\widetilde{\gamma}, \widetilde{\nu})$ and $u_0 \in \Lambda$.

Proof (1) Suppose that $e[0] : U \to \mathbb{R} \times \Lambda$ is a pre-envelope of $(\widetilde{\gamma}, \widetilde{\nu})$. By Theorem 2, $e[0]$ satisfies $\widetilde{A}(e[0](u)) = 0$ for all $u \in U$, thus $-t(u)\ell(\lambda(u)) = 0$ for all $u \in U$. Since (γ, ν) has no inflection points, we have $t(u) = 0$ for all $u \in U$. We take $U = \Lambda$ and let $\lambda(u) = u$ which satisfies the variability condition. Then $e[0] : \Lambda \to \mathbb{R} \times \Lambda$ is given by $e[0](u) = (0, u)$. Moreover, $\widetilde{\gamma} \circ e[0](u) = \gamma(u)$ is an envelope of $(\widetilde{\gamma}, \widetilde{\nu})$.

(2) Suppose that $e[\theta] : U \to \mathbb{R} \times \Lambda$ is a pre-θ-enveloid of $(\widetilde{\gamma}, \widetilde{\nu})$. By Theorem 2, $e[\theta]$ satisfies

$$\sin\theta \widetilde{\beta}(e[\theta](u))t'(u) - \cos\theta \widetilde{A}(e[\theta](u))\lambda'(u) + \sin\theta \widetilde{B}(e[\theta](u))\lambda'(u) = 0$$

for all $u \in U$. By the curvature of $(\widetilde{\gamma}, \widetilde{\nu})$, we have

$$\sin\theta t'(u) + \cos\theta t(u)\ell(\lambda(u))\lambda'(u) + \sin\theta \beta(\lambda(u))\lambda'(u) = 0,$$

then we have

$$t'(u) = (-\cot\theta t(u)\ell(\lambda(u)) - \beta(\lambda(u)))\lambda'(u)$$

for all $u \in U$. Let $t(u) = y$ and $\lambda(u) = x$, we have

$$\frac{dy}{dx} = -\cot\theta y \ell(x) - \beta(x).$$

By solving the above ordinary differential equation with the initial value $y(x_0) = 0$, where $x_0 = \lambda(u_0)$, $u_0 \in \Lambda$, we have the solution

$$y = -e^{-\int_{x_0}^{x} \cot\theta \ell(x) dx} \int_{x_0}^{x} \beta(x) e^{\int_{x_0}^{x} \cot\theta \ell(x) dx} dx.$$

We take $U = \Lambda$ and let $\lambda(u) = u$ which satisfies the variability condition, then we have

$$t(u) = -e^{-\int_{u_0}^{u} \cot\theta \ell(u) du} \int_{u_0}^{u} \beta(u) e^{\int_{u_0}^{u} \cot\theta \ell(u) du} du.$$

Thus $e[\theta] : \Lambda \to \mathbb{R} \times \Lambda$ is given by

$$e[\theta](u) = \left(-e^{-\int_{u_0}^{u} \cot\theta \ell(u) du} \int_{u_0}^{u} \beta(u) e^{\int_{u_0}^{u} \cot\theta \ell(u) du} du, u \right).$$

Moreover,

$$\widetilde{\gamma} \circ e[\theta](u) = \gamma(u) - \left(e^{-\int_{u_0}^{u} \cot\theta \ell(u) du} \int_{u_0}^{u} \beta(u) e^{\int_{u_0}^{u} \cot\theta \ell(u) du} du \right) \mu(u)$$

is a θ-enveloids of $(\widetilde{\gamma}, \widetilde{\nu})$. ∎

Proposition 8 Let $(\gamma,\nu): I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with curvature (ℓ,β) and without inflection points. For any $t_0 \in I$, the θ-involutoid $Inv(\gamma,t_0)[\theta]: I \to \mathbb{R}^2$ $(\theta \neq 0)$ is a front. More precisely, $(Inv(\gamma,t_0)[\theta], \cos\theta\nu - \sin\theta\mu): I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature

$$\ell_I[\theta](t) = \ell(t),\ \beta_I[\theta](t) = \frac{\omega[\theta](t)}{\sin\theta}\ell(t),$$

where

$$\omega[\theta](t) = e^{-\int_{t_0}^t \cot\theta\ell(t)dt}\int_{t_0}^t \beta(t)e^{\int_{t_0}^t \cot\theta\ell(t)dt}dt.$$

Proof By the Frenet formula of (γ,ν), we have

$$\begin{aligned}\frac{d}{dt}Inv(\gamma,t_0)[\theta](t) &= \beta(t)\mu(t) + (\cot\theta\ell(t)\omega[\theta](t) - \beta(t))\mu(t) + \omega[\theta](t)\ell(t)\nu(t)\\ &= \omega[\theta](t)\ell(t)(\cot\theta\mu(t) + \nu(t)).\end{aligned}$$

Then we have

$$\left(\frac{d}{dt}Inv(\gamma,t_0)[\theta](t)\right)\cdot(\cos\theta\nu(t) - \sin\theta\mu(t)) = 0,$$

thus $(Inv(\gamma,t_0)[\theta], \cos\theta\nu - \sin\theta\mu)$ is a Legendre curve. Moreover, we denote

$$\nu_I[\theta](t) = \cos\theta\nu(t) - \sin\theta\mu(t),$$

then we define

$$\mu_I[\theta](t) = J(\nu_I[\theta](t)) = \sin\theta\nu(t) + \cos\theta\mu(t).$$

Thus we have the curvature

$$\begin{aligned}\ell_I[\theta](t) &= \left(\frac{d}{dt}\nu_I[\theta](t)\right)\cdot\mu_I[\theta](t)\\ &= (\ell(t)(\cos\theta\mu(t) + \sin\theta\nu(t)))\cdot\mu_I[\theta](t) = \ell(t),\\ \beta_I[\theta](t) &= \left(\frac{d}{dt}Inv(\gamma,t_0)[\theta](t)\right)\cdot\mu_I[\theta](t)\\ &= (\omega[\theta](t)\ell(t)(\cot\theta\mu(t) + \nu(t)))\cdot(\cos\theta\mu(t) + \sin\theta\nu(t))\\ &= \frac{\omega[\theta](t)}{\sin\theta}\ell(t).\end{aligned}$$

□

We can see $t_1 \in I$ is a singular point of the θ-involutoid $Inv(\gamma,t_0)[\theta]$ if $\omega[\theta](t_1) = 0$.

Next we give the relationships between evolutoids and involutoids.

Proposition 9 Let $t_0 \in I$ and $(\gamma,\nu): I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with curvature (ℓ,β) and without inflection points. If $\theta \neq 0$, we have

(1) $\mathcal{E}v(Inv(\gamma,t_0)[\pi-\theta])[\theta](t) = \gamma(t)$,

(2) $Inv(\mathcal{E}v(\gamma)[\theta],t_0)[\pi-\theta](t) = \gamma(t) + (\delta(t) - (\beta(t)/\ell(t))\sin\theta)(\cos\theta\mu(t) + \sin\theta\nu(t))$,

where

$$\delta(t) = e^{\int_{t_0}^t \cot\theta\ell_{\mathcal{E}}[\theta](t)dt}\int_{t_0}^t \beta_{\mathcal{E}}[\theta](t)e^{-\int_{t_0}^t \cot\theta\ell_{\mathcal{E}}[\theta](t)dt}dt.$$

Proof (1) Since $\omega[\theta](t) = e^{-\int_{t_0}^t \cot\theta\ell(t)dt}\int_{t_0}^t \beta(t)e^{\int_{t_0}^t \cot\theta\ell(t)dt}dt$, we have

$$\omega[\pi-\theta](t) = e^{\int_{t_0}^t \cot\theta\ell(t)dt}\int_{t_0}^t \beta(t)e^{-\int_{t_0}^t \cot\theta\ell(t)dt}dt.$$

By Proposition 8, we have

$$\begin{aligned}
&\mathcal{E}v[\theta](Inv(\gamma,t_0)[\pi-\theta])(t)\\
&= Inv(\gamma,t_0)[\pi-\theta](t) - \left(\frac{\beta_I[\pi-\theta](t)}{\ell_I[\pi-\theta](t)}\right)\sin\theta(\cos\theta\mu_I[\pi-\theta](t) + \sin\theta\nu_I[\pi-\theta](t))\\
&= \gamma(t) - \omega[\pi-\theta](t)\mu(t) + \frac{\omega[\pi-\theta](t)\ell(t)}{\ell(t)}(\cos^2\theta\mu(t) - \cos\theta\sin\theta\nu(t)\\
&\quad + \cos\theta\sin\theta\nu(t) + \sin^2\theta\mu(t))\\
&= \gamma(t).
\end{aligned}$$

(2) By the definition of θ-involutoids and the curvature of θ-evolutoids, we have

$$\begin{aligned}
&Inv(\mathcal{E}v(\gamma)[\theta],t_0)[\pi-\theta](t)\\
&= \mathcal{E}v(\gamma)[\theta](t) - \delta(t)\mu_{\mathcal{E}}[\theta](t)\\
&= \gamma(t) - \frac{\beta(t)}{\ell(t)}\sin\theta(\cos\theta\mu(t) + \sin\theta\nu(t)) - \delta(t)(-\cos\theta\mu(t) - \sin\theta\nu(t))\\
&= \gamma(t) + \left(\delta(t) - \frac{\beta(t)}{\ell(t)}\sin\theta\right)(\cos\theta\mu(t) + \sin\theta\nu(t)).
\end{aligned}$$

□

For the special case when $\delta(t) - (\beta(t)/\ell(t))\sin\theta = 0$, we have $Inv(\mathcal{E}v(\gamma)[\theta],t_0)[\pi-\theta](t) = \gamma(t)$.

Remark 6 When $\theta = 0$, $\mathcal{E}v(\gamma)[0](t) = \gamma(t)$ and $Inv(\gamma,t_0)[\pi](t) = \gamma(t)$. Therefore, we have

(1) $\mathcal{E}v(Inv(\gamma,t_0)[\pi])[0](t) = \gamma(t)$,
(2) $Inv(\mathcal{E}v(\gamma)[0],t_0)[\pi](t) = \gamma(t)$.

Remark 7 When $\theta = \pi/2$, $\mathcal{E}v(\gamma)[\pi/2](t) = \mathcal{E}v(\gamma)(t)$ and $Inv(\gamma,t_0)[\pi/2](t) = Inv(\gamma,t_0)(t)$. Therefore, we have

(1) $\mathcal{E}v(Inv(\gamma,t_0))(t) = \gamma(t)$,
(2) $Inv(\mathcal{E}v(\gamma),t_0)(t) = \gamma(t) - (\beta(t_0)/\ell(t_0))\nu(t)$,

see [11].

5 Normal envelopes

Normal envelopes ($\pi/2$-enveloids) are the special cases of θ-enveloids. As the corollaries of Proposition 1 and Theorem 2, we give the basic properties of normal envelopes.

Corollary 1 Let $(\gamma,\nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ, m, β, A, B). If $n : U \to I \times \Lambda,\ n(u) = (t(u), \lambda(u))$ is a pre-normal envelope and $N_\gamma = \gamma \circ n : U \to \mathbb{R}^2$ is a normal envelope of (γ,ν), respectively. Then $(N_\gamma, \mu \circ n) : U \to \mathbb{R}^2 \times S^1$ is a Legendre curve with the curvature

$$\begin{aligned} \ell_N(u) &= t'(u)\ell(n(u)) + \lambda'(u)m(n(u)), \\ \beta_N(u) &= -\lambda'(u)A(n(u)). \end{aligned}$$

Corollary 2 Let $(\gamma,\nu) : I\times\Lambda \to \mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ, m, β, A, B) and $n : U \to I \times \Lambda,\ n(u) = (t(u), \lambda(u))$ be a smooth curve which satisfies the variability condition. Then n is a pre-normal envelope of (γ,ν) if and only if

$$t'(u)\beta(n(u)) + \lambda'(u)B(n(u)) = 0$$

for all $u \in U$.

Then we give the relationships between envelopes and normal envelopes of one-parameter families of Legendre curves by using pre-envelopes and pre-normal envelopes.

Proposition 10 Let $(\gamma,\nu) : I\times\Lambda \to \mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ, m, β, A, B). Suppose that $e : U \to I \times \Lambda$ is a pre-envelope and $n : U \to I\times\Lambda$ is a pre-normal envelope of (γ,ν), respectively. If e and n intersect at a point $u_0 \in U$, then u_0 is a singular point of the normal envelope $N = \gamma \circ n$.

Proof Let $e(u) = (t_1(u), \lambda_1(u))$ and $n(u) = (t_2(u), \lambda_2(u))$. Suppose that e and n intersect at u_0, we have

$$e(u_0) = (t_1(u_0), \lambda_1(u_0)) = (t_2(u_0), \lambda_2(u_0)) = n(u_0).$$

By Theorem 2, we have $A(e(u_0)) = 0$. Then by Corollary 1, we have

$$\beta_N(u_0) = -\lambda_2'(u_0)A(n(u_0)) = 0.$$

Thus u_0 is a singular point of the normal envelope N. □

Example 3 Let $(\gamma,\nu) : [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves given by

$$\gamma(t,\lambda) = (\cos t + \lambda, \sin t),\ \ \nu(t,\lambda) = (\cos t, \sin t).$$

The curvature of (γ,ν) is given by

$$(\ell, m, \beta, A, B)(t,\lambda) = (1, 0, 1, \cos t, -\sin t).$$

Since $A(t,\lambda) = \cos t$, it follows that $e : \mathbb{R} \to [0, 2\pi) \times \mathbb{R},\ e(u) = (\pi/2, u), (3\pi/2, u)$ are pre-envelopes of (γ,ν) respectively. Therefore, the envelopes $E = \gamma \circ e : \mathbb{R} \to \mathbb{R}^2$ of (γ,ν) are given by $E(u) = \gamma \circ e(u) = (u, 1), (u, -1)$ respectively. Moreover, let $n^c : \mathbb{R} \to [0, 2\pi) \times \mathbb{R},\ n^c(u) = (2 \arctan \exp(cu), u)$, where c is a constant. Then n^c satisfies the variability condition and

$$t'(u)\beta(n^c(u)) + \lambda'(u)B(n^c(u)) = 2\frac{\exp(cu)}{1 + \exp(2cu)} - \sin(2 \arctan \exp(cu)).$$

Let $\alpha = \arctan \exp(cu)$, then $\tan \alpha = \exp(cu)$ and

$$\sin(2 \arctan \exp(cu)) = \sin 2\alpha = \frac{2 \tan \alpha}{1 + \tan^2 \alpha} = 2\frac{\exp(cu)}{1 + \exp(2cu)}.$$

Thus $t'(u)\beta(n^c(u)) + \lambda'(u)B(n^c(u)) = 0$ for all $u \in U$. By Corollary 2, n^c is a one-parameter family of pre-normal envelopes of (γ, ν) and the one-parameter family of normal envelopes $N^c = \gamma \circ n^c : \mathbb{R} \to \mathbb{R}^2$ of (γ, ν) is given by

$$N^c(u) = \gamma \circ n^c(u) = (\cos(2 \arctan \exp(cu)) + u, \sin(2 \arctan \exp(cu))).$$

For the pre-envelope $e(u) = (\pi/2, u)$ and the pre-normal envelope $n^1(u) = (2 \arctan \exp(u), u)$, we have e and n^1 intersect at $u = 0$, see Fig. 5. By Proposition 10, $u = 0$ is a singular point of N^1, see Fig. 6.

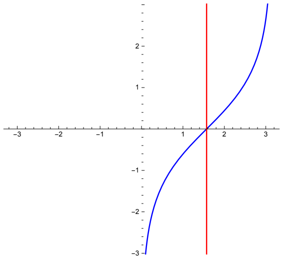

Fig. 5 The pre-envelope e (the red curve) and the pre-normal envelope n^1 (the blue curve).

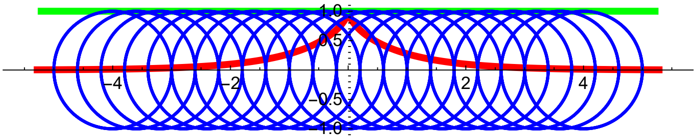

Fig. 6 γ (the blue curves), the envelope E (the green curve) and the normal envelope N^1 (the red curve).

Proposition 11 Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves. Suppose that $e : U \to I \times \Lambda$ is a pre-envelope and $n : U \to I \times \Lambda$ is a pre-normal envelope of (γ, ν), respectively. If e and n are regular and tangent to each other at $u_0 \in U$. Then u_0 is a singular point of the envelope $E = \gamma \circ e$.

Proof Let $e(u) = (t_1(u), \lambda_1(u))$ and $n(u) = (t_2(u), \lambda_2(u))$. Suppose that e tangent to n at u_0, we have a constant k such that

$$(t_1'(u_0), \lambda_1'(u_0)) = k(t_2'(u_0), \lambda_2'(u_0)), \ (t_1(u_0), \lambda_1(u_0)) = (t_2(u_0), \lambda_2(u_0)).$$

By Proposition 10, we have $N'(u_0) = (0, 0)$. Moreover,

$$E'(u_0) = t_1'(u_0)\gamma_t(t_1(u_0), \lambda_1(u_0)) + \lambda_1'(u_0)\gamma_\lambda(t_1(u_0), \lambda_1(u_0)) = kN'(u_0).$$

Therefore, we have $E'(u_0) = (0, 0)$. □

Example 4 Let $(\gamma, \nu) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves given by

$$\gamma(t, \lambda) = \left(\frac{\lambda^3}{3} - \frac{t\lambda}{\sqrt{1 + \lambda^2}}, \frac{\lambda^4}{4} + \frac{t}{\sqrt{1 + \lambda^2}} \right),$$

$$\nu(t,\lambda) = \left(-\frac{1}{\sqrt{1+\lambda^2}}, -\frac{\lambda}{\sqrt{1+\lambda^2}}\right).$$

The curvature of (γ,ν) is given by

$$(\ell, m, \beta, A, B)(t,\lambda) = \left(0, \frac{1}{1+\lambda^2}, -1, -\lambda^2\sqrt{1+\lambda^2} + t\frac{1}{1+\lambda^2}, 0\right).$$

Since $A(t,\lambda) = -\lambda^2\sqrt{1+\lambda^2} + t(1+\lambda^2)^{-1}$, we have $e : \mathbb{R} \to \mathbb{R}\times\mathbb{R}$, $e(u) = (u^2(1+u^2)^{3/2}, u)$ is a pre-envelope of (γ,ν). Therefore, the envelope $E = \gamma\circ e : \mathbb{R} \to \mathbb{R}^2$ of (γ,ν) is given by

$$E(u) = \gamma\circ e(u) = \left(\frac{u^3}{3} - u^3(1+u^2), \frac{u^4}{4} + u^2(1+u^2)\right).$$

Moreover, let $n^c : \mathbb{R} \to \mathbb{R}\times\mathbb{R}$, $n^c(u) = (c, u)$, where c is a constant. Then n^c satisfies the variability condition and

$$t'(u)\beta(n^c(u)) + \lambda'(u)B(n^c(u)) = 0$$

for all $u \in \mathbb{R}$. By Corollary 2, n^c is a one-parameter family of pre-normal envelopes of (γ,ν) and the one-parameter family of normal envelopes $N^c = \gamma\circ n^c : \mathbb{R} \to \mathbb{R}^2$ of (γ,ν) is given by

$$N^c(u) = \gamma\circ n^c(u) = \left(\frac{u^3}{3} - \frac{cu}{\sqrt{1+u^2}}, \frac{u^4}{4} + \frac{c}{\sqrt{1+u^2}}\right).$$

We take the pre-envelope $e(u) = (u^2(1+u^2)^{3/2}, u)$ and the pre-normal envelope $n^0(u) = (0, u)$ as an example. By a direct calculation,

$$e'(u) = (2u(1+u^2)^{3/2} + 3u^3\sqrt{1+u^2}, 1),\ n^{0'}(u) = (0,1),$$
$$e'(0) = (0,1),\ n^{0'}(0) = (0,1),$$

and

$$e(0) = (0,0),\ n^0(0) = (0,0).$$

Thus e and n^0 are regular and tangent to each other at $u = 0$, see Fig. 7. By Proposition 11, $u = 0$ is a singular point of E, see Fig. 8.

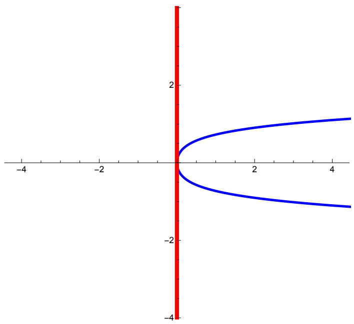

Fig. 7 The pre-envelope e (the blue curve) and the pre-normal envelope n^0 (the red curve).

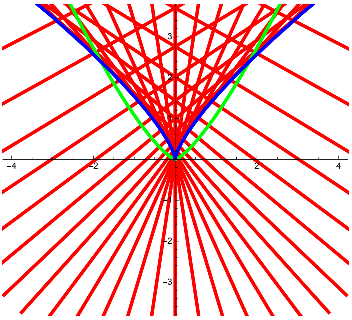

Fig. 8 γ **(the red curves), the envelope** E **(the blue curve) and the normal envelope** N^0 **(the green curve).**

Proposition 12 Let $(\gamma,\nu): I\times\Lambda \to \mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ,m,β,A,B). Suppose that $e: U\to I\times\Lambda$ is a pre-envelope and $n: U\to I\times\Lambda$ is a pre-normal envelope of (γ,ν), respectively. We assume that e and n are regular. If $\gamma_t(e(u_0))$ and $\gamma_\lambda(e(u_0))$ are linearly independent, e and n intersect with each other but do not tangent to each other at $u_0\in U$, then u_0 is a regular point of the envelope $E=\gamma\circ e$.

Proof Let $e(u)=(t_1(u),\lambda_1(u))$ and $n(u)=(t_2(u),\lambda_2(u))$. Suppose that e and n intersect with each other at u_0, we have $(t_1(u_0),\lambda_1(u_0))=(t_2(u_0),\lambda_2(u_0))$. Since e and n do not tangent to each other at u_0, we have that $(t_1'(u_0),\lambda_1'(u_0))$ and $(t_2'(u_0),\lambda_2'(u_0))$ are linearly independent. If u_0 is a singular point of E, by Proposition 1, we have

$$t_1'(u_0)\beta(t_1(u_0),\lambda_1(u_0))+\lambda_1'(u_0)B(t_1(u_0),\lambda_1(u_0))=0.$$

By Corollary 2, we have

$$t_2'(u_0)\beta(t_2(u_0),\lambda_2(u_0))+\lambda_2'(u_0)B(t_2(u_0),\lambda_2(u_0))=0.$$

Since $(t_1'(u_0),\lambda_1'(u_0))$ and $(t_2'(u_0),\lambda_2'(u_0))$ are linearly independent, we have

$$\begin{aligned}\beta(t_1(u_0),\lambda_1(u_0)) &= \gamma_t(t_1(u_0),\lambda_1(u_0))\cdot\mu(t_1(u_0),\lambda_1(u_0))=0,\\ B(t_1(u_0),\lambda_1(u_0)) &= \gamma_\lambda(t_1(u_0),\lambda_1(u_0))\cdot\mu(t_1(u_0),\lambda_1(u_0))=0.\end{aligned}$$

It means that $\gamma_t(t_1(u_0),\lambda_1(u_0))$ and $\gamma_\lambda(t_1(u_0),\lambda_1(u_0))$ are linearly dependent, which is contrary to the assumption. □

We give two examples of normal envelopes and envelopes of one-parameter families of Legendre curves that constructed by two Legendre curves.

Example 5 (cf. [24]) Let $(\gamma,\nu):[0,2\pi)\times[0,2\pi)\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves given by

$$\begin{aligned}\gamma(t,\lambda) &= \begin{pmatrix}\cos\lambda\\ \sin\lambda\end{pmatrix}+\begin{pmatrix}\cos\lambda & -\sin\lambda\\ \sin\lambda & \cos\lambda\end{pmatrix}\begin{pmatrix}\cos^3 t-1\\ \sin^3 t\end{pmatrix},\\ \nu(t,\lambda) &= \begin{pmatrix}\cos\lambda & -\sin\lambda\\ \sin\lambda & \cos\lambda\end{pmatrix}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix},\end{aligned}$$

see Fig. 9. The curvature of (γ,ν) is given by

$$(\ell,m,\beta,A,B)(t,\lambda)=(-1,1,3\sin t\cos t,2\cos^2 t-1,\sin t\cos t).$$

Let $n_k:[0,2\pi)\to[0,2\pi)\times[0,2\pi), n_k(u)=(k\pi/2,u)$ $(k=0,1,2,3)$. By Corollary 2, n_k is a family of pre-normal envelopes of (γ,ν). Therefore, the normal envelopes $N_k=\gamma\circ n_k:[0,2\pi)\to\mathbb{R}^2$ of (γ,ν) are given by

$$(\cos u,\sin u),\ (-\sin u,\cos u),\ (-\cos u,\sin u),\ (\sin u,-\cos u),$$

respectively. Actually, the normal envelopes N_k are orthogonal to each curve of the family at singular points, see Fig. 10. Moreover, let $e_k:[0,2\pi)\to[0,2\pi)\times[0,2\pi)$, $e_k(u)=((2k\pi+\pi)/4,u)$ $(k=0,1,2,3)$. By Theorem 2, e_k is a family of pre-envelopes of (γ,ν). Therefore the envelopes $E_k=\gamma\circ e_k:[0,2\pi)\to\mathbb{R}^2$ of (γ,ν) are given by

$$E_k(u)=\left(\frac{1}{2}\cos\left(u+\frac{2k\pi+\pi}{4}\right),\frac{1}{2}\sin\left(u+\frac{2k\pi+\pi}{4}\right)\right),$$

where $k=0,1,2,3$, see Fig. 10.

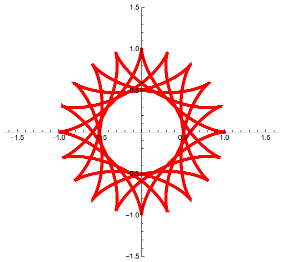

Fig. 9 One-parameter family of astroids γ (the red curves), see [24].

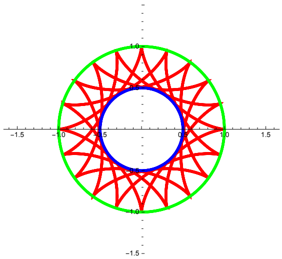

Fig. 10 γ (the red curves), the envelopes (the blue curves) and the normal envelopes (the green curves).

Example 6 Let $(\gamma,\nu):[0,2\pi)\times[0,2\pi)\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves given by

$$\gamma(t,\lambda)=\begin{pmatrix}\cos\lambda\\ \sin\lambda\end{pmatrix}+\begin{pmatrix}\cos\lambda & -\sin\lambda\\ \sin\lambda & \cos\lambda\end{pmatrix}\begin{pmatrix}1-\cos^3 t\\ \sin^3 t\end{pmatrix},$$

$$\nu(t,\lambda)=\begin{pmatrix}\cos\lambda & -\sin\lambda\\ \sin\lambda & \cos\lambda\end{pmatrix}\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix},$$

see Fig. 11. The curvature of (γ,ν) is given by

$$(\ell,m,\beta,A,B)(t,\lambda)=(1,1,-3\sin t\cos t,1-2\cos^2 t+2\cos t,-2\sin t+\sin t\cos t).$$

Since $A(t,\lambda) = 1 - 2\cos^2 t + 2\cos t$, it follows that $e : [0, 2\pi) \to [0, 2\pi) \times [0, 2\pi), e(u) = (\arccos((1 - \sqrt{3})/2), u)$ is a pre-envelope of (γ, ν) and the envelope $E = \gamma \circ e : [0, 2\pi) \to \mathbb{R}^2$ of (γ, ν) is given by

$$E(u) = \gamma \circ e(u) = \begin{pmatrix} \cos u \\ \sin u \end{pmatrix} + \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix} \begin{pmatrix} 1 - (\frac{1-\sqrt{3}}{2})^3 \\ (\frac{\sqrt{3}}{2})^{3/2} \end{pmatrix}.$$

Moreover, let $n_i : [0, 2\pi) \to [0, 2\pi) \times [0, 2\pi), n_1(u) = (0, u), n_2(u) = (\pi, u)$ respectively. By Corollary 2, n_1 and n_2 are pre-normal envelopes of (γ, ν) and the normal envelopes of (γ, ν) are given by $N_1(u) = \gamma \circ n_1(u) = (\cos u, \sin u)$ and $N_2(u) = \gamma \circ n_2(u) = (3\cos u, 3\sin u)$, respectively. Note that N_1 and N_2 are orthogonal to each curve of the family at singular points, see Fig. 12.

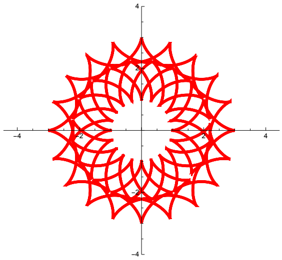

Fig. 11 One-parameter family of frontals γ (the red curves).

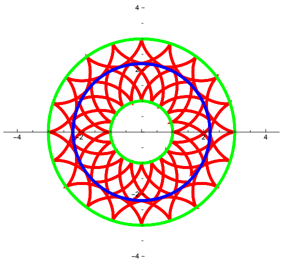

Fig. 12 γ (the red curves), the envelope E (the green curve) and the normal envelopes N_1 and N_2 (the green curves).

Evolutes and involutes of curves have been studied comprehensively both in regular condition and in singular condition [6, 8, 10, 11, 16, 25, 26]. Now we give the definitions of evolute and involute of a one-parameter family of Legendre curves.

Definition 5 Let $(\gamma, \nu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ, m, β, A, B). Suppose that $\ell(t, \lambda) \neq 0$ for all $(t, \lambda) \in I \times \Lambda$, then the evolute of (γ, ν) is given by

$$\mathcal{E}v(\gamma)(t, \lambda) = \gamma(t, \lambda) - \frac{\beta(t, \lambda)}{\ell(t, \lambda)} \nu(t, \lambda).$$

By a direct calculation, $(\mathcal{E}v(\gamma), \mu) : I \times \Lambda \to \mathbb{R}^2 \times S^1$ is a one-parameter family of Legendre curves with curvature $(\ell_{\mathcal{E}}, m_{\mathcal{E}}, \beta_{\mathcal{E}}, A_{\mathcal{E}}, B_{\mathcal{E}})$, where

$$\ell_{\mathcal{E}}(t,\lambda)=\ell(t,\lambda),\ m_{\mathcal{E}}(t,\lambda)=m(t,\lambda),\ \beta_{\mathcal{E}}(t,\lambda)=\frac{\beta_t\ell-\beta\ell_t}{\ell^2}(t,\lambda),$$
$$A_{\mathcal{E}}(t,\lambda)=B(t,\lambda)-\frac{\beta m}{\ell}(t,\lambda),\ B_{\mathcal{E}}(t,\lambda)=-A(t,\lambda)+\frac{\beta_\lambda\ell-\beta\ell_\lambda}{\ell^2}(t,\lambda).$$

Definition 6 Let $(\gamma,\nu):I\times\Lambda\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ,m,β,A,B). Then the involute of (γ,ν) at $t_0\in I$ is given by

$$Inv(\gamma,t_0)(t,\lambda)=\gamma(t,\lambda)-\left(\int_{t_0}^{t}\beta(t,\lambda)dt\right)\mu(t,\lambda).$$

By a direct calculation, $(Inv(\gamma,t_0),\mu):I\times\Lambda\to\mathbb{R}^2\times S^1$ is a one-parameter family of Legendre curves with curvature $(\ell_I,m_I,\beta_I,A_I,B_I)$, where

$$\ell_I(t,\lambda)=\ell(t,\lambda),\ m_I(t,\lambda)=m(t,\lambda),\ \beta_I(t,\lambda)=-\ell(t,\lambda)\int_{t_0}^{t}\beta(t,\lambda)dt,$$
$$A_I(t,\lambda)=B(t,\lambda)-\int_{t_0}^{t}\beta_\lambda(t,\lambda)dt,\ B_I(t,\lambda)=-A(t,\lambda)-m(t,\lambda)\int_{t_0}^{t}\beta(t,\lambda)dt.$$

Proposition 13 Let $(\gamma,\nu):I\times\Lambda\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ,m,β,A,B). We assume that $\ell(t,\lambda)\neq 0$ for all $(t,\lambda)\in I\times\Lambda$. If $n:U\to I\times\Lambda,\ n(u)=(t(u),\lambda(u))$ is a pre-normal envelope, $N_\gamma=\gamma\circ n:U\to\mathbb{R}^2$ is a normal envelope of (γ,ν) respectively and

$$t'(u)\left(\frac{\beta_t\ell-\beta\ell_t}{\ell^2}\right)\circ n(u)-\lambda'(u)A(n(u))+\lambda'(u)\left(\frac{\beta_\lambda\ell-\beta\ell_\lambda}{\ell^2}\right)\circ n(u)=0$$

for all $u\in U$, then n is also a pre-normal envelope of $(\mathcal{E}v(\gamma),\mu)$. Moreover, $N_{\mathcal{E}v(\gamma)}(u)=Inv(N_\gamma,u_0)(u)$ for all $u\in U$, where $N_{\mathcal{E}v(\gamma)}$ is the normal envelope of $(\mathcal{E}v(\gamma),\mu)$, $Inv(N_\gamma,u_0)$ is the involute of N_γ at $u_0\in U$ and u_0 satisfies $\beta(n(u_0))=0$.

Proof Since

$$t'(u)\left(\frac{\beta_t\ell-\beta\ell_t}{\ell^2}\right)\circ n(u)-\lambda'(u)A(n(u))+\lambda'(u)\left(\frac{\beta_\lambda\ell-\beta\ell_\lambda}{\ell^2}\right)\circ n(u)=0$$

for all $u\in U$, we have

$$t'(u)\beta_{\mathcal{E}}(n(u))+\lambda'(u)B_{\mathcal{E}}(n(u))=0.$$

By Corollary 2, n is a pre-normal envelope of $(\mathcal{E}v(\gamma),\mu)$. The normal envelope of $(\mathcal{E}v(\gamma),\mu)$ is given by

$$N_{\mathcal{E}v(\gamma)}(u)=\mathcal{E}v(\gamma)\circ n(u)=\gamma(n(u))-\frac{\beta(n(u))}{\ell(n(u))}\nu(n(u)).$$

On the other hand, the involute of N_γ at u_0 is given by

$$Inv(N_\gamma,u_0)(u)=\gamma(n(u))-\left(\int_{u_0}^{u}\lambda'(u)A(n(u))du\right)\nu(n(u)).$$

Since

$$t'(u)\left(\frac{\beta_t\ell-\beta\ell_t}{\ell^2}\right)\circ n(u)-\lambda'(u)A(n(u))+\lambda'(u)\left(\frac{\beta_\lambda\ell-\beta\ell_\lambda}{\ell^2}\right)\circ n(u)=0,$$

we have

$$\lambda'(u)A(n(u))=t'(u)\left(\frac{\beta_t\ell-\beta\ell_t}{\ell^2}\right)\circ n(u)+\lambda'(u)\left(\frac{\beta_\lambda\ell-\beta\ell_\lambda}{\ell^2}\right)\circ n(u)$$

for all $u\in U$. By $\beta(n(u_0))=0$, we have

$$\int_{u_0}^{u}\lambda'(u)A(n(u))du=\frac{\beta(n(u))}{\ell(n(u))}$$

for all $u\in U$. Therefore, we have $N_{\mathcal{E}v(\gamma)}(u)=Inv(N_\gamma,u_0)(u)$ for all $u\in U$. □

Proposition 14 Let $(\gamma,\nu):I\times\Lambda\to\mathbb{R}^2\times S^1$ be a one-parameter family of Legendre curves with curvature (ℓ,m,β,A,B). If $n:U\to I\times\Lambda$, $n(u)=(t(u),\lambda(u))$ is a pre-normal envelope of (γ,ν), $N_\gamma=\gamma\circ n:U\to\mathbb{R}^2$ is a normal envelope without inflection points and

$$(t'(u)\ell(n(u))+\lambda'(u)m(n(u)))\left(\int_{t_0}^{t}\beta dt\right)\circ n(u)+\lambda'(u)A(n(u))=0$$

for all $u\in U$, then n is also a pre-normal envelope of $(Inv(\gamma,t_0),\mu)$. Moreover, $N_{Inv(\gamma,t_0)}(u)=\mathcal{E}v(N_\gamma)(u)$ for all $u\in U$, where $N_{Inv(\gamma,t_0)}$ is the normal envelope of $(Inv(\gamma,t_0),\mu)$ and $\mathcal{E}v(N_\gamma)$ is the evolute of N_γ.

Proof Since

$$(t'(u)\ell(n(u))+\lambda'(u)m(n(u)))\left(\int_{t_0}^{t}\beta dt\right)\circ n(u)+\lambda'(u)A(n(u))=0$$

for all $u\in U$, we have

$$t'(u)\beta_I(n(u))+\lambda'(u)B_I(n(u))=0.$$

By Corollary 2, n is a pre-normal envelope of $(Inv(\gamma,t_0),\mu)$. The normal envelope of $(Inv(\gamma,t_0),\mu)$ is given by

$$N_{Inv(\gamma,t_0)}(u)=Inv(\gamma,t_0)\circ n(u)=\gamma(n(u))-\left(\int_{t_0}^{t}\beta dt\right)\circ n(u)\mu(n(u)).$$

Since N_γ has no inflection points, by Corollary 1, we have

$$t'(u)\ell(n(u))+\lambda'(u)m(n(u))\neq 0$$

for all $u\in U$. Then the evolute of N_γ is given by

$$\mathcal{E}v(N_\gamma)(u)=\gamma(n(u))-\frac{-\lambda'(u)A(n(u))}{t'(u)\ell(n(u))+\lambda'(u)m(n(u))}\mu(n(u)).$$

Since

$$(t'(u)\ell(n(u))+\lambda'(u)m(n(u)))\left(\int_{t_0}^{t}\beta dt\right)\circ n(u)+\lambda'(u)A(n(u))=0,$$

we have

$$\frac{-\lambda'(u)A(n(u))}{t'(u)\ell(n(u))+\lambda'(u)m(n(u))}=\left(\int_{t_0}^{t}\beta dt\right)\circ n(u)$$

for all $u\in U$. Therefore, we have $N_{Inv(\gamma,t_0)}(u)=\mathcal{E}v(N_\gamma)(u)$ for all $u\in U$. □

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Declarations

- Conflicts of interest/Competing interests
  The authors declare that there is no conflicts of interests in this work.
- Availability of data and material
  All data generated or analysed during this study are included in this article.
- Code availability (Not applicable).
- Authors' contributions
  Conceptualization, E. Li, D. Pei and M. Takahashi; Writing-Original Preparation, E. Li; Writing-Review and Editing, D. Pei and M. Takahashi; Funding Acquisition, D. Pei and M. Takahashi. All authors red and approved the final manuscript.
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